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THE EVENNESS CONJECTURE IN EQUIVARIANT UNITARY BORDISM

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Abstract

The evenness conjecture for the equivariant unitary bordism groups states that these bordism groups are free modules over the unitary bordism ring in even dimensional generators. In this paper we review the cases on which the conjecture is known to hold and we highlight the properties that permit to prove the conjecture in these cases.

Introduction

The G-equivariant unitary bordism groups for G a compact Lie group are the bordism groups of G-equivariant tangentially stable almost complex manifolds, also known as G-equivariant unitary manifolds. These are closed G-manifolds M for which a stable tangent bundle $TM \oplus \underline{\mathbb{R}}^k$, where $\underline{\mathbb{R}}^k$ denotes the trivial bundle $\mathbb{R}^k \times M$ with trivial G-action, can be endowed with the structure of a G-equivariant complex bundle. Two tangentially stable almost complex G-structures are identified if after stabilization with further G-trivial $\underline{\mathbb{C}}$ summands the structures become G-homotopic through complex G-structures. Being unitary is inherited to the fixed points sets. Whenever H is a closed subgroup of G the fixed points M^H are also tangentially stable almost complex, and moreover a N_H-tubular neighborhood around M^H in M possesses a complex N_H structure May [1996, §XVIII, Prop. 3.2].

For a cofibration of G-spaces $Y \to X$, the geometric G-equivariant unitary bordism groups $\Omega_n^G(X, Y)$ are the G-bordism classes of G-equivariant n-dimensional manifolds with map $(M^n, \partial M^n) \to (X, Y)$.

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The G-equivariant unitary bordism groups of a point Ω_*^G become a ring under the cartesian product of manifolds with the diagonal G-action, and therefore a module of the unitary bordism ring Ω_* whenever we consider a unitary manifold as a trivial G-manifold. Milnor [1960] and Novikov [1960], using Thom's remarkable calculation of the unoriented bordism groups Thom [1954], showed that the unitary bordism ring is a polinomial ring $\Omega_* = \mathbb{Z}[x_{2i} : i \geq 1]$ with one generator in each even degree. In this work we will be interested in the Ω_*-module structure of the equivariant unitary bordism groups Ω_*^G.

Explicit calculations carried out by Landweber [1972] in the cyclic case and by Stong [1970] in the abelian p-group case permitted them to conclude that in these cases the equivariant unitary bordism group Ω_*^G is a free Ω_*-module in even dimensional generators. Ossa [1972] generalized this result to any finite abelian group and Löffler [1974] and Comezaña in May [1996, §XXVIII, Thm. 5.1] showed that this also holds whenever G is a compact abelian Lie group. Explicit calculations done for the Diehadral groups D_{2p} with p prime by Ángel, Gómez, and B. Uribe [n.d.] for groups of order pq with p and q different primes by Lazarov [1972] and for metacyclic groups by Rowlett [1980] show that for these groups this phenomenon also occurs. We believe that this property should hold in the G-equivariant unitary bordism groups for any compact Lie group G, in the same way that the coefficients for G-equivariant K-theory are trivial in odd degrees and a free module over the integers in even degrees.

The theme of this work is the

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which states that the G-equivariant unitary bordism group is a free Ω_*-modules in even dimensional generators whenever G is a compact Lie group. Rowlett explicitly mentions this conjecture in his work of 1980 Rowlett [ibid.] and later Comezaña in his work of 1996 May [1996, §XXVIII.5]. We also believe that this conjecture holds in general and we do hope that this paper will help spreading it to the mathematical community for its eventual proof.

In this work we survey the original proofs of the known cases of the evenness conjecture for finite groups. We start in Section 1 with the definition of the equivariant unitary bordism groups for pairs of families and the long exact sequence associated to them. In Section 2 we review the decomposition of equivariant complex vector bundles restricted to fixed point sets done by Ángel, Gómez, and B. Uribe [n.d.] and how this decomposition allows to write the equivariant unitary bordism groups of adjacent pair of families as the bordism groups of equivariant classifying spaces. In Section 3 we review the proofs of the evenness conjecture for done by Landweber [1972] for cyclic groups, by Stong [1970] for abelian p-groups and by Ossa [1972] for general finite abelian groups. In Section 4 we review the proof of the evenness conjecture for metacyclic groups done by Rowlett [1980] and we finish with in Section 5 with some conclusions.

1 Equivariant unitary bordism for families of subgroups

To study the equivariant bordism groups Conner and Floyd introduced the study of bordism groups of manifolds with prescribed isotropy groups Conner and Floyd [1966, §5]. A family of subgroups $\mathfrak{F}$ of G is a set of subgroups of G which is closed under taking subgroups and under conjugation. The classifying space for the family $E\mathfrak{F}$ is a G-space which is terminal in the category of $\mathfrak{F}$-numerable G-spaces tom Dieck [1987, §1, Thm 6.6] and characterized by the following properties on fixed point sets: $E\mathfrak{F}^H \simeq *$ if $H \in \mathfrak{F}$ and $E\mathfrak{F}^H = \varnothing$ if $H \notin \mathfrak{F}$. This classifying space may be constructed in such a way that whenever $\mathfrak{F}' \subset \mathfrak{F}$, the induced map $E\mathfrak{F}' \to E\mathfrak{F}$ is a G-cofibration.

The equivariant unitary bordism groups for pairs of families may be defined as follows

$$\Omega_*^G[\mathfrak{F}, \mathfrak{F}'](X, A) := \Omega_*^G(X \times E\mathfrak{F}, X \times E\mathfrak{F}' \cup A \times E\mathfrak{F}'),$$

see tom Dieck [1972, p. 310], or alternatively they may be defined in a geometric way is in Stong [1970, §2].

A $(\mathfrak{F}, \mathfrak{F}')$ *free geometric unitary bordism element* of (X, A) is an equivalence class of 4-tuples (M, M_0, M_1, f), where:

- M is an n-dimensional G-manifold endowed with tangentially stable almost G structure which is moreover $\mathfrak{F}$-free, i.e. such that all isotropy groups $G_m = \{g \in G | gm = m\}$ for $m \in M$ belong to $\mathfrak{F}$, and such that $f : M \to X$ is G-equivariant; and

- M_0, M_1 are compact submanifolds of the boundary of M, with $\partial M = M_0 \cup M_1$, $M_0 \cap M_1 = \partial M_0 = \partial M_1$ having tangentially stable almost complex structure induced from M, both G-invariant, such that $f(M_1) \subset A$ and M_0 if $\mathfrak{F}'$-free, i.e. all isotropy groups of M_0 belong to $\mathfrak{F}'$.

Two four tuples (M, M_0, M_1, f) and (M', M_0', M_1', f) are equivalent if there is a 5-tuple (V, V^+, V_0, V_1, F) where

- V is a $\mathfrak{F}$-free manifold and $F : V \to X$ is a G-equivariant map;

- The boundary of V is the union of M, M' and V^+ with $M \cap V^+ = \partial M$, $M' \cap V^+ = \partial M'$, $M \cap M' = \varnothing$, $V^+ \cap (M \cup M') = \partial V^+$, with V inducing the tangentially stable almost complex G-structure on M and the opposite one on M'; V^+ is G-invariant and F restricts to f in M and to f' on M'; and

- V^+ is the union of the G-invariant submanifolds V_0, V_1 with intersection a submanifold V^- in their boundaries, such that $\partial V_i = M_i \cup V^- \cup M_i'$, $M_i \cap V^- = \partial M_i$, $M_i' \cap V^- = \partial M_i'$ with V_0 is $\mathfrak{F}'$-free and $F(V_1) \subset A$.

Definition 1.1. The set of equivalence classes of $(\mathfrak{F}, \mathfrak{F}')$-free geometric unitary bordism elements of (X, A), consisting of clases (M, M_0, M_1, f) where the dimension of M is n, and under the operation of disjoint union, forms an abelian group denoted by

$$\Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X, A).$$

Call these groups the geometric G-equivariant unitary bordism groups of the pair (X, A) restricted to the pair of families $\mathfrak{F}' \subset \mathfrak{F}$.

Note that if N is a tangentially stable almost complex closed manifold, we can define $N \cdot (M, M_0, M_1, f) = (N \times M, N \times M_0, N \times M_1, f \circ \pi_M)$ thus making $\Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X, A)$ a module over the unitary bordism ring Ω_*.

The covariant functor $\Omega_*^G\{\mathfrak{F}, \mathfrak{F}'\}$ defines a G-equivariant homology theory Stong [1970, Prop. 2.1], the boundary map on A

$$\begin{aligned} \delta : \Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X, A) &\to \Omega_{n-1}^G\{\mathfrak{F}, \mathfrak{F}'\}(A, \varnothing) \\ (M, M_0, M_1, f) &\mapsto (M_1, \partial M_1, \varnothing, f|_{M_1}) \end{aligned}$$

defines the long exact sequence in homology for pairs

$$\cdots \Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X, A) \xrightarrow{\delta} \Omega_{n-1}^G\{\mathfrak{F}, \mathfrak{F}'\}(A, \varnothing) \to \Omega_{n-1}^G\{\mathfrak{F}, \mathfrak{F}'\}(X, \varnothing) \to \cdots,$$

and for families $\mathfrak{F}'' \subset \mathfrak{F}' \subset \mathfrak{F}$, choosing the boundary which is $\mathfrak{F}'$-free

$$\begin{aligned} \partial : \Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X, A) &\to \Omega_{n-1}^G\{\mathfrak{F}', \mathfrak{F}''\}(X, A) \\ (M, M_0, M_1, f) &\mapsto (M_0, \varnothing, \partial M_0, f|_{M_0}) \end{aligned}$$

one obtains by Stong [ibid., Prop. 2.2] the long exact sequence in homology for families

$$\cdots \Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X, A) \xrightarrow{\partial} \Omega_{n-1}^G\{\mathfrak{F}', \mathfrak{F}''\}(X, A) \to \Omega_{n-1}^G\{\mathfrak{F}, \mathfrak{F}''\}(X, A) \to \cdots.$$

The bordism condition restricted to the non-relative case $\Omega_*^G\{\mathfrak{F}, \mathfrak{F}'\}(X)$ can be read as the set bordism classes of maps $f : M \to X$ such that M is $\mathfrak{F}$-free and ∂M is $\mathfrak{F}'$-free with M endowed with a tangentially stable almost complex G-structure. Two become equivalent if there exists a G-manifold $F : V \to X$ which is $\mathfrak{F}$-free such that $\partial V = M \cup M' \cup V^+$ and $M \cap V^+ = \partial M$, $M' \cap V^+ = \partial M'$, $M \cap M' = \varnothing$, $V^+ \cap (M \cup M') = \partial V^+$, with the property that F restricts to f on M and to f' on M' and with V^+ $\mathfrak{F}'$-free.

In tom Dieck [1972, Satz 3] it is shown that the canonical map that one can define

$$\mu : \Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X, A) \to \Omega_n^G[\mathfrak{F}, \mathfrak{F}'](X, A) \tag{1.2}$$

becomes a natural isomorphism of homology theories.

A key fact about the $(\mathfrak{F},\mathfrak{F}')$-free geometric unitary bordism elements of X is the following result proven in Conner and Floyd [1966, Lemma 5.2]. Whenever $(M^n, \partial M^n, f)$ is a $(\mathfrak{F},\mathfrak{F}')$-free geometric unitary bordism element of X and W^n a compact manifold with boundary regularly embedded in the interior of M^n and invariant under the G-action, such that $G_m \in \mathfrak{F}'$ for all $m \in M^n \backslash W^n$, then $[M^n, \partial M^n, f] = [W^n, \partial W_n, f|_{W^n}]$ in $\Omega_n^G\{\mathfrak{F},\mathfrak{F}'\}(X)$.

Whenever the pair of families $\mathfrak{F}' \subset \mathfrak{F}$ differ by a fixed group A, i.e. $\mathfrak{F}\backslash\mathfrak{F}' = (A)$ with (A) the set of subgroups of G conjugate to A, then the pair $(\mathfrak{F},\mathfrak{F}')$ is called *adjacent pair of families of groups*. In the case that A is normal in G a $(\mathfrak{F},\mathfrak{F}')$-free geometric unitary bordism class $[M, \partial M, f]$ of X is equivalent to $\sum_{j=1}^{l}[U_j, \partial U_j, f|_{U_j}]$ where the U_j's are disjoint G-equivariant tubular neighborhoods of the M_j^A's and these sets are the connected components of the A-fixed point set M^A. Since the normal bundle of the fixed point set M_j^A may be classified with a map to an appropriate classifying space, the groups $\Omega_*^G\{\mathfrak{F},\mathfrak{F}'\}(X)$ become isormophic to the direct sum of G/A-free equivariant unitary bordism groups of the prroduct of X^A with an appropriate classifying space (see Ángel, Gómez, and B. Uribe [n.d., Thm. 4.5]). To introduce this result we need to understand how the fixed points of universal equivariant bundles behave. This is the subject of the next section.

2 Equivariant vector bundles and fixed points

2.1 Complex representations. Let G be a compact Lie group and A a closed and normal subgroup of G fitting in the exact sequence

$$1 \to A \to G \to Q \to 1.$$

Let $\rho : A \to U(V_\rho)$ be an irreducible unitary representation of A, denote by $\mathrm{Irr}(A)$ the set of isomorphism classes of irreducible representations of A and let W be a finite dimensional complex G-representation. Then we have an isomorphism of A-representations

$$\bigoplus_{\rho\in\mathrm{Irr}(A)} V_\rho \otimes \mathrm{Hom}_A(V_\rho, W) \xrightarrow{\cong} W.$$

The group G acts on the set of A-representations

$$(g\cdot\rho)(a) := \rho(g^{-1}ag)$$

and therefore it acts on $\mathrm{Irr}(A)$. Denote by $G_\rho := \{g \in G | g\cdot\rho \cong \rho\}$ the isotropy group of the isomorphism class of ρ and denote $Q_\rho := G_\rho/A$.

If $g \cdot \rho \cong \rho$ then there exists $M \in U(V_\rho)$ such that $g \cdot \rho(a) = M^{-1}\rho(a)M$. Since this matrix M is unique up to a central element, we obtain a homomorphism $f : G_\rho \to PU(V_\rho)$ which fits in the following diagram

$$\begin{array}{ccc} A & \xhookrightarrow{\iota} & G_\rho \\ \rho \downarrow & & \downarrow f \\ U(V_\rho) & \xrightarrow{p} & PU(V_\rho). \end{array}$$

thus making V_ρ into a projective G_ρ-representation.

Define the $\mathbb{S}^1$-central extension $\widetilde{G}_\rho := f^*U(V_\rho)$ of G_ρ which fits into following diagram

$$\begin{array}{ccccc} & & \mathbb{S}^1 & & \mathbb{S}^1 \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{\widetilde{\iota}} & \widetilde{G}_\rho & \xrightarrow{\widetilde{f}} & U(V_\rho) \\ \downarrow = & & \downarrow & & \downarrow \\ A & \xrightarrow{\iota} & G_\rho & \xrightarrow{f} & PU(V_\rho). \end{array}$$

endowing V_ρ with the structure of a $\widetilde{G}_\rho$-representation where $\mathbb{S}^1$ acts by multiplication with scalars.

The vector space $\mathrm{Hom}_A(V_\rho, W)$ is also a $\widetilde{G}_\rho$-representation where for $\phi \in \mathrm{Hom}_A(V_\rho, W)$ and $\widetilde{g} \in \widetilde{G}_\rho$ we set

$$(\widetilde{g} \bullet \phi)(v) := g\phi(\widetilde{f}(\widetilde{g})^{-1}v).$$

It follows that A acts trivially on $\mathrm{Hom}_A(V_\rho, W)$ and moreover the lements of $\mathbb{S}^1$ act by multiplication of their inverse.

Hence $V_\rho \otimes \mathrm{Hom}_A(V_\rho, W)$ is a G_ρ representation, where $\mathrm{Hom}_A(V_\rho, W)$ is a $\widetilde{Q}_\rho := \widetilde{G}_\rho/A$ representation where $\mathbb{S}^1$ acts by multiplication of the inverse. Here $\widetilde{Q}_\rho$ is an $\mathbb{S}^1$-central extension of Q_ρ.

Since the isotropy group G_ρ contains the connected component of the identity in G, the index $[G : G_\rho]$ is finite and we may induce the G_ρ-representation $V_\rho \otimes \mathrm{Hom}_A(V_\rho, W)$ to G thus obtaining the following result.

Theorem 2.1. *There is a canonical isomorphism of G-representations*

$$\bigoplus_{\rho \in G \backslash \mathrm{Irr}(A)} Ind_{G_\rho}^{G} \left(V_\rho \otimes \mathrm{Hom}_A(V_\rho, W)\right) \cong W$$

where ρ runs over representatives of the orbits of the action of G on $Irr(A)$.

2.2 Equivariant complex bundles. The previous result generalizes to equivariant complex vector bundles, but prior to showing this generalization we need to recall the multiplicative induction map introduced in Bix and tom Dieck [1978, §4]. Let H be a closed subgroup of the compact Lie group G. The right adjoint to the restriction functor r_H^G from G-spaces to H-spaces is called the multiplicative induction functor and takes an H-space Y and returns the G-space

$$m_H^G(Y) := \mathrm{map}(G, Y)^H$$

of H-equivariant maps from G to Y, with G considered as an H-space by left multiplication. The G-action on $m_H^G(Y)$ is given by $(g \cdot f)(k) := f(kg)$, it is homeomorphic to the space of sections of the projection map $G \times_H Y \to G/H$ and, in the case that G/H is finite, it is homeomorphic to $[G : H]$ copies of Y.

There is a homeomorphism

$$\mathrm{map}(X, m_H^G(Y))^G \xrightarrow{\cong} \mathrm{map}(r_H^G(X), Y)^H, \quad F \mapsto (x \mapsto F(x)(1_G))$$

whose inverse maps f to $m_H^G(f) \circ p_H^G$ where $p_H^G : X \to m_H^G(r_H^G(X))$, $p_H^G(x)(g) = gx$, is the unit of the adjunction.

Now consider a G-space X on which the closed and normal subgroup A acts trivially. Take a G-equivariant complex vector bundle $p : E \to X$ and assume that E has an hermitian metric in such a way that G acts through unitary matrices on the complex fibers. For a complex A-respresentation $\rho : A \to U(V_\rho)$ denote by $\mathbb{V}_\tau$ the trivial A-vector bundle $\pi_2 : V_\rho \times X \to X$.

The complex vector bundle $\mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is a $\widetilde{Q}_\rho$-equivariant complex vector bundle where $\mathbb{S}^1$-acts on the fibers bymultiplication of the inverse, $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is a G_ρ-equivariant complex vector bundle and

$$(p_{G_\rho}^G)^* \left(m_{G_\rho}^G(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)) \right) \to X$$

is a G-equivariant complex vector bundle over X.

Theorem 2.2. *Ángel, Gómez, and B. Uribe [n.d., Thm. 2.7] Let G be a compact Lie group, A a closed and normal subgroup, X a G-space on which A acts trivially and $E \to X$ a G-equivariant complex vector bundle. Then there is an isomorphism of G-equivariant complex vector bundles*

$$\bigoplus_{\rho \in G \backslash \mathrm{Irr}(A)} (p_{G_\rho}^G)^* \left(m_{G_\rho}^G(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)) \right) \xrightarrow{\cong} E$$

where ρ runs over representatives of the orbits of the G-action on the set of isomorphism classes of A-irreducible representations.

With the same hypothesis as in the previous theorem, there is an induced decomposition in equivariant K-theory

$$K^*_G(X) \cong \bigoplus_{\rho \in G\backslash \operatorname{Irr}(A)} {}^{\widetilde{Q}_\rho} K^*_{Q_\rho}(X), \quad E \mapsto \bigoplus_{\rho \in G\backslash \operatorname{Irr}(A)} \operatorname{Hom}_A(\mathbb{V}_\rho, E)$$

where ${}^{\widetilde{Q}_\rho} K^*_{Q_\rho}(X)$ is the $\widetilde{Q}_\rho$-twisted Q_ρ- equivariant K-theory of X which is built out of the Grothendieck group of $\widetilde{Q}_\rho$-equivariant complex vector bundles over X on which the central $\mathbb{S}^1$ acts by multilpication of the fibers.

2.3 Classifying spaces. The decomposition described above can also be written at the level of classifying spaces; let us set up the notation first. Let G be a compact Lie group and $\widetilde{G}$ a $\mathbb{S}^1$-central group extension of G. Let $\widetilde{\mathbf{C}}^\infty$ be a countable direct sum of all complex irreducible $\widetilde{G}$ representations on which $\mathbb{S}^1$ acts by multiplication of their inverse. Denote by ${}^{\widetilde{G}}B_G U(n)$ the Grasmannian of n-planes of $\widetilde{\mathbf{C}}^\infty$ and denote by ${}^{\widetilde{G}}\gamma_G U(n)$ the canonical n-plane bundle over ${}^{\widetilde{G}}B_G U(n)$. The complex vector bundle

$$\mathbb{C}^n \to {}^{\widetilde{G}}\gamma_G U(n) \to {}^{\widetilde{G}}B_G U(n)$$

is a universal $\widetilde{G}$-twisted G-equivariant complex vector bundle of rank n. Denote by $\gamma_G U(n) \to B_G U(n)$ the universal G-equivariant complex vector bundle of rank n.

Take a closed subgroup A of G, let N_A denote the normalizer of A in G and $W_A := N_A/A$. Consider the fixed point set $B_G U(n)^A$ and the restriction $\gamma_G U(n)|_{B_G U(n)^A}$ of the universal bundle to this fixed point set. Take $\rho \in \operatorname{Irr}(A)$ and by the arguments above we have that

$$\operatorname{Hom}_A(\mathbb{V}_\rho, \gamma_G U(n)|_{B_G U(n)^A})$$

is a $(\widetilde{W_A})_\rho$-twisted $(W_A)_\rho$-equivariant complex bundle, but since the space $B_G U(n)^A$ is not necessarily connected, it may not have constant rank. Therefore Theorem 2.2 implies the following equivariant homotopy equivalence.

Theorem 2.3. *Ángel, Gómez, and B. Uribe [n.d., Thms. 3.3 & 3, 5] There are W_A-equivariant homotopy equivalences*

$$\coprod_{n=0}^{\infty} \gamma_G U(n)^A \simeq \left(\coprod_{n=0}^{\infty} \gamma_{W_A} U(n_1) \right) \times \prod_{\substack{\rho \in W_A\backslash \operatorname{Irr}(A) \\ \rho \neq 1}} m^{W_A}_{(W_A)_\rho} \left(\coprod_{n_\rho=0}^{\infty} {}^{(\widetilde{W_A})_\rho} B_{(W_A)_\rho} U(n_\rho) \right),$$

$$\coprod_{n=0}^{\infty} B_G U(n)^A \simeq \prod_{\rho \in W_A\backslash \operatorname{Irr}(A)} m^{W_A}_{(W_A)_\rho} \left(\coprod_{n_\rho=0}^{\infty} {}^{(\widetilde{W_A})_\rho} B_{(W_A)_\rho} U(n_\rho) \right).$$

If G is abelian then A is normal, G acts trivially on $\mathrm{Irr}(A)$ and all the irreducible representations are 1-dimensional. Therefore we get a G/A-homotopy equivalence

$$\gamma_G U(n)^A \simeq \coprod_{\substack{(n_\rho)_{\rho\in\mathrm{Irr}(A)} \\ \sum_\rho n_\rho = n}} \left(\gamma_{G/A} U(n_1) \times \prod_{\substack{\rho\in\mathrm{Irr}(A) \\ \rho\neq 1}} B_{G/A} U(n_\rho) \right).$$

In order to get a similar formula for the case on which G is not abelian we need to add up some notation and make some choices. Let $\mathcal{P}(n, A)$ be the set of arrangements of non-negative integers $(n_\rho)_{\rho\in\mathrm{Irr}(A)}$ such that

$$\sum_{\rho\in\mathrm{Irr}(A)} n_\rho |\rho| = n,$$

then non-equivariantly there is a homotopy equivalence

$$B_G U(n)^A \simeq \coprod_{(n_\rho)\in\mathcal{P}(n,A)} \prod_{\rho\in\mathrm{Irr}(A)} \left({}^{(\widetilde{W_A})_\rho} B_{(W_A)_\rho} U(n_\rho) \right).$$

The group W_A acts on $\mathcal{P}(n, A)$ on the right permuting the arrangements, i.e. the action of $g \in W_A$ on the arrangement (n_ρ) is the arrangement $(n_\rho) \cdot g := (n_{g\cdot\rho})$ meaning that it has the number $n_{g\cdot\rho}$ in the coordinate ρ. Denote by $(W_A)_{(n_\rho)}$ the isotropy group of the arrangement (n_ρ). Rearranging the terms we obtain the following W_A-equivariant homotopy equivalence

(2.4) $$B_G U(n)^A \simeq$$

$$\coprod_{(n_\rho)\in\mathcal{P}(n,A)/W_A} W_A \underset{(W_A)_{(n_\rho)}}{\times} \left(\prod_{\rho\in(W_A)_{(n_\rho)}\backslash\mathrm{Irr}(A)} m^{(W_A)_{(n_\rho)}}_{(W_A)_\rho\cap(W_A)_{(n_\rho)}} \left({}^{(\widetilde{W_A})_\rho} B_{(W_A)_\rho} U(n_\rho) \right) \right)$$

where (r_ρ) runs over representatives of the orbits of the action of W_A on $\mathcal{P}(n, A)$, and ρ runs over representatives of the orbits of the action of $(W_A)_{(n_\rho)}$ on $\mathrm{Irr}(A)$.

For the calculation of the equivariant unitary bordism of adjacent families of groups we need to consider only the arrangements of non-negative integers (n_ρ) such that the number associated to the trivial representation is zero, i.e. $n_1 = 0$. Denote by $\overline{\mathcal{P}}(n, A)$ the set of arrangements (n_ρ) such that $n_1 = 0$ and define the W_A space:

(2.5) $$C_{N_A,A}(k) :=$$

$$\coprod_{(n_\rho)\in\overline{\mathcal{P}}(k,A)/W_A} W_A \underset{(W_A)_{(n_\rho)}}{\times} \left(\prod_{\substack{\rho\in(W_A)_{(n_\rho)}\backslash\mathrm{Irr}(A) \\ \rho\neq 1}} m^{(W_A)_{(n_\rho)}}_{(W_A)_\rho\cap(W_A)_{(n_\rho)}} \left({}^{(\widetilde{W_A})_\rho} B_{(W_A)_\rho} U(n_\rho) \right) \right)$$

Therefore we have the following W_A-homotopy equivalence

$$\gamma_G U(n)^A \simeq \bigsqcup_{k=0}^{n} \gamma_{W_A} U(n-k) \times C_{N_A,A}(k) \tag{2.6}$$

such that in the case that G is abelian we have the simple formula

$$C_{G,A}(k) = \bigsqcup_{(n_\rho)\in\overline{\mathcal{P}}(k,A)} \prod_{\substack{\rho\in \mathrm{Irr}(A)\\ \rho\neq 1}} B_{G/A}U(n_\rho). \tag{2.7}$$

Now we are ready to state the relation between the G-equivariant unitary bordism groups of adjacent pair of families of groups and the classifying spaces defined above.

Theorem 2.8. *Ángel, Gómez, and B. Uribe [n.d., Cor. 4.6] Let G be a finite group, X a G-space and $(\mathfrak{F}, \mathfrak{F}')$ an adjacent pair of families differing by the conjugacy class of the subgroup A, then there is an isomorphism*

$$\Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X) \cong \bigoplus_{0\leq 2k\leq n} \Omega_{n-2k}^{W_A}\{\{1\}\}(X^A \times C_{N_A,A}(k))$$

where $\{1\}$ is the family of subgroups of W_A which only contains the trivial group.

Take a bordism class $[M, \partial M, f : M \to X]$ in $\Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X)$ and note that $M^A \cap M^{gAg^{-1}} = \varnothing$ whenever g does not belong to N_A. Then choose a N_A-equivariant tubular neighborhood U of M^A such that its G-orbit $G\cdot U$ is a G-equivariant tubular neighborhood of $G \cdot M^A$ and such that

$$G \underset{N_A}{\times} U \xrightarrow{\cong} G \cdot U, \quad [(g,u)] \to gu$$

is a G-equivariant diffeomorphism. The assignment

$$[M, \partial M, f : M \to X] \mapsto [U, \partial U, f|_U : U \to X]$$

induces an isomorphism

$$\Omega_n^G\{\mathfrak{F}, \mathfrak{F}'\}(X) \xrightarrow{\cong} \Omega_n^{N_A}\{\mathfrak{F}|_{N_A}, \mathfrak{F}'|_{N_A}\}(X).$$

Let M^A_{n-2k} denote the component of M^A which is a $(n-2k)$-dimensional W_A-free manifold and such that $M^A = \bigcup_{0\leq 2k\leq n} M^A_{n-2k}$. The tubular neighborhood U is N_A-equivariantly diffeomorphic to $\bigcup_{0\leq 2k\leq n} D(\nu_{n-2k})$ where $\nu_{n-2k} \to M^A_{n-2k}$ is the normal bundle of the inclusion $M^A_{n-2k} \to M$. Since the trivial A-representation does not appear on the fibers of the normal bundles, by Theorem 2.3 and formula (2.6) we know that the bundle ν_{n-2k} is classified by a W_A-equivariant map $h_{n-2k} : M^A_{n-2k} \to C_{N_A,A}(k)$. The bordism class $[M^A_{n-2k}, f|_{M^A_{n-2k}} \times h_{n-2k} : M^A_{n-2k} \to X^A \times C_{N_A,A}(k)]$ belongs to $\Omega^{W_A}_{n-2k}\{\{1\}\}(X^A \times C_{N_A,A}(k))$ and the assignment

$$[M, \partial M, f : M \to X] \mapsto \bigoplus_{0 \le 2k \le n} [M^A_{n-2k}, f|_{M^A_{n-2k}} \times h_{n-2k} : M^A_{n-2k} \to X^A \times C_{N_A,A}(k)]$$

induces the desired isomorphism.

3 The evenness conjecture for finite abelian groups

In this section we will outline the main ingredients that used Landweber [1972] in the cyclic group case, Stong [1970] in the p-group case and Ossa [1972] in the general case to show that evenness conjecture holds for finite abelian groups. The conjecture also holds for compact abelian groups, Löffler [1974] showed it for the homotopic G-equivariant unitary bordism groups in the case that G is a unitary torus, and Comezaña in May [1996, §XXVIII] generalized it to any compact abelian group. Comezaña furthermore showed that the map from the G-equivariant unitary bordism groups to the homotopic ones is injective whenever G is compact abelian thus proving the evenness conjecture for any compact abelian group. In this work we will address the finite group case.

Prior to addressing the study of the G-equivariant unitary bordism groups for finite abelian groups we need to recall some results on the unitary bordism groups.

Thom's remarkable Theorem 1954 shows that the unitary bordism groups Ω_* can be calculated as the stable homotopy groups $\lim_k \pi_{n+k}(MU(k))$ of the Thom spaces $MU(k)$ of the canonical complex vector bundles over $BU(k)$. Milnor [1960, Thm. 3] showed that the these stable homotopy groups are zero if n is odd and free abelian if n is even with a number of generators equal to the number of partitions of $n/2$. Indepently Novikov [1960, Thm. 4] showed that as a ring the unitary bordism groups are isomorphic to the ring of polynomials over the integers with generators x_{2i} of degree $2i$ for $i \geq 1$. The spectrum MU that the Thom spaces $MU(k)$ defines permitted Atiyah [1961] to define the homotopy unitary bordism groups $MU_*(X)$ and the homotopy unitary cobordism groups $MU^*(X)$ of a space X as a generalized homology and cohomology theory respectively. Thom's theorem implies that for X a CW-complex the unitary bordism groups over X are equivalent to the homotopic ones $\Omega_*(X) \cong MU_*(X)$ via the Thom-Pontrjagin map.

The spectral sequence of Atiyah and Hirzebruch [1961] (cf. Kochman [1996, §4.2]) applied to the unitary bordism groups of a CW-complex X produces a spectral sequence which converges to $\Omega_*(X)$ and whose second page is $E^2_{p,q} \cong H_p(X; \Omega_q)$; let us call this spectral sequence the *bordism spectral sequence*. The Thom homomorphism

$$\mu : \Omega_*(X) \to H_*(X;\mathbb{Z}), \;\; [M, f : M \to X] \mapsto f_*[M],$$

which takes a unitary bordism element in X and maps it to he image under f of the fundamental class $[M] \in H_*(M;\mathbb{Z})$, is a natural transformation of homology theories and is also the edge homomorphism $\Omega_*(X) \to E^2_{*,0} \cong H_*(X;\mathbb{Z})$ of the spectral sequence.

Whenever X is a CW-complex whose homology $H_*(X;\mathbb{Z})$ is free abelian then by Conner and Smith [1969, Lemma 3.1] the bordism spectral sequence collapses, the unitary bordism group $\Omega_*(X)$ is a free Ω_*-module and the homomorphism induced by Thom map

$$\widetilde{\mu} : \mathbb{Z} \underset{\Omega_*}{\otimes} \Omega_*(X) \to H_*(X;\mathbb{Z})$$

is an isomorphism.

Applying the bordism spectral sequence to the unitary bordism groups of $BU(n)$ it is shown in Kochman [1996, Prop. 4.3.3] that $\Omega_*(BU(n))$ is a free Ω_*-module with basis

$$\Omega_*(BU(n)) \cong \Omega_*\{\alpha_{k_1}\alpha_{k_2}\dots\alpha_{k_n} : k_1 \leq \dots \leq k_n\}$$

where $\alpha_{k_1}\alpha_{k_2}\dots\alpha_{k_n}$ is the unitary bordism class of the bordism element

$$(\mathbb{C}P^{k_1} \times \dots \times \mathbb{C}P^{k_n}, F : \mathbb{C}P^{k_1} \times \dots \times \mathbb{C}P^{k_n} \to BU(n))$$

where the map F classifies the canonical rank n complex vector bundle over the product of projective spaces.

In Conner and Smith [1969, Prop. 3.6] it is shown that if X is a finite CW-complex such that the Thom homomorphism is surjective then the bordism spectral sequence collapses. Whenever BG is the classifying space of a finite group G Landweber [1971, Thm. 3] showed that the following conditions are equivalent:

- The Thom homomorphism $\mu : \Omega_*(BG) \to H_*(BG;\mathbb{Z})$ is surjective.
- The bordism spectral sequence collapses.
- G has periodic cohomology, i.e. every abelian subgroup of G is cyclic.
- $H^n(BG;\mathbb{Z}) = 0$ for all odd n.
- The projective dimension of $\Omega_*(BG)$ as a Ω_*-module is 1 or 0.

The previous result implies that whenever we consider the cyclic group $G = \mathbb{Z}/k$ of order k, the bordism classes $[L^{2n+1}(k), \iota : L^{2n+1}(k) \to B\mathbb{Z}_k]$ of the Lens spaces $L^{2n+1}(k) := S^{2n+1}_k/(\mathbb{Z}/k)$, where S^{2n+1}_k denotes the sphere of unit vectors in $\mathbb{C}^{n+1}$

with the $\mathbb{Z}/k$-action given by multiplication of the root of unity $e^{\frac{2\pi i}{k}}$, generate $\Omega_*(B\mathbb{Z}_k)$ as a Ω_*-module.

One property of finite abelian groups that will be used is the following. If A is a subgroup of a finite abelian group G and Γ is a product of classifying spaces of the form $B_G U(k)$, then by Theorem 2.3 the fixed point set Γ^A is a product of classifying spaces of the form $B_{G/A} U(l)$. This fact permits to use an induction hypothesis when calculating the equivariant unitary bordism groups of products of spaces of the form $B_G U(k)$.

Now when can start with the proof of the evenness conjecture for finite abelian groups. First we will handle the case of cyclic p-groups following Landweber [1972], then we will review the case of general abelian p-groups following Stong [1970] and we will show the general case using a simple argument on localization shown in Ossa [1972].

3.1 Cyclic p-groups. Let G be a cyclic group of order p^s a power of the prime p. Let $\Gamma := \prod_{i=1}^{l} B_G U(k_i)$ be a product of spaces of the form $B_G U(k)$ and $\mathfrak{F}_t = \{H \subset G : |H| \leq p^t\}$ the family of of subgroups or order bounded by p^t; the family $\mathfrak{F}_s$ is the family of all subgroups of G and therefore $\Omega_*^G(\,) = \Omega_*^G\{\mathfrak{F}_s\}(\,)$. Let us split $\Omega_*^G(\,) = \Omega_+^G(\,) \oplus \Omega_-^G(\,)$ where $\Omega_+^G(\,)$ denotes the even degree bordism groups and $\Omega_-^G(\,)$ the odd degree ones. We will prove by induction on the size of the group that for any $0 \leq t < s$ the following properties hold:

- $\Omega_*^G\{\mathfrak{F}_s, \mathfrak{F}_t\}(\Gamma)$ is a free Ω_*-module in even dimensional generators.
- $\Omega_+^G\{\mathfrak{F}_t, \mathfrak{F}_{t-1}\}(\Gamma)$ is a free Ω_*-module.
- The boundary homomorphism is surjective

$$\Omega_+^G\{\mathfrak{F}_s, \mathfrak{F}_t\}(\Gamma) \xrightarrow{\partial} \Omega_-^G\{\mathfrak{F}_t, \mathfrak{F}_{t-1}\}(\Gamma).$$

Let us see that these properties imply that $\Omega_*^G(\Gamma)$ is a free Ω_*-module in even dimensional generators. Since $\Omega_*^G\{\mathfrak{F}_s, \mathfrak{F}_0\}(\Gamma)$ is a free Ω_*-module in even dimensional generators, the long exact sequence associated to the families of groups $\mathfrak{F}_0 \subset \mathfrak{F}_s$ induce the exact sequence

$$0 \to \Omega_+^G\{\mathfrak{F}_0\}(\Gamma) \to \Omega_+^G(\Gamma) \to \Omega_+^G\{\mathfrak{F}_s, \mathfrak{F}_0\}(\Gamma) \xrightarrow{\partial} \Omega_-^G\{\mathfrak{F}_0\}(\Gamma) \to \Omega_-^G(\Gamma) \to 0.$$

The unitary bordism group of free actions $\Omega_*^G\{\mathfrak{F}_0\}(\Gamma)$ is isomorphic to $\Omega_*(BG \times \prod_{i=1}^{l} BU(k_i))$ since both $EG \times B_G U(k_i)$ and $EG \times BU(k_i)$ classify G-equivariant complex vector bundles of rank k_i over free G-spaces. The unitary bordism groups of

$BU(k_i)$ are free Ω_*-modules in even dimensional generators, and therefore by the Kunneth theorem we have that

$$\Omega^G_*\{\mathfrak{F}_0\}(\Gamma) \cong \Omega_*(BG) \underset{\Omega_*}{\otimes} \Omega_*\left(\prod_{i=1}^{l} BU(k_i)\right).$$

Hence we have that $\Omega^G_+\{\mathfrak{F}_0\}(\Gamma)$ is a free Ω_*-module in even degrees and that $\Omega^G_-\{\mathfrak{F}_0\}(\Gamma)$ is all p-torsion. Consider a unitary bordism class defined by the map $h : M \to \prod_{i=1}^{l} BU(k_i)$ and denote by $E := E_1 \oplus \cdots \oplus E_l$ with E_j the complex vector bundle that the map $\pi_j \circ h : M \to BU(k_j)$ defines. Take the ball $B^{2n+2}_{p^s}$ of vectors in $\mathbb{C}^{n+1}$ with norm less than 1 endowed with the action of G given by multiplication by $e^{\frac{2\pi i}{p^s}}$ and consider the G-equivariant $\prod_{i=1}^{l} U(k_i)$ complex bundle that the product $B^{2n+2}_{p^s} \times E \to B^{2n+2}_{p^s} \times M$ defines.

This G-equivariant $\prod_{i=1}^{l} U(k_i)$ complex bundle is classified by a G-equivariant map

$$f : B^{2n+2}_{p^s} \times M \to \Gamma$$

and its G-equivarian unitary bordism class $[B^{2n+2}_{p^s} \times M, f]$ belongs to $\Omega^G_+\{\mathfrak{F}_s, \mathfrak{F}_0\}(\Gamma)$. Its boundary is $[S^{2n+1}_{p^s} \times M, f|_{S^{2n+1}_{p^s} \times M}]$ and it belongs to $\Omega^G_-\{\mathfrak{F}_0\}(\Gamma)$. By the Kunneth isomorphism described above we know that the unitary bordism classes $[S^{2n+1}_{p^s} \times M, f|_{S^{2n+1}_{p^s} \times M}]$ generate $\Omega^G_-\{\mathfrak{F}_0\}(\Gamma)$ and therefore the boundary homomorphism $\Omega^G_+\{\mathfrak{F}_s, \mathfrak{F}_0\}(\Gamma) \xrightarrow{\partial} \Omega^G_-\{\mathfrak{F}_0\}(\Gamma)$ is surjective. This implies that $\Omega^G_-(\Gamma)$ is trivial.

Since $\Omega^G_+\{\mathfrak{F}_0\}(\Gamma) \cong \Omega_+(\prod_{i=1}^{l} BU(k_i))$ is a free Ω_*-module, and by hypothesis $\Omega^G_+\{\mathfrak{F}_s, \mathfrak{F}_0\}$ also, then it implies that $\Omega^G_+(\Gamma)$ is a free Ω_*-module.

In particular we have that the G-equivariant unitary bordism group Ω^G_* is a free Ω_*-module in even dimensional generators.

Now let us sketch the proof of the properties cited above. Let us assume that the properties hold for cyclic groups of order less than p^s and let us proceed by induction on the families of subgroups of G. For the adjacent pair of families $(\mathfrak{F}_s, \mathfrak{F}_{s-1})$ differing by the group G, we know by Theorem 2.8 that $\Omega^G_*\{\mathfrak{F}_s, \mathfrak{F}_{s-1}\}(\Gamma)$ is a direct sum of groups $\Omega_*(\Gamma^G \times \Gamma')$ where both Γ^G and Γ' are products of classifying spaces of unitary groups. Therefore $\Omega^G_*\{\mathfrak{F}_s, \mathfrak{F}_{s-1}\}(\Gamma)$ is a free Ω_*-module in even dimensional generators and we have started our induction.

Now let us assume that the properties hod for the pair of families $(\mathfrak{F}_s, \mathfrak{F}_j)$ for $s > j \geq t$. Therefore we get the following exact sequence of groups

$$\begin{aligned} 0 \to \Omega^G_+\{\mathfrak{F}_t, \mathfrak{F}_{t-1}\}(\Gamma) \to \Omega^G_+\{\mathfrak{F}_s, \mathfrak{F}_{t-1}\}(\Gamma) \to \Omega^G_+\{\mathfrak{F}_s, \mathfrak{F}_t\}(\Gamma) \xrightarrow{\partial} \\ \Omega^G_-\{\mathfrak{F}_t, \mathfrak{F}_{t-1}\}(\Gamma) \to \Omega^G_-\{\mathfrak{F}_s, \mathfrak{F}_{t-1}\}(\Gamma) \to 0. \end{aligned} \tag{3.1}$$

Since the pair of families $(\mathfrak{F}_t, \mathfrak{F}_{t-1})$ differ by the cyclic group H or order p^t, G/H is a cyclic group of order p^{s-t}, and Γ^H is a product of classifying spaces of the form $B_{G/H}U(k)$, then by Theorem 2.8 there is an isomorphism

$$\Omega^G_*\{\mathfrak{F}_t, \mathfrak{F}_{t-1}\}(\Gamma) \cong \bigoplus_{k\geq 0} \Omega^{G/H}_{*-2k}\{\mathfrak{F}_0\}(\Gamma^H \times C_{G,H}(k))$$

where both Γ^H and $C_{G,H}(k)$ are disjoint unions of products of classifying spaces of the form $B_{G/H}U(k)$.

Therefore we know that $\Omega^G_+\{\mathfrak{F}_t, \mathfrak{F}_{t-1}\}(\Gamma)$ is a free Ω_*-module and by the induction hypothesis we know that the boundary map

$$\Omega^{G/H}_+\{\mathfrak{F}_{s-t}, \mathfrak{F}_0\}(\Gamma^H \times C_{G,H}(k)) \xrightarrow{\partial} \Omega^{G/H}_-\{\mathfrak{F}_0\}(\Gamma^H \times C_{G,H}(k)) \tag{3.2}$$

is surjective. A bordism class in $\Omega^G_-\{\mathfrak{F}_t, \mathfrak{F}_{t-1}\}(\Gamma)$ can be represented by a class $[D(E), f : D(E) \to \Gamma]$ where $D(E)$ is the disk bundle of a G-equivariant vector bundle $E \to M$ over a manifold M on which H acts trivially and G/H acts freely, and such that the trivial representation of H does not appear on the fibers of E. This bundle is classified by a G/H-equivariant map $h : M \to C_{G,H}(k)$ for some k, and the bordism class $[M, f|_M \times h : M \to \Gamma^H \times C_{G,H}(k)]$ lives in $\Omega^{G/H}_-\{\mathfrak{F}_0\}(\Gamma^H \times C_{G,H}(k))$. By the surjectivicty of (3.2) there is a bordism class $[Z, F \times \tilde{h} : Z \to \Gamma^H \times C_{G,H}(k)]$ in $\Omega^{G/H}_+\{\mathfrak{F}_{s-t}, \mathfrak{F}_0\}(\Gamma^H \times C_{G,H}(k))$ such that $\partial Z = M$, $F|_M = f|_M$ and $\tilde{h}|_M = h$. Let $p : V \to Z$ denote the G-equivariant vector bundle over Z that the map $\tilde{h}$ defines and note that the bordism class $[D(V), F \circ p : D(V) \to \Gamma]$ defines an element in $\Omega^G_*\{\mathfrak{F}_s, \mathfrak{F}_t\}(\Gamma)$ since the trivial H-representation does not appear on the fibers of V and the action of G/H over M is free. The boundary of $D(V)$ is the union of the sphere bundle $S(V)$ and $D(V)|_M = D(E)$, but since $S(V)$ is $\mathfrak{F}_{t-1}$-free we have that

$$\begin{aligned}\partial[D(V), F \circ p : D(V) \to \Gamma] &= [D(E), p|_E \circ f|_M : D(E) \to \Gamma] \\ &= [D(E), f : D(E) \to \Gamma]\end{aligned}$$

and therefore the boundary map

$$\Omega^G_+\{\mathfrak{F}_s, \mathfrak{F}_t\}(\Gamma) \xrightarrow{\partial} \Omega^G_-\{\mathfrak{F}_t, \mathfrak{F}_{t-1}\}(\Gamma)$$

is surjective.

By the long exact sequence of (3.1) we deduce that $\Omega^G_*\{\mathfrak{F}_s, \mathfrak{F}_{t-1}\}(\Gamma)$ is a free Ω_*-module on even-dimensional generators and we conclude that the properties also hold for the pair of families $(\mathfrak{F}_s, \mathfrak{F}_{t-1})$.

Therefore the evenness conjecture holds for cyclic p-groups Landweber [1972, Thm. 1’].

3.2 General p-groups. The argument to show the evenness conjecture for general p-groups is more elaborate than the one done above for cyclic p-groups. We will follow the original proof of Stong [1970] on which the author uses very cleverly the Thom isomorphism and the long exact sequence for pairs of spaces in order to understand the long exact sequence for a pair of families once restricted to a special kind of actions on manifolds. Here we shorten the original proof and we highlight its main ingredients.

Let $G = H \times \mathbb{Z}/q$ with $q = p^s$ such that all elements in H have order less or equal than p^s and let

$$\Gamma := \prod_{i=1}^{l} B_G U(k_i)$$

be a product of spaces of the form $B_G U(k)$. We will show by induction on the order of the group G that the bordism group $\Omega_*^G(\Gamma)$ is a free Ω_*-module on even dimensional generators. Therefore let us assume that $\Omega_*^K(\Gamma')$ is a free Ω_*-module in even dimensional generators for all p-groups of order less than the order of G and Γ' any product of classifying spaces of the form $B_K U(l)$.

Following the notation of Stong [ibid.] let us consider the following families of subgroups of G:

- $\mathfrak{F}_a$ is the family of all subgroups of G,
- $\mathfrak{F}_s$ is the family of subgroups whose intersection with $\mathbb{Z}/q$ is proper, i.e. $\mathfrak{F}_s := \{W \subset H \times Z/q \colon \{1\} \times Z/q \not\subset W\}$,
- $\mathfrak{F}_f$ is the family of subgroups whose intersection with $\mathbb{Z}/q$ is the unit subgroup, i.e. $\mathfrak{F}_f := \{W \subset H \times Z/q \colon \{1\} \times Z/q \cap W = \{(1,1)\}$

A manifold M is $\mathfrak{F}_s$-free if for every $x \in M$ the isotropy group $(\mathbb{Z}/q)_x \neq \mathbb{Z}/q$, and it is $\mathfrak{F}_f$-free if the restriction of the action to $\mathbb{Z}/q$ is free.

The classifying space $E\mathfrak{F}_f$ has a free $\mathbb{Z}/q$-action and can be understood as the universal H-equivariant $\mathbb{Z}/q$-principal bundle $E_H\mathbb{Z}/q$ Lück and Uribe [2014, Thm 11.4] . Hence $E\mathfrak{F}_f = E_H\mathbb{Z}/q$ and its quotient $E\mathfrak{F}_f/(\mathbb{Z}/q) = B_H\mathbb{Z}/q$ is the classifynig space of H-equivariant $\mathbb{Z}/q$-principal bundles. By the isomorphism of (1.2), and since the action of $\mathbb{Z}/q$ is free, we get the following isomorphisms (see Stong [1970, Prop. 3.1]):

$$\Omega_*^G\{\mathfrak{F}_f\}(X) \cong \Omega_*^G(X \times E_H\mathbb{Z}/q) \cong \Omega_*^H(X \times_{\mathbb{Z}/q} E_H\mathbb{Z}/q). \tag{3.3}$$

Since both spaces $E_H\mathbb{Z}/q \times B_G U(k_i)$ and $E_H\mathbb{Z}/q \times B_H U(k_i)$ classify $H \times \mathbb{Z}/q$-equivariant $U(k_i)$-principal bundles over spaces with free $\mathbb{Z}/q$-action, we may take the maps

$$B_H U(k_i) \to B_G U(k_i) \to B_H U(k_i),$$

where the left hand side map classifies the G-equivariant complex bundles such that the action of $\mathbb{Z}/q$ is trivial over the total space of the bundle, and the right hand side is the one that forgets the $\mathbb{Z}/q$-action, thus producing G-equivariant homotopy equivalences.

$$E_H\mathbb{Z}/q \times E_H U(k_i) \xrightarrow{\simeq} E_H\mathbb{Z}/q \times E_G U(k_i) \xrightarrow{\simeq} E_H\mathbb{Z}/q \times E_H U(k_i).$$

If we denote by

$$\Gamma' := \prod_{i=1}^{l} B_H U(k_i)$$

and the map $\iota : \Gamma' \to \Gamma$ is the one that classifies trivial $\mathbb{Z}/q$-bundles over H-spaces, then the argument above implies that the following isomorphism holds (see Stong [ibid., Prop. 3.2]) :

$$\Omega_*^H\left(B_H\mathbb{Z}/q \times \Gamma'\right) \cong \Omega_*^G\left(E_H\mathbb{Z}/q \times \Gamma'\right) \underset{\cong}{\xrightarrow{\iota_*}} \Omega_*^G\{\mathfrak{F}_f\}(\Gamma). \tag{3.4}$$

Let T be the generator of the group $\mathbb{Z}/q$ and denote by $\mathbb{Z}/p^t$ the subgroup generated by $T^{p^{s-t}}$. A manifold is $\mathfrak{F}_f$-free if and only if $T^{p^{s-1}}$ acts freely and therefore a $(\mathfrak{F}_a, \mathfrak{F}_f)$-manifold M can be localized to the normal bundle of the fixed point set $M^{\mathbb{Z}/p}$ of the subgroup $\mathbb{Z}/p$. The normal bundle is a G-equivariant complex bundle over the trivial $\mathbb{Z}/p$ space and once it is classified to the appropriate spaces $C_{G,\mathbb{Z}/p}(k)$ of (2.5) we obtain the isomorphism (see Stong [ibid., Prop. 3.4]):

$$\Omega_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(X) \cong \bigoplus_{0\leq 2k \leq *} \Omega_{*-2k}^{G/(\mathbb{Z}/p)}\left(X^{\mathbb{Z}/p} \times C_{G,\mathbb{Z}/p}(k)\right). \tag{3.5}$$

Applying the previous isomorphism to $\Gamma = \prod_{i=1}^{l} B_G U(k_i)$ we obtain that

$$\Omega_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma)$$

is a free Ω_*-module in even dimensional generators since both $\Gamma^{\mathbb{Z}/p}$ and $C_{G,\mathbb{Z}/p}(k)$ are products of spaces of the form $B_{G/(\mathbb{Z}/p)}U(l)$ and by induction we assumed that the evenness conjecture was true for groups of order less than the one of G and spaces of this type. Therefore the long exact sequence for the pair of families $(\mathfrak{F}_a, \mathfrak{F}_f)$ becomes:

$$0 \to \Omega_+^G\{\mathfrak{F}_f\}(\Gamma) \to \Omega_+^G(\Gamma) \to \Omega_+^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma) \xrightarrow{\partial} \Omega_-^G\{\mathfrak{F}_f\}(\Gamma) \to \Omega_-^G(\Gamma) \to 0. \tag{3.6}$$

A $(\mathfrak{F}_a, \mathfrak{F}_s)$-free manifold M once restricted to the action of $\mathbb{Z}/q$ becomes a $\mathbb{Z}/q$-manifold on which the boundary has no fixed points of the whole group. Therefore the manifold can be localized on the normal bundle of the fixed point set $M^{\mathbb{Z}/q}$ and the information of th normal bundle can be recorded with appropriate maps to the classifying spaces $C_{G,\mathbb{Z}/q}(k)$ of (2.5). Following the same proof as in Theorem 2.8 one obtains the following isomorphism (see Stong [1970, Prop. 3.3]):

$$\Omega_*^G\{\mathfrak{F}_a,\mathfrak{F}_s\}(X) \cong \bigoplus_{0\le 2k\le *} \Omega^H_{*-2k}\left(X^{\mathbb{Z}/q}\times C_{G,\mathbb{Z}/q}(k)\right). \tag{3.7}$$

Since both $\Gamma^{\mathbb{Z}/q}$ and $C_{G,\mathbb{Z}/q}(k)$ are products of spaces of the form $B_H U(l)$, by the induction hypothesis we obtain that $\Omega_*^G\{\mathfrak{F}_a,\mathfrak{F}_s\}(\Gamma)$ is a free Ω_*-module in even dimensional generators.

In order to understand the image of the boundary map of (3.6) Stong restricted the equivariant bordism groups to manifolds with a special type of G action. Stong noticed that the image of the boundary map could be determined by restricting to manifolds on which the $\mathbb{Z}/q$-fixed points are of codimension 2 and therefore he studied the class of *special G actions*.

Definition 3.8. Let $G = H\times\mathbb{Z}/q$ be a finite abelian group. The class of *special G actions* is the collection of G-equivariant unitary manifolds M satisfying:

- The restriction to a $\mathbb{Z}/q$-action is semi-free, i.e. for each $x\in M$ the isiotropy group $(\mathbb{Z}/q)_x$ is either $\mathbb{Z}/q$ or $\{1\}$.
- The set $M^{\mathbb{Z}/q}$ of fixed point sets has codimension 2 in M and $\mathbb{Z}/q$ acts in the normal bundle of $M^{\mathbb{Z}/q}$ so that the generator T of $\mathbb{Z}/q$ acts by multiplication by $e^{\frac{2\pi i}{q}}$, or the fixed point set $M^{\mathbb{Z}/q}$ is empty.

The class of special G actions is sufficiently large to permit all constructions done in Section 1, and for a pair of families $(\mathfrak{F},\mathfrak{F}')$ in G we denote by $\overline{\Omega}_*^G\{\mathfrak{F},\mathfrak{F}'\}$ the equivariant homology theory defined by using only special G actions. The inclusion of special G actions in the class of all G actions defines natural transformations of homology theories

$$I_*\colon \overline{\Omega}_*^G\{\mathfrak{F},\mathfrak{F}'\}\to\Omega_*^G\{\mathfrak{F},\mathfrak{F}'\}$$

preserving the relations between these functors. The G-equivariant unitary bordism groups of special G actions satisfy the following properties:

(i) The natural transformation

$$I_*\colon \overline{\Omega}_*^G\{\mathfrak{F}_f\}\xrightarrow{\cong}\Omega_*^G\{\mathfrak{F}_f\}$$

is an equivalence since every $\mathfrak{F}_f$ action is a special G action.

(ii) The inclusion $(\mathfrak{F}_a, \mathfrak{F}_f) \subset (\mathfrak{F}_a, \mathfrak{F}_s)$ induces an isomorphism

$$\overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(X) \xrightarrow{\cong} \overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_s\}(X)$$

since $\mathfrak{F}_s$-free special G actions are $\mathfrak{F}_f$-free.

(iii) From the equation (3.7) we get the isomorphism

$$\overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_s\}(X) \cong \Omega_{*-2}^H(X^{\mathbb{Z}/q} \times B_H U(1)),$$

thus implying that $\overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_s\}(X)$ maps isomorphically to a direct summand in $\Omega_*^G\{\mathfrak{F}_a, \mathfrak{F}_s\}(X)$.

(iv) For $\Gamma := \prod_{i=1}^l B_G U(k_i)$ the induction hypothesis implies that $\overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma)$ is a free Ω_*-module in even dimensional generators. Therefore the canonical maps

$$\overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma) \to \Omega_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma) \to \Omega_*^G\{\mathfrak{F}_a, \mathfrak{F}_s\}(\Gamma)$$

imply that $\overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma)$ also maps isomorphically to a direct summand in $\Omega_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma)$.

Let us now concentrate in understanding the five term exact sequence restricted to special G actions

(3.9) $$0 \to \overline{\Omega}_+^G\{\mathfrak{F}_f\}(\Gamma) \to \overline{\Omega}_+^G(\Gamma) \to \overline{\Omega}_+^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma) \xrightarrow{\partial} \overline{\Omega}_-^G\{\mathfrak{F}_f\}(\Gamma) \to \overline{\Omega}_-^G(\Gamma) \to 0.$$

Note that the map $\iota_* : \Gamma' \to \Gamma$ induces the commutative diagram

$$\begin{array}{ccccc}
\Omega_{*-2}^H(\Gamma^{\mathbb{Z}/q} \times B_H U(1)) & \xrightarrow{\cong} & \overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma) & \xrightarrow{\partial} & \overline{\Omega}_{*-1}^G\{\mathfrak{F}_f\}(\Gamma) \\
\uparrow & & \iota_*\uparrow & & \iota_*\uparrow\cong \\
\Omega_{*-2}^H(\Gamma' \times B_H U(1)) & \xrightarrow{\cong} & \overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma') & \xrightarrow{\partial} & \overline{\Omega}_{*-1}^G\{\mathfrak{F}_f\}(\Gamma')
\end{array}$$

where the middle homomorphism ι_* maps isomorphically $\overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma')$ into a direct summand in $\overline{\Omega}_*^G\{\mathfrak{F}_a, \mathfrak{F}_f\}(\Gamma)$ since Γ' is mapped to one connected component of the fixed point set $\Gamma^{\mathbb{Z}/q}$. Therefore the image of the boundary homomorphism ∂ is the same same in both cases.

In what follows we will study the induced boundary homomorphism

(3.10) $$\Omega_{*-2}^H(\Gamma' \times B_H U(1)) \to \overline{\Omega}_{*-1}^G\{\mathfrak{F}_f\}(\Gamma') \cong \Omega_{*-1}^H(\Gamma' \times B_H \mathbb{Z}/q)$$

using the Thom isomorphism, the long exact sequence for pairs and a particular model for $E_H\mathbb{Z}/q$.

Let $\mathbf{C}^\infty_H$ be a countable direct sum of all complex irreducible H-representations and consider the $\mathbb{Z}/q$ action on $\mathbf{C}^\infty_H$ such that the generator T of $\mathbb{Z}/q$ acts by mutliplication of $e^{\frac{2\pi i}{q}}$. The sphere $S(\mathbf{C}^\infty_H)$ of vectors of norm 1 is an $G = \mathbb{Z}/q \times H$ space on which $\mathbb{Z}/q$ acts freely and moreover is a $\mathfrak{F}_f$-free space. Since the non empty fixed point sets are infinitely dimensional spheres we know that this sphere $S(\mathbf{C}^\infty_H)$ is a model for $E_H\mathbb{Z}/q$. The Grassmannian $Gr_1\mathbf{C}^\infty_H$ is a model for $B_HU(1)$ since $\mathbb{Z}/q$ acts trivially on the one dimensional vector spaces, and $\mathbb{Z}/q$ acts on the fibers of the canonical line bundle $\gamma_H U(1) \to B_H U(1)$ by multiplication of $e^{\frac{2\pi i}{q}}$. To simplify the notation denote by

$$\gamma_1 := \gamma_H U(1)$$

and note that $S(\mathbf{C}^\infty_H) \cong S(\gamma_1)$ where $S(\gamma_1)$ denotes the sphere bundle of γ_1.

Consider now the line bundle $\gamma_1^{\otimes q}$ over $B_HU(1)$ which is q-fold tensor product of γ_1. The diagonal map

$$\Delta : \gamma_1 \to \gamma_1^{\otimes q}, \quad v \mapsto v \otimes \cdots \otimes v$$

is a q to 1 map on the fibers of the line bundles and therefore it induces an H-equivariant homeomorphism

$$S(\gamma_1)/(\mathbb{Z}/q) \cong S(\gamma_1^{\otimes q}).$$

Therefore we have that we may take either $S(\gamma_1)/(\mathbb{Z}/q)$ or $S(\gamma_1^{\otimes q})$ as a model for $B_H\mathbb{Z}/q$. The Thom isomorphism

$$\Omega^H_*((D(\gamma_1^{\otimes q}), S(\gamma_1^{\otimes q})) \times \Gamma') \cong \Omega^H_{*-2}(B_HU(1) \times \Gamma')$$

together with the long exact sequence for the pair $(D(\gamma_1^{\otimes q}), S(\gamma_1^{\otimes q}))$ and the induction hypothesis provides a four term exact sequence

$$\begin{aligned} 0 \to \Omega^H_+(\Gamma' \times S(\gamma_1^{\otimes q})) \to \Omega^H_+(\Gamma' \times B_HU(1)) \to \\ \Omega^H_+(\Gamma' \times B_HU(1)) \to \Omega^H_-(\Gamma' \times S(\gamma_1^{\otimes q})) \to 0, \end{aligned}$$

where the right hand side homomorphism is precisely the one of (3.10). Therefore we obtain that the boundary homomorphism of (3.10) is surjective, and since by the induction hypothesis $\Omega^H_+(\Gamma' \times B_HU(1))$ is a free Ω_*-module, we conclude that $\Omega^H_+(\Gamma' \times S(\gamma_1^{\otimes q}))$ is also a free Ω_*-module.

Therefore we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
0 \longrightarrow & \Omega_+^G\{\mathfrak{F}_f\}(\Gamma) & \longrightarrow & \Omega_+^G(\Gamma) & \longrightarrow & \Omega_+^G\{\mathfrak{F}_a,\mathfrak{F}_f\}(\Gamma) & \xrightarrow{\partial} & \Omega_-^G\{\mathfrak{F}_f\}(\Gamma) & \longrightarrow 0 \\
& \uparrow\cong & & \uparrow & & \uparrow & & \uparrow\cong & \\
0 \longrightarrow & \overline{\Omega}_+^G\{\mathfrak{F}_f\}(\Gamma) & \longrightarrow & \overline{\Omega}_+^G(\Gamma) & \longrightarrow & \overline{\Omega}_+^G\{\mathfrak{F}_a,\mathfrak{F}_f\}(\Gamma) & \xrightarrow{\partial} & \overline{\Omega}_-^G\{\mathfrak{F}_f\}(\Gamma) & \longrightarrow 0 \\
& \iota_*\uparrow\cong & & \uparrow & & \uparrow & & \iota_*\uparrow\cong & \\
0 \longrightarrow & \overline{\Omega}_+^G\{\mathfrak{F}_f\}(\Gamma') & \longrightarrow & \overline{\Omega}_+^G(\Gamma') & \longrightarrow & \overline{\Omega}_+^G\{\mathfrak{F}_a,\mathfrak{F}_f\}(\Gamma') & \xrightarrow{\partial} & \overline{\Omega}_-^G\{\mathfrak{F}_f\}(\Gamma') & \longrightarrow 0,
\end{array}$$

thus implying that $\Omega_-^G(\Gamma) = 0$ and that $\Omega_+^G(\Gamma)$ is a free Ω_*-module since both $\Omega_+^G\{\mathfrak{F}_f\}(\Gamma)$ and $\Omega_+^G\{\mathfrak{F}_a,\mathfrak{F}_f\}(\Gamma)$ are free Ω_*-modules.

Therefore the evenness conjecture holds for finite abelian p-groups.

3.3 The general case. The proof of the evenness conjecture for general finite abelian groups was done by Ossa [1972] and is based on the proof of Stong for p-groups and appropriate localizations at different primes. For a finite abelian group K denote by $Z_K := \mathbb{Z}[1/|K|]$ the localization of the integers at the ideal generated by the order of K.

Let $G = K \times L$ with K and L finite abelian with $|K|$ and $|L|$ relatively prime and consider the homomorphism $\Omega_*^{K\times L}\{\mathfrak{F}\} \to \Omega_*^L\{\mathfrak{F}\}$ which forgets the K action and $\mathfrak{F}$ is any family of subgroups of L . Let us show that the localized homomorphism

$$\Omega_*^{K\times L}\{\mathfrak{F}\}(\Gamma) \otimes Z_K \to \Omega_*^L\{\mathfrak{F}\}(\Gamma) \otimes Z_K$$

is an isomorphism whenever $\Gamma := \prod_{i=1}^l B_G U(k_i)$. Let us proceed by induction over L and over the family $\{\mathfrak{F}\}$.

For the trivial family $\mathfrak{F} = \{\{1\}\}$ we obtain the isomorphism

$$\Omega_*(BK \times BL \times \prod_i BU(k_i)) \otimes Z_K \xrightarrow{\cong} \Omega_*(BL \times \prod_i BU(k_i)) \otimes Z_K$$

since $\Omega_*(BK) \otimes Z_K \cong \Omega_* \otimes Z_K$.

Whenever the adjacent pair of families $(\mathfrak{F}, \mathfrak{F}')$ differ by $H \subset L$ we obtain the homomorphism of long exact sequences

$$\begin{array}{ccccccc}
\longrightarrow \Omega_*^{K\times L}\{\mathfrak{F}'\}(\Gamma) & \longrightarrow & \Omega_*^{K\times L}\{\mathfrak{F}\}(\Gamma) & \longrightarrow & \Omega_*^{K\times L/H}\{\{1\}\}(\Gamma^H \times \Gamma') & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\longrightarrow \Omega_*^{L}\{\mathfrak{F}'\}(\Gamma) & \longrightarrow & \Omega_*^{L}\{\mathfrak{F}\}(\Gamma) & \longrightarrow & \Omega_*^{L/H}\{\{1\}\}(\Gamma^H \times \Gamma') & \longrightarrow
\end{array}$$

with Γ' a disjoint union of products of spaces of the form $B_{K\times L/H}U(l)$. Tensoring with Z_K induces an isomorphism on the left vertical arrow by the induction hypothesis on the families and an isomorphism on the right vertical arrow by the induction hypothesis on the group L/H. The 5-lemma implies the desired isomorphism.

Now let $\mathfrak{F}$ be any family of subgroups of K and denote by $\mathfrak{F}\times\Phi$ the family of subgroups of G whose elements are groups $J \times H$ with $J \in \mathfrak{F}$ and H any subgroup of L. Let us show by induction on $\mathfrak{F}$ and on the group K that the localized module

$$\Omega_*^{K\times L}\{\mathfrak{F}\times\Phi\}(\Gamma)\otimes Z_K$$

is a free $\Omega_*\otimes Z_K$-module. Whenever $\mathfrak{F}$ is the trivial family we have shown above that

$$\Omega_*^{K\times L}\{\{1\}\times\Phi\}(\Gamma)\otimes Z_K \xrightarrow{\cong} \Omega^L(\Gamma)\otimes Z_K$$

is an isomorphism.

If the adjacent pair of families $(\mathfrak{F},\mathfrak{F}')$ differ by the subgroup J, then we obtain the long exact sequence

$$\cdots\to\Omega_*^{K\times L}\{\mathfrak{F}'\times\Phi\}(\Gamma)\to\Omega_*^{K\times L}\{\mathfrak{F}\times\Phi\}(\Gamma)\to\Omega_*^{K/J}\{\{1\}\}(\Gamma^{J\times L}\times\Gamma'')\to\cdots$$

where Γ'' is a disjoint union of spaces of the form $B_{K/J}U(l)$. Tensoring with Z_K gives us free $\Omega_*\otimes Z_K$-modules on the left hand side by the induction on families and on the right hand side by the induction on the group K. Therefore the middle term is also a free $\Omega_*\otimes Z_K$-module.

Therefore we have proved that if $\Omega^L(\Gamma)$ is free Ω_*-module then $\Omega^{K\times L}(\Gamma)\otimes Z_K$ is a free $\Omega_*\otimes Z_K$-module. Let us now write $G = P_1\times\cdots\times P_k$ with P_i its sylow p_i-subgroup. Since the evenness conjecture holds for p-groups, we have that $\Omega_*^{P_i}(\Gamma)$ is a free Ω_*-module and therefore $\Omega_*^G(\Gamma)\otimes\mathbb{Z}[1/[G:P_i]]$ is a free $\Omega_*\otimes\mathbb{Z}[1/[G:P_i]]$-module. Since the numbers $[G:P_i]$ are relatively prime it follows that $\Omega_*^G(\Gamma)$ is a free Ω_*-module.

Therefore the evenness conjecture holds for finite abelian groups.

4 The equivariant unitary bordism groups for non abelian groups

The evenness conjecture has been shown to be true for the dihedral groups D_{2p} with p-prime by Ángel, Gómez, and B. Uribe [n.d.], for groups of order pq where p and q are different primes by Lazarov [1972] and for the more general case of metacyclic groups by Rowlett [1980]. In these cases the group G is a semidirect product $\mathbb{Z}/r\rtimes\mathbb{Z}/s$ of cyclic groups with r and s relatively prime, and the study of the equivariant unitary bordism groups is also carried out calculating the equivariant unitary bordism groups of adjacent pair of families of subgroups as is done in the cyclic group case of Section 3.1.

The main tool used by Rowlett to study the metacyclic case is the equivariant unitary spectral sequence constructed by himself in Rowlett [1978, Prop. 2.1]. Suppose that A is a normal subgroup of G and that $Q = G/A$. A family $\mathfrak{F}$ of subgroups of A is called G-invariant if it is closed under conjugation by elements of G. Consider a pair $(\mathfrak{F}, \mathfrak{F}')$ of G-invariant families of A and note that $\Omega_*^A\{\mathfrak{F}, \mathfrak{F}'\}$ becomes a Q-module in the following way. Consider an A-manifold M with action $\theta : A \times M \to M$ and take an element $g \in G$. Define a new action on M by the map $g_*\theta : A \times M \to M$, $g_*(a, m) := \theta(g^{-1}ag, m)$ and denote the action of g on the bordism class $[M, \theta]$ by $\overline{g}[M, \theta] := [M, g_*\theta]$. This action is trivial on elements of A and therefore it boils down to an action of Q. Then there is a first quadrant spectral sequence E^r converging to $\Omega_*^G\{\mathfrak{F}, \mathfrak{F}'\}$ whose second page is

$$E^2_{p,q} \cong H_p(Q, \Omega_q^A\{\mathfrak{F}, \mathfrak{F}'\}).$$

In the case that both groups A and Q are cyclic of relative prime order, the action of Q on $\Omega_+^A\{\mathfrak{F}, \mathfrak{F}'\}$ factors through a permutation action on the free generators and therefore the second page is not difficult to calculate. If we take the family $\mathfrak{F}_A$ of all subgroups of A, the second page of the spectral sequence becomes $H_q(Q, \Omega_q^A)$, and since Ω_*^A is a free Ω-module in even dimensional generators, then we obtain that $\Omega_+^G\{\mathfrak{F}\}$ is a free Ω_*-module. Moreover, the same explicit construction carried out in Section 3.1 can be adopted in this case to show that the long exact sequence associated to the pair of families $\{\mathfrak{F}_a, \mathfrak{F}_A\}$, with $\mathfrak{F}_a$ the family of all subgroups, becomes

$$0 \to \Omega_+^G\{\mathfrak{F}_A\} \to \Omega_+^G \to \Omega_+^G\{\mathfrak{F}_a, \mathfrak{F}_A\} \xrightarrow{\partial} \Omega_-^G\{\mathfrak{F}_A\} \to 0.$$

The same argument as in (3.5) shows that $\Omega_*^G\{\mathfrak{F}_a, \mathfrak{F}_A\}$ is a free Ω_*-module in even dimensional generators and therefore we conclude that Ω_-^G is zero and Ω_+^G a free Ω_*-module.

The spectral sequence defined above can also be used in order to understand the torsion free part of the G-equivariant unitary groups for any abelian group. Take any subgroup A of G and let $(\mathfrak{F}_A, \mathfrak{F}_A')$ be the adjacent pair of families of G which differ by the conjugacy class of A. Tensoring with the rationals we obtain an isomorphism

$$\Omega_*^G\{\mathfrak{F}_A, \mathfrak{F}_A'\} \otimes \mathbb{Q} \cong \Omega_*^A\{\mathfrak{F}_A, \mathfrak{F}_A'\}^{W_A} \otimes \mathbb{Q}$$

where the right hand side consists of the W_A-invariant part. Since $\Omega_*^A\{\mathfrak{F}_A, \mathfrak{F}_A'\}$ is a free Ω_*-module in even dimensional generators we obtain the isomorphism

$$\Omega_*^G \otimes \mathbb{Q} \cong \bigoplus_{(A)} \Omega_*^A\{\mathfrak{F}_A, \mathfrak{F}_A'\}^{W_A} \otimes \mathbb{Q}$$

where (A) runs over the conjugacy classes of subgroups of G (see Rowlett [ibid., Thm. 1.1], c.f. tom Dieck [1973, Thm. 1]). In particular the torsion-free component of Ω_*^G is all of even degree.

Apart from the non-abelian groups which are metacyclic, there is no other finite non-abelian group on which the evenness conjecture has been shown to hold.

The main difficulty lies in the understanding of the equivariant bordism groups $\Omega_*^G\{\mathfrak{F}\}({}^{\widetilde{G}}B_G U($ of the classifying spaces ${}^{\widetilde{G}}B_G U(n)$ associated to $\mathbb{S}^1$-central extensions $\widetilde{G}$ of G for different families $\mathfrak{F}$ of subgroups. These bordism groups are the ones appearing once we try to calculate the equivariant unitary bordism groups for adjacent pair of families. Any development on the understanding of these equivariant unitary bordism groups will shed a light on the proof of the evenness conjecture for a bigger class of groups.

5 Conclusion

The evenness conjecture for equivariant unitary bordism has been an important question in algebraic topology for more than forty years. The conjecture has been proved to hold only for compact abelian Lie groups and finite metacyclic groups, for all other groups the conjecture remains open. We do hope that the present summary of known results will help settle the conjecture in the near future.

References

A. Ángel, J. M. Gómez, and B. Uribe (n.d.). "Equivariant complex bundles, fixed points and equivariant unitary bordism". arXiv: 1710.00879 (cit. on pp. 1236, 1239, 1241, 1242, 1244, 1256).

M. F. Atiyah (1961). "Bordism and cobordism". *Proc. Cambridge Philos. Soc.* 57, pp. 200–208. MR: 0126856 (cit. on p. 1245).

M. F. Atiyah and F. Hirzebruch (1961). "Vector bundles and homogeneous spaces". In: *Proc. Sympos. Pure Math., Vol. III*. American Mathematical Society, Providence, R.I., pp. 7–38. MR: 0139181 (cit. on p. 1245).

Michael Bix and Tammo tom Dieck (1978). "Characteristic numbers of G-manifolds and multiplicative induction". *Trans. Amer. Math. Soc.* 235, pp. 331–343. MR: 0478182 (cit. on p. 1241).

P. E. Conner and E. E. Floyd (1966). "Maps of odd period". *Ann. of Math. (2)* 84, pp. 132–156. MR: 0203738 (cit. on pp. 1237, 1239).

P. E. Conner and Larry Smith (1969). "On the complex bordism of finite complexes". *Inst. Hautes Études Sci. Publ. Math.* 37, pp. 117–221. MR: 0267571 (cit. on p. 1246).

Tammo tom Dieck (1972). "Orbittypen und äquivariante Homologie. I". *Arch. Math. (Basel)* 23, pp. 307–317. MR: 0310919 (cit. on pp. 1237, 1238).

– (1973). "Equivariant homology and Mackey functors". *Math. Ann.* 206, pp. 67–78. MR: 0375285 (cit. on p. 1257).

– (1987). *Transformation groups*. Vol. 8. De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, pp. x+312. MR: 889050 (cit. on p. 1237).

S. O. Kochman (1996). *Bordism, stable homotopy and Adams spectral sequences*. Vol. 7. Fields Institute Monographs. American Math. Society, Providence, RI, pp. xiv+272. MR: 1407034 (cit. on pp. 1245, 1246).

Peter S. Landweber (1971). "Complex bordism of classifying spaces". *Proc. Amer. Math. Soc.* 27, pp. 175–179. MR: 0268885 (cit. on p. 1246).

– (1972). "Equivariant bordism and cyclic groups". *Proc. Amer. Math. Soc.* 31, pp. 564–570. MR: 0296969 (cit. on pp. 1236, 1245, 1247, 1249).

Connor Lazarov (1972). "Actions of groups of order pq". *Trans. Amer. Math. Soc.* 173, pp. 215–230. MR: 0309131 (cit. on pp. 1236, 1256).

Peter Löffler (1974). "Bordismengruppen unitärer Torusmannigfaltigkeiten". *Manuscripta Math.* 12, pp. 307–327. MR: 0348776 (cit. on pp. 1236, 1245).

Wolfgang Lück and Bernardo Uribe (2014). "Equivariant principal bundles and their classifying spaces". *Algebr. Geom. Topol.* 14.4, pp. 1925–1995. MR: 3331607 (cit. on p. 1250).

J. P. May (1996). *Equivariant homotopy and cohomology theory*. Vol. 91. CBMS Regional Conference Series in Mathematics. With contributors M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, pp. xiv+366. MR: 1413302 (cit. on pp. 1235, 1236, 1245).

J. Milnor (1960). "On the cobordism ring Ω^* and a complex analogue. I". *Amer. J. Math.* 82, pp. 505–521. MR: 0119209 (cit. on pp. 1236, 1245).

S. P. Novikov (1960). "Some problems in the topology of manifolds connected with the theory of Thom spaces". *Soviet Math. Dokl.* 1, pp. 717–720. MR: 0121815 (cit. on pp. 1236, 1245).

Erich Ossa (1972). "Unitary bordism of abelian groups". *Proc. Amer. Math. Soc.* 33, pp. 568–571. MR: 0293666 (cit. on pp. 1236, 1245, 1247, 1255).

Russell J. Rowlett (1978). "The fixed-point construction in equivariant bordism". *Trans. Amer. Math. Soc.* 246, pp. 473–481. MR: 515553 (cit. on p. 1257).

– (1980). "Bordism of metacyclic group actions". *Michigan Math. J.* 27.2, pp. 223–233. MR: 568643 (cit. on pp. 1236, 1256).

R. E. Stong (1970). "Complex and oriented equivariant bordism". In: *Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969)*. Markham, Chicago, Ill., pp. 291–316. MR: 0273644 (cit. on pp. 1236–1238, 1245, 1247, 1250–1252).

René Thom (1954). "Quelques propriétés globales des variétés différentiables". *Comment. Math. Helv.* 28, pp. 17–86. MR: 0061823 (cit. on pp. 1236, 1245).

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