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FLUIDS, WALLS AND VANISHING VISCOSITY

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Abstract

The vanishing viscosity problem consists of understanding the limit, or limits, of solutions of the Navier-Stokes equations, with viscosity ν, as ν tends to zero. The Navier-Stokes equations are a model for real-world fluids and the parameter ν represents the ratio of friction, or resistance to shear, and inertia. Ultimately, the relevant question is whether a real-world fluid with very small viscosity can be approximated by an ideal fluid, which has no viscosity. In this talk we will be primarily concerned with the classical open problem of the vanishing viscosity limit of fluid flows in domains with boundary. We will explore the difficulty of this problem and present some known results. We conclude with a discussion of criteria for the vanishing viscosity limit to be a solution of the ideal fluid equations.

Introduction

The vanishing viscosity problem is a classical one in fluid dynamics. In its simplest form, the question is to understand under which circumstances the behavior of real world fluids can be well-approximated by that of ideal, or frictionless, fluids. Said differently, when can small viscosity be realistically neglected? The purpose of this article is to discuss some of the current knowledge concerning this problem.

More precisely, we will be focusing on incompressible Newtonian fluids. In addition, we are specifically interested in the interaction of fluid flows with rigid boundaries. Finally, without ignoring the physics, we will be primarily concerned with the mathematical treatment of this problem. We will ignore the important issues surrounding the computational modeling of such flows.

We begin our discussion with the physical description of the small viscosity flow regime and, in particular, with Ludwig Prandtl's contributions. After discussing the discrepancy between slightly viscous and non-viscous flow near a solid boundary, we will explore what

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is known if this discrepancy propagates into the bulk of the fluid, and we conclude with criteria under which the small viscosity limit may still be considered ideal, despite this discrepancy.

Hereafter, *viscosity* is understood as the inverse of the Reynolds number, a non-dimensional constant which parameterizes incompressible flow. Roughly, the Reynolds number captures the relative importance of inertia and friction, with large Reynolds number flows dominated by inertia. It incorporates the scale and characteristic speed of interest, along with the kinematic viscosity of the fluid.

1 L. Prandtl and boundary layers

"... the behaviour of a fluid of small viscosity μ may, on account of boundary layer separation, be completely different to that of a (hypothetical) fluid of no viscosity at all." Acheson [p. 30 1990].

A fundamental part of the study of fluid motion is understanding the interaction of fluids with solid objects. A natural point of departure for this discussion is the fact that the interaction of incompressible flow with a solid object is completely different if the flow has very small viscosity or none at all. This fact was observed in experiments, long before a consensus physical theory for it became available, see Acheson [p. 264 ibid.].

The relevant physical theory was proposed in a 7.5-page paper delivered at the Third ICM in Heidelberg, in 1904, by Ludwig Prandtl [1905]. At the time, Prandtl was a young fluid dynamicist transitioning from the University of Hannover to Gottingen. This remarkable short paper contains several new ideas, among which are the foundations of Boundary Layer Theory. In this section we briefly present its main ideas. We refer the reader to the classical text Schlichting [1960] for a broad discussion.

Prandtl assumes that a viscous fluid does not slip along the boundary, something which was still controversial at the time. Further, Prandtl's model describes the flow as two separate, yet interacting parts: in one part, far from the boundary, the flow can be treated as inviscid, and satisfies, in particular, the conservation laws of ideal fluid theory. The second part, localized near the rigid boundary, is where viscous effects are important. Prandtl refers to the region near the boundary as a *transition* or *boundary layer* and he suggests that, within this layer, the tangential velocity varies rapidly in the normal direction, while the normal velocity is slowly varying; together, they vanish at the boundary and interpolate the inviscid flow in the bulk of the fluid domain. Prandtl derives a system of partial differential equations which is an asymptotic model for the flow in the boundary layer; these equations are now known as the *Prandtl equations*. Furthermore, he estimates the thickness of the boundary layer as being of the order of the square-root of viscosity. Lastly, Prandtl notes that the boundary layer may separate, or detach, from the boundary and entrain into the bulk of the flow, even for flows with small, yet positive, viscosity.

Let us be more precise. The standard mathematical model for incompressible viscous flows is given by the ν-Navier-Stokes equations, which we write as

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + \mathbf{f}, & \text{in } (0,+\infty)\times\mathfrak{D}; \\ \operatorname{div}\mathbf{u} = 0, & \text{in } [0,+\infty)\times\mathfrak{D}. \end{cases} \tag{1-1}$$

Above, $\mathfrak{D}$ is the physical domain of the fluid, $\mathbf{u} = \mathbf{u}(t,\mathbf{x}) = (u_1,u_2,u_3)(t,\mathbf{x})$ represents the fluid velocity at time $t \geq 0$ and at the point $\mathbf{x} \in \mathfrak{D}$, and $p = p(t,\mathbf{x})$ is the scalar pressure. The viscosity is $\nu > 0$, and $\mathbf{f}$ represents a given external force, which we assume to vanish throughout this paper. The *no slip* boundary condition translates to

$$\mathbf{u} = 0 \text{ on } (0,+\infty)\times\partial\mathfrak{D}. \tag{1-2}$$

The equations for inviscid, or ideal, fluid flow are known as the Euler equations and are given by setting $\nu = 0$ in (1-1). The boundary condition for the Euler equations in a domain with boundary is the *non-penetration* boundary condition:

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } (0,+\infty)\times\partial\mathfrak{D}, \tag{1-3}$$

where $\mathbf{n}$ is the unit normal exterior to $\partial\mathfrak{D}$.

The mismatch between (1-2) and (1-3) is at the heart of the difficulty in addressing the small viscosity problem.

For simplicity, let us assume that $\mathfrak{D}$ is the two-dimensional half-plane $\mathbb{H} = \{\mathbf{x} = (x,y) \in \mathbb{R}^2 \mid y > 0\}$. Prandtl's theory applies to situations where the flow $\mathbf{u} = (u,v)$ can be described as a boundary layer flow $\mathbf{u}^{BL}$ inside a layer of thickness $\delta = \delta(\nu)$, superimposed to a mainstream, inviscid flow $\mathbf{U} = (U,V)$ at a distance greater than δ from the boundary. Asymptotic matching may be used to derive an approximate model for the behavior of the flow in the boundary layer. Rescaling the problem with respect to δ and introducing the variable $Y = y/\delta$ yields $\delta = \sqrt{\nu}$ and leads to the boundary layer equations below.

Let $\mathbf{u}^P = (u^P, v^P)$. The Prandtl equations are given by

$$\begin{cases} \partial_t u^P + (\mathbf{u}^P\cdot\widetilde{\nabla})u^P = -\partial_x p^P + \partial_Y^2 u^P, & \text{in } (0,+\infty)\times\mathbb{H}; \\ \partial_Y p^P = 0, & \text{in } [0,+\infty)\times\mathbb{H}; \\ \widetilde{\operatorname{div}}\,\mathbf{u}^P = 0, & \text{in } [0,+\infty)\times\mathbb{H}, \end{cases} \tag{1-4}$$

where $\widetilde{\nabla} = (\partial_x, \partial_Y)$ and $\widetilde{\operatorname{div}}\,\mathbf{u}^P = \widetilde{\nabla}\cdot\mathbf{u}^P$.

We expect that the solution $\mathbf{u} = (u, v)$ of (1-1) is well-approximated, for small $\nu > 0$, by

$$\mathbf{u} = (u, v) \sim \begin{cases} \mathbf{U} = (U, V), \text{ if } y >> \sqrt{\nu}, \\ \mathbf{u}^P = (u^P, \sqrt{\nu} v^P), \text{ if } 0 < y << \sqrt{\nu}. \end{cases} \tag{1-5}$$

The boundary conditions for the Prandtl solution should be:

$$\mathbf{u}^P(t, x, 0) = (0, 0) \text{ and } \lim_{Y \to +\infty} \mathbf{u}^P(t, x, Y) = (U(t, x, 0), 0), \quad \text{on } (0, +\infty) \times \mathbb{R}. \tag{1-6}$$

In addition, since the Prandtl pressure is independent of the vertical variable Y we have, taking the limit $Y \to \infty$, that the Prandtl pressure is matched to the Euler pressure at the boundary. Therefore,

$$p^P(t, x, Y) = p^E(t, x, 0), \quad \text{in } (0, +\infty) \times \mathbb{H},$$

so that

$$-\partial_x p^P = (\partial_t U + U \partial_x U)\big|_{(t,x,0)}, \quad \text{in } (0, +\infty) \times \mathbb{H}. \tag{1-7}$$

Finally, the initial data for the Prandtl equations is chosen so that the approximation in (1-5) is verified at $t = 0$. We write:

$$\mathbf{u}_0^P(x, Y) \equiv \mathbf{u}^P(0, x, Y). \tag{1-8}$$

In summary, given an initial flow $\mathbf{u}_0$, Prandtl's boundary layer theory, therefore, is based on the *ansatz* that the viscous flow whose initial velocity is $\mathbf{u}_0$ is well-approximated by the inviscid flow with the same initial velocity, far from the boundary, superimposed with an interpolating field near the boundary. As Prandtl himself notes, the agreement of this theory with experiments happens only in very particular situations, such as special laminar flows. The key observation is that the boundary layer is a thin region of intense shear next to the boundary.

This theory is not valid, even for laminar flows, when *boundary layer separation* occurs. This is when the boundary layer *detaches* from the boundary and affects the inviscid downstream flow. It is the case with flow past a cylinder, flow past a finite flat plate, past a corner, etc.

From a mathematical point-of-view, Prandtl's boundary layer theory introduces two natural questions. First, under which conditions are the Prandtl equations well-posed, and, second, given solutions of Prandtl's equations, when can the validity of the large Reynolds number asymptotics be rigorously verified. These two questions are obviously related, but much more is known regarding the former.

The rigorous study of the Prandtl equations began with the work of O. Oleinik, in the 1960s, where the steady and time-dependent problems were studied, in several scenarios, see for instance Oleĭnik [1963, 1966] and Oleĭnik and Samokhin [1999]. The following result is particularly noteworthy.

Theorem 1.1. (O. A. Oleinik, 1967, Oleĭnik [1966]) *Let $u_0^P = u^P(0,x,Y)$ satisfy the monotonicity condition $\partial_Y u_0^P > 0$ in $\mathbb{H}$, and assume that $U = U(t,x,0) > 0$ for all $t \geq 0$, $x \in \mathbb{R}$. Then there exists a local, classical, solution of the Prandtl equations* (1-4)*, subject to the initial and boundary conditions* (1-8)*,* (1-6)*, with pressure given by* (1-7).

The proof of this theorem is based on a clever time-dependent change of variables called *Crocco* transform. The condition $U = U(t,x,0) > 0$ is the "no back-flow" condition, known to prevent boundary layer separation, as does the monotonicity condition. We observe, however, that there is no proof that the Prandtl approximation holds under these conditions. We refer the reader to Kelliher [2017] and Constantin, Kukavica, and Vicol [2015] for a discussion of the inviscid limit in this context.

A different, energy-based proof of Oleinik's theorem, still assuming the Oleinik monotonicity condition and no back-flow, was obtained in Masmoudi and Wong [2015] and in Alexandre, Y.-G. Wang, Xu, and Yang [2015], see also Kukavica, Masmoudi, Vicol, and Wong [2014].

Well-posedness of Prandtl's equations has also been studied in the analytic setting. R. Caflisch and M. Sammartino studied flows in a half-plane and proved, see Sammartino and Caflisch [1998b], that both Euler and Prandtl are locally well-posed for real analytic data. In Sammartino and Caflisch [1998a] they went on to show that, under the assumption of real-analyticity of the initial Euler and Prandtl velocities, the solution of the Navier-Stokes equations is well-approximated as in (1-5), assuming the initial data for the Navier-Stokes equations satisfies the same asymptotics which, in certain contexts, reads as $\mathbf{u}_0^\nu = \mathbf{u}_0^\nu(x,y) = \mathbf{u}_0^E(x,y) + (u_0^P, \sqrt{\nu} v_0^P)(x,Y) + \mathcal{O}(\sqrt{\nu})$. In addition, results on local well-posedness for Prandtl, assuming only tangential analyticity have been obtained, see Lombardo, Cannone, and Sammartino [2003] and Kukavica and Vicol [2013].

The problem in the analytic setting is, therefore, well-understood. However, as pointed out in Grenier, Guo, and Nguyen [2015], analytic regularity is too much to expect in real-world flows. In Maekawa [2013, 2014] it was shown that the Prandtl approximation is valid if the initial vorticity is compactly supported away from the boundary. More precisely, the author assumes that the Navier-Stokes and Euler initial velocities are the same and their initial curl is supported far from the boundary and, additionally, the curl is Sobolev regular. The author establishes local-in-time well-posedness for the Prandtl equations and shows that, in L^∞, the Prandtl approximation is valid. Note that, in particular, the initial velocity is assumed to be analytic in a neighborhood of the boundary, but this analyticity is lost at positive time and the author carefully estimates how.

The mathematical difficulty in treating the Prandtl equations stems from the loss of one derivative in x, which cannot be recovered due to lack of diffusion in the horizontal direction. One realization of this difficulty is the fact that the Prandtl equations have been shown to be linearly ill-posed in Sobolev spaces, see Gérard-Varet and Dormy [2010]. Finite-time blow-up for smooth solutions of Prandtl's equation goes back to E and Engquist [1997], see also Kukavica, Vicol, and F. Wang [2017] for a more physically motivated example.

As we have already observed, the mismatch between the no-slip and non-penetration boundary conditions is largely responsible for the difficulty in studying the vanishing viscosity limit. Assuming settings for which Prandtl's *ansatz*, that the difference between viscous and inviscid flow is confined to a small region near the boundary, holds true then, at small viscosity, this mismatch corresponds to the formation of a thin layer of intense shear near the boundary. In Grenier, Guo, and Nguyen [2015] this was explored by explicitly connecting the validity of the Prandtl asymptotic model to the stability of viscous shear flows. More precisely, the authors of Grenier, Guo, and Nguyen [ibid.] conjecture that shear flows are typically unstable for the Navier-Stokes equations and, therefore, that the Prandtl approximation does not hold in Sobolev spaces; their conjecture is the subject of ongoing work, see Grenier, Guo, and Nguyen [2016].

In the shear layer discussed above, vorticity, the curl of velocity, tends to be very large. If we consider, instead, the infinite Reynolds number limit, then the mismatch between viscous and ideal flow boundary conditions will lead to a *vortex sheet*, understood as an idealization of a thin region of intense shear, forming at the boundary. This was noted more precisely in Kelliher [2008, 2017], where (a variant of) the result below was proved.

A word on notation: $L^2_\sigma(\mathfrak{D})$ refers to divergence-free vector fields whose components are square-integrable. We recall that a vector field in $L^2_\sigma(\mathfrak{D})$ has a well-defined trace of normal component at $\partial\mathfrak{D}$. In addition, if $\mathbf{x} = (x, y)$ then $\mathbf{x}^\perp \equiv (-y, x)$, and $\nabla^\perp \equiv (-\partial_y, \partial_x)$.

Fix $T > 0$ and assume $\mathfrak{D}$ is connected and simply connected.

Proposition 1.2. (See Kelliher [2008, 2017].) *For each $\nu > 0$, let $\mathbf{u}^\nu \in L^2_\sigma(\mathfrak{D}) \cap H^1_0(\mathfrak{D})$. Assume that there exists $\mathbf{v} \in L^2_\sigma(\mathfrak{D})$ such that $\mathbf{u}^\nu \rightharpoonup \mathbf{v}$ weakly in $L^2_\sigma(\mathfrak{D})$ as $\nu \to 0$, and that the trace of $\mathbf{v}$, at $\partial\mathfrak{D}$, is well-defined. Then, if $!^\nu \equiv \operatorname{curl} \mathbf{u}^\nu$, it follows that*

1. *If $\mathfrak{D} \subset \mathbb{R}^2$, then $!^\nu \rightharpoonup \operatorname{curl} \mathbf{v} - (\mathbf{v} \cdot \tau)\mu$, weak-$*$ $(H^1(\mathfrak{D}))^*$, as $\nu \to 0$, where $\tau = \mathbf{n}^\perp$ and μ is the 1-dimensional Hausdorff measure on $\partial\mathfrak{D}$.*

2. *If $\mathfrak{D} \subset \mathbb{R}^3$, then $!^\nu \rightharpoonup \operatorname{curl} \mathbf{v} + (\mathbf{v} \times \mathbf{n})\mu$, weak-$*$ $(H^1(\mathfrak{D}))^*$, as $\nu \to 0$, where μ is the 2-dimensional Hausdorff measure on $\partial\mathfrak{D}$.*

This result can be derived from Stokes' theorem in a straightforward manner.

Proposition 1.2 implies that, if $\mathbf{u}^\nu \rightharpoonup \mathbf{u}^E$, with $\mathbf{u}^E$ a solution of the Euler equations, and if the trace of tangential component of $\mathbf{u}^E$ is well-defined on $\partial\mathfrak{D}$, then a vortex sheet will form on the boundary, with strength given by $\mathbf{u}^E \cdot \tau$.

2 Vortex sheets in ideal fluid flow

As we have seen, in the vanishing viscosity limit, thin shear layers arise near the boundary and are expected be unstable, detach and affect the bulk of the fluid. In these thin shear layers vorticity is very intense and concentrated, and it is the nature of ideal flow that vorticity can neither be created nor destroyed. An idealization of these thin shear layers are the "Helmholtz surfaces of discontinuity" (see Prandtl [1905]), also known as vortex sheets, which are surfaces across which the velocity field has a tangential discontinuity, while the normal component is continuous. The tangential discontinuity of the flow is read at the level of vorticity as a Dirac delta supported on the surface of discontinuity. In this section we will discuss the evolution of vortex sheets in ideal fluids. We emphasize that, at this point, the problems are purely mathematical, as no real-world fluid has zero viscosity, nor do vortex sheets exist in nature.

2.1 Full plane. We will continue to carry out our discussion in two dimensions. We introduce the vorticity $\omega \equiv \operatorname{curl} \mathbf{u} = \nabla^\perp \cdot \mathbf{u}$ as a new dynamical variable. Taking the curl $\equiv \nabla^\perp\cdot$ of the Euler equations yields:

$$\text{(2-1)} \qquad \begin{cases} \partial_t \omega + (\mathbf{u}\cdot\nabla)\omega = 0, & \text{in } (0,+\infty)\times\mathfrak{D}, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } [0,+\infty)\times\mathfrak{D}, \\ \operatorname{curl} \mathbf{u} \equiv \nabla^\perp\cdot\mathbf{u} = \omega, & \text{in } [0,+\infty)\times\mathfrak{D}. \end{cases}$$

The study of ideal fluid flow from the point-of-view of the evolution of vorticity is called *vortex dynamics*. It is a particularly useful stance given that vorticity is transported. This is true even in three dimensional space, although the transport is more complicated. We note that, if $\mathfrak{D}$ is a simply connected domain, then (2-1) is actually a closed evolution equation for the dynamic variable ω. In this case, the *elliptic system* formed by the last two equations in (2-1) is explicitly solvable:

$$\text{(2-2)} \qquad \mathbf{u}(t,\cdot) = \nabla^\perp(\Delta_0^{\mathfrak{D}})^{-1}\omega(t,\cdot) \equiv K^{\mathfrak{D}}[\omega(t,\cdot)],$$

where $\Delta_0^{\mathfrak{D}}$ is the homogeneous Dirichlet Laplacian on $\mathfrak{D}$. This is called the *Biot-Savart law* and the kernel in the integral operator $K^{\mathfrak{D}}[\cdot]$ is the *Biot-Savart kernel* $K^{\mathfrak{D}} = K^{\mathfrak{D}}(\mathbf{x},\mathbf{y})$.

Let $\mathbf{u}_0$ be a vector field which is irrotational on either side of a given, smooth, curve $\mathcal{C}_0 \subset \mathbb{R}^2$ and such that there is a jump in the tangential component across $\mathcal{C}_0$. It is easy to see, in this case, that the vorticity $\omega_0 = \operatorname{curl} \mathbf{u}_0$ is a measure concentrated on the curve $\mathcal{C}_0$, with density, or *vortex sheet strength* $\gamma_0 = [\mathbf{u}_0]_{\mathcal{C}_0}$, that is, $\omega_0 = \gamma_0 \delta_{\mathcal{C}_0}$.

There are two points-of-view used to describe the evolution of a vortex sheet. One gives rise to the *explicit description*, where we seek a time-dependent parameterization of the curve of discontinuity of the flow. To begin with, we parameterize the initial curve and assume that the vortex sheet structure is preserved under the flow, so that $\omega(t,\cdot) = \gamma_t \delta_{\mathcal{C}_t}$, a reasonable assumption given that vorticity is transported by $\mathbf{u} = \mathbf{u}(t,\cdot)$. This *ansatz* leads to the Birkhoff–Rott equations, derived explicitly by G. Birkhoff [1962], implicit in the work of N. Rott [1956]. Using the identification $\mathbb{R}^2 \sim \mathbb{C}$, $z = x + iy$, the Birkhoff–Rott equations are written as

$$\frac{\partial}{\partial t}\overline{z}(t,\Gamma) = \frac{1}{2\pi i}\, pv \int \frac{1}{z(t,\Gamma) - z(t,\Gamma')}\, d\Gamma'. \tag{2-3}$$

The parameter Γ is called the *circulation variable*.

The Birkhoff–Rott equations encode both the motion of the sheet and the time-dependent evolution of the density, or sheet strength, or yet, the magnitude of the jump in tangential velocity across the sheet. The density can be recovered through $\gamma_t(\cdot) = (\partial_\Gamma z(t,\cdot))^{-1}$.

The study of vortex sheet motion through the Birkhoff–Rott equations has a long history. As an idealization of intense thin shear layers, it is expected that vortex sheets develop a complicated motion through spontaneous generation of small scales. This can be illustrated by performing a periodic perturbation on a stationary flat vortex sheet and observing the exponentially growing modes that ensue, see Marchioro and Pulvirenti [1994]. The linear instability observed in the Birkhoff–Rott equations is called *Kelvin-Helmholtz instability* and it manifests itself macroscopically as a tendency of the sheet to roll-up into spirals.

In C. Sulem, P.-L. Sulem, Bardos, and Frisch [1981] short time existence was established assuming the initial vortex sheet and sheet strength were real analytic, since real analyticity implies exponential decay of high Fourier modes. In Moore [1978] and Moore [1979] sophisticated asymptotic calculations were performed which suggested the appearance of a singularity in finite time for analytic vortex sheets and, moreover, he described the expected singularity as a blow-up in curvature. Moore's calculations were rigorously confirmed by Caflisch and Orellana [1986, 1989], who showed that, for an analytic perturbation of amplitude $\mathcal{O}(\varepsilon)$ the time-of-existence is $\mathcal{O}(\log \varepsilon)$. For further work see Duchon and Robert [1988] and Lebeau [2002]. The state-of-the-art result regarding ill-posedness is due to S. Wu [2002, 2006].

An alternative point-of-view in the description of vortex sheet evolution is to embed the discontinuity curve and density in a solution of the Euler equations, whose evolution should carry the information along. In this *implicit description* we make no assumption on the structure of the solution at future time. The tools used when taking this approach are

PDE methods, and the relevant information is the regularity space in which the equations are studied. From this standpoint a Dirac delta or a general bounded Radon measure are indistinguishable. We refer to initial velocities whose curl is a bounded Radon measure as *vortex sheet initial data*.

Let us proceed with a precise definition of a weak solution in a general fluid domain $\mathfrak{D}$.

Definition 2.1. Let $\mathbf{u}_0 \in L^2_{\mathrm{loc}}(\mathfrak{D})$ and assume that $\omega_0 = \operatorname{curl} \mathbf{u}_0 \in \mathfrak{BM}(\mathfrak{D})$. The vector field $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2_{\mathrm{loc}}(\mathfrak{D}))$ is a *weak solution* of the incompressible Euler equations in $\mathfrak{D}$, (1-1) with $\nu = 0$, with initial data $\mathbf{u}_0$, if the following conditions hold.

1. For every divergence-free test vector field $\Phi = \Phi(t, \mathbf{x}) \in C^\infty_c(\mathbb{R}_+ \times \mathfrak{D})$ the identity below holds true:

$$\int_0^{+\infty} \int_{\mathfrak{D}} \{\partial_t \Phi \cdot \mathbf{u} + [(\mathbf{u} \cdot \nabla)\Phi] \cdot \mathbf{u}\} \, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathfrak{D}} \Phi(0, \mathbf{x}) \cdot \mathbf{u}_0(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0, \tag{2-4}$$

2. $\operatorname{div} \mathbf{u}(t, \cdot) = 0$ in $\mathfrak{D}'(\mathbb{R}_+ \times \mathfrak{D})$,

3. if $\partial\mathfrak{D} \neq \varnothing$ then $\mathbf{u} \cdot \mathbf{n} = 0$ in the trace sense on $\partial\mathfrak{D}$, a.e. t.

The study of weak solutions of the Euler equations with vortex sheet initial data was pioneered by R. DiPerna and A. Majda in a series of papers, see DiPerna and Majda [1987a,b, 1988], where they developed the framework and criteria to establish existence. J.-M. Delort [1991], proved the existence of a weak solution with vortex sheet initial data provided the vorticity has a *distinguished sign*. Let us briefly recall Delort's result, for $\mathfrak{D} = \mathbb{R}^2$, and revisit the proof.

Theorem 2.2. (Delort [ibid.]) *Let* $\mathbf{u}_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ *and assume that* $\omega_0 = \operatorname{curl} \mathbf{u}_0 \in \mathfrak{BM}_{c,+}(\mathbb{R}^2)$. *Then there exists a weak solution in the sense of Definition 2.1.*

The proof is obtained by means of a compensated compactness argument. We discuss an alternative proof, given by Schochet [1995], which involves rewriting the weak formulation in terms of vorticity and then symmetrizing the integration kernels which arise. This has turned out to be a very useful technique, the source of a number of additional results.

The weak vorticity formulation is obtained by first multiplying, formally, the vorticity equation (2-1) by a test function $\varphi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}^2)$, then integrating by parts so as to throw all derivatives onto the test function, thus finding

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^2} [(\partial_t \varphi)\omega + (\mathbf{u} \cdot \nabla\varphi)\omega] \, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathbb{R}^2} \varphi(0, \mathbf{x})\omega_0(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0.$$

One then recalls that the velocity $\mathbf{u}$ can be recovered from the vorticity ω by means of the *Biot-Savart* law (2-2) which, in the full plane, reads:

$$\mathbf{u} = \mathbf{u}(t, \mathbf{x}) = \int_{\mathbb{R}^2} K(\mathbf{x} - \mathbf{y})\omega(t, \mathbf{y}) \, d\mathbf{y},$$

with

$$K(\mathbf{z}) = \frac{\mathbf{z}^\perp}{2\pi|\mathbf{z}|^2} \equiv \nabla^\perp(\Delta^{-1}).$$

Indeed, since $\operatorname{div} \mathbf{u} = 0$ it follows that $\mathbf{u} = \nabla^\perp \psi$ and, therefore, $\omega = \operatorname{curl} \mathbf{u} = \nabla^\perp \cdot \nabla^\perp \psi = \Delta\psi$. We then substitute $\mathbf{u}$ for the Biot-Savart law in the nonlinear term of the weak vorticity formulation, we symmetrize with respect to $\mathbf{x}$ and $\mathbf{y}$ and use the anti-symmetry of the kernel K to get:

$$\begin{aligned}
\int_{\mathbb{R}_+} \int_{\mathbb{R}^2} (\mathbf{u} \cdot \nabla\varphi)\omega \, d\mathbf{x}dt &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} K(\mathbf{x} - \mathbf{y})\omega(t, \mathbf{y}) \, d\mathbf{y} \right) \cdot \nabla\varphi(t, \mathbf{x})\omega(t, \mathbf{x}) \, d\mathbf{x}dt \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} K(\mathbf{x} - \mathbf{y})\omega(t, \mathbf{x}) \, d\mathbf{x} \right) \cdot \nabla\varphi(t, \mathbf{y})\omega(t, \mathbf{y}) \, d\mathbf{y}dt \\
&\equiv \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} H_\varphi(t, \mathbf{x}, \mathbf{y})\omega(t, \mathbf{x})\omega(t, \mathbf{y}) \, d\mathbf{x}d\mathbf{y}dt,
\end{aligned}$$

where the *auxiliary test function* H_φ is given by

(2-5)
$$H_\varphi(t, \mathbf{x}, \mathbf{y}) = K(\mathbf{x} - \mathbf{y}) \cdot \frac{(\nabla\varphi(t, \mathbf{x}) - \nabla\varphi(t, \mathbf{y}))}{2} \equiv \frac{(\mathbf{x} - \mathbf{y})^\perp}{|\mathbf{x} - \mathbf{y}|} \cdot \frac{\nabla\varphi(t, \mathbf{x}) - \nabla\varphi(t, \mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|}.$$

With this notation the weak vorticity formulation becomes

(2-6)
$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^2} (\partial_t\varphi)\omega \, d\mathbf{x}dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} H_\varphi(t, \mathbf{x}, \mathbf{y})\omega(t, \mathbf{x})\omega(t, \mathbf{y}) \, d\mathbf{x}d\mathbf{y}dt + \int_{\mathbb{R}^2} \varphi(0, \mathbf{x})\omega_0(\mathbf{x}) \, d\mathbf{x} = 0.$$

The main advantage of this formulation is that it makes sense for any flow whose vorticity is a bounded Radon measure which is *continuous*, that is, which does not contain an atomic part. To see this one first observes that, for any test function $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$, the auxiliary test function H_φ is a *bounded* function on $\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$. In addition, H_φ is discontinuous *only* on the set $\{(t, \mathbf{x}, \mathbf{y}) \mid \mathbf{x} = \mathbf{y}\}$. Now, if the measure $\omega(t, \cdot)$ is a continuous bounded measure on $\mathbb{R}^2$, then the tensor product $\omega(t, \cdot) \otimes \omega(t, \cdot)$ is a continuous bounded measure on $\mathbb{R}^2 \times \mathbb{R}^2$, which can be integrated against $H_\varphi(t, \cdot, \cdot)$. Moreover, if

$\omega^n \rightharpoonup \omega$ weak-$*$ in $L^\infty(\mathbb{R}_+; \mathfrak{BM}(\mathbb{R}^2))$, and if, uniformly in n and $t \geq 0$, $\omega^n(t,\cdot)$ does not attribute mass to points (no atomic part, uniformly in n), that is, if

$$\sup_n \sup_t \sup_{\mathbf{x}} \int_{B(\mathbf{x};r)} |\omega^n(t,\mathbf{y})|\, d\mathbf{y} \to 0 \text{ as } r \to 0, \tag{2-7}$$

then the weak limit commutes with the nonlinearity.

To prove Delort's theorem one chooses a smooth approximation of the initial data and solves the Euler equations with the smooth initial data. This procedure yields a sequence of exact solutions $(\mathbf{u}^n, \omega^n = \operatorname{curl} \mathbf{u}^n)$ satisfying uniform bounds in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2_{\text{loc}}(\mathbb{R}^2))$ for velocity and in $L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2))$ for vorticity. The distinguished sign of vorticity, together with the respective bounds above, yield the condition that $\omega^n(t,\cdot)$ does not attribute mass to points, uniformly in n, in the sense of (2-7). Thus, upon extracting a weakly convergent subsequence, one can pass to the weak limit in all the terms of the weak vorticity formulation (2-6). To conclude the proof one shows that (2-6) is equivalent to items 1. and 2. in Definition 2.1.

We note in passing that it is a well-known fact that sequences of *nonnegative* bounded Radon measures, uniformly bounded in H^{-1}_{loc}, satisfy (2-7).

Delort obtained his existence result for an initial velocity in $L^2_{\text{loc}}(\mathbb{R}^2)$ whose curl belongs to $L^p_c(\mathbb{R}^2) + \mathfrak{BM}_{c,+}(\mathbb{R}^2)$, for some $1 < p \leq \infty$, that is, whose vorticity is such that only the singular part is of distinguished sign. In Vecchi and S. Wu [1993] existence was extended to $p = 1$ using the Dunford-Pettis theorem for uniformly integrable functions.

Versions of Delort's theorem have been obtained for other approximations, such as solutions of the Navier-Stokes equations in the full plane, see Majda [1993], numerical approximations using the vortex blob method, see Liu and Xin [1995], approximations generated by truncation, see Lions [1996] and by central difference schemes, see Lopes Filho, Nussenzveig Lopes, and Tadmor [2000]. An alternative proof, highlighting the compensated compactness aspect of the result, was given in Evans and Müller [1994].

It should be noted that a comparison between the Birkhoff–Rott (explicit) and weak Euler (implicit) mathematical models for the evolution of vortex sheets was carried out in Lopes Filho, Nussenzveig Lopes, and Schochet [2007], where it was shown that a weak solution of the Euler equations whose vorticity is a Dirac delta on a curve of finite length $\mathcal{C}_t$ and with density γ_t is a solution of the Birkhoff–Rott equations if and only if the density is integrable along the curve $\mathcal{C}_t$. This establishes a restricted equivalence between the two descriptions.

2.2 Domains with boundary. In view of the focus of this paper, it is necessary to consider how vortex sheet initial data flow interacts with a rigid boundary. Delort [1991] studied flows with vortex sheet initial data of distinguished sign in bounded domains with smooth boundary, and he established existence of a weak solution in much the same way as for the full plane. Delort explored the fact that his proof, for the full plane, was *local*, since the test vector fields were assumed to have compact support; this made it possible to use the same proof for a bounded domain. However, ideal flows in bounded domains must satisfy the non-penetration condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial \mathfrak{D}$, see (1-3). This is a *linear* condition for velocity (understood in the trace sense), which is trivially continuous with respect to weak convergence in $L^\infty(\mathbb{R}_+; L^2_\sigma(\mathfrak{D}))$. Hence, using the same strategy as for the full plane, namely smoothing out initial data and exactly solving the Euler equations, with the smooth data, *in the bounded domain* $\mathfrak{D}$, yields a sequence of approximations $\mathbf{u}^n$ which converge weak-$*$ in $L^\infty(\mathbb{R}_+; L^2(\mathfrak{D}))$, which are all tangent to $\mathfrak{D}$, so that the weak limit satisfies the non-penetration condition in the trace sense.

The simplest two-dimensional fluid domain with a boundary is the half-plane, $\mathbb{H} \equiv \mathbb{R}^2_+ = \{\mathbf{x} = (x, y) \mid y > 0\}$. If we use the same arguments as Delort, but for the (unbounded) half-plane, we also obtain a weak solution with vortex sheet initial data assuming the (singular part of the) initial vorticity is of distinguished sign. Now, the *image method* is well-known in fluid dynamics as a means to extend to the full plane an ideal fluid flow in a half-plane: one simply reflects the half-plane flow, by mirror-symmetry, with respect to the boundary of the half-plane. In $\mathbb{H}$ this means that the first component of the velocity field is even with respect to y, and the second is odd. This symmetry induces the corresponding vorticity to be odd with respect to y, so that, necessarily, it must change sign. If, however, one attempts to use the image method for the weak solution obtained using Delort's proof, one *does not* find a weak solution in the full plane, with this odd vorticity. The reason is that the non-penetration condition along the boundary of $\mathbb{H}$, $y = 0$, is assumed only in the trace sense, and this is too weak for the image method to work. This observation is at the heart of the main result in Lopes Filho, Nussenzveig Lopes, and Xin [2001], where Delort's theorem is extended to include flows whose vorticity is an odd bounded Radon measure, single-signed on each side of a line, plus an arbitrary L^p_c function, $p \geq 1$.

Theorem 2.3. (See Lopes Filho, Nussenzveig Lopes, and Xin [ibid.].) *Let* $\mathbf{u}_0 \in L^2(\mathbb{R}^2)$ *be a divergence-free vector field such that* $\omega_0 = \operatorname{curl} \mathbf{u}_0 \in \mathfrak{BM}_c(\mathbb{R}^2) + L^p_c(\mathbb{R}^2)$*, for some* $p \geq 1$*. Assume that* ω_0 *is odd with respect to* y *and single-signed on* $\mathbb{H}$*. Then there exists a weak solution of the incompressible 2D Euler equations, in the sense of Definition 2.1, with initial data* $(\mathbf{u}_0, \omega_0)$.

A key point in the proof of Theorem 2.3 is the *a priori* estimate below, estabished in Lopes Filho, Nussenzveig Lopes, and Xin [ibid.]. Fix $T > 0$, $L > 0$. There exists $C = C(T, L, \|\mathbf{u}_0\|_{L^2} \|\omega_0\|_{\mathfrak{BM}}) > 0$ such that, if $\mathbf{n} = (0, -1)$, then

$$\int_0^T \int_{-L}^{L} |\mathbf{u}^n \cdot \mathbf{n}^\perp|^2(t, x, 0)\, \mathrm{d}x \mathrm{d}t \leq C. \tag{2-8}$$

The new estimate (2-8) implies, together with the estimates with which Delort worked, that, for any compact subset $\mathcal{K} \subset \mathbb{R}^2$,

$$\sup_n \int_0^T \left(\sup_{\mathbf{x}\in\mathcal{K}} \int_{B(\mathbf{x};r)} |\omega^n(\mathbf{y},t)| \mathrm{d}\mathbf{y} \right) \mathrm{d}t \le C |\log r|^{-1/2} \to 0 \text{ as } r \to 0.$$

This is a slightly weaker condition than (2-7), but still sufficient to pass to the limit in the nonlinear term. Moreover, (2-8) makes it possible to establish the validity of the image method for weak solutions in the half plane, since it allows one to prove that Delort-type weak solutions actually satisfy a stronger notion of weak solution, called *boundary-coupled weak solution*, for which the non-penetration condition is assumed in a stronger way than simply the trace sense.

Let us now seek a weak vorticity formulation in a general bounded domain $\mathfrak{D}$ with boundary. Assume that $\mathfrak{D}$ has $k \ge 0$ disjoint holes, so that $\partial\mathfrak{D} = \Gamma_0 \cup_{i=1}^k \Gamma_i$, with Γ_0 being the outer boundary and Γ_i, $i = 1, \dots, k$ being the boundaries of each of the holes. We will need an analogue of the Biot-Savart law, that is, a means of writing the velocity in terms of its curl. In domains with non-trivial topology, in order to recover velocity from vorticity, it is necessary to assign the *circulation* around each hole, γ_i:

$$\gamma_i \equiv \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n}^\perp \mathrm{d}S, \quad i = 1, \dots, k. \tag{2-9}$$

If the vector field $\mathbf{u}$ is divergence-free and its curl is a bounded Radon measure then the circulation is well-defined. In fact, it is enough that $\operatorname{div}\mathbf{u} \in \mathfrak{BM}(\mathfrak{D})$ and $\operatorname{curl}\mathbf{u} \in \mathfrak{BM}(\mathfrak{D})$ for the entire tangential component $\mathbf{u}\cdot\mathbf{n}^\perp$ to be a well-defined distribution on $\partial\mathfrak{D}$, see Iftimie, Lopes Filho, Nussenzveig Lopes, and Sueur [2017]. An analogous fact had already been noted previously by Chen and Frid [2003].

We will also make use of the *harmonic measures* $\mathbf{w}_i$, solutions of the following boundary-value problem:

$$\begin{cases} \Delta \mathbf{w}_i = 0, & \text{in } \mathfrak{D}, \\ \mathbf{w}_i = \delta_{i\ell}, & \text{on } \Gamma_\ell,\ \ell = 1, \dots, k. \\ \mathbf{w}_i = 0, & \text{on } \Gamma_0. \end{cases}$$

Lastly, let $\mathbf{H}_i$ denote a basis of *harmonic vector fields*, so that each $\mathbf{H}_i$ is divergence-free, curl-free, and the circulation of $\mathbf{H}_i$ around Γ_ℓ is $\delta_{i\ell}$.

With this notation we can express velocity in terms of vorticity and circulations.

Proposition 2.4. (See Iftimie, Lopes Filho, Nussenzveig Lopes, and Sueur [2017].) *Let $\omega \in \mathfrak{BM}(\mathfrak{D})$ and fix γ_i, $i = 1, \dots, k$. If $\mathbf{u} \in L^2(\mathfrak{D})$ is a divergence-free vector field such that* $\operatorname{curl}\mathbf{u} = \omega$ *and for which the circulations of* $\mathbf{u}$ *around Γ_i are γ_i, then*

$$\mathbf{u} = K^{\mathfrak{D}}[\omega] + \sum_{i=1}^k \left(\gamma_i + \int_{\mathfrak{D}} \mathbf{w}_i \omega \, \mathrm{d}\mathbf{x} \right) \mathbf{H}_i .$$

It is possible to express the weak velocity formulation in Definition 2.1 in terms of vorticity, using the symmetrization technique. To this end we observe that, up to a constant, there is a one-to-one correspondence between scalar functions $\varphi \in C_c^\infty(\mathbb{R}_+ \times \overline{\mathfrak{D}})$, constant *in a neighborhood* of $\partial\mathfrak{D}$, with possibly *different* constants *on each* Γ_i, and test vector fields $\Phi \in C_c^\infty(\mathbb{R}_+ \times \mathfrak{D})$, divergence-free, given through the map $\varphi \mapsto \Phi = \nabla^\perp \varphi$.

We introduce:

$$H_\varphi^{\mathfrak{D}} = H_\varphi^{\mathfrak{D}}(t, \mathbf{x}, \mathbf{y}) = \frac{1}{2}\Big(K^{\mathfrak{D}}(\mathbf{x}, \mathbf{y}) \cdot \nabla\varphi(t, \mathbf{x}) + K^{\mathfrak{D}}(\mathbf{y}, \mathbf{x}) \cdot \nabla\varphi(t, \mathbf{y}) \Big).$$

Let $\mathcal{Y} \equiv \{\varphi \in C_c^\infty(\mathbb{R}_+ \times \overline{\mathfrak{D}}) \mid \varphi$ is constant *in a neighborhood* of each $\Gamma_i\}$, as in Iftimie, Lopes Filho, Nussenzveig Lopes, and Sueur [2017].

Proposition 2.5. (See Iftimie, Lopes Filho, Nussenzveig Lopes, and Sueur [ibid.].) *For all $\varphi \in \mathcal{Y}$, $H_\varphi^{\mathfrak{D}}$ is bounded on $\mathbb{R}_+ \times \overline{\mathfrak{D}} \times \overline{\mathfrak{D}}$, continuous if $\mathbf{x} \neq \mathbf{y}$ and vanishes on $\mathbb{R}_+ \times \partial(\mathfrak{D} \times \mathfrak{D})$, $\mathbf{x} \neq \mathbf{y}$.*

The main result in Iftimie, Lopes Filho, Nussenzveig Lopes, and Sueur [ibid.] is:

Theorem 2.6. *The vector field $\mathbf{u} \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathfrak{D}))$, such that*

$$\operatorname{curl} \mathbf{u} = \omega \in L^\infty(\mathbb{R}_+; \mathcal{BM}(\mathfrak{D}))$$

and whose circulations around Γ_i are γ_i, $i = 0, \dots, k$, is a weak solution of the Euler equations, with initial data $\mathbf{u}_0 \in L^2(\mathfrak{D})$, if and only if the following identity holds, for all $\varphi \in \mathcal{Y}$:

$$\begin{aligned}
\text{(2-10)} \quad & \int_0^\infty \int_{\mathfrak{D}} \partial_t\varphi\, \omega \, \mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty \gamma_0(t)\partial_t\varphi(t, \cdot)\big|_{\Gamma_0} \mathrm{d}t \\
& + \sum_{i=1}^k \int_0^\infty \gamma_i(t)\partial_t\varphi(t, \cdot)\big|_{\Gamma_i} \mathrm{d}t + \int_{\mathfrak{D}} \varphi(0, \cdot)\omega_0 \, \mathrm{d}\mathbf{x} - \gamma_0(0)\varphi(0, \cdot)\big|_{\Gamma_0} \\
& + \sum_{i=1}^k \gamma_i(0)\varphi(0, \cdot)\big|_{\Gamma_i} + \int_0^\infty \int_{\mathfrak{D}} \int_{\mathfrak{D}} H_\varphi^{\mathfrak{D}}(t, \mathbf{x}, \mathbf{y})\omega(t, \mathbf{x})\omega(t, \mathbf{y}) \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}t \\
& + \sum_{i=1}^k \int_0^\infty \left(\gamma_i + \int_{\mathfrak{D}} \mathbf{w}_i(\mathbf{y})\omega(t, \mathbf{y}) \, \mathrm{d}\mathbf{y} \right) \int_{\mathfrak{D}} \mathbf{H}_i(\mathbf{x}) \cdot \nabla\varphi(t, \mathbf{x})\omega(t, \mathbf{x}) \, \mathrm{d}\mathbf{x}\mathrm{d}t = 0.
\end{aligned}$$

If $\mathbf{u}_0 \in L^2(\mathfrak{D})$, $\operatorname{curl} \mathbf{u}_0 = \omega_0 \in \mathcal{BM}_+(\mathfrak{D})$ then it follows immediately from Delort's theorem and Theorem 2.6 that identity (2-10) holds true for all $\varphi \in \mathcal{Y}$.

Now, for smooth flows, circulation around material curves – curves which are transported by the velocity field – is a conserved quantity; this is known as Kelvin's circulation theorem. This includes circulation around the boundaries of the holes, Γ_i. Theorem 2.6 deals with non-smooth flows and highlights that, at this level of regularity, it is possible that circulation *may not be conserved*.

Identity (2-10) is the *weak vorticity formulation* in a bounded domain. It is equivalent to the weak formulation in Definition 2.1, yet it *explicitly* incorporates the possibility of violation of Kelvin's circulation theorem, something not apparent in the weak velocity formulation (2-4).

We introduce a stronger notion of weak solution, adapted from Lopes Filho, Nussenzveig Lopes, and Xin [2001]. Let $\overline{\mathfrak{Y}} = \{\varphi \in C_c^\infty(\mathbb{R}_+ \times \overline{\mathfrak{D}}) \mid \varphi \text{ is constant on each } \Gamma_i\}$.

Definition 2.7. Let $\omega_0 \in \mathfrak{BM}(\mathfrak{D}) \cap H^{-1}(\mathfrak{D})$ and $\gamma_{i,0} \in \mathbb{R}$, $i = 1, \dots, k$. We say that $(\omega, \gamma_1, \dots, \gamma_k)$, $\omega \in L^\infty(\mathbb{R}_+; \mathfrak{BM}(\mathfrak{D}) \cap H^{-1}(\mathfrak{D}))$, $\gamma_i \in L^\infty(\mathbb{R}_+)$, $\gamma_0(\cdot) = \int_{\mathfrak{D}} \omega + \sum_{i=1}^k \gamma_i(\cdot)$, is a *boundary-coupled weak solution* of the Euler equations in $\mathfrak{D}$, with initial data ω_0, $\mathbf{u}_0 = K^{\mathfrak{D}}[\omega_0] + \sum_{i=1}^k \left(\gamma_{i,0} + \int_{\mathfrak{D}} \mathbf{w}_i \omega_0\right) \mathbf{H}_i$, if, for every $\varphi \in \overline{\mathfrak{Y}}$, the identity (2-10) holds true.

Assume that $\mathbf{u}$ is a weak solution which is a weak-$*$ limit, in $L^\infty(\mathbb{R}_+; L^2(\mathfrak{D}))$, of a sequence $\{\mathbf{u}^n\}$ of *exact* smooth solutions with initial data $\{\mathbf{u}_0^n\}$ tending to $\mathbf{u}_0$.

Theorem 2.8. (See Iftimie, Lopes Filho, Nussenzveig Lopes, and Sueur [2017].) *If* $\omega = \operatorname{curl} \mathbf{u} \in \mathfrak{BM}_{c,+}(\mathfrak{D}) \cap H^{-1}(\mathfrak{D})$ *then*

1. $\gamma_i(t) \geq \gamma_{i,0}$;
2. *If* γ_i *is conserved, for all* $i = 1, \dots, k$*, then the solution is boundary-coupled.*

If $\omega_0 \in L^1(\mathfrak{D}) \cap H^{-1}(\mathfrak{D})$ *then* γ_i *is conserved,* $i = 1, \dots, k$ *and the solution is boundary-coupled.*

The role of the conservation of circulation is to ensure that there is no *vorticity concentration* on $\partial\mathfrak{D}$, that is, that ω^n does not attribute mass to the boundary, uniformly in n, see (2-7).

Boundary-coupled weak solutions have additional interesting properties. The net force exerted by the fluid on the boundary is defined, for smooth solutions, as

$$\int_{\partial\mathfrak{D}} p\mathbf{n} \, \mathrm{d}S,$$

where $p = p(t, \mathbf{x})$ is the scalar pressure. Weak solutions, however, lack sufficient smoothness to have a well-defined net force. A weak formulation of the net force, consistent with the definition for smooth flows, can be shown to be a well-defined object *if and only if* the weak solution is boundary-coupled, see Iftimie, Lopes Filho, Nussenzveig Lopes, and Sueur [ibid.]. The same is true of the net torque.

A weak solution is said to satisfy the weak-strong uniqueness property if, given a strong solution, any weak solution with the same initial data must coincide with it. It is noted in Wiedemann [2017] that boundary-coupled weak solutions who satisfy an energy inequality also satisfy the weak-strong uniqueness property. Moreover, such boundary-coupled weak solutions are, in fact, *dissipative weak solutions*, a notion introduced by P.-L. Lions [1996].

We close this section by observing that, while it is clearly desirable to produce boundary-coupled weak solutions, there is no analogue of (2-8), which was key to prove the existence of boundary-coupled weak solutions in the half-plane, for bounded domains. It remains an open problem whether boundary-coupled weak solutions exist in bounded domains, for vortex sheet initial data, even with a distinguished sign.

3 Vanishing viscosity limit and convergence criteria

In this section we return to the problem of vanishing viscosity, or the infinite Reynolds number limit, in a bounded domain with rigid boundary. Here we are concerned with the *mathematical problem* of whether a vector field which is a limit of vanishing viscosity is, in some sense, a solution of the inviscid equations. Hindsight gathered from our previous discussion suggests that, in view of the mismatch between the no slip (1-2) and non-penetration (1-3) boundary conditions, the key issue is to control the production of vorticity at the boundary as $\nu \to 0$.

Let us assume hereafter that $\mathfrak{D}$ is a bounded, connected and simply connected domain with smooth boundary. We will restrict our discussion to two dimensional fluid flow since, from the point-of-view of rigorous mathematical analysis, the Euler and Navier-Stokes equations are better understood in $2D$. We will point out which, among the results we will discuss, have extensions to $3D$.

We begin by recalling the Lighthill principle, which relates the flux of vorticity through the boundary to the tangential derivative of pressure. To see this we assume that the ν-Navier-Stokes equations are valid *up to* the boundary of the domain and we use the no slip condition to deduce, formally, that

$$0 = \nabla p^\nu + \nu \Delta \mathbf{u}^\nu \text{ on } \mathbb{R}_+ \times \partial\mathfrak{D}.$$

Next note that $\Delta \mathbf{u}^\nu = \nabla^\perp \omega^\nu$ and take the inner product with $\mathbf{n}^\perp$ to deduce that

$$\frac{\partial \omega^\nu}{\partial \mathbf{n}} = -\frac{1}{\nu}\frac{\partial p^\nu}{\partial \mathbf{n}^\perp}, \text{ on } (0,+\infty) \times \partial\mathfrak{D}. \tag{3-1}$$

The vorticity formulation of the ν-Navier-Stokes equations is

$$\begin{cases} \partial_t \omega^\nu + (\mathbf{u}^\nu \cdot \nabla)\omega^\nu = -\nabla p^\nu + \nu \Delta \omega^\nu, & \text{in } (0,+\infty) \times \mathfrak{D}; \\ \operatorname{div} \mathbf{u}^\nu = 0, & \text{in } [0,+\infty) \times \mathfrak{D}; \\ \operatorname{curl} \mathbf{u}^\nu = \omega^\nu, & \text{in } [0,+\infty) \times \mathfrak{D}, \end{cases} \tag{3-2}$$

subject to the boundary condition (3-1), and given an initial data.

The Lighthill principle (3-1) was derived *formally* and it does not appear to be particularly useful in general, since estimates on the tangential derivative of the pressure are not usually available.

For certain flows with symmetry, however, (3-1) proves to be sufficient to establish the vanishing viscosity limit. For two-dimensional flows with circular symmetry, see Matsui [1994], Bona and J. Wu [2002], Lopes Filho, Mazzucato, and Nussenzveig Lopes [2008], and Lopes Filho, Mazzucato, Nussenzveig Lopes, and Taylor [2008]. Flows with plane-parallel symmetry were studied in Mazzucato and Taylor [2008], Mazzucato, Niu, and X. Wang [2011], and Gie, Kelliher, Lopes Filho, Mazzucato, and Nussenzveig Lopes [2017] and paralell-pipe flows were discussed in Mazzucato and Taylor [2011], Han, Mazzucato, Niu, and X. Wang [2012], and Gie, Kelliher, Lopes Filho, Mazzucato, and Nussenzveig Lopes [2017].

In general, if there is no mismatch, that is, if the Euler velocity happens to vanish at the boundary at all times $t > 0$, then trivial energy estimates yield convergence of the Navier-Stokes solutions to the inviscid solution as viscosity vanishes.

Let us discuss the general problem; we are interested in *criteria* for the vanishing viscosity limit to hold, that is, conditions under which the limit of solutions of ν-Navier-Stokes, $\nu \to 0$, are solutions, in some sense, of the Euler equations. The baseline result of this nature is known as the *Kato condition*, which we state below.

Fix $T > 0$.

Theorem 3.1. (Kato [1984].) *Let* $\mathbf{u}^\nu \in L^\infty((0,T); L^2_\sigma(\mathfrak{D})) \cap L^2((0,T); H^1_0(\mathfrak{D}))$ *be a Leray-Hopf solution of the* ν*-Navier-Stokes equations in* $\mathfrak{D} \subset \mathbb{R}^d$, $d = 2, 3$, $\nu > 0$, *with initial data* $\mathbf{u}_0 \in L^2_\sigma(\mathfrak{D})$, $\mathbf{u}_0 \cdot \mathbf{n} = 0$ *on* $\partial\mathfrak{D}$. *Assume that there exists a smooth solution* $\mathbf{u}^0$ *of the Euler equations in* $\mathfrak{D}$*, satisfying the non-penetration boundary condition, with initial data* $\mathbf{u}_0$. *Then* $\mathbf{u}^\nu \to \mathbf{u}^0$ *strongly in* $L^\infty((0,T); L^2(\mathfrak{D}))$ *if and only if*

$$\nu \int_0^T \int_{\Gamma^\nu} |D\mathbf{u}^\nu|^2 \, d\mathbf{x} \to 0 \text{ as } \nu \to 0, \tag{3-3}$$

where Γ^ν *is a region near the boundary of thickness* $\mathcal{O}(\nu)$.

Remark 3.2. Kato proved the equivalence between vanishing viscosity and several other statements. Among these, he showed that the Kato condition (3-3) holds if and only if

$\mathbf{u}^\nu \rightharpoonup \mathbf{u}^0$ weak-$*$ in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathfrak{D}))$, paving the way towards understanding the vanishing viscosity limit as a weak limit.

The proof of the Kato criterion is by energy methods, with the necessity of (3-3) deriving immediately from the energy inequality for Leray-Hopf solutions of Navier-Stokes. To show that (3-3) is sufficient Kato introduced the *Kato correctors*, which are certain cut-off functions near the boundary. He then uses these correctors on the Euler solution, "correcting it" at a distance δ, from the boundary. The error term is estimated using the smoothness of the Euler solution. The choice $\delta = \mathcal{O}(\nu)$ allows to estimate terms involving $D\mathbf{u}^\nu$, which are not *a priori* bounded, provided (3-3) holds true.

The Kato criterion has been revisited by several authors. For instance, in Temam and X. Wang [1997] and X. Wang [2001] the full gradient is replaced by the tangential derivative along the boundary; in Kelliher [2007] the gradient is substituted by vorticity. It should be noted that the "Kato layer" is not a physical boundary layer, only a mathematical device.

The key issue in what follows is the fact that the Kato criterion assumes the underlying Euler flow to be smooth. It has been the main point of these notes that this is not what is expected, typically, in the vanishing viscosity limit. We have argued that, due to the mismatch between no slip and non-penetration boundary conditions, vortex sheets arise naturally in the infinite Reynolds number limit. These structures are idealizations of thin shear layers near the boundary. Experiments and the Prandtl asymptotic boundary layer model suggest that these thin layers are unstable and may detach from the boundary, entraining the bulk of the fluid. This justifies the study of the inviscid problem with vortex sheet initial data. Now, solutions of the Euler equations with vortex sheet initial data are far from smooth. Yet these nonsmooth solutions are precisely what is expected at the vanishing viscosity limit! We conclude this discussion with a different criterion for the vanishing viscosity limit to hold, in the two-dimensional case – one which allows the limiting flow to have vortex sheet regularity.

Let $\mathfrak{D} \subset \mathbb{R}^2$ be a smooth, connected and simply connected, bounded domain.

Theorem 3.3. (See Constantin, Lopes Filho, Nussenzveig Lopes, and Vicol [2017].) *Fix $T > 0$. Let $\{\nu_n\}$ be a sequence of positive real numbers such that $\nu_n \to 0$ and choose $\mathbf{u}_0 \in L^2_\sigma(\mathfrak{D})$, $\mathbf{u}_0 \cdot \mathbf{n} = 0$ on $\partial\mathfrak{D}$.*

Let $\mathbf{u}^n \in L^\infty(0,T; L^2(\mathfrak{D})) \cap L^2(0,T; H^1_0(\mathfrak{D}))$ be a Leray-Hopf solution of the ν_n-Navier-Stokes equations, subject to the no slip boundary condition, and with initial data $\mathbf{u}_0$.

Set $\omega^n = \omega^n(t,\cdot) = \operatorname{curl} \mathbf{u}^n \equiv \nabla^\perp \cdot \mathbf{u}^n(t,\cdot)$.

Suppose, additionally, that:

1. *$\mathbf{u}^n \rightharpoonup \mathbf{u}^\infty$ weak-$*$ in $L^\infty(0,T; L^2(\mathfrak{D}))$;*
2. *$\{\omega^n\}$ is uniformly bounded in $L^\infty(0,T; L^1_{\text{loc}}(\mathfrak{D}))$;*

3. For any $\mathfrak{K} \subset\subset \mathfrak{D}$ *we have*

$$\sup_n \int_0^T \left(\sup_{\mathbf{x}\in\mathfrak{K}} \int_{B(\mathbf{x};r)} |\omega^n(\mathbf{y},t)| \mathrm{d}\mathbf{y} \right) \mathrm{d}t \to 0 \text{ as } r \to 0.$$

Then $\mathbf{u}^\infty$ *is a weak solution of the Euler equations in* $\mathfrak{D}$ *with initial data* $\mathbf{u}_0$.

Theorem 3.3 is inspired on a result contained in Constantin and Vicol [2017]. Its proof is adapted from Schochet's proof of Delort's theorem, see Schochet [1995].

Assumptions 2. and 3. encode the expected behavior of vortex sheets, yet there is no proof that they hold true along viscous approximations in general. In light of our previous discussion, however, these are very natural hypotheses, in contrast with what is assumed in the Kato criterion.

Items 2. and 3. are strictly *local* hypotheses; nothing is assumed about the behavior near the boundary. Surprisingly, the limit flow $\mathbf{u}^\infty$ does satisfy the non-penetration boundary condition, but only in the trace sense in $L^\infty((0,T); H^{-1/2}(\partial\mathfrak{D}))$; this highlights just how unsatisfactory is the weak formulation. Of course, it is hopeless to obtain a boundary-coupled weak solution in this way.

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