ON THE STRUCTURE OF MEASURES CONSTRAINED BY LINEAR PDES

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Abstract

The aim of this note is to present some recent results on the structure of the singular part of measures satisfying a PDE constraint and to describe some applications.

1 Introduction

We describe recent advances obtained by the authors and collaborators concerning the structure of singularities in measures satisfying a linear PDE constraint. Besides its own theoretical interest, understanding the structure of singularities of PDE-constrained measures turns out to have several (sometimes surprising) applications in the calculus of variations, geometric measure theory, and metric geometry.

Let $\mathcal{A}$ be a k'th-order linear constant-coefficient differential operator acting on $\mathbb{R}^N$-valued functions, i.e.

$$\mathcal{A}u := \sum_{|\alpha| \le k} A_\alpha \partial^\alpha u, \qquad u \in C^\infty(\Omega; \mathbb{R}^N),$$

where $A_\alpha \in \mathbb{R}^n \otimes \mathbb{R}^N$ are linear maps from $\mathbb{R}^N$ to $\mathbb{R}^n$ and $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}$ for every multindex $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$.

The starting point of the investigation is the following:

Question 1.1. *Let $\mu \in \mathfrak{M}(\Omega, \mathbb{R}^N)$ be an $\mathbb{R}^N$-valued Radon measure on an open set $\Omega \subset \mathbb{R}^d$ and let μ be $\mathcal{A}$-free, i.e. μ solves the system of linear PDEs*

$$(1\text{-}1) \qquad \mathcal{A}\mu := \sum_{|\alpha| \le k} A_\alpha \partial^\alpha \mu = 0 \qquad \textit{in the sense of distributions.}$$

What can be said about the singular part[1] of μ?

In answering the above question a prominent role is played by the *wave cone* associated with the differential operator $\mathcal{A}$:

$$\Lambda_{\mathcal{A}} := \bigcup_{|\xi|=1} \ker \mathbb{A}^k(\xi) \subset \mathbb{R}^N \qquad \text{with} \qquad \mathbb{A}^k(\xi) = (2\pi i)^k \sum_{|\alpha|=k} A_\alpha \xi^\alpha,$$

where we have set $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$.

Roughly speaking, $\Lambda_{\mathcal{A}}$ contains all the amplitudes along which the system (1-1) is *not elliptic*. Indeed if we assume that $\mathcal{A}$ is homogeneous, $\mathcal{A} = \sum_{|\alpha|=k} A_\alpha \partial^\alpha$, then it is immediate to see that $\lambda \in \mathbb{R}^N$ belongs to $\Lambda_{\mathcal{A}}$ if and only if there exists a non-zero $\xi \in \mathbb{R}^d \setminus \{0\}$ such that $\lambda h(x \cdot \xi)$ is $\mathcal{A}$-free for all smooth functions $h \colon \mathbb{R} \to \mathbb{R}$. In other words, "one-dimensional" oscillations and concentrations are possible only if the amplitude (direction) belongs to the wave cone. For this reason the wave cone plays a crucial role in the compensated compactness theory for sequences of $\mathcal{A}$-free maps, see Murat [1978, 1979], Tartar [1979], Tartar [1983], and DiPerna [1985], and in convex integration theory, see for instance Chiodaroli, De Lellis, and Kreml [2015], Chiodaroli, Feireisl, Kreml, and Wiedemann [2017], De Lellis and Székelyhidi [2009, 2012], De Lellis and Székelyhidi [2013], Székelyhidi and Wiedemann [2012], and Isett [2016] and the references cited therein. However, all these references are concerned with oscillations only, not with concentrations.

Since the singular part of a measure can be thought of as containing "condensed" concentrations, it is quite natural to conjecture that $|\mu|^s$-almost everywhere the polar vector $\frac{d\mu}{d|\mu|}$ belongs to $\Lambda_{\mathcal{A}}$. This is indeed the case and the main result of De Philippis and Rindler [2016]:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $\mathcal{A}$ be a k'th-order linear constant-coefficient differential operator as above, and let $\mu \in \mathfrak{M}(\Omega; \mathbb{R}^N)$ be an $\mathcal{A}$-free Radon measure on Ω with values in $\mathbb{R}^N$. Then,*

$$\frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathcal{A}} \qquad \text{for } |\mu|^s\text{-a.e. } x \in \Omega.$$

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[1] If not specified, the terms "singular" and "absolutely continuous" always refer to the Lebesgue measure. We also recall that, thanks to the Radon-Nikodym theorem, a vector-valued measure μ can be written as

$$\mu = \frac{d\mu}{d|\mu|} d|\mu| = g \mathcal{L}^d + \frac{d\mu}{d|\mu|} d|\mu|^s$$

where $|\mu|$ is the total variation measure, $g \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\mathcal{L}^d$ is the Lebesgue measure.

Remark 1.3.   Theorem 1.2 is also valid in the situation

(1-2) $$\mathcal{A}\mu = \sigma \qquad \text{for some } \sigma \in \mathcal{M}(\Omega; \mathbb{R}^n).$$

This can be reduced to the setting of Theorem 1.2 by defining

$$\tilde{\mu} := (\mu, \sigma) \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{N+n})$$

and $\tilde{\mathcal{A}}$ (with an additional 0'th-order term) such that (1-2) is equivalent to $\tilde{\mathcal{A}}\tilde{\mu} = 0$. It is easy to check that $\Lambda_{\tilde{\mathcal{A}}} = \Lambda_{\mathcal{A}} \times \mathbb{R}^n$ and that for $|\mu|$-almost every point $\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}$ is proportional to $\frac{\mathrm{d}\mu}{\mathrm{d}|\tilde{\mu}|}$.

One interesting feature of Theorem 1.2 is that it gives information about the directional structure of μ at singular points (the "shape of singularities"). Indeed, it is not hard to check that for all "elementary" $\mathcal{A}$-free measures of the form

(1-3) $$\mu = \lambda\nu, \qquad \text{where} \qquad \lambda \in \Lambda_{\mathcal{A}}, \; \nu \in \mathcal{M}^+(\mathbb{R}^d),$$

the scalar measure ν is necessarily translation invariant along directions that are orthogonal to the *characteristic set*

$$\Xi(\lambda) := \left\{ \xi \in \mathbb{R}^d \; : \; \lambda \in \ker \mathbb{A}(\xi) \right\}.$$

Note that $\Xi(\lambda)$ turns out to be a subspace of $\mathbb{R}^d$ whenever $\mathcal{A}$ is a first-order operator. In this case, the translation invariance of ν in the directions orthogonal to $\Xi(\lambda)$ is actually the best information one can get from (1-3).

In the case of operators of order $k > 1$, due to the lack of linearity of the map $\xi \mapsto \mathbb{A}^k(\xi)$ for $k > 1$, the structure of elementary $\mathcal{A}$-free measures is more complicated and not yet fully understood.

In the next sections we will describe some applications of Theorem 1.2 to the following problems:

- The description of the singular part of derivatives of BV- and BD-maps.

- Lower semicontinuity for integral functionals defined on measures.

- Characterization of generalized gradient Young measures.

- The study of the sharpness of the Rademacher's theorem .

- Cheeger's conjecture on Lipschitz differentiability spaces.

In Section 7 we will sketch the proof of Theorem 1.2.

2   Structure of singular derivatives

Let $f : \mathbb{R}^\ell \otimes \mathbb{R}^d \to [0, \infty)$ be a linear growth integrand with $f(A) \sim |A|$ for $|A|$ large. Consider the following variational problem:

$$\int_\Omega f(\nabla u) \, \mathrm{d}x \to \min, \qquad u \in \mathrm{C}^1(\Omega; \mathbb{R}^\ell) \text{ with given boundary conditions.}$$

It is well known that in order to apply the Direct Method of the calculus of variations one has to relax the above problem to a setting where it is possible to obtain both compactness of minimizing sequences and lower semicontinuity of the functional with respect to some topology, usually the weak(*) topology in some function space. Due to the linear growth of the integrand the only easily available estimate on a minimizing sequence (u_k) is an a-priori bound on the L^1-norm of their derivativesx:

$$\sup_k \int_\Omega |\nabla u_k| \, \mathrm{d}x < \infty.$$

It is then quite natural to relax the functional to the space $\mathrm{BV}(\Omega, \mathbb{R}^\ell)$ of functions of *bounded variation*, i.e. those functions $u \in \mathrm{L}^1(\Omega; \mathbb{R}^\ell)$ whose distributional gradient is a matrix-valued Radon measure. A fine understanding of the possible behavior of measures arising as derivatives is then fundamental to study the weak* lower semicontinuity of the functional as well as its relaxation to the space BV.

In this respect, in Ambrosio and Giorgi [1988] Ambrosio and De Giorgi proposed the following conjecture:

Question 2.1.   *Is the singular part of the derivative of a function $u \in \mathrm{BV}(\Omega; \mathbb{R}^\ell)$, which is usually denoted by $D^s u$, always of rank one? Namely, is it true that*

$$\frac{\mathrm{d} D^s u}{\mathrm{d} |D^s u|}(x) = a(x) \otimes b(x)$$

for $|D^s u|$-a.e. x and some $a(x) \in \mathbb{R}^\ell$, $b(x) \in \mathbb{R}^d$?

Their conjecture was motivated by the fact that this structure is trivially true for the so-called *jump part* of $D^s u$ (which is always of the form $[u] \otimes n \, \mathcal{H}^{d-1} \llcorner J$, where J is the $\mathcal{H}^{d-1}$-rectifiable jump set, n is a normal on J, and $[u]$ is the jump height in direction n); see Ambrosio, Fusco, and Pallara [2000, Chapter 3] for a complete reference concerning functions of bounded variations.

A positive answer to the above question was given by Alberti in Alberti [1993] with his celebrated *rank-one theorem*. It was recognized quickly that this result has a central place

in the calculus of variations and importance well beyond, in particular because it implies that locally all singularities in BV-functions are necessarily *one-directional*. Indeed, after a blow-up (i.e. magnification) procedure at $|D^s u|$-almost every point, the blow-up limit measure depends only on a single direction and is translation-invariant with respect to all orthogonal directions. This is not surprising for jumps, but it is a strong assertion about all other singularities in the *Cantor part* of $D^s u$, i.e., the remainder of $D^s u$ after subtracting the jump part.

While Alberti's original proof is geometric in nature, one can also interpret the theorem as a result about singularities in PDEs: BV-derivatives Du satisfy the PDE

$$\operatorname{curl} Du = 0 \qquad \text{in the sense of distributions,}$$

which can be written with a linear constant-coefficient PDE operator $\mathcal{A} := \sum_{j=1}^{d} A_j \partial_j$ as $\mathcal{A}\mu = 0$ in the sense of distributions.

Besides its intrinsic theoretical interest, the rank-one theorem also has many applications in the theory of BV-functions, for instance for lower semicontinuity and relaxation Ambrosio and Dal Maso [1992], Fonseca and Müller [1993], and Kristensen and Rindler [2010], integral representation theorems Bouchitté, Fonseca, and Mascarenhas [1998], Young measure theory Kristensen and Rindler [2012] and Rindler [2014], and the study of continuity equations with BV-vector fields Ambrosio [2004]. We refer to Ambrosio, Fusco, and Pallara [2000, Chapter 5] for further history.

At the the end of this section we will see that Alberti's rank-one theorem is a straightforward consequence of Theorem 1.2. Let us also mention that recently a very short proof of the Alberti rank-one theorem has been given by Massaccesi and Vittone in Massaccesi and Vittone [2016].

In problems arising in the theory of geometrically-linear elasto-plasticity Suquet [1978], Suquet [1979], and Temam and Strang [1980/81] one often needs to consider a larger space of functions than the space of functions of bounded variations. Indeed, in this setting energies usually only depend on the *symmetric part* of the gradient and one has to consider the following type of variational problem:

$$\int_\Omega f(\mathcal{E}u)\,\mathrm{d}x \to \min, \qquad u \in \mathrm{C}^1(\Omega; \mathbb{R}^d) \text{ with given boundary conditions,}$$

where $\mathcal{E}u = (\nabla u + \nabla u^T)/2 \in (\mathbb{R}^d \otimes \mathbb{R}^d)_{\mathrm{sym}}$ is the symmetric gradient $((\mathbb{R}^d \otimes \mathbb{R}^d)_{\mathrm{sym}}$ being canonically isomorphic to the space of symmetric $(d \times d)$-matrices) and f is a linear-growth integrand with $f(A) \sim |A|$ for $|A|$ large. In this case, for a minimizing sequence (u_k) one can only obtain that

$$\sup_k \int_\Omega |\mathcal{E}u_k|\,\mathrm{d}x < \infty$$

and, due to the failure of Korn's inequality in L^1 Ornstein [1962], Conti, Faraco, and Maggi [2005], and Kirchheim and Kristensen [2016], this is not enough to ensure that $\sup_k \int |\nabla u_k| \, dx < \infty$. One then introduces the space $BD(\Omega)$ of functions of *bounded deformation*, i.e. those functions $u \in L^1(\Omega; \mathbb{R}^d)$ such that the symmetrized distributional derivative exists as a Radon measure, i.e.,

$$Eu := \frac{1}{2}(Du + Du^T) \in \mathcal{M}(\Omega; (\mathbb{R}^d \otimes \mathbb{R}^d)_{\text{sym}}),$$

see Ambrosio, Coscia, and Dal Maso [1997], Temam [1983], and Temam and Strang [1980/81]. Clearly, $BV(\Omega, \mathbb{R}^d) \subset BD(\Omega)$ and the inclusion is strict Conti, Faraco, and Maggi [2005] and Ornstein [1962]. Note that for $u \in BV(\Omega; \mathbb{R}^d)$ as a consequence of Alberti's rank-one theorem one has

$$\frac{dE^s u}{d|E^s u|}(x) = a(x) \odot b(x),$$

where $a \odot b = (a \otimes b + b \otimes a)/2$ is the symmetrized tensor product. One is then naturally led to the following conjecture:

Question 2.2. *Is it true that for every function $u \in BD(\Omega)$ it holds that*

$$\frac{dE^s u}{d|E^s u|}(x) = a(x) \odot b(x)$$

for $|E^s u|$-a.e. x and some $a(x), b(x) \in \mathbb{R}^d$?

Again, besides its theoretical interest, it has been well known that a positive answer of the above question would have several applications to the study of lower semicontinuity and relaxation of functionals defined on BD, see the next section, as well as in establishing the absence of a singular part for minimizers, see for instance Francfort, Giacomini, and Marigo [2015, Remark 4.8].

Let us conclude this section by showing how both a positive answer to Question 2.2 and a new proof of Alberti's rank-one theorem can easily be obtained by applying Theorem 1.2 to suitable differential operators:

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^d$ be open. Then:*

(i) If $u \in BV(\Omega; \mathbb{R}^\ell)$, then

$$\frac{dD^s u}{d|D^s u|}(x) = a(x) \otimes b(x) \qquad \text{for } |D^s u|\text{-a.e. } x \text{ and some } a(x) \in \mathbb{R}^\ell, \, b(x) \in \mathbb{R}^d.$$

(ii) If $u \in \mathrm{BD}(\Omega)$, then

$$\frac{\mathrm{d}E^s u}{\mathrm{d}|E^s u|}(x) = a(x) \odot b(x) \qquad \text{for } |E^s u|\text{-a.e. } x \text{ and some } a(x), b(x) \in \mathbb{R}^d.$$

Proof. Observe that $\mu = Du$ is curl-free,

$$0 = \operatorname{curl} \mu = \left(\partial_i \mu_j^k - \partial_j \mu_i^k \right)_{i,j=1,\dots,d\,;\,k=1,\dots,\ell}.$$

Then, assertion (i) above follows from

$$\Lambda_{\mathrm{curl}} = \left\{ a \otimes \xi \ : \ a \in \mathbb{R}^\ell,\ \xi \in \mathbb{R}^d \setminus \{0\} \right\},$$

which can be proved by an easy computation.

In the same way, if $\mu = Eu$, then μ satisfies the *Saint-Venant compatibility conditions*,

$$0 = \operatorname{curl\,curl} \mu := \left(\sum_{i=1}^d \partial_{ik}\mu_i^j + \partial_{ij}\mu_i^k - \partial_{jk}\mu_i^i - \partial_{ii}\mu_j^k \right)_{j,k=1,\dots,d}.$$

It is now a direct computation to check that

$$\Lambda_{\mathrm{curl\,curl}} = \left\{ a \odot \xi \ : \ a \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d \setminus \{0\} \right\}.$$

This shows assertion (ii) above. $\qquad\square$

3   Functionals on measures

The theory of integral functionals with linear-growth integrands defined on vector-valued measures satisfying PDE constraints is central to many questions of the calculus of variations. In particular, their relaxation and lower semicontinuity properties have attracted a lot of attention, see for instance Ambrosio and Dal Maso [1992], Fonseca and Müller [1993, 1999], Fonseca, Leoni, and Müller [2004], Kristensen and Rindler [2010], Rindler [2011], and Baía, Chermisi, Matias, and Santos [2013]. Based on Theorem 1.2 one can unify and extend many of these results.

Concretely, let $\Omega \subset \mathbb{R}^d$ be an open and bounded set and consider the functional

$$(3\text{-}1) \qquad \mathcal{F}[\mu] := \int_\Omega f\left(x, \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^d}(x) \right) \mathrm{d}x + \int_\Omega f^\infty\left(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu|^s}(x) \right) \mathrm{d}|\mu|^s(x),$$

defined for finite vector Radon measures $\mu \in \mathfrak{M}(\Omega; \mathbb{R}^N)$ with values in $\mathbb{R}^N$ and satisfying

$$\mathcal{A}\mu = 0 \qquad \text{in the sense of distributions.}$$

Here, $f: \Omega \times \mathbb{R}^N \to [0, \infty)$ is a Borel integrand that has *linear growth at infinity*, i.e.,

$$|f(x, A)| \leq M(1 + |A|) \qquad \text{for all } (x, A) \in \Omega \times \mathbb{R}^N.$$

We also assume that the *strong recession function* of f exists, which is defined as

$$(3\text{-}2) \qquad f^\infty(x, A) := \lim_{\substack{x' \to x \\ A' \to A \\ t \to \infty}} \frac{f(x', tA')}{t}, \qquad (x, A) \in \overline{\Omega} \times \mathbb{R}^N.$$

The (weak*) lower semicontinuity properties of $\mathcal{F}$ depend on (generalized) *convexity* properties of the integrand in its second variable. For this, we need the following definition: A Borel function $h: \mathbb{R}^N \to \mathbb{R}$ is called $\mathcal{Q}^k$-*quasiconvex* ($\mathcal{Q}^k = \sum_{|\alpha|=k} A_\alpha \partial^\alpha$ being the principal part of $\mathcal{Q}$) if

$$h(F) \leq \int_Q h(F + w(y)) \, dy$$

for all $F \in \mathbb{R}^N$ and all Q-periodic $w \in C^\infty(Q; \mathbb{R}^N)$ such that $\mathcal{Q}^k w = 0$ and $\int_Q w \, dy = 0$, where $Q := (0, 1)^d$ is the open unit cube in $\mathbb{R}^d$; see Fonseca and Müller [1999] for more on this class of integrands. For $\mathcal{Q} = \text{curl}$ this notion is equivalent to the classical *quasiconvexity* as introduced by Morrey [1952].

It has been known for a long time that $\mathcal{Q}^k$-quasiconvexity of $f(x, \cdot)$ is a necessary condition for the sequential weak* lower semicontinuity of $\mathcal{F}$ on $\mathcal{Q}$-free measures. As for the sufficiency, we can now prove the following general lower semicontinuity theorem, which is taken from Arroyo-Rabasa, Philippis, and Rindler [2017] (where also more general results can be found):

Theorem 3.1. *Let $f: \Omega \times \mathbb{R}^N \to [0, \infty)$ be a continuous integrand with linear growth at infinity such that f is uniformly Lipschitz in its second argument, f^∞ exists as in (3-2), and $f(x, \cdot)$ is $\mathcal{Q}^k$-quasiconvex for all $x \in \Omega$. Further assume that there exists a modulus of continuity $\omega: [0, \infty) \to [0, \infty)$ (increasing, continuous, $\omega(0) = 0$) such that*

$$(3\text{-}3) \quad |f(x, A) - f(y, A)| \leq \omega(|x - y|)(1 + |A|) \qquad \text{for all } x, y \in \Omega, \ A \in \mathbb{R}^N.$$

Then, the functional $\mathcal{F}$ is sequentially weakly* lower semicontinuous on the space

$$\mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{Q} := \{\mu \in \mathcal{M}(\Omega; \mathbb{R}^N) \ : \ \mathcal{Q}\mu = 0\}.$$

Remark 3.2. As special cases of Theorem 3.1 we get, among others, the following well-known results:

(i) For $\mathcal{Q} = \text{curl}$, one obtains BV-lower semicontinuity results in the spirit of Ambrosio and Dal Maso [1992] and Fonseca and Müller [1993].

(ii) For $\mathcal{A} = \operatorname{curl}\operatorname{curl}$, the second order operator expressing the Saint-Venant compatibility conditions, we re-prove the lower semicontinuity and relaxation theorem in the space of functions of bounded deformation (BD) from Rindler [2011].

(iii) For first-order operators $\mathcal{A}$, a similar result was proved in Baía, Chermisi, Matias, and Santos [2013].

The proof of Theorem 3.1 essentially follows by combining Theorem 1.2 with the main theorem of Kirchheim and Kristensen [2016], which establishes that the restriction of f^∞ to the linear space spanned by the wave cone is in fact *convex* at all points of $\Lambda_{\mathcal{A}}$ (in the sense that a supporting hyperplane exists). In this way we gain classical convexity for f^∞ at singular points, which can be exploited via the theory of generalized Young measures developed in DiPerna and Majda [1987], Alibert and Bouchitté [1997], and Kristensen and Rindler [2012] and also briefly discussed in the next section.

One can also show relaxation results, where f is not assumed to be $\mathcal{A}^k$-quasiconvex in the second argument and the task becomes to compute the largest weakly* lower semicontinuous functional below $\mathcal{F}$; see Arroyo-Rabasa, Philippis, and Rindler [2017] for more details.

4 Characterization of generalized Young measures

Young measures quantitatively describe the asymptotic oscillations in L^p-weakly converging sequences. They were introduced in Young [1937, 1942a,b] and later developed into an important tool in modern PDE theory and the calculus of variations in Tartar [1979], Tartar [1983], Ball [1989], and Ball and James [1987] and many other works. In order to deal with concentration effects as well, DiPerna & Majda extended the framework to so-called "generalized" Young measures, see DiPerna and Majda [1987], Alibert and Bouchitté [1997], Kružík and Roubíček [1997], Fonseca, Müller, and Pedregal [1998], Sychev [1999], and Kristensen and Rindler [2012]. In the following we will refer also to these objects simply as "Young measures". We recall some basic theory, for which proofs and examples can be found in Alibert and Bouchitté [1997], Kristensen and Rindler [2012], and Rindler [2011].

Let again $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For $f \in C(\overline{\Omega} \times \mathbb{R}^N)$ we define

$$\mathbf{E}(\Omega; \mathbb{R}^N) := \big\{ f \in C(\overline{\Omega} \times \mathbb{R}^N) \ : \ f^\infty \text{ exists in the sense (3-2)} \big\}.$$

A *(generalized) Young measure* $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N) \subset \mathbf{E}(\Omega; \mathbb{R}^N)^*$ on the open set $\Omega \subset \mathbb{R}^d$ with values in $\mathbb{R}^N$ is a triple $\nu = (\nu_x, \lambda_\nu, \nu_x^\infty)$ consisting of

(i) a parametrized family of probability measures $(\nu_x)_{x \in \Omega} \subset \mathfrak{M}_1(\mathbb{R}^N)$, called the *oscillation measure*;

(ii) a positive finite measure $\lambda_\nu \in \mathbf{M}_+(\overline{\Omega})$, called the *concentration measure*; and

(iii) a parametrized family of probability measures $(\nu_x^\infty)_{x \in \overline{\Omega}} \subset \mathbf{M}_1(\mathbb{S}^{N-1})$, called the *concentration-direction measure*,

for which we require that

(iv) the map $x \mapsto \nu_x$ is *weakly* measurable* with respect to $\mathcal{L}^d$, i.e. the function $x \mapsto \langle f(x, \cdot), \nu_x \rangle$ is $\mathcal{L}^d$-measurable for all bounded Borel functions $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$,

(v) the map $x \mapsto \nu_x^\infty$ is weakly* measurable with respect to λ_ν, and

(vi) $x \mapsto \langle |\bullet|, \nu_x \rangle \in \mathrm{L}^1(\Omega)$.

The *duality pairing* between $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$ and $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N)$ is given as

$$\langle f, \nu \rangle := \int_\Omega \langle f(x, \cdot), \nu_x \rangle \, \mathrm{d}x + \int_{\overline{\Omega}} \langle f^\infty(x, \cdot), \nu_x^\infty \rangle \, \mathrm{d}\lambda_\nu(x)$$

$$:= \int_\Omega \int_{\mathbb{R}^N} f(x, A) \, \mathrm{d}\nu_x(A) \, \mathrm{d}x + \int_{\overline{\Omega}} \int_{\partial \mathbb{B}^N} f^\infty(x, A) \, \mathrm{d}\nu_x^\infty(A) \, \mathrm{d}\lambda_\nu(x).$$

If $(\gamma_j) \subset \mathbf{M}(\overline{\Omega}; \mathbb{R}^N)$ is a sequence of Radon measures with $\sup_j |\gamma_j|(\overline{\Omega}) < \infty$, then we say that the sequence (γ_j) *generates* a Young measure $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N)$, in symbols $\gamma_j \xrightarrow{\mathbf{Y}} \nu$, if for all $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$ it holds that

$$f\left(x, \frac{\mathrm{d}\gamma_j}{\mathrm{d}\mathcal{L}^d}(x)\right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(x, \frac{\mathrm{d}\gamma_j^s}{\mathrm{d}|\gamma_j^s|}(x)\right) |\gamma_j^s|(\mathrm{d}x)$$

$$\xrightarrow{*} \langle f(x, \cdot), \nu_x \rangle \mathcal{L}^d \llcorner \Omega + \langle f^\infty(x, \cdot), \nu_x^\infty \rangle \lambda_\nu(\mathrm{d}x) \qquad \text{in } \mathbf{M}(\overline{\Omega}).$$

Here, γ_j^s is the singular part of γ_j with respect to Lebesgue measure.

It can be shown that if $(\gamma_j) \subset \mathbf{M}(\overline{\Omega}; \mathbb{R}^N)$ is a sequence of measures with $\sup_j |\gamma_j|(\overline{\Omega}) < \infty$ as above, then there exists a subsequence (not relabeled) and a Young measure $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N)$ such that $\gamma_j \xrightarrow{\mathbf{Y}} \nu$, see Kristensen and Rindler [2012].

When considering generating sequences (γ_j) as above that satisfy a differential constraint like curl-freeness (i.e. the generating sequence is a sequence of *gradients*), the following question arises:

Question 4.1. *Can one characterize the class of Young measures generated by sequences satisfying some (linear) PDE constraint?*

In applications, such results provide valuable information on the allowed oscillations and concentrations that are possible under this differential constraint, which usually constitutes a strong restriction. Characterization theorems are of particular use in the relaxation of minimization problems for non-convex integral functionals, where one passes from a functional defined on functions to one defined on Young measures. A characterization theorem then allows one to restrict the class of Young measures over which to minimize. This strategy is explained in detail (for classical Young measures) in Pedregal [1997].

The first general classification results are due to Kinderlehrer and Pedregal [1991, 1994], who characterized classical *gradient* Young measures, i.e. those generated by gradients of $W^{1,p}$-bounded sequences, $1 < p \leq \infty$. Their theorems put such gradient Young measures in duality with quasiconvex functions. For generalized Young measures the corresponding result was proved in Fonseca, Müller, and Pedregal [1998] (also see Kała-majska and Kružík [2008]) and numerous other characterization results in the spirit of the Kinderlehrer–Pedregal theorems have since appeared, see for instance Kružík and Roubíček [1996], Fonseca and Müller [1999], Fonseca and Kružík [2010], and Benešová and Kružík [2016].

The characterization of generalized BV-Young measures, i.e. those ν generated by a sequence (Du_j) of the BV-derivatives of maps $u_j \in \mathrm{BV}(\Omega; \mathbb{R}^\ell))$ was first achieved in Kristensen and Rindler [2012]. A different, "local" proof was given in Rindler [2014], another improvement is in Kirchheim and Kristensen [2016, Theorem 6.2]. All of these arguments crucially use Alberti's rank-one theorem.

The most interesting case beyond BV is again the case of functions of bounded deformation (BD), which were introduced above: In plasticity theory Suquet [1978], Suquet [1979], and Temam and Strang [1980/81], one often deals with sequences of uniformly L^1-bounded symmetric gradients $\mathcal{E}u_j := (\nabla u_j + \nabla u_j^T)/2$. In order to understand the asymptotic oscillations and concentrations in such sequences $(\mathcal{E}u_j)$ one needs to characterize the (generalized) Young measures ν generated by them. We call such ν *BD-Young measures* and write $\nu \in \mathbf{BDY}(\Omega)$, since all BD-functions can be reached as weak* limits of sequences (u_j) as above.

In this situation the following result can be shown, see De Philippis and Rindler [2017]:

Theorem 4.2. *Let $\nu \in \mathbf{Y}(\Omega; (\mathbb{R}^d \otimes \mathbb{R}^d)_{\mathrm{sym}})$ be a (generalized) Young measure. Then, ν is a BD-Young measure, $\nu \in \mathbf{BDY}(\Omega)$, if and only if there exists $u \in \mathrm{BD}(\Omega)$ with*

$$\langle \mathrm{id}, \nu_x \rangle \mathcal{L}_x^d + \langle \mathrm{id}, \nu_x^\infty \rangle (\lambda_\nu \llcorner \Omega)(\mathrm{d}x) = Eu$$

and for all symmetric-quasiconvex $h \in \mathrm{C}((\mathbb{R}^d \otimes \mathbb{R}^d)_{\mathrm{sym}})$ with linear growth at infinity, the Jensen-type inequality

$$h\left(\langle \mathrm{id}, \nu_x \rangle + \langle \mathrm{id}, \nu_x^\infty \rangle \frac{\mathrm{d}\lambda_\nu}{\mathrm{d}\mathcal{L}^d}(x)\right) \leq \langle h, \nu_x \rangle + \langle h^\#, \nu_x^\infty \rangle \frac{\mathrm{d}\lambda_\nu}{\mathrm{d}\mathcal{L}^d}(x).$$

holds at $\mathcal{L}^d$-almost every $x \in \Omega$, where $h^\#$ is defined via

$$h^\#(A) := \limsup_{\substack{A' \to A \\ t \to \infty}} \frac{h(tA')}{t}, \qquad A \in (\mathbb{R}^d \otimes \mathbb{R}^d)_{\text{sym}}.$$

One application of this result (in the spirit of Young's original work Young [1942a,b, 1980]) is the following: For a suitable integrand $f : \Omega \times (\mathbb{R}^d \otimes \mathbb{R}^d)_{\text{sym}} \to \mathbb{R}$, the minimum principle

$$(4\text{-}1) \qquad\qquad \langle f, \nu \rangle \to \min, \quad \nu \in \mathbf{BDY}(\Omega).$$

can be seen as the *extension-relaxation* of the minimum principle

$$(4\text{-}2) \qquad \int_\Omega f(x, \mathcal{E}u(x)) \, dx + \int_\Omega f^\infty\left(x, \frac{dE^s u}{d|E^s u|}(x)\right) d|E^s u| \to \min$$

over $u \in \text{BD}(\Omega)$. The point is that (4-2) may not be solvable if f is not symmetric-quasiconvex, whereas (4-1) always has a solution. In this situation, Theorem 4.2 then gives (abstract) restrictions on the Young measures to be considered in (4-1). Another type of relaxation involving the symmetric-quasiconvex envelope of f is investigated in Arroyo-Rabasa, Philippis, and Rindler [2017] within the framework of general linear PDE side-constraints.

The necessity part of Theorem 4.2 follows from a lower semicontinuity or relaxation theorem like the one in Rindler [2011]. For the sufficiency part (which is quite involved), one first characterizes so-called *tangent Young measures*, which are localized versions of Young measures. There are two types: regular and singular tangent Young measures, depending on whether regular (Lebesgue measure-like) effects or singular effects dominate around the blow-up point. We stress that the argument crucially rests on the BD-analogue of Alberti's rank-one theorem, see Theorem 2.3 (ii). Technically, in one of the proof steps to establish Theorem 4.2 we need to create "artificial concentrations" by compressing symmetric gradients in one direction. This is only possible if we know precisely what these singularities look like.

A characterization results for Young measures under general linear PDE constraints is currently not available (there is a partial result in the work Baía, Matias, and Santos [2013], but limited to first-order operators and needing additional technical assumptions). The reason is that currently not enough is known about the directional structure of $\mathcal{A}$-free measures at singular points.

5   The converse of Rademacher's theorem

Rademacher's theorem asserts that a Lipschitz function $f \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^\ell)$ is diffferentiable $\mathcal{L}^d$-almost everywhere. A natural question, which has attracted considerable attention, is to understand how sharp this result is. The following questions have been folklore in the area for a while:

Question 5.1 (Strong converse of Rademacher's theorem). *Given a Lebesgue null set $E \subset \mathbb{R}^d$ is it possible to find some $\ell \geq 1$ and a Lipschitz function $f \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^\ell)$ such that f is not differentiable in any point of E?*

Question 5.2 (Weak converse of Rademacher's theorem). *Let $v \in \mathfrak{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that every Lipschitz function is differentiable v-almost everywhere. Is v necessarily absolutely continuous with respect to $\mathcal{L}^d$?*

Clearly, a positive answer to Question 5.1 implies a positive answer to Question 5.2. Let us also stress that in answering Question 5.1, an important role is played by the dimension ℓ of the target set, see point (ii) below, while this does not have any influence on Question 5.2, see Alberti and Marchese [2016]. We refer to Alberti, Csörnyei, and Preiss [2005, 2010b] and Alberti and Marchese [2016] for a detailed account on the history of these problems and here we simply record the following facts:

(i) For $d = 1$ a positive answer to Question 5.1 is due to Zahorski [1946].

(ii) For $d \geq 2$ there exists a null set E such that every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^\ell$ with $\ell < d$ is differentiable in at least one point of E. This is was proved by Preiss in Preiss [1990] for $d = 2$ and later extended by Preiss and Speight in Preiss and Speight [2015] to every dimension.

(iii) For $d = 2$ a positive answer to Question 5.1 has been given by Alberti, Csörnyei and Preiss as a consequence of their deep result concerning the structure of null sets in the plane Alberti, Csörnyei, and Preiss [2005, 2010b,a]. Namely, they show that for every null set $E \subset \mathbb{R}^2$ there exists a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is not differentiable at any point of E.

(iv) For $d \geq 2$ an extension of the result described in point (iii) above, i.e. that for every null set $E \subset \mathbb{R}^d$ there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that f is not differentiable at any point of E, has been announced in 2011 by Jones [2011].

Let us now show how Question 5.2 is related to Question 1.1. In Alberti and Marchese [2016, Theorem 1.1] Alberti & Marchese have shown the following result:

Theorem 5.3 (Alberti–Marchese). *Let $\nu \in \mathfrak{M}_+(\mathbb{R}^d)$ be a positive Radon measure. Then, there exists a vector space-valued ν-measurable map $V(\nu, x)$ (the decomposability bundle of ν) such that:*

(i) *Every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable in the directions of $V(\nu, x)$ at ν-almost every x.*

(ii) *There exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that for ν-almost every x and every $v \notin V(\nu, x)$ the derivative of f at x in the direction of v does not exist.*

Thanks to the above theorem, Question 5.2 is then equivalent to the following:

Question 5.4. *Let $\nu \in \mathfrak{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that $V(\nu, x) = \mathbb{R}^d$ for ν-almost every x. Is ν absolutely continuous with respect to $\mathcal{L}^d$?*

The link between the above question and Theorem 1.2 is due to the following result, again due to Alberti & Marchese, see Alberti and Marchese [2016, Corollary 6.5] and De Philippis and Rindler [2016, Lemma 3.1][2].

Lemma 5.5. *Let $\nu \in \mathfrak{M}_+(\mathbb{R}^d)$ be a positive Radon measure. Then the following are equivalent:*

1. *The decomposability bundle of ν is of full dimension, i.e. $V(\nu, x) = \mathbb{R}^d$ for ν-almost every x.*

2. *There exist $\mathbb{R}^d$-valued measures $\mu_1, \ldots, \mu_d \in \mathfrak{M}(\mathbb{R}^d; \mathbb{R}^d)$ with measure-valued divergences $\operatorname{div} \mu_i \in \mathfrak{M}(\mathbb{R}^d; \mathbb{R})$ such that $\nu \ll |\mu_i|$ for $1 = 1, \ldots, d$ and[3]*

$$(5\text{-}1) \qquad \operatorname{span}\left\{\frac{\mathrm{d}\mu_1}{\mathrm{d}|\mu_1|}(x), \ldots, \frac{\mathrm{d}\mu_d}{\mathrm{d}|\mu_d|}(x)\right\} = \mathbb{R}^d \qquad \text{for } \nu\text{-a.e. } x.$$

With the above lemma at hand, a positive answer to Question 5.4 (and thus to Question 5.2) follows from Theorem 1.2 in a straightforward fashion:

Theorem 5.6. *Let $\nu \in \mathfrak{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that every Lipschitz function is differentiable ν-almost everywhere. Then, ν is absolutely continuous with respect to $\mathcal{L}^d$.*

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[2]In the cited references the results are stated in terms of normal currents. By the trivial identifications of the space of normal currents with the space of measure-valued vector fields whose divergence is a measure it is immediate to see that they are equivalent to our Lemma 5.5

[3]Note that since $\nu \ll |\mu_i|$ for all $i = 1, \ldots, d$, in item (ii) above all the Radon-Nikodym derivatives $\frac{\mathrm{d}\mu_i}{\mathrm{d}|\mu_i|}$ $i = 1, \ldots, d$ exist for ν-a.e. x.

Proof. Let ν be a measure such that $V(\nu, x) = \mathbb{R}^d$ for ν-almost every x an let μ_i be the measures provided by Lemma 5.5 (ii). Let us consider the matrix-valued measure

$$\boldsymbol{\mu} := \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_d \end{pmatrix} \in \mathfrak{M}(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d).$$

Note that $\operatorname{div} \boldsymbol{\mu} \in \mathfrak{M}(\mathbb{R}^d; \mathbb{R}^d)$, where div is the row-wise divergence operator. Since, by direct computation,

$$\Lambda_{\operatorname{div}} = \{ M \in \mathbb{R}^d \otimes \mathbb{R}^d \ : \ \operatorname{rank} M \leq d - 1 \},$$

Theorem 1.2 and Remark 1.3 imply that rank $\left(\frac{d\boldsymbol{\mu}}{d|\boldsymbol{\mu}|} \right) \leq d - 1$ for $|\boldsymbol{\mu}|^s$-almost every point. Hence, by (5-1), ν is singular with respect to $|\boldsymbol{\mu}|^s$. On the other hand, since $\nu \ll |\mu_i|$ for all $i = 1, \ldots, d$, we get $\nu^s \ll |\boldsymbol{\mu}|^s$. Hence, we conclude $\nu^s = 0$, as desired. □

Let us conclude this section by remarking that the weak converse of Rademacher's theorem, i.e. a positive answer to Question 5.2, has some consequences for the structure of Ambrosio–Kirchheim metric currents Ambrosio and Kirchheim [2000], see the work of Schioppa [2016b]. In particular, it allows one to prove the top-dimensional case of the flat chain conjecture proposed by Ambrosio and Kirchheim in Ambrosio and Kirchheim [2000].

6   Cheeger's conjecture

Among the many applications of Rademacher's theorem, it allows one to pass from "non-infintesimal" information (the existence of certain Lipschitz maps) to infinitesimal information. For instance, one can easily establish the following fact:

There is no bi-Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^\ell$ if $d \neq \ell$.

Indeed, if this were the case, Rademacher's theorem would imply (at a differentiability point) the existence of a bijective *linear* map from $\mathbb{R}^d$ to $\mathbb{R}^\ell$ with $d \neq \ell$.

While the above statement is an immediate consequence of the theorem on the invariance of dimension (asserting that there are no bijective continuous maps from $\mathbb{R}^d$ to $\mathbb{R}^\ell$ if $d \neq \ell$), the point here is that the almost everywhere result allows to pass from a non-linear statement (the existence of a bi-Lipschitz map) to a linear one, whose rigidity can be proved by elementary methods.

This line of thought has been adopted in the study of rigidity of several metric structures. For instance, the natural generalization of Rademacher's theorem in the context of Carnot

groups, which was established by Pansu in Pansu [1989], allows one to show that there are no bi-Lipscitz embeddings of a Carnot group into $\mathbb{R}^{\ell}$ if the former is non-commutative.

The fact that (a suitable notion of) differentiability of Lipschitz functions allows to develop a first-order calculus on metric spaces and in turn to obtain non-embedding results has been recognised by Cheeger in his seminal paper Cheeger [1999] and later studied by several authors.

Let us briefly introduce the theory of Cheeger as it has been axiomatized by Keith in Keith [2004]. Note that it is natural to generalize Rademacher's theorem to the setting of *metric measure spaces* since we need to talk about Lipschitz functions (a metric concept) and almost everywhere (a measure-theoretic concept).

Let (X, ρ, μ) be a *metric measure space*, that is, (X, ρ) is a separable, complete metric space and $\mu \in \mathfrak{M}_+(X)$ is a positive Radon measure on X. We call a pair (U, φ) such that $U \subset X$ is a Borel set and $\varphi \colon X \to \mathbb{R}^d$ is Lipschitz, a *d-dimensional chart*, or simply a *d-chart*. A function $f \colon X \to \mathbb{R}$ is said to be *differentiable with respect to a d-chart* (U, φ) at $x_0 \in U$ if there exists a unique (co-)vector $df(x_0) \in \mathbb{R}^d$ such that

$$(6\text{-}1) \qquad \limsup_{x \to x_0} \frac{|f(x) - f(x_0) - df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{\rho(x, x_0)} = 0.$$

Definition 6.1. *A metric measure space (X, ρ, μ) is a* Lipschitz differentiability space *if there exists a countable family of $d(i)$-charts (U_i, φ_i) $(i \in \mathbb{N})$ such that $X = \bigcup_i U_i$ and any Lipschitz map $f \colon X \to \mathbb{R}$ is differentiable with respect to every (U_i, φ_i) at μ-almost every point $x_0 \in U_i$.*

In this terminology, the main result of Cheeger [1999] asserts that every doubling metric measure space (X, ρ, μ) satisfying a Poincaré inequality is a Lipschitz differentiability space.

In the same paper, Cheeger conjectured that the push-forward of the reference measure μ under every chart φ_i has to be absolutely continuous with respect to the Lebesgue measure, see Cheeger [ibid., Conjecture 4.63]:

Question 6.2. *For every d-chart (U, φ) in a Lipschitz differentiability space, does it hold that $\varphi_\#(\mu \llcorner U) \ll \mathcal{L}^d$?*

Some consequences of this fact concerning the existence of bi-Lipschitz embeddings of X into some $\mathbb{R}^N$ are detailed in Cheeger [ibid., Section 14], also see Cheeger and Kleiner [2006, 2009].

Let us assume that $(X, \rho, \mu) = (\mathbb{R}^d, \rho_{\mathcal{E}}, \nu)$ with $\rho_{\mathcal{E}}$ the Euclidean distance and ν a positive Radon measure, is a Lipschitz differentiability space when equipped with the (single) identity chart (note that it follows a-posteriori from the validity of Cheeger's conjecture

that no mapping into a higher-dimensional space can be a chart in a Lipschitz differentiability structure of $\mathbb{R}^d$). In this case the validity of Cheeger's conjecture reduces to the validity of the (weak) converse of Rademacher's theorem, which we stated above in Theorem 5.6.

One can also prove the assertion of Cheeger's conjecture directly. Indeed, from the work of Bate [2015], and Alberti and Marchese [2016] an analogue of Lemma 5.5 for $\varphi_\#(\mu \llcorner U)$ in place of μ follows, see also Schioppa [2016a,b]. This allows one to conclude as in the proof of Theorem 5.6 to get:

Theorem 6.3. *Let (X, ρ, μ) be a Lipschitz differentiability space and let (U, φ) be a d-dimensional chart. Then, $\varphi_\#(\mu \llcorner U) \ll \mathfrak{L}^d$.*

The details can be found in De Philippis, Marchese, and Rindler [2017].

We conclude this section by mentioning that the weak converse of Rademacher's theorem also has some consequences concerning the structure of measures on metric measure spaces with Ricci curvature bounded from below, see Kell and Mondino [2016] and Gigli and Pasqualetto [2016].

7 Sketch of the proof of Theorem 1.2

In this section we shall give some details concerning the proof of Theorem 1.2. For simplicity we will only consider the case in which $\mathcal{A}$ is a *first-order homogeneous* operator, namely we will assume that μ satisfies

$$\mathcal{A}\mu = \sum_{j=1}^{d} A_j \partial_j \mu = 0 \qquad \text{in the sense of distributions.}$$

Note that in this case we have

$$\Lambda_\mathcal{A} = \bigcup_{|\xi|=1} \ker \mathbb{A}(\xi), \qquad \mathbb{A}(\xi) = \mathbb{A}^1(\xi) = 2\pi i \sum_{j=1}^{d} A_j \xi_j.$$

Let

$$E := \left\{ x \in \Omega \ : \ \frac{d\mu}{d|\mu|}(x) \notin \Lambda_\mathcal{A} \right\},$$

and let us assume by contradiction that $|\mu|^s(E) > 0$.

Employing a fundamental technique of geometric measure theory, one can "zoom in" around a generic point of E. Indeed, one can show that for $|\mu|^s$-almost every point $x_0 \in E$

there exists a sequence of radii $r_k \downarrow 0$ such that

$$\underset{k \to \infty}{\text{w*-lim}} \frac{(T^{x_0,r_k})_\# \mu}{|\mu|(B_{r_k}(x_0))} = \underset{k \to \infty}{\text{w*-lim}} \frac{(T^{x_0,r_j})_\# \mu^s}{|\mu|^s(B_{r_k}(x_0))} = P_0 \nu,$$

where $T^{x,r} : \mathbb{R}^d \to \mathbb{R}^d$ is the dilation map $T^{x,r}(y) := (y - x)/r$, $T^{x,r}_\#$ denotes the push-forward operator[4], $\nu \in \text{Tan}(x_0, |\mu|) = \text{Tan}(x_0, |\mu|^s)$ is a non-zero tangent measure in the sense of Preiss [1987],

$$\lambda_0 = \frac{d\mu}{d|\mu|}(x_0) \notin \Lambda_{\mathcal{A}},$$

and the limit is to be understood in the weak* topology of Radon measures (i.e. in duality with compactly supported continuous functions). Moreover, one easily checks that

$$\sum_{j=1}^{d} A_j \frac{d\mu}{d|\mu|}(x_0) \, \partial_j \nu = 0 \qquad \text{in the sense of distributions.}$$

By taking the Fourier transform of the above equation, we get

$$[\mathbb{A}(\xi)\lambda_0] \, \hat{\nu}(\xi) = 0, \qquad \xi \in \mathbb{R}^d.$$

where $\hat{\nu}(\xi)$ is the Fourier transform of ν in the sense of distributions (actually, ν does not need to be a tempered distribution, hence some care is needed, see below for more details). Having assumed that $\lambda_0 \notin \Lambda_{\mathcal{A}}$, i.e. that

$$\mathbb{A}(\xi)\lambda_0 \neq 0 \quad \text{for all } \xi \neq 0,$$

this implies supp $\hat{\nu} = \{0\}$ and thus $\nu \ll \mathcal{L}^d$. The latter fact, however, is not by itself a contradiction to $\nu \in \text{Tan}(x_0, |\mu|^s)$. Indeed, Preiss [ibid.] provided an example of a purely singular measure that has only multiples of Lebesgue measure as tangents (we also refer to O'Neil [1995] for a measure that has *every* measure as a tangent at almost every point).

   The above reasoning provides a sort of *rigidity* property for $\mathcal{A}$-measures: If, for a constant polar vector $\lambda_0 \notin \Lambda_{\mathcal{A}}$ and a measure $\nu \in \mathbb{M}_+(\mathbb{R}^d)$, the measure $\lambda_0 \nu$ is $\mathcal{A}$-free, then necessarily $\nu \ll \mathcal{L}^d$. However, as we commented above, this is not enough to conclude. In order to prove the theorem we need to strengthen this rigidity property (absolute continuity of the measures $\lambda_0 \nu$ with $\lambda_0 \notin \Lambda_{\mathcal{A}}$) to a stability property which can be roughly stated as follows:

   $\mathcal{A}$-free measures μ with $|\mu|(\{x : \frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathcal{A}}\}) \ll 1$ *have small singular part.*

---

[4]That is, for any measure σ and Borel set B, $[(T^{x,r})_\# \sigma](B) := \sigma(x + rB)$

In this respect note that since $\lambda_0 \notin \Lambda_{\mathcal{Q}}$ implies that $\mathbb{A}(\xi)\lambda_0 \neq 0$ for $\xi \neq 0$, one can hope for some sort of "elliptic regularization" that forces not only $\nu \ll \mathcal{L}^d$ but also

$$\mu_k := \frac{(T^{x_0,r_k})_\# \mu}{|\mu|^s(B_{r_k}(x_0))} \ll \mathcal{L}^d,$$

at least for small r_k. This is actually the case: Inspired by Allard's strong constancy lemma in Allard [1986], we can show that the ellipticity of the system at the limit (i.e. that $\mathbb{A}(\xi)\lambda_0 \neq 0$) improves the weak* convergence of (μ_k) to convergence in the total variation norm, i.e.

(7-1) $$|\mu_k - \lambda_0 \nu|(B_{1/2}) \to 0.$$

Since the singular part of μ_k is asymptotically predominant around x_0, see (7-2) below, this latter fact implies that

$$|\mu_k^s - \lambda_0 \nu|(B_{1/2}) \to 0,$$

which easily gives a contradiction to $\nu \ll \mathcal{L}^d$ and concludes the proof.

Let us briefly sketch how (7-1) is obtained. For $\chi \in \mathfrak{D}(B_1)$, $0 \leq \chi \leq 1$, consider the measures $\lambda_0 \chi \nu_k$, where

$$\nu_k := \frac{(T^{x_0,r_k})_\# |\mu|^s}{|\mu|^s(B_{r_k}(x_0))},$$

and note that, since we can assume that for the chosen x_0 it holds that

(7-2) $$\frac{|\mu|^a(B_{r_k}(x_0))}{|\mu|^s(B_{r_k}(x_0))} \to 0, \qquad \fint_{B_{r_k}(x_0)} \left| \frac{d\mu}{d|\mu|}(x) - \frac{d\mu}{d|\mu|}(x_0) \right| d|\mu|^s(x) \to 0,$$

we have that

(7-3) $$|\lambda_0 \chi \nu_k - \chi \mu_k|(\mathbb{R}^d) \leq |\lambda_0 \nu_k - \mu_k|(B_1) \to 0.$$

Using the $\mathcal{Q}$-freeness of μ_k (which trivially follows from the one of μ) we can derive an equation for $\chi \nu_k$:

(7-4) $$\sum_{j=1}^d A_j \lambda_0 \partial_j(\chi \nu_k) = \sum_{j=1}^d A_j \partial_j(\lambda_0 \chi \nu_k - \chi \mu_k) + \sum_{j=1}^d A_j \mu_k \partial_j \chi.$$

Since we are essentially dealing with a-priori estimates, in the following we treat measures as if they were smooth L^1-functions; this can be achieved by a sufficiently fast regularization, see De Philippis and Rindler [2016] for more details.

Taking the Fourier transform of equation (7-4) (note that we are working with compactly supported functions) we obtain

$$(7\text{-}5) \qquad \mathbb{A}(\xi)\lambda_0\widehat{\chi v_k}(\xi) = \mathbb{A}(\xi)\widehat{V_k}(\xi) + \widehat{R_k}(\xi),$$

where

$$(7\text{-}6) \qquad V_k := \lambda_0\chi v_k - \chi\mu_k \qquad \text{satisfies} \qquad |V_k|(\mathbb{R}^d) \to 0$$

and

$$(7\text{-}7) \qquad R_k := \sum_{j=1}^{d} A_j\mu_k\partial_j\chi \qquad \text{satisfies} \qquad \sup_k |R_k|(\mathbb{R}^d) \le C.$$

Scalar multiplying (7-5) by $\overline{\mathbb{A}(\xi)P_0}$, adding $\widehat{\chi v_k}$ to both sides and rearranging the terms, we arrive to

$$(7\text{-}8) \qquad \begin{aligned} \widehat{\chi v_k}(\xi) &= \frac{\overline{\mathbb{A}(\xi)\lambda_0}\mathbb{A}(\xi)\widehat{V_k}(\xi)}{1 + |\mathbb{A}(\xi)\lambda_0|^2} + \frac{\overline{\mathbb{A}(\xi)\lambda_0} \cdot \widehat{R_k}(\xi)}{1 + |\mathbb{A}(\xi)\lambda_0|^2} + \frac{\widehat{\chi v_k}(\xi)}{1 + |\mathbb{A}(\xi)\lambda_0|^2} \\ &=: T_0(V_k) + T_1(R_k) + T_2(\chi v_k), \end{aligned}$$

where

$$\begin{aligned} T_0[V] &= \mathcal{F}^{-1}\big[m_0(\xi)\hat{V}(\xi)\big], \\ T_1[R] &= \mathcal{F}^{-1}\big[m_1(\xi)(1 + 4\pi^2|\xi|^2)^{-1/2}\hat{R}(\xi)\big], \\ T_2[u] &= \mathcal{F}^{-1}\big[m_2(\xi)(1 + 4\pi^2|\xi|^2)^{-1}\hat{u}(\xi)\big], \end{aligned}$$

and we have set

$$\begin{aligned} m_0(\xi) &:= (1 + |\mathbb{A}(\xi)\lambda_0|^2)^{-1}\overline{\mathbb{A}(\xi)\lambda_0}\mathbb{A}(\xi), \\ m_1(\xi) &:= (1 + |\mathbb{A}(\xi)\lambda_0|^2)^{-1}(1 + 4\pi^2|\xi|^2)^{1/2}\overline{\mathbb{A}(\xi)\lambda_0}, \\ m_2(\xi) &:= (1 + |\mathbb{A}(\xi)\lambda_0|^2)^{-1}(1 + 4\pi^2|\xi|^2). \end{aligned}$$

We now note that since $\lambda_0 \notin \Lambda_\mathcal{Q}$, by homogeneity there exists $c > 0$ such that $|\mathbb{A}(\xi)\lambda_0| \ge c|\xi|$ (this is the ellipticity condition we mentioned at the beginning). Hence, the symbols m_i, $i = 1, 2, 3$, satisfy the assumptions of the Hörmander–Mihlin multiplier theorem Grafakos [2014, Theorem 5.2.7], i.e. there exists constants $K_\beta > 0$ such that

$$|\partial^\beta m_i(\xi)| \le K_\beta|\xi|^{-|\beta|} \qquad \text{for all } \beta \in \mathbb{N}^d.$$

This implies that T_0 is a bounded operator from L^1 to $L^{1,\infty}$ and thus, thanks to (7-6), we get

$$\|T_0(V_k)\|_{L^{1,\infty}} \le C|V_k|(\mathbb{R}^d) \to 0. \tag{7-9}$$

Moreover,

$$\langle T_0(V_k), \varphi \rangle = \langle V_k, T_0^*(\varphi) \rangle \to 0 \qquad \text{for every } \varphi \in \mathfrak{D}(\mathbb{R}^d). \tag{7-10}$$

where T_0^* is the adjoint operator of T_0. We also observe

$$T_1 = Q_{m_1} \circ (\mathrm{Id} - \Delta)^{-1/2} \qquad \text{and} \qquad T_2 = Q_{m_2} \circ (\mathrm{Id} - \Delta)^{-1},$$

where Q_{m_1} and Q_{m_2} are the Fourier multipliers operators associated with the symbols m_1 and m_2, respectively. In particular, again by the Hörmander–Mihlin multiplier theorem, these operators are bounded from L^p to L^p for every $p \in (1, \infty)$. Moreover, $(\mathrm{Id}-\Delta)^{-s/2}$ is a compact operator from[5] $L_c^1(B_1)$ to L^q for some $q = q(d, s) > 1$, see for instance De Philippis and Rindler [2016, Lemma 2.1]. In conclusion, by (7-7) and $\sup_k |\chi v_k|(\mathbb{R}^d) \le C$,

$$\{T_1(R_k) + T_2(\chi v_k)\}_{k \in \mathbb{N}} \qquad \text{is pre-compact in } L^1(B_1). \tag{7-11}$$

Hence, combining equation (7-8) with (7-9), (7-10) and (7-11) implies that

$$\chi v_k = u_k + w_k,$$

where $u_k \to 0$ in $L^{1,\infty}$, $u_k \overset{*}{\rightharpoonup} 0$ in the sense of distributions and (w_k) is pre-compact in $L^1(B_1)$. Since $\chi v_k \ge 0$,

$$u_k^- := \max\{-u_k, 0\} \le |w_k|,$$

so that the sequence (u_k^-) is pre-compact in $L^1(B_1)$. Since $u_k \to 0$ in $L^{1,\infty}$, Vitali's convergence theorem implies that $u_k^- \to 0$ in $L^1(B_1)$ which, combined with $u_k \overset{*}{\rightharpoonup} 0$, easily yields that $u_k \to 0$ in $L^1(B_1)$, see De Philippis and Rindler [ibid., Lemma 2.2]. In conclusion, (χv_k) is pre-compact in $L^1(B_1)$. Together with (7-1) and the weak* convergence of μ_k to $\lambda_0 \nu$, this implies

$$|\mu_k - \lambda_0 \nu|(B_{1/2}) \to 0,$$

which concludes the proof.

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[5] Here we denote by $L_c^1(B_1)$ the space of L^1-functions vanishing outside B_1.

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