PROC. INT. CONG. OF MATH. – 2018
Rio de Janeiro, Vol. 3 (2599–2614)

INTEGRABLE COMBINATORICS

PHILIPPE DI FRANCESCO

Abstract

We explore various combinatorial problems mostly borrowed from physics, that share the property of being continuously or discretely integrable, a feature that guarantees the existence of conservation laws that often make the problems exactly solvable. We illustrate this with: random surfaces, lattice models, and structure constants in representation theory.

1 Introduction

In this note we deal with combinatorial *objects*, mostly provided by physical systems or models. These are: random surfaces, lattice models, and structure constants. We will illustrate how to solve the various problems, mostly of exact or asymptotic enumeration, via a panel of techniques borrowed from pure combinatorics as well as statistical physics. The *tools* utilized are: generating functions, transfer matrices, bijections, matrix integrals, determinants, field theory, etc.

We have organized this collection of problems according to some common or analogous properties, essentially related to their underlying symmetries. Among them the most powerful is the notion of *integrability*. The latter appears under many different guises. The first form is continuous: Existence of conservation laws, flat connections, commuting transfer matrices, links to the Yang-Baxter equation, infinite dimensional algebra symmetries. The second form is discrete: Existence of discrete integrals of motion in discrete time.

What kind of results did we obtain? Solving a system completely usually entails a complete understanding of correlation functions within the model. This can be achieved by explicit diagonalization of the transfer matrix or Hamiltonians, explicit computation of generating functions, or derivation of complete systems of equations for averaged quantities. As usual in statistical physics, one also investigates the asymptotic (or thermodynamic) properties of the systems, leading to such results as asymptotic enumeration,

I acknowledge support by the Morris and Gertrude Fine foundation.
MSC2010: primary 05Axx; secondary 82B23, 05E05, 05E10, 82B41.

identification of phases and their separations, identification of underlying field theoretical descriptions of fluctuations.

One of the main features common to all the problems listed above is some kind of connection to discrete *paths* or *trees*, the two simplest and most fundamental combinatorial objects. The constructs of this note place these two main characters in new non-standard contexts which shed some new light on their deep significance. Together they form the basis of the notion of combinatorial integrability, i.e. the properties shared by combinatorial problems that connect them to discrete or continuous integrable systems.

The paper is organized as follows. In Section 2, we explore discretized models of random surfaces, whether Lorentzian in 1+1 dimensions (Section 2.1), or Euclidian in 2 dimensions (Section 2.2). Both type of models display integrability respectively via commuting transfer matrices and discrete integrals of motion, which allows to solve them explicitly.

In Section 3, we first describe the 6 vertex model and its many combinatorial wonders (Sect. 3.1), among which a description of Alternating Sign Matrices (ASM), and links to special types of plane partitions, as well as the geometry of nilpotent matrix varieties.

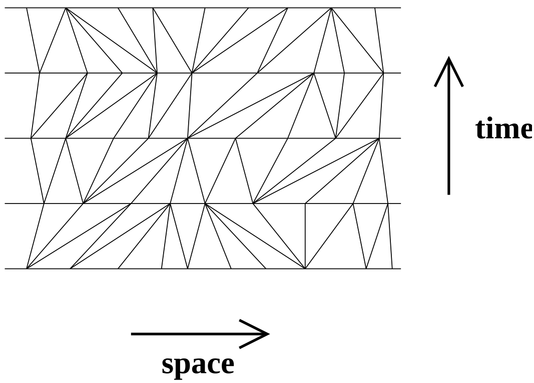
Section 4 focuses on Lie algebraic structures with a description of Whittaker vectors (Section 4.1) using path models, and of graded multiplicities in tensor products occurring in inhomogeneous quantum spin chains with Lie symmetry (Section 4.2). The description of the latter involves a construction of difference operators that generalize the celebrated Macdonald operators, and can be understood within the context of polynomial representations of Double-Affine Hecke Algebras (DAHA), quantum toroidal algebras, and Elliptic Hall Algebras (EHA).

Finally we gather some important open problems in Section 5, which we think should shape the future of integrable combinatorics.

Acknowledgments. Much of the work described in this note is the fruit of collaborations with various authors: R. Behrend, O. Golinelli, E. Guitter, C. Itzykson, R. Kedem, M. Lapa, N. Reshetikhin, R. Soto Garrido, P. Zinn-Justin, J.-B. Zuber. I also benefited from many important discussions with fellow mathematicians and physicists: O. Babelon, A. Borodin, J.-E. Bourgine, I. Cherednik, I. Corwin, S. Fomin, M. Gekhtmann, M. Jimbo, R. Kenyon, M. Kontsevich, Y. Matsuo, G. Musiker, A. Negut, A. Okounkov, V. Pasquier, M. Shapiro, O. Schiffman, V. Retakh.

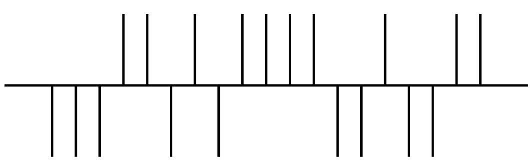
2 Random surfaces

2.1 1+1-dimensional Lorentzian triangulations and (continuous) integrability. Lorentzian triangulations Di Francesco, Guitter, and Kristjansen [2000] are used as a discrete model for quantum gravity in one (space)+1 (time) dimension. Pure gravity deals with fluctuations of such bare space-times, while matter theories include for instance particle systems in interction defined on such space-times. General relativity expresses the relation between those fluctuations and in particular the associated fluctuations of the metric, area and curvature of the space-time and the matter stress tensor. The model for a fluctuating 1+1D space-time is an arrangement of triangles organized into time slices as depicted below:



Fluctuations of space are represented by random arrangements of triangles in each time slice, while the time direction remains regular. These triangulations are best described in the dual picture by considering triangles as vertical half-edges and pairs of triangles that share a time-like (horizontal) edge as vertical edges between two consecutive time-slices. We may now concentrate on the transition between two consecutive time-slices which typically looks like:

(2-1)



with say i half-edges on the bottom and j on the top (here for instance we have $i = 9$ and $j = 10$). To take into account the *area* and *curvature* of space-time, we may introduce a Boltzmann weight g per triangle (i.e. per trivalent vertex in the dual picture) and a weight a per pair of consecutive triangles in a time-slice pointing in the same direction (both up or both down). The total weight of a configuration is the product of all local weights pertaining ot it. It is easy to see that these weights correspond to a transfer operator $\mathcal{T}(g, a)$ which describes the configurations of one time-slice with a total i of up-pointing triangles and j of down-pointing ones. The matrix element between states i and j reads:

$$T(g,a)_{i,j} = (ag)^{i+j} \sum_{k=0}^{\mathrm{Min}(i,j)} \binom{i}{k}\binom{j}{k} a^{-2k} \qquad (i,j \geq 0)$$

Equivalently, the double generating function for matrix elements of $\mathcal{T}(g,a)$ reads:

$$f_{T(g,a)}(z,w) = \sum_{i,j\geq 0} T(g,a)_{i,j} z^i w^j = \frac{1}{1-ga(z+w)-g^2(1-a^2)zw} \tag{2-2}$$

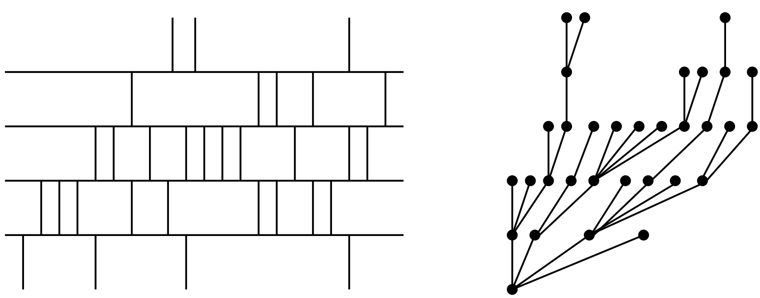
This model turns out to provide one of the simplest examples of quantum integrable system, with an infinite family of commuting transfer matrices. Indeed, we have:

Theorem 2.1 (Di Francesco, Guitter, and Kristjansen [2000]). *The transfer matrices $T(g,a)$ and $T(g',a')$ commute if and only if the parameters (g,a,g',a') are such that $\varphi(g,a) = \varphi(g',a')$ where:*

$$\varphi(g,a) = \frac{1-g^2(1-a^2)}{ag}$$

This and the explicit generating function (2-2) were extensively used in Di Francesco, Guitter, and Kristjansen [ibid.] to diagonalize $\mathcal{T}(g,a)$ and to compute correlation functions of boundaries in random Lorentzian triangulations.

For suitable choices of boundary conditions, the dual random Lorentzian triangulations introduced above may be viewed as random plane trees. This is easily realized by gluing all the bottom vertices of successive parallel vertical edges (no interlacing with the neighboring time slices). A typical such example reads:



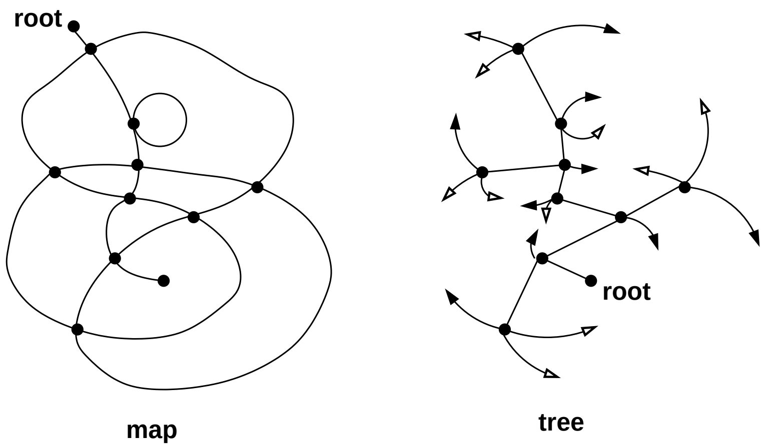
Note that the tree is naturally rooted at its bottom vertex.

To summarize, we have unearthed some integrable structure attached naturally to plane trees, one of the most fundamental objects of combinatorics. Note that in tree language the weights are respectively g^2 per edge, and a per pair of consecutive descendent edges and per pair of consecutive leaves at each vertex (from left to right).

2.2 2-dimensional Euclidian tessellations and (discrete) integrability. As opposed to Lorentzian gravity, the 2D Euclidian theory involves fluctuations of both space and time, allowing for space-times that look like random surfaces of arbitrary genus. Those are discretized by tessellations. A powreful tool for enumerating those maps was provided by matrix integrals, allowing to keep track of the area, as well as the genus via the size N of the matrices (see Ref. Di Francesco, Ginsparg, and J. Zinn-Justin [1995] and references therein). In a parallel way, the field-theoretical descriptions of the (critical) continuum limit of two-dimensional quantum gravity (2DQG) have blossomed into a more complete picture with identification of relevant operators and computation of their correlation functions Di Francesco and Kutasov [1991]. This was finally completed by an understanding in terms of the intersection theory of the moduli space of curves with punctures and fixed genus Kontsevich [1991]. Remarkably, in all these approaches a common integrable structure is always present. It takes the form of commuting flows in parameter space. However, a number of issues were left unadressed by the matrix/field theoretical approaches. What about the intrinsic geometry of the random surfaces? Correlators must be integrated w.r.t. the position of their insertions, leaving us only with topological invariants of the surfaces. But how to keep track for instance of the geodesic distances between two marked points of a surface, while at the same time summing over all surface fluctuations?

Answers to these questions came from a better combinatorial understanding of the structure of the (planar) tessellations involved in the discrete models. And, surprisingly, yet another form of integrability appeared. Following pioneering work of Schaeffer [1997], it was observed that all models of discrete 2DQG with a matrix model solution (at least in genus 0) could be expressed as statistical models of (decorated) trees, and moreover, the decorations allowed to keep track of geodesic distances between some faces of the tessellations. Marked planar tessellations are known as rooted planar maps in combinatorics. They correspond to connected graphs (with vertices, edges, faces) embedded into the Riemann sphere. Such maps are usually represented on a plane with a distinguished face "at infinity", and a marked edge adjacent to that face. The degree of a vertex is the number of distinct half-edges adjacent to it, the degree of a face is the number of edges forming its boundary.

Consider the example of tetravalent (degree 4) planar maps with 2 univalent (degree 1) vertices, one of which is singled out as the root. The Schaeffer bijection associates to each of these a unique rooted tetravalent (with inner vertices of degree 4) tree called blossom-tree, with two types of leaves (black and white), and such that there is exactly one black leaf attached to each inner vertex:



This is obtained by the following cutting algorithm: travel clockwise along the bordering edges of the face at infinity, starting from the root. For each traversed edge, cut it if and only if after the cut, the new graph remains connected, and replace the two newly formed half-edges by a black and a white leaf respectively in clockwise order. Once the loop is traveled, this has created a larger face at infinity. Repeat the procedure until the graph has only one face left: it is the desired blossom-tree, which we reroot at the other univalent vertex, while the original root is transformed into a white leaf.

This bijection allows to keep track of the geodesic distance between the 2 univalent vertices. Defining $R_n(g)$ to be the generating function for maps with geodesic distance $\leq n$ between the two univalent vertices, we have the following recursion relation Bouttier, Di Francesco, and Guitter [2003]:

$$R_n(g) = 1 + gR_n(g)\left(R_{n+1}(g) + R_n(g) + R_{n-1}(g)\right) \tag{2-3}$$

easily derived by inspecting the environment of the vertex attached to the root of the tree when it exists. It is supplemented by boundary conditions $R_{-1}(g) = 0$ and $\lim_{n\to\infty} R_n(g) = R(g) = \frac{1-\sqrt{1-12g}}{6g}$, the generating function of maps with no geodesic distance constraint.

Equation (2-3), viewed as governing the evolution of the quantity $R_n(g)$ in the discrete time variable n, is a classical *discrete integrable system*. By this we mean that it has a *discrete integral of motion*, expressed as follows. The function $\phi(x, y)$ defined by

$$\phi(x, y) = xy(1 - g(x + y)) - x - y \tag{2-4}$$

is such that for any solution S_n of the recursion relation (2-3), the quantity $\phi(S_n, S_{n+1})$ is independent of n. In other words, the quantity $\phi(S_n, S_{n+1})$ is conserved modulo (2-3).

(This is easily shown by factoring $\phi(S_n, S_{n+1}) - \phi(S_{n-1}, S_n)$.). This conservation law gives in particular a relation of the form:

$$\phi(R_n(g), R_{n+1}(g)) = \lim_{m\to\infty} \phi(R_m(g), R_{m+1}(g)) = \phi(R(g), R(g))$$

. It turns out that we can solve explicitly for $R_n(g)$:

Theorem 2.2 (Bouttier, Di Francesco, and Guitter [ibid.]). *The generating function $R_n(g)$ for rooted tetravalent planar maps with two univalent vertices at geodesic distance at most n from each other reads:*

$$R_n(g) = R(g)\frac{(1-x(g)^{n+1})(1-x(g)^{n+4})}{(1-x(g)^{n+2})(1-x(g)^{n+3})}$$

where $x(g)$ is the unique solution of the equation: $x + \frac{1}{x} + 4 = \frac{1}{gR(g)^2}$ with a power series expansion of the form $x(g) = g + O(g^2)$.

The form of the solution in Theorem 2.2 is that of a discrete soliton with tau-function $\tau_n = 1 - x(g)^n$. Imposing more general boundary conditions on the equation (2-3) leads to elliptic solutions of the same flavor. The solution above and its generalizations to many classes of planar maps Di Francesco [2005] have allowed for a better understanding of the critical behavior of surfaces and their intrinsic geometry. Recent developments include planar three-point correlations, as well as higher genus results.

To summarize, we have seen yet another integrable structure emerge in relation to (decorated) trees. This is of a completely different nature from the one discussed in Section 2.1, where a quantum integrable structure was attached to rooted planar trees. Here we have a discrete classical integrable system, with soliton-like solutions.

3 Lattice models

3.1 The six-vertex model and beyond. The Six Vertex (6V) model is the archetypical example of 2D integrable lattice model. It is defined on domains of the square lattice $\mathbb{Z}^2$, with configurations obtained by orienting all the nearest neigbor edges in such a way that there are exactly to ingoing and two outgoing edges incident to each vertex in the interior of the domain (ice rule). This gives rise to $\binom{4}{2} = 6$ local vertex configurations, to which one usually attaches Boltzmann weights. The integrability of the model becomes manifest if we parametrize these weights with rapidities (spectral parameters) that are derived from the relevant R-matrix solution of the Yang-Baxter equation. This ensures that the system has an infinite set of commuting transfer matrices, similarly to Section 2.1. This property ensures that the transfer matrix is explicitly diagonalizable by means of Bethe Ansatz

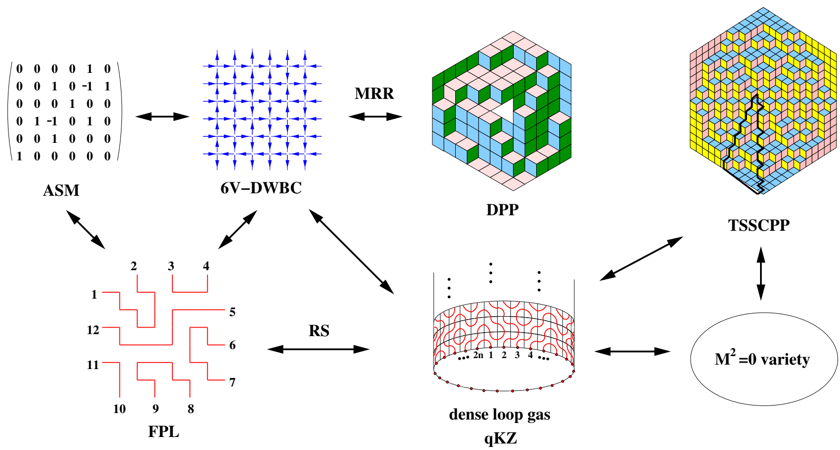


Figure 1: The combinatorial family of ASMs. From left to right: ASM, 6V-DWBC and FPL , all in bijection; dense O(n) loop gas: its groundstate/limiting probability vector satisfies the qKZ equation, the components measure FPL correlations (RS conjecture); DPP: their refined evaluation matches that of ASMs; TSSCPP: their refined enumeration matches a sum rule for qKZ solutions at generic q and $z_i = 1$; Variety $M^2 = 0$: its degree/multidegree matches solutions of qKZ for $q = 1$.

techniques. Note that a certain limit of the transfer matrix yields the Hamiltonian of the anisotropic XXZ spin chain.

A remarkable web of connections between many combinatorial objects relates to the 6V model, as shown in Figure 1. The configurations of the 6V model on a square grid of size n with the so-called Domain Wall Boundary Conditions (DWBC) that all boundary horizontal arrows are pointed towards the grid while all boundary vertical arrows point outward, are in bijection with Alternating Sign Matrices (ASM), namely matrices with entries in $\{-1, 0, 1\}$ with alternance of 1's and -1's along each row and column, and with row and column sums all equal to 1. This observation allowed Kuperberg [1996] to come up with an elegant proof of the ASM conjecture for the number of $n \times n$ ASMs, soon after the combinatorial proof of Zeilberger [1996]. Another bijection related the configurations of the 6V model with DWBC to so-called Fully Packed Loops (FPL) obtained by coloring edges of the square grid in such a way that exactly 2 edges incident to each inner vertex are colored, while every other boundary edge is colored. The colored edges form closed loops or open paths connecting boundary edges by pairs (such a pattern of connections is equivalent to non-crossing partitions or link patterns). The latter remark prompted the celebrated Razumov and Stroganov [2004] conjecture that FPL configurations with prescribed boundary edge connections form the Perron-Frobenius eigenvector of the XXZ spin chain at its *combinatorial point* (when all Boltzmann weights are 1), when expressed in the link pattern basis (in the O(n) model formulation of the spin chain), later proved by Cantini and Sportiello [2011]. Among the many developments around the conjecture, we used the link between the combinatorial problem and solutions of the quantum Knizhnik-Zamolodchikov (qKZ) equation for the O(n) model Di Francesco and P. Zinn-Justin [2005] and P. Zinn-Justin and Di Francesco [2008], which led us to connections with the geometry of the variety of square zero matrices Di Francesco and P. Zinn-Justin [2006]. Beyond bijections other sets of combinatorial objects have the same cardinality A_n. These are the Totally Symmetric Self-Complementary Plane Partitions (TSSCPP) on one hand and the Descending Plane Partitions (DPP) on the other. Both classes of objects can be formulated as the rhombus tilings of particular domains of the triangular lattice with particular symmetries. We found a proof Behrend, Di Francesco, and P. Zinn-Justin [2012] and Behrend, Di Francesco, and P. Zinn-Justin [2013] of the Mills-Robbins-Rumsey refined ASM-DPP conjecture Mills, Robbins, and Rumsey [1983] using generating functions similar to (2-2), however no bijection is known to this day.

It turns out that, among other formulations, the 6V model with DWBC may be expressed as a model of osculating paths, namely non-intersecting paths with unit steps $(1, 0)$ and $(0, 1)$, from the W border edges to the N border edges of the grid with allowed "kissing" or *osculating* vertices visited by two paths that do not cross. The latter path formulation allows to predict the arctic curve phenomenon (i.e. the sharp separation between ordered and disordered phases) for random ASMs Colomo and Sportiello [2016] as well as for random ASMs with a vertical reflection symmetry Di Francesco and Lapa [2018].

4 Lie algebras, quantum spin chains and CFT

In this section, we present combinatorial problems/approaches to algebra representation theory.

4.1 Whittaker vectors and path models. Whittaker vectors Kostant [1978] are fundamental objects in the representation theory of Lie algebras expressed in terms of Chevalley generators $\{e_i, f_i, h_i\}_{i=\epsilon}^r$ and relations (here $\epsilon = 1$ for finite algebras, and $\epsilon = 0$ for affine algebras). They are instrumental in constructing Whittaker functions, which are eigenfunctions for the quantum Toda operators, namely Schroedinger operators with kinetic and potential terms coded by the root system of the algebra. Given a Verma module $V_\lambda = \mathcal{U}(\{f_i\})|\lambda\rangle$ with highest weight vector $|\lambda\rangle$, a Whittaker vector $v_{\mu,\lambda}$ with parameters μ_i is an element of the completion of V_λ (an infinite series in V_λ) that satisfies the relations $e_i v_{\lambda,\mu} = \mu_i v_i$ for all i. It is unique upon a choice of normalization. In Ref.Di Francesco, Kedem, and Turmunkh [2017] we developed a general approach to the computation of Whittaker vectors by expanding them on the "words" of the form $f_{i_1} f_{i_2} \cdots f_{i_k}|\lambda\rangle$ for arbitrary $i_j \in [\epsilon, r]$ and $k \geq 0$. The latter are of course not linearly independent, but we found some extremely nice and simple expression for their coefficients $c_{i_1,\ldots,i_k}$ in the expression of $v_{\lambda,\mu} = \sum_{k\geq 0} \sum_{i_1,\ldots,i_k \in [\epsilon,r]} c_{i_1,\ldots,i_k}\, f_{i_1} f_{i_2} \cdots f_{i_k}|\lambda\rangle$ (the normalization is chosen so that the empty word has coefficient $c_{0,0,\ldots,0} = 1$). We made the observation that the set of vectors of the form $f_{i_1} f_{i_2} \cdots f_{i_k}|\lambda\rangle$ is in bijection with the set of paths $p \in \mathcal{P}$ on the positive cone Q_+ of the root lattice, from the origin to some root $\beta = (\beta_i)_{i\in[\epsilon,r]}$ where β_i is the number of occurrences of f_i in the vector (or of the letter i in the word). Indeed, the steps of p are taken successively in the directions $i_k, i_{k-1}, \ldots, i_1$ in $[\epsilon, r]$. We denote by $|p\rangle = f_{i_1} f_{i_2} \cdots f_{i_k}|\lambda\rangle$. We have the following general result.

Theorem 4.1 (Di Francesco, Kedem, and Turmunkh [ibid.]). *For finite or affine Lie algebras, the Whittaker vector $v_{\lambda,\mu}$ is expressed as:*

$$v_{\lambda,\mu} = \sum_{\beta\in Q_+} \prod_i \mu_i^{\beta_i} \sum_{\text{paths } p:0\to\beta} w(p)|p\rangle$$

where the weight $w(p)$ is a product of local weights:

$$w(p) = \prod_{\substack{\gamma\in Q_+^* \\ \gamma \text{ vertex of } p}} \frac{1}{v(\gamma)}, \qquad v(\gamma) = (\lambda+\rho|\gamma) - \frac{1}{2}(\gamma|\gamma)$$

This construction was shown in Di Francesco, Kedem, and Turmunkh [ibid.] to extend to the A type quantum algebras $\mathcal{U}_q(\mathfrak{sl}_{r+1})$ with local weights depending on both the vertex and the direction of the step from the vertex.

This new formulation of Whittaker vectors yields a very simple proof for the fact that the corresponding Whittaker function obeys the quantum Toda equation (classical case), the Lamé-like deformed Toda equation (affine, non-critical case) or the q-difference Toda equation (quantum case).

The q-Whittaker functions are known to be a degenerate limit of Macdonald polynomials, when $t \to 0$ or ∞. This suggests to look for a possible path formulation of Macdonald polynomials.

4.2 Fusion product, Q-system cluster algebra and Macdonald theory.

4.2.1 Graded characters and quantum Q-system. We now turn to the combinatorial problem of finding the fusion coefficients $\mathrm{Mult}_q(\otimes KR_{\alpha,n}^{\otimes n_{\alpha,n}}; V_\lambda)$ for graded tensor product decompositions of so-called Kirillov and Reshetikhin [1987] (KR) modules $KR_{\alpha,n}$ ($\alpha \in [1,r]; n \in \mathbb{N}$) of a Lie algebra into irreducibles. The grading, inherited from the loop algebra Feigin and Loktev [1999] (fusion product) turns out to have many equivalent formulations: as energy of the corresponding crystal, as linearized energy in the Bethe Ansatz solutions of the corresponding inhomogeneous isotropic XXX quantum spin chain (the physical system at the origin of the problem, from which so-called fermionic formulas for graded multiplicities were derived Hatayama, Kuniba, Okado, Takagi, and Yamada [1999]). Recently, we have found yet another interpretation of this grading, as being provided by the canonical quantum deformation of the cluster algebra of the so-called Q-system for the algebra Di Francesco and Kedem [2014].

The latter is a recursive system for scalar variables $Q_{\alpha,n}$ $\alpha = 1, 2..., r$, $n \in \mathbb{Z}$. For the case of A_r it takes the form:

$$Q_{\alpha,n+1}\, Q_{\alpha,n-1} = (Q_{\alpha,n})^2 - Q_{\alpha+1,n}\, Q_{\alpha-1,n}$$

with boundary conditions $Q_{0,n} = Q_{r+1,n} = 1$ for all $n \in \mathbb{Z}$. It is satisfied by the KR characters $Q_{\alpha,n} = \chi_{KR_{\alpha,n}}(\mathbf{x})$. This is a discrete integrable system: there exist r algebraically independent polynomial quantities of the Q's that are conserved modulo the system, which we can view as describing evolution of the variables $Q_{\alpha,n}$ in discrete time n Di Francesco and Kedem [2010, 2018]. Taking advantage of this property, we were able to solve such systems by means of (strongly) non-intersecting lattice paths (the solution involves also a new continued fraction rearrangement theory Di Francesco and Kedem [2010] and Di Francesco [2011a]).

Such systems exist for all finite and affine algebras, and were shown to be particular sets of mutations in some cluster algebras Kedem [2008] and Di Francesco and Kedem [2009]. As such, they admit a natural quantization into a q-deformed, non-commutative Q-system, coined the quantum Q-system. For the case A_r it reads:

$$Q_{\alpha,n}\, Q_{\beta,n+1} = q^{\lambda_{\alpha,\beta}}\, Q_{\beta,n+1}\, Q_{\alpha,n} \tag{4-1}$$

$$q^{\lambda_{\alpha,\alpha}}\, Q_{\alpha,n+1}\, Q_{\alpha,n-1} = (Q_{\alpha,n})^2 - q\, Q_{\alpha+1,n}\, Q_{\alpha-1,n} \tag{4-2}$$

where $\lambda_{\alpha,\beta} = (C^{-1})_{\alpha,\beta}$, C the Cartan matrix of the algebra, and with the boundary conditions $Q_{0,n} = Q_{r+1,0} = 1$, $Q_{r+2,n} = 0$ for all $n \in \mathbb{Z}$. The non-commuting variables $Q_{\alpha,n}$ play the role of quantized KR characters. The path solutions of the classical Q-system admit a non-commutative version using non-commutative continued fractions Di Francesco and Kedem [2011].

For simplicity let us perform a change of variables. Define $A = Q_{r+1,1}$ and the degree operator Δ such that $\Delta\, Q_{\alpha,n} = q^{\alpha n} Q_{\alpha,n}\, \Delta$. Then the new variables $M_{\alpha,n} := q^{-\frac{1}{2}\lambda_{\alpha,\alpha}(n+r+1)}\, Q_{\alpha,n} \Delta^{\frac{\alpha}{r+1}}$ are subject to the new "M-system":

$$M_{\alpha,n}\, M_{\beta,n+1} = q^{\mathrm{Min}(\alpha,\beta)}\, M_{\beta,n+1}\, M_{\alpha,n} \tag{4-3}$$

$$q^{\alpha}\, M_{\alpha,n+1}\, M_{\alpha,n-1} = (M_{\alpha,n})^2 - M_{\alpha+1,n}\, M_{\alpha-1,n} \tag{4-4}$$

with boundary conditions $M_{0,n} = 1$ and $M_{r+1,n} = A^n \Delta$.

Theorem 4.2. *We have the following representation of the M-system via difference operators acting on the ring of symmetric functions of $N = r+1$ variables $(x_1, \ldots, x_N)$:*

$$M_{\alpha,n} = \sum_{\substack{I\subset[1,N]\\ |I|=\alpha}} x_I^n \prod_{\substack{i\in I\\ j\notin I}} \frac{x_i}{x_i - x_j}\Gamma_I \tag{4-5}$$

where $x_I = \prod_{i\in I} x_i$, $\Gamma_I = \prod_{i\in I}\Gamma_i$, and Γ_i is the multiplicative q-shift operator on the i-th variable: $(\Gamma_i\, f)(x_1, x_2, \ldots, x_N) = f(x_1, \ldots, x_{i-1}, q x_i, x_{i+1}, \ldots, x_N)$, and with moreover $A = x_1 x_2 \cdots x_N$, and $\Delta = \Gamma_1\Gamma_2\cdots\Gamma_N$.

Let $\chi_{\mathbf{n}}(q;\mathbf{x})$, $\mathbf{n} = \{n_{\alpha,n}\}_{\alpha\in[1,r];n\in[1,k]}$, $\mathbf{x} = (x_1, \ldots, x_N)$, denote the graded character: $\chi_{\mathbf{n}}(q;\mathbf{x}) = \sum_\lambda \mathrm{Mult}_q(\otimes KR_{\alpha,n}^{\otimes n_{\alpha,n}}; V_\lambda)\chi_\lambda(\mathbf{x})$, i.e. the generating function for graded multiplicities, where the irreducible characters $\chi_\lambda(\mathbf{x}) = s_\lambda(\mathbf{x})$ are the Schur functions.

Theorem 4.3 (Di Francesco and Kedem [2018]). *The graded character for the tensor product $\otimes KR_{\alpha,n}^{\otimes n_{\alpha,n}}$ reads*

$$\chi_{\mathbf{n}}(q^{-1};\mathbf{x}) = q^{-a(\mathbf{n})} \prod_{j=k}^{1}\prod_{\alpha=1}^{r} (M_{\alpha,j})^{n_{\alpha,j}} \cdot 1$$

with $a(\mathbf{n}) = \frac{1}{2}\sum_{i,j,\alpha,\beta} n_{\alpha,i}\mathrm{Min}(i,j)\mathrm{Min}(\alpha,\beta)n_{\beta,j} - \frac{1}{2}\sum_{i,\alpha} i\alpha\, n_{\alpha,i}$.

The results above were so far only derived for the A case, but they can be extended to B, C, D types Di Francesco and Kedem [n.d.].

4.2.2 From Cluster algebra to quantum toroidal and Elliptic Hall algebras. The form of the difference operator (4-5) is reminiscent of the celebrated Macdonald operators Macdonald [1995], for which the Macdonald polynomials form a complete family of eigenvectors. These were best understood in the context of Double Affine Hecke Algebra Cherednik [2005], in the functional representation. We actually found that a certain action of the natural $SL_2(\mathbb{Z})$ symmetry of DAHA produces the following *generalized Macdonald* difference operators in the functional representation:

$$\mathfrak{M}_{\alpha,n} = \sum_{\substack{I\subset[1,N]\\|I|=\alpha}} x_I^n \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j}\Gamma_I \tag{4-6}$$

We note the relation $M_{\alpha,n} = \lim_{t\to\infty} t^{-\alpha(N-\alpha)}\,\mathfrak{M}_{\alpha,n}$. We may therefore identify the quantum cluster algebra solution of the Q-system with the $t \to \infty$ (so-called dual q-Whittaker) limit of generalized Macdonald operators.

These operators allow to construct a representation of the so-called Ding–Iohara–Miki Ding and Iohara [1997] and Miki [2007] (DIM) or quantum toroidal $\mathfrak{gl}_1$ algebra as follows Di Francesco and Kedem [2017b]. Introduce the currents:

$$e(z) := \frac{q^{1/2}}{1-q}\sum_{n\in\mathbb{Z}} q^{n/2} z^n \mathfrak{M}_{1,n} \quad \text{and} \quad \mathfrak{f}(z) := e(z)\big|_{q\to q^{-1},t\to t^{-1}} \tag{4-7}$$

as well as the series

$$\psi^{\pm}(z) := \prod_{i=1}^{N} \frac{(1-q^{-\frac{1}{2}}t(zx_i)^{\pm1})(1-q^{\frac{1}{2}}t^{-1}(zx_i)^{\pm1})}{(1-q^{-\frac{1}{2}}(zx_i)^{\pm1})(1-q^{\frac{1}{2}}(zx_i)^{\pm1})} \in \mathbb{C}[[z^{\pm1}]] \tag{4-8}$$

Theorem 4.4 (Di Francesco and Kedem [ibid.])**.** *The currents and series* $(e, \mathfrak{f}, \psi^{\pm})$ *form a level* $(0,0)$ *representation of the DIM algebra.*

In particular, we have the following exchange relation:

$$g(z,w)\,e(z)\,e(w) + g(w,z)\,e(w)\,e(z) = 0, \;\; g(z,w) = (z-qw)(z-t^{-1}w)(z-q^{-1}tw)$$

In the $t \to \infty$ limit this reduces to the $\mathcal{U}_{\sqrt{q}}(\widehat{\mathfrak{sl}_2})$ upper Borel subalgebra relation, while the DIM relations go to some interesting degeneration, directly connected to the quantum

Q-system cluster algebra Di Francesco and Kedem [2018, 2017b]. Algebra relations allow in particular to derive a quantum determinant formula for $M_{\alpha,n}$ as a polynomial of the $M_n := M_{1,n}$'s. Let us introduce the currents $m_\alpha(z) := \sum_{n\in\mathbb{Z}} z^n M_{\alpha,n}$, and in particular $m(z) := m_1(z) = \lim_{t\to\infty} t^{1-N} \frac{1-q}{q^{1/2}} e(z)$, then:

Theorem 4.5 (Di Francesco and Kedem [2017a]).

$$m_\alpha(z) = \left\{\left(\prod_{1\le i<j\le\alpha}\left(1-q\frac{u_j}{u_i}\right)\right) m(u_1)m(u_2)\cdots m(u_\alpha)\right\}\Big|_{(u_1u_2\cdots u_\alpha)^n} \tag{4-9}$$

where the subscript stands for the coefficient of $(u_1u_2\cdots u_\alpha)^n$.

Note also that the function of $\mathbf{u}$ in (4-9) is a multi-current generating series. Let us define $M_{a_1,\dots,a_\alpha}$ to be the coefficient of $u_1^{a_1}\cdots u_\alpha^{a_\alpha}$ in this series. There is a very nice expression for $M_{a_1,\dots,a_\alpha}$ as a quantum determinant, involving a sum over Alternating Sign Matrices. This is because the quantity $\prod_{i<j} v_i + \lambda v_j$ is the λ-determinant $\lambda\det(V_n)$ (as defined by Robbins and Rumsey [1986]) of the Vandermonde matrix $V_n := (v_i^{n-j})_{1\le i,j\le n}$. We denote by ASM_n the set of ASM of size n. The inversion number of an ASM is the quantity $I(A) = \sum_{i>k,j<\ell} A_{i,j}A_{k,\ell}$. We also denote by $N(A)$ the number of -1's in A. Let us also define the column vector $v = (n-1, n-2, .., 1, 0)^t$, and for each ASM A we denote by $m_i(A) := (Av)_i$. Then we have the explicit formula, obtained by taking $\lambda = -q$ for the λ-determinant of the $\alpha\times\alpha$ Vandermonde matrix V_α:

$$\prod_{1\le i<j\le\alpha}(v_i - qv_j) = \sum_{A\in ASM_n}(-q)^{I(A)-N(A)}(1-q)^{N(A)}\prod_{i=1}^{n} v_i^{m_i(A)}$$

Combining this with (4-9), we deduce the following compact expression for the quantum determinant:

Theorem 4.6. *The quantum determinant of the matrix* $\big(M_{a_j+i-j}\big)_{1\le i,j\le\alpha}$ *reads:*

$$M_{a_1,\dots,a_\alpha} = \Big|\big(M_{a_j+i-j}\big)_{1\le i,j\le\alpha}\Big|_q \tag{4-10}$$

$$= \sum_{A\in ASM_\alpha}(-q)^{I(A)-N(A)}(1-q)^{N(A)}\prod_{i=1}^{\alpha} M_{a_i+\alpha-i-m_i(A)} \tag{4-11}$$

Finally, we use a known isomorphism Schiffmann and Vasserot [2011] between the Spherical DAHA with the Elliptic Hall algebra (EHA) to make the connection between generalized Macdonald operators and a functional representation of the EHA Di Francesco and Kedem [2017a]. From this connection, we obtain new algebraic relations between the operators $\mathfrak{M}_{\alpha,n}$, inherited from the "skinny triangle" relations of Burban and Schiffmann [2012].

To conclude, the results of this section are so far valid for the A type only. It would be interesting to investigate the (t-deformed) algebraic structures hiding behind the B,C,D cases as well.

5 Open problems

In this note, we described various instances of discrete or continuous integrability in combinatorial problems. A recurrent theme is the ability to rephrase said combinatorial problems in terms of paths or trees.

Paths are very important objects. Equipped with non-commutative weights, paths allow to describe non-commutative monomials in finitely generated non-commutative algebras. We have encountered a few instances of this in the present note. A crucial question remains open: how to deal with *families* of non-intersecting non-commutative paths? We have found specific answers in the cases where the non-commutativity is "under control", e.g. in the case of quantum path weights with specific q-commuting relations Di Francesco [2011b]. More general non-commuting weighted paths can be described via the theory of quasideterminants I. Gelfand, S. Gelfand, Retakh, and Wilson [2005], however it remains to find a good theory for non-intersecting non-commutative paths, and perhaps a non-commutative version of the Lindström-Gessel-Viennot (LGV) determinant formula.

Interacting paths are a combinatorial version of interacting fermions. Starting from NILP, we can turn on some interaction, by for instance allowing paths to touch without crossing (osculating paths) and attaching a contact energy to such instances. As shown in the case of the 6V/ASM model, such interactions still allow for solving, together with the machinery of integrable lattice models. As another indication, he so-called tangent method for determining phase transitions in large random tilings (arctic curves) seems to apply to interacting paths as well. The determinant form of the partition function for the 6V model with DWBC would point to the fact that there should exist determinant formulas for interacting paths that generalize LGV. It seems that a number of interacting path problems are still open, and a systematic study is in order.

References

Roger E. Behrend, Philippe Di Francesco, and P. Zinn-Justin (2012). "On the weighted enumeration of alternating sign matrices and descending plane partitions". *J. Combin. Theory Ser. A* 119.2, pp. 331–363. MR: 2860598 (cit. on p. 2607).

Roger E. Behrend, Philippe Di Francesco, and Paul Zinn-Justin (2013). "A doubly-refined enumeration of alternating sign matrices and descending plane partitions". *J. Combin. Theory Ser. A* 120.2, pp. 409–432. arXiv: 1202.1520. MR: 2995049 (cit. on p. 2607).

J. Bouttier, Philippe Di Francesco, and E. Guitter (2003). "Geodesic distance in planar graphs". *Nuclear Phys. B* 663.3, pp. 535–567. MR: 1987861 (cit. on pp. 2604, 2605).

Igor Burban and Olivier Schiffmann (2012). "On the Hall algebra of an elliptic curve, I". *Duke Math. J.* 161.7, pp. 1171–1231. MR: 2922373 (cit. on p. 2613).

Luigi Cantini and Andrea Sportiello (2011). "Proof of the Razumov-Stroganov conjecture". *J. Combin. Theory Ser. A* 118.5, pp. 1549–1574. MR: 2771600 (cit. on p. 2607).

Ivan Cherednik (2005). *Double affine Hecke algebras*. Vol. 319. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, pp. xii+434. MR: 2133033 (cit. on p. 2611).

F. Colomo and A. Sportiello (2016). "Arctic curves of the six-vertex model on generic domains: the tangent method". *J. Stat. Phys.* 164.6, pp. 1488–1523. MR: 3541190 (cit. on p. 2608).

Philippe Di Francesco (2005). "Geodesic distance in planar graphs: an integrable approach". *Ramanujan J.* 10.2, pp. 153–186. MR: 2194521 (cit. on p. 2605).

– (2011a). "Discrete integrable systems, positivity, and continued fraction rearrangements". *Lett. Math. Phys.* 96.1-3, pp. 299–324. MR: 2788915 (cit. on p. 2609).

– (2011b). "Quantum A_r Q-system solutions as q-multinomial series". *Electron. J. Combin.* 18.1, Paper 176, 17. MR: 2836811 (cit. on p. 2613).

Philippe Di Francesco, P. Ginsparg, and J. Zinn-Justin (1995). "2D gravity and random matrices". *Phys. Rep.* 254.1-2, p. 133. MR: 1320471 (cit. on p. 2603).

Philippe Di Francesco, E. Guitter, and C. Kristjansen (2000). "Integrable 2D Lorentzian gravity and random walks". *Nuclear Phys. B* 567.3, pp. 515–553. MR: 1741653 (cit. on pp. 2600, 2602).

Philippe Di Francesco and Rinat Kedem (n.d.). work in progress (cit. on p. 2611).

– (2009). "Q-systems as cluster algebras. II. Cartan matrix of finite type and the polynomial property". *Lett. Math. Phys.* 89.3, pp. 183–216. MR: 2551179 (cit. on p. 2610).

– (2010). "Q-systems, heaps, paths and cluster positivity". *Comm. Math. Phys.* 293.3, pp. 727–802. MR: 2566162 (cit. on p. 2609).

– (2011). "Non-commutative integrability, paths and quasi-determinants". *Adv. Math.* 228.1, pp. 97–152. MR: 2822229 (cit. on p. 2610).

– (2014). "Quantum cluster algebras and fusion products". *Int. Math. Res. Not. IMRN* 10, pp. 2593–2642. MR: 3214279 (cit. on p. 2609).

– (Apr. 2017a). "(t,q) Q-systems, DAHA and quantum toroidal algebras via generalized Macdonald operators". arXiv: 1704.00154 (cit. on p. 2612).

– (2017b). "Quantum Q systems: from cluster algebras to quantum current algebras". *Lett. Math. Phys.* 107.2, pp. 301–341. arXiv: 1606.09052. MR: 3600918 (cit. on pp. 2611, 2612).

– (2018). "Difference equations for graded characters from quantum cluster algebra". *Transform. Groups*, pp. 1–34. arXiv: 1505.01657 (cit. on pp. 2609, 2610, 2612).

Philippe Di Francesco, Rinat Kedem, and Bolor Turmunkh (2017). "A path model for Whittaker vectors". *J. Phys. A* 50.25, pp. 255201, 35. MR: 3659140 (cit. on pp. 2608, 2609).

Philippe Di Francesco and D. Kutasov (1991). "Correlation functions in 2D string theory". *Phys. Lett. B* 261.4, pp. 385–390. MR: 1110973 (cit. on p. 2603).

Philippe Di Francesco and Matthew F. Lapa (2018). "Arctic Curves in path models from the Tangent Method". *J. Phys. A* 51, pp. 155202, 63. arXiv: 1711.03182 (cit. on p. 2608).

Philippe Di Francesco and P. Zinn-Justin (2005). "Around the Razumov-Stroganov conjecture: proof of a multi-parameter sum rule". *Electron. J. Combin.* 12, Research Paper 6, 27. MR: 2134169 (cit. on p. 2607).

– (2006). "Inhomogeneous model of crossing loops and multidegrees of some algebraic varieties". *Comm. Math. Phys.* 262.2, pp. 459–487. arXiv: 0412031. MR: 2200268 (cit. on p. 2607).

Jintai Ding and Kenji Iohara (1997). "Generalization of Drinfeld quantum affine algebras". *Lett. Math. Phys.* 41.2, pp. 181–193. MR: 1463869 (cit. on p. 2611).

B. Feigin and S. Loktev (1999). "On generalized Kostka polynomials and the quantum Verlinde rule". In: *Differential topology, infinite-dimensional Lie algebras, and applications*. Vol. 194. Amer. Math. Soc. Transl. Ser. 2. Amer. Math. Soc., Providence, RI, pp. 61–79. MR: 1729359 (cit. on p. 2609).

Israel Gelfand, Sergei Gelfand, Vladimir Retakh, and Robert Lee Wilson (2005). "Quasideterminants". *Adv. Math.* 193.1, pp. 56–141. MR: 2132761 (cit. on p. 2613).

G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada (1999). "Remarks on fermionic formula". In: *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*. Vol. 248. Contemp. Math. Amer. Math. Soc., Providence, RI, pp. 243–291. MR: 1745263 (cit. on p. 2609).

Rinat Kedem (2008). "Q-systems as cluster algebras". *J. Phys. A* 41.19, pp. 194011, 14. MR: 2452184 (cit. on p. 2610).

A. N. Kirillov and N. Yu. Reshetikhin (1987). "Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras". *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 160.Anal. Teor. Chisel i Teor. Funktsiĭ. 8, pp. 211–221, 301. MR: 906858 (cit. on p. 2609).

M. L. Kontsevich (1991). "Intersection theory on the moduli space of curves". *Funktsional. Anal. i Prilozhen.* 25.2, pp. 50–57, 96. MR: 1142208 (cit. on p. 2603).

Bertram Kostant (1978). "On Whittaker vectors and representation theory". *Invent. Math.* 48.2, pp. 101–184. MR: 507800 (cit. on p. 2608).

Greg Kuperberg (1996). "Another proof of the alternating-sign matrix conjecture". *Internat. Math. Res. Notices* 3, pp. 139–150. MR: 1383754 (cit. on p. 2607).

I. G. Macdonald (1995). *Symmetric functions and Hall polynomials*. Second. Oxford Mathematical Monographs. With contributions by A. Zelevinsky, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, pp. x+475. MR: 1354144 (cit. on p. 2611).

Kei Miki (2007). "A (q,γ) analog of the $W_{1+\infty}$ algebra". *J. Math. Phys.* 48.12, pp. 123520, 35. MR: 2377852 (cit. on p. 2611).

W. H. Mills, David P. Robbins, and Howard Rumsey Jr. (1983). "Alternating sign matrices and descending plane partitions". *J. Combin. Theory Ser. A* 34.3, pp. 340–359. MR: 700040 (cit. on p. 2607).

A. V. Razumov and Yu. G. Stroganov (2004). "Combinatorial nature of the ground-state vector of the $\mathrm{O}(1)$ loop model". *Teoret. Mat. Fiz.* 138.3, pp. 395–400. MR: 2077318 (cit. on p. 2607).

David P. Robbins and Howard Rumsey Jr. (1986). "Determinants and alternating sign matrices". *Adv. in Math.* 62.2, pp. 169–184. MR: 865837 (cit. on p. 2612).

Gilles Schaeffer (1997). "Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees". *Electron. J. Combin.* 4.1, Research Paper 20, 14. MR: 1465581 (cit. on p. 2603).

O. Schiffmann and E. Vasserot (2011). "The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials". *Compos. Math.* 147.1, pp. 188–234. MR: 2771130 (cit. on p. 2612).

Doron Zeilberger (1996). "Proof of the alternating sign matrix conjecture". *Electron. J. Combin.* 3.2. The Foata Festschrift, Research Paper 13, approx. 84. MR: 1392498 (cit. on p. 2607).

P. Zinn-Justin and Philippe Di Francesco (2008). "The quantum Knizhnik–Zamolodchikov equation, completely symmetric self-complementary plane partitions, and alternating-sign matrices". *Teoret. Mat. Fiz.* 154.3, pp. 387–408. arXiv: 0703015. MR: 2431554 (cit. on p. 2607).

Received 2017-11-21.

PHILIPPE DI FRANCESCO

Department of Mathematics
University of Illinois at Urbana-Champaign
1409 West Green Street
Urbana, IL 61801
USA

and

Institut de Physique Théorique du Commissariat à l'Energie Atomique
Unité de Recherche associée du CNRS
Université Paris-Saclay
CEA Saclay/IPhT/Bat 774
F-91191 Gif sur Yvette Cedex
France
philippe@illinois.edu