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INTERACTION OF SOLITONS FROM THE PDE POINT OF VIEW

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Abstract

We review recent results concerning the interactions of solitary waves for several universal nonlinear dispersive or wave equations. Though using quite different techniques, these results are partly inspired by classical papers based on the inverse scattering theory for integrable models.

1 Introduction

Pioneering numerical experiments of Fermi, Pasta, and Ulam [1955] in 1955, and of Zabusky and Kruskal [1965] in 1965, revealed unexpected phenomena related to the interactions of nonlinear waves[1]. Shortly thereafter, the inverse scattering theory and its generalizations, developed by many influential mathematicians such as Ablowitz, Kaup, Newell, and Segur [1974], Gardner, C. S. Greene, Kruskal, and Miura [1967], Gardner, J. M. Greene, Kruskal, and Miura [1974], Lax [1968], Miura [1976], Miura, Gardner, and Kruskal [1968], and Zakharov and Shabat [1971], provided a rigorous ground and a unified approach to these observations. It led very rapidly to an accurate and deep understanding of remarkable properties of several universal nonlinear models, referred to as *completely integrable*, such as for example, the Korteweg-de Vries equation, the one dimensional cubic Schrödinger equation and the sine-Gordon equation. It has created a very active and inspiring field of research since then[2]. Among the most notable achievements of this theory, we mention

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[1] We refer to Chapter 8 of Dauxois and Peyrard [2010] for details on this discovery and on the relation between the model considered in Fermi, Pasta, and Ulam [1955] and the KdV equation. It is quite rightly suggested in Dauxois and Peyrard [2010] to recognize the work of M. Tsingou, contributor to the numerical computations of Fermi, Pasta, and Ulam [1955].

[2] See for example Lamb [1980] and Chapter 7 of Dauxois and Peyrard [2010] for synthetic presentations of the inverse scattering transform.

(i) the existence of *infinitely many conservation laws*;

(ii) the *purely elastic nature of the collision* of any number of solitary waves, which means that the interacting solitary waves recover their exact shape and velocity after a collision. Solitary waves enjoying such remarkable property were called *solitons*[3];

(iii) the *decomposition into solitons*, saying that from any solution should emerge in large time a sum of nonlinear states, such as solitons, plus a dispersive part.

These rigorous mathematical facts are known to be physically relevant in numerous contexts, though sometimes under less extreme forms. For example, in several practical applications or for more elaborate nonlinear models, the collision of nonlinear waves is not purely elastic and some loss of energy takes place during collisions[4]. This reveals that the inverse scattering theory is restricted to models with specific algebraic structure and despite many extensions to nearly integrable systems (see *e.g.* Kivshar and Malomed [1989]), it cannot be applied to general nonlinear models.

In view of the beautiful achievements of the integrability theory but also of its inevitable limitations, it appeared necessary to investigate similar questions for general nonlinear models with solitary waves using tools from the theory of partial differential equations. In these notes, we review some results on interactions of solitary waves obtained for models that are not close to any known integrable equation, such as the generalized Korteweg-de Vries equation, the nonlinear Schrödinger equation in any space dimension, the ϕ^4 equation and the nonlinear wave equation.

Mainly in the 80s, the solitary wave theory, proving existence, uniqueness, symmetry and stability properties of nonlinear waves, was successfully developed using the elliptic theory, ODE analysis and general variational arguments, at least for *ground states* (see Section 3). More recently, *asymptotic stability* results appeared (see Section 4). Then, energy type arguments extending the elliptic theory have allowed to consider several solitary waves in *weak interactions*, *i.e.* cases where the soliton dynamics is only slightly perturbed by the interactions. Pushing the perturbative analysis one step forward, some examples of *strong interactions* have also been exhibited; the solitons are still distant, but their dynamics is substantially modified by the interactions (see Section 5). Next, we review the few recent cases where a version of the *soliton resolution conjecture* was proved for non-integrable wave models in Section 6. Finally, we discuss in Section 7 some situations where collisions were proved to be *inelastic*.

This review points out that despite some impressive and surprizing recent progress, notably on the soliton resolution conjecture, most of the questions raised above on the interaction of solitary waves remain open for general nonlinear models.

[3] This term is now commonly used for solitary waves even in the non-integrable context.

[4] We refer to Craig, Guyenne, Hammack, Henderson, and Sulem [2006] for a discussion on this topic.

2 Integrable equations

In this section, we briefly highlight some results from the inverse scattering theory that inspired mathematical research much beyond their range of applicability.

2.1 KdV solitons and multi-solitons. For the Korteweg-de Vries equation[5]

$$\partial_t u + \partial_x(\partial_x^2 u + u^2) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

the inverse scattering transform led to a very striking property which is the existence of exact multi-soliton solutions (see *e.g.* Hirota [1971], Miura [1976], and Whitham [1974]).

Let $Q(x) = \frac{3}{2}\cosh^{-2}(\frac{x}{2})$ be the unique positive even solution of $Q'' + Q^2 = Q$, and for $c > 0$, let $Q_c(x) = cQ(\sqrt{c}x)$. Then, for any $c > 0$, $\sigma \in \mathbb{R}$, the function defined by $u(t,x) = Q_c(x - ct - \sigma)$ is a solution of (1), called *soliton*, traveling with speed c.

Solutions containing an arbitrary number of such solitons (called *multi-solitons*) have been obtained by the inverse scattering theory.

Theorem 1 (Multi-solitons for KdV, Hirota [1971] and Miura [1976])**.** *Let $K \in \mathbb{N}$, $K \geq 2$. Let $0 < c_K < \cdots < c_1$ and $\sigma_1^-, \ldots, \sigma_K^- \in \mathbb{R}$. There exist $\sigma_1^+, \ldots, \sigma_K^+ \in \mathbb{R}$ and an explicit solution u of* (1) *such that*

$$\lim_{t \to \pm\infty} \left\| u(t) - \sum_{k=1}^{K} Q_{c_k}\left(\cdot - c_k t - \sigma_k^{\pm}\right) \right\|_{H^1} = 0.$$

The most remarkable fact is that all the solitons recover exactly the same sizes and speeds after the collision. Moreover, the values of σ_k^+ are explicit. It it interesting to recall that the multi-soliton behavior, even in the simple case of two solitons, differs qualitatively according to the relative sizes of the solitons. We refer to Lemma 2.3 in Lax [1968] for a definition of the three *Lax categories* of two-solitons and to Zabusky and Kruskal [1965] for a previous formal discussion. In particular, if their sizes are close (*i.e.* $c_1 \sim c_2$), the two solitons never cross, but rather repulse each other at a large distance (this is category (c) in Lax [1968]). See Sections 7.1 to 7.3.

2.2 Decomposition into solitons for KdV. The multi-soliton behavior is fundamental for general solutions of the KdV equation as shown by the following decomposition result.

Theorem 2 (Decomposition into solitons, Eckhaus and Schuur [1983] and Schuur [1986])**.** *Let u_0 be a C^4 function such that for any $j \in \{0, \ldots, 4\}$, for all $x \in \mathbb{R}$, $\left|(\partial^j u_0/\partial x^j)(x)\right| \lesssim$*

[5] We refer to Chapter 1 of Dauxois and Peyrard [2010] for historical facts on this equation and its applications to Physics.

$\langle x \rangle^{-10}$. *Let u be the solution of* (1) *corresponding to u_0. Then, there exist $K \in \mathbb{N}$, $\sigma_1, \dots, \sigma_K \in \mathbb{R}$ and $c_1 > \cdots > c_K > 0$ such that, for all $x > 0$,*

$$\lim_{t \to +\infty} \left\{ u(t,x) - \sum_{k=1}^{K} Q_{c_k}(x - c_k t - \sigma_k) \right\} = 0.$$

This result has a rich history, see Ablowitz, Kaup, Newell, and Segur [1974], Cohen [1979], Dauxois and Peyrard [2010], Dodd, Eilbeck, Gibbon, and Morris [1982], Eckhaus and Schuur [1983], Kruskal [1974], Lax [1968], Schuur [1986], and Zabusky and Kruskal [1965] and the references therein. Note that if some space decay is necessary to apply the inverse scattering transform, the decay assumption on the initial data in the above result is not optimal. Note also that the asymptotic behavior of the solution is described for $x > 0$ (see results in Schuur [1986] for slight improvement). For the region $x < 0$, see Deift, Venakides, and Zhou [1994], Eckhaus and Schuur [1983], and Schuur [1986] and references therein.

Last, we mention that the modified KdV equation (i.e. the KdV equation with a cubic nonlinearity) is also an integrable model that enjoys most of the properties of the KdV equation, like the infinitely many conservation laws and the existence of pure multi-soliton solutions (see *e.g.* Miura [1976]). Actually, it even has a richer family of exceptional solutions: breather solutions (see Alejo and Muñoz [2013], Lamb [1980], and Wadati [1973]) and dipole solitons, *i.e.* special multi-solitons where solitons are distant like $C \log t$ (see Gorshkov and Ostrovsky [1981], Karpman and Solovev [1981], and Wadati and Ohkuma [1982]). This complicates any possible soliton resolution conjecture on this equation (see Schuur [1986]).

2.3 One dimensional cubic NLS.

The 1D cubic nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + |u|^2 u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{2}$$

is also an integrable equation, widely studied for its numerous physical applications and remarkable mathematical properties. See *e.g.* Dauxois and Peyrard [2010], Deift and Trubowitz [1979], Deift and Zhou [1993], Dodd, Eilbeck, Gibbon, and Morris [1982], Faddeev and Takhtajan [2007], Novokšenov [1980], Olmedilla [1987], Yang [2010], Zabusky and Kruskal [1965], Zakharov and Manakov [1976], and Zakharov and Shabat [1971].

Here, we denote $Q(x) = \sqrt{2}\cosh^{-1}(x)$ the unique positive even solution of $Q'' + Q^3 = Q$, and for any $c > 0$, $Q_\lambda(x) = \sqrt{c}\,Q(\sqrt{c}x)$. Then, for any $c > 0$, $\beta \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and $\gamma \in \mathbb{R}$,

$$u(t,x) = Q_c(x - \beta t - \sigma)e^{i\Gamma(t,x)}, \quad \Gamma(t,x) = \frac{1}{2}(\beta \cdot x) - \frac{1}{4}|\beta|^2 t + ct + \gamma,$$

is a solitary wave of (2), moving on the line $x = \sigma + \beta t$ and also called *soliton*.

As the KdV equation, the 1D cubic NLS admits explicit multi-solitons. However, the possible behaviors of multi-solitons is richer for NLS. In addition to multi-solitons distant like Ct, which is the generic situation, the equation also admits multi-solitons where the distance between some solitons is $C \log t$ (see Olmedilla [1987] and Zakharov and Shabat [1971]; this requires solitons of exactly the same size, like for mKdV) and solutions where some solitons are staying at a finite distance from each other for all time (see Yang [2010] and Zakharov and Shabat [1971]). As for mKdV, the presence of such multi-solitons complicates any general decomposition result but does not prevent it. For such questions, we refer to the recent work Borghese, Jenkins, and McLaughlin [2016] and its references.

2.4 The sine-Gordon equation. The sine-Gordon equation

$$\partial_t^2 u - \partial_x^2 u + \sin u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

was also widely studied as a physically relevant and completely integrable model (see *e.g.* Dauxois and Peyrard [2010], Dodd, Eilbeck, Gibbon, and Morris [1982], and Lamb [1980]). This equation has an explicit kink solution $S(x) = 4 \arctan(e^x)$. It also has other exceptional solutions, like time-periodic *wobbling kinks* (see Cuenda, Quintero, and Sánchez [2011] and Segur [1983]), and *breathers* (see Lamb [1980]).

2.5 Other integrable models and nearly integrable models. For the derivative NLS equation, we refer to Jenkins, J. Liu, Perry, and Sulem [2017] and its references. For the KP-I equation, see Lamb [1980]. For integrable models set on the torus, see Kuksin [2000] and references therein.

Several *nearly integrable equations* have also been studied in the context of the theory of inverse scattering. We refer to Dauxois and Peyrard [2010], Deift and Zhou [2002], Kivshar and Malomed [1989], and Yang [2010] and to the references therein.

2.6 Formal works and numerical simulations. Note that shortly after the development of the inverse scattering and the discovery of explicit multi-solitons, other approaches appeared, like in Ei and Ohta [1994], Gorshkov and Ostrovsky [1981], and Karpman and Solovev [1981], to investigate possible multi-soliton behaviors for integrable or non-integrable models. Such papers focus on the modulation equations of the parameters of the solitons and lack the analysis of the error terms, but they aim at justifying formally multi-solitons behaviors beyond any integrability property or proximity to integrable equations. In particular, as for the rigorous results presented in Section 5 below, they are *asymptotic results*, restricted to cases where the distances between the various solitons are large enough.

Theoretical and numerical works have been developed in parallel. As mentioned in the Introduction, the subject started with two fundamental numerical experiments presented in Fermi, Pasta, and Ulam [1955] and Zabusky and Kruskal [1965]. Since then, there has been an intense activity on studying solitary waves interactions from the numerical point of view. We refer to Craig, Guyenne, Hammack, Henderson, and Sulem [2006] which compares KdV multi-solitons, the water wave problem from the numerical point of view and real experiments on waves generated in water tanks. For Klein-Gordon equations, we refer to Ablowitz, Kruskal, and Ladik [1979]. We also refer to Bona, Pritchard, and Scott [1980], Dauxois and Peyrard [2010], Hammack, Henderson, Guyenne, and Yi [2004], Li and Sattinger [1999], Shih [1980], and Yang [2010] and references therein. One of the main questions studied by numerical experiments is the elastic versus inelastic character of the collisions of nonlinear waves.

3 Nonlinear models with solitary waves

In these notes, we consider four typical nonlinear models and work with the notion of solution in the energy space.

3.1 The generalized Korteweg-de Vries equation.

Consider the following 1D model, for any integer $p \geq 2$,

$$\partial_t u + \partial_x(\partial_x^2 u + u^p) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{3}$$

As seen before, the case $p = 2$ corresponds to the KdV equation and $p = 3$ to the mKdV equation, which are both completely integrable.

The mass and energy

$$\int u^2(t), \quad \int \left(\frac{1}{2}u_x^2(t) - \frac{1}{p+1}u^{p+1}(t)\right)$$

are formally conserved for solutions of (3). We refer to Kenig, Ponce, and Vega [1993] for the local well-posedness of the Cauchy problem in the energy space H^1 (see also Kato [1983]). For $1 < p < 5$, all solutions in H^1 are global and bounded, and the problem is called *sub-critical*. For $p = 5$, the problem is mass critical (blow up solution do exist, see Martel, Merle, and Raphaël [2014] and references therein) and $p > 5$ correspond to the super-critical case. The notion of criticality corresponds to the scaling invariance of equation (3): indeed, if $u(t,x)$ is solution then for any $c > 0$, $u_c(t,x) = c^{\frac{1}{p-1}} u(c^{\frac{3}{2}} t, c^{\frac{1}{2}} x)$ is also solution and $\|u_c(t)\|_{L^2} = c^{\frac{1}{p-1} - \frac{1}{4}} \|u(t)\|_{L^2}$.

Let Q be the unique (up to sign change if p is odd) non-trivial even solution of $Q'' + Q^p = Q$ on $\mathbb{R}$, explicitely given by

$$Q(x) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \cosh^{-\frac{2}{p-1}}\left(\frac{p-1}{2}x\right).$$

For $c > 0$, let $Q_c(x) = c^{\frac{1}{p-1}} Q(c^{\frac{1}{2}} x)$. Note that these formulas for Q and Q_c generalize the previous ones given for $p = 2$ and $p = 3$. As before, solitary waves (also called solitons by abuse of terminology) are solutions of (3) of the form $u(t,x) = Q_c(x - ct - \sigma)$, for any $c > 0$ and $\sigma \in \mathbb{R}$.

The *orbital stability of solitons* with respect to small perturbations in the energy space H^1 is known in the sub-critical case.

Theorem 3 (Stability of the soliton for sub-critical gKdV Benjamin [1972], Bona [1975], Cazenave and Lions [1982], and Weinstein [1985, 1986])**.** *Let $1 < p < 5$. For all $\epsilon > 0$, there exists $\delta > 0$, such that if $\|u_0 - Q\|_{H^1} \leq \delta$, then the solution u of (3) with initial data u_0 satisfies, for all $t \in \mathbb{R}$, $\|u(t, . + \sigma(t)) - Q\|_{H^1} \leq \epsilon$, for some function σ.*

In contrast, solitons are unstable in the critical and super-critical case $p \geq 5$. Note that the instability phenomenon is quite different in the critical case (linear stability holds and the nonlinear instability is related to the scaling parameter) and in the super-critical case (linear exponential instability). See Bona, Souganidis, and Strauss [1987], Cazenave [2003], Grillakis, Shatah, and Strauss [1987], Martel and Merle [2000], and Pego and Weinstein [1992].

3.2 The nonlinear Schrödinger equation.

Recall the nonlinear NLS equation

$$i\partial_t u - \Delta u - |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{4}$$

We consider the case $p > 1$ for $d = 1, 2$, and $1 < p < \frac{d+2}{d-2}$ for $d \geq 3$. For $d = 1$ and $p = 3$, the model is completely integrable, as seen before. Note that, similarly as for the gKdV equation, $p = 1 + \frac{4}{d}$ corresponds to L^2 criticality, while for $d \geq 3$, $p = \frac{d+2}{d-2}$ corresponds to $\dot{H}^1$ criticality.

The mass, energy and momentum

$$\int |u(t)|^2, \quad \int \left(\frac{1}{2}|\nabla u(t)|^2 - \frac{1}{p+1}|u(t)|^{p+1}\right), \quad \Im\left(\int \nabla u(t)\bar{u}(t)\right)$$

are formally conserved for solutions of (4). We refer to Cazenave [2003], Ginibre and Velo [1979], and Tao [2006] for the local well-posedness of the Cauchy problem in the energy space H^1.

We denote by Q the unique positive radially symmetric H^1 solution of $\Delta Q + |Q|^{p-1}Q = Q$ on $\mathbb{R}^d$ (the function Q is called the *ground state*; see existence and uniqueness results in Berestycki and Lions [1983], Cazenave [2003], Kwong [1989], and Tao [2006]). For $c > 0$, let $Q_c(x) = c^{\frac{1}{p-1}} Q(c^{\frac{1}{2}} x)$. Note that this is a further generalization of the notation for gKdV, for any space dimension $d \geq 1$. For $d \geq 2$, ground states are no longer explicit, but their properties are well-understood (see references above). Then, for any $c > 0$, $\beta \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$, the function u defined by

$$u(t,x) = Q_c(x - \beta t - \sigma) e^{i\Gamma(t,x)} \quad \text{where} \quad \Gamma(t,x) = \frac{1}{2}(\beta \cdot x) - \frac{1}{4}|\beta|^2 t + ct + \gamma,$$

is a traveling wave of (4), with speed β.

The stability and instability properties of solitary waves of NLS are similar: stability in the L^2 sub-critical case, and instability in the critical and super-critical cases. We refer to Cazenave [2003], Cazenave and Lions [1982], Grillakis [1990], Grillakis, Shatah, and Strauss [1987], and Weinstein [1985] for details.

3.3 The ϕ^4 equation. We consider the ϕ^4 model (see *e.g.* Dauxois and Peyrard [2010] and Manton and Sutcliffe [2004])

$$\partial_t^2 \phi - \partial_x^2 \phi - \phi + \phi^3 = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{5}$$

Recall that the energy

$$E(\phi, \partial_t \phi) = \int \frac{1}{2}|\partial_t \phi|^2 + \frac{1}{2}|\partial_x \phi|^2 + \frac{1}{4}\left(1 - |\phi|^2\right)^2$$

is formally conserved along the flow. The *kink*, defined by $H(x) = \tanh\left(x/\sqrt{2}\right)$ is the unique (up to sign change), bounded, odd solution of the equation $-H'' = H - H^3$ on $\mathbb{R}$. We recall that the orbital stability of the kink with respect to small perturbations in the energy space has been proved in Henry, Perez, and Wreszinski [1982] using mainly the energy conservation. This model is analogue to the sine-Gordon equation, but it is not completely integrable and breathers solutions or woobling kinks are not expected to exist.

3.4 The energy critical nonlinear wave equation. For space dimensions $d \geq 3$, we consider the following nonlinear wave equation,

$$\partial_t^2 u = \Delta u + |u|^{\frac{4}{d-2}} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{6}$$

We denote

$$E(u,v) = \int \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}v^2 - \frac{1}{6}|u|^6\right)$$

so that the energy of a solution $(u, \partial_t u)$ of (6), defined by $E(u, \partial_t u)$, is formally conserved by the flow. Concerning the Cauchy problem in $\dot{H}^1 \times L^2$ for the energy critical wave equation, we refer to Kenig and Merle [2008] and the references given therein. As before, the notion of criticality is related to the scaling invariance: if $u(t,x)$ is a solution, then for any $\lambda > 0$,

$$u_\lambda(t,x) = \frac{1}{\lambda^{\frac{d-2}{2}}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right) \quad \text{is also solution and} \quad \|\nabla u_\lambda\|_{L^2} = \|\nabla u\|_{L^2}.$$

Here, solitary waves are stationary solutions $W \in \dot{H}^1$ satisfying $\Delta W + |W|^{\frac{4}{d-2}} W = 0$, and traveling waves obtained as Lorentz transforms of such solutions. For $\boldsymbol{\ell} \in \mathbb{R}^d$, $|\boldsymbol{\ell}| < 1$, we denote

$$W_{\boldsymbol{\ell}}(x) = W\left(\left(\frac{1}{\sqrt{1-|\boldsymbol{\ell}|^2}} - 1\right)\frac{\boldsymbol{\ell}(\boldsymbol{\ell}\cdot x)}{|\boldsymbol{\ell}|^2} + x\right),$$

so that $u(t,x) = W_{\boldsymbol{\ell}}(x - \boldsymbol{\ell} t)$ is solution of (6). As for the NLS equation, we consider only *ground states* solitary waves, *i.e.* solutions of the above elliptic equation explicitely given by

$$W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}.$$

As solutions of the evolution equation (6), they are unstable with respect to perturbation of the initial data with one direction of exponential instability (see Duyckaerts and Merle [2008] and Grillakis [1990]).

4 Asymptotic stability

We recall briefly some results of asymptotic stability of solitons.

4.1 Asymptotic stability for gKdV solitons.

Theorem 4 (Asymptotic stability of the gKdV soliton in H^1, Martel and Merle [2001]). *Let $p = 2, 3, 4$. For any $\beta > 0$, there exists $\delta = \delta(\beta) > 0$ such that the following is true. Let $u_0 \in H^1$ be such that $\|u_0 - Q\|_{H^1} \le \delta$. Then, the global solution u of* (3) *with initial data u_0 satisfies*

$$\lim_{t\to+\infty} \|u(t) - Q_{c^+}(\cdot - \sigma(t))\|_{H^1(x>\beta t)} = 0,$$

for some $c^+ > 0$ with $|c^+ - 1| \lesssim \delta$ and some C^1 function σ such that $\lim_{+\infty} \sigma' = c^+$.

We refer to Pego and Weinstein [1994] for the first result of asymptotic stability of gKdV solitons. Theorem 1 claims strong convergence in H^1 in the region $x > \beta t$. Strong convergence in $H^1(\mathbb{R})$ is never true since it would imply by stability that u is exactly a soliton. The region where convergence is obtained in Theorem 1 is sharp since one can construct a solution which behaves asymptotically as $t \to +\infty$ as the sum $Q(x-t) + Q_c(x-ct)$, where $0 < c \neq 1$ is arbitrary (see Martel [2005], Miura [1976], and Wadati and Toda [1972]). In particular, choosing $c \ll 1$, the H^1 norm of $Q_c(x-ct)$ is small, and this soliton travels on the line $x = ct$. This explains the necessity of a positive β in the convergence result. We also refer to the survey Tao [2009]. For $p = 4$, the result has been completed in Kenig and Martel [2009] and Tao [2007] showing that the residue scatters in a Besov space close to the critical Sobolev space $\dot{H}^{-1/6}$. For $p = 3$, we refer to Germain, Pusateri, and Rousset [2016] for a full asymptotic stability statement.

4.2 Asymptotic stability for NLS equations. In the context of the nonlinear Schrödinger equation, pioneering results on asymptotic stability of traveling waves are Buslaev and Perelman [1992, 1995] and Soffer and Weinstein [1990, 1992]. These papers initiated the method of separating modes and using dispersive estimates with potential, under assumptions on the spectrum of the linearized operator.

This question has then been extensively studied, for the NLS equation with or without potential and for various nonlinearities, see *e.g.* Buslaev and Sulem [2003], Cuccagna [2014], Gustafson, Nakanishi, and Tsai [2004], Nakanishi and Schlag [2011], Rodnianski, Schlag, and Soffer [2005], Rodnianski, Schlag, and Soffer [2003], Schlag [2006], and Schlag [2007, 2009] as typical papers, and their references. Most of these works require specific assumptions, like spectral assumptions or suitable dispersive estimates for equations with unknown potential, a suitable Fermi Golden Rule or flatness conditions on the nonlinearities at 0. It follows that no result of asymptotic stability is fully proved for any pure power NLS equation without potential with stable solitons, except for the integrable cubic 1D NLS treated in Cuccagna and Pelinovsky [2014].

In larger dimensions, or higher order nonlinearities, the solitons are unstable. The notion of *conditional asymptotic stability* and the construction of *center stable manifolds* then become relevant. For the focusing 3D cubic NLS equation (which is an $\dot{H}^{\frac{1}{2}}$ critical equation with exponentially unstable solitons) the theory has been especially well-developed, at least in the radial case, in Beceanu [2008, 2012], Costin, Huang, and Schlag [2012], Nakanishi and Schlag [2011], Schlag [2006], and Schlag [2009]. In particular, spectral assumptions implying the desired dispersive estimates for the linearized equation around the soliton have been checked, first numerically and then rigorously by computer assisted proof (see Costin, Huang, and Schlag [2012] and references therein).

4.3 Asymptotic stability of the ϕ^4 kink. The asymptotic stability of the kink H by the ϕ^4 flow (5) is known in the case of odd perturbations in the energy space. Note that for odd initial data, the corresponding solution of (5) is odd. Rewrite $\phi = H + u$. Then, one has

$$\partial_t^2 u - \partial_x^2 u - u + 3H^2 u + 3Hu^2 + u^3 = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{7}$$

Theorem 5 (Asymptotic stability of the kink by odd perturbations, Kowalczyk, Martel, and Muñoz [2017])**.** *There exists $\delta > 0$ such that for any odd $(u_0, u_1) \in H^1 \times L^2$ with $\|(u_0,u_1)\|_{H^1 \times L^2} \leq \delta$, the solution $(u, \partial_t u)$ of (7) with initial data (u_0, u_1) satisfies, for any bounded interval $I \subset \mathbb{R}$,*

$$\lim_{t \to \pm\infty} \|(u, \partial_t u)(t)\|_{H^1(I) \times L^2(I)} = 0.$$

As for gKdV, if a solution u of (7) satisfies $\lim_{t \to +\infty} \|(u, \partial_t u)(t)\|_{H^1 \times L^2} = 0$, then by orbital stability Henry, Perez, and Wreszinski [1982], $u(t) \equiv 0$, for all $t \in \mathbb{R}$. Thus the local result is in some sense optimal.

For previous related results, we refer to Kopylova and Komech [2011a,b] where the asymptotic stability of the kink is studied for the 1D equation $\partial_t^2 u - \partial_x^2 u + F(u) = 0$, under specific assumptions on F (not including the ϕ^4 model) and to Cuccagna [2008], where the stability and asymptotic stability of the one dimensional kink for the ϕ^4 model, subject to *localized three dimensional* perturbations is studied. We also refer the references in Cuccagna [2008], Kopylova and Komech [2011a,b], and Kowalczyk, Martel, and Muñoz [2017] for related works on scattering of small solutions to Klein-Gordon equations. See also the review Kowalczyk, Martel, and Muñoz [2016-2017] and references therein.

4.4 Blow up profile for the critical wave equation. Recall that Kenig and Merle [2008] provides a classification of all possible behaviors (blow up or scattering) of solutions of (6) whose initial data (u_0, u_1) satisfies $E(u_0, u_1) < E(W, 0)$. Next, Duyckaerts and Merle [2008] studies the *threshold case* $E(u_0, u_1) = E(W, 0)$ and constructs the stable manifold around W. Then, Duyckaerts, Kenig, and Merle [2011, 2012] proved the following result for solutions slightly above the threshold.

Theorem 6 (Blow up profile for 3D critical NLW, Duyckaerts, Kenig, and Merle [2011, 2012])**.** *Let $d = 3$. There exists $\delta > 0$ such that if u is a solution of* (6) *blowing up in finite time $T > 0$ and satisfying the bound*

$$\sup_{[0,T)} \left(\|\nabla u(t)\|_{L^2} + \frac{1}{2} \|\partial_t u(t)\|_{L^2} \right) \leq \|\nabla W\|_{L^2} + \delta,$$

then

$$\lim_{t\uparrow T}\left\|(u(t),\partial_t u(t))-(v_0,v_1)\mp\left(\frac{1}{\lambda^{\frac12}(t)}W_{\boldsymbol\ell}\left(\frac{\cdot-\sigma(t)}{\lambda(t)}\right),-\frac{1}{\lambda^{\frac32}(t)}(\boldsymbol\ell\cdot\nabla W_{\boldsymbol\ell})\left(\frac{\cdot-\sigma(t)}{\lambda(t)}\right)\right)\right\|_{L^2}=0$$

for some σ, λ *and* $\boldsymbol\ell\in\mathbb{R}^3$, $|\boldsymbol\ell|<1$ *and* $(v_0,v_1)\in\dot H^1\times L^2$.

We see that the family $\{\pm W_{\boldsymbol\ell}\}$ is the universal blow up profile. We refer to the original papers for more results and details.

We refer to Krieger, Nakanishi, and Schlag [2014, 2013] for classification results of solutions with energy at most slightly above the one of the ground state, and to Krieger, Schlag, and Tataru [2009] and Jendrej [2017] for contructions of solutions with prescribed blow up rates (*type II* blow up). We also refer to Martel and Merle [2002], Martel, Merle, and Raphaël [2014], and Merle and Raphael [2004, 2005] and references therein for previous results of blow up profile in the case of mass critical gKdV and NLS equations. Concerning blow up, see also the review Raphaël [2014] and the references therein.

5 Asymptotic multi-solitons

In this section, we discuss results of existence of *asymptotic multi-solitons* for non-integrable models, inspired by Theorem 1 and other explicit constructions of multi-solitons for integrable models, but limited to one direction of time. In particular, these results are valid in asymptotic situations where the distances between all the solitary waves are large enough.

5.1 Multi-solitons with weak interactions. As a rough idea, weak interaction means that the trajectories of the solitary waves are not affected asymptotically.

Theorem 7 (Existence and uniqueness of gKdV multi-solitons, Martel [2005]). *Let* $p=2,3,4$ *or* 5. *Let* $K\geq 2$, $0<c_K<\cdots<c_1$, *and* $\sigma_1,\ldots,\sigma_K\in\mathbb{R}$. *Let* $\mathbf{R}=\sum_{k=1}^K R_k$ *where*

$$R_k(t,x)=Q_{c_k}(x-\sigma_k-c_kt).$$

There exists a unique H^1 *solution of (3) such that* $\lim_{t\to-\infty}\|u(t)-\mathbf{R}(t)\|_{H^1}=0$. *Moreover, there exists* $\kappa>0$ *such that, for all* $t\leq 0$, $\|u(t)-\mathbf{R}(t)\|_{H^1}\lesssim e^{-\kappa|t|}$.

Such result shows that the multi-soliton behavior is universal, at least in one direction of time. Observe that it says nothing about the behavior the solution as $t\to+\infty$. The uniqueness statement in the energy space is relevant even in the integrable case since the inverse scattering theory requires *a priori* more space decay. The stability of the multi-soliton structure was studied in the sub-critical case in Martel, Merle, and Tsai [2002].

We observe that a similar existence result also holds for the gKdV super-critical equation ($p > 5$), with a specific classification related to the exponential instability, see Combet [2011] and Côte, Martel, and Merle [2011].

For the NLS equation, we recall the following existence result.

Theorem 8 (Existence of NLS multi-solitary waves, Côte, Martel, and Merle [2011], Martel and Merle [2006], and Merle [1990])**.** *Let $d \geq 1$. Let $p > 1$ for $d = 1, 2$ and $1 < p < \frac{d+2}{d-2}$ for $d \geq 3$. Let $K \geq 2$ and for any $k \in \{1, \dots, K\}$, let $c_k > 0$, $\beta_k \in \mathbb{R}^d$, $\sigma_k \in \mathbb{R}^d$ and $\gamma_k \in \mathbb{R}$. Assume that, for any $k \neq k'$, $\beta_k \neq \beta_{k'}$. Let $\mathbf{R} = \sum_{k=1}^{K} R_k$ where*

$$R_k(t,x) = Q_{c_k}(x - \sigma_k - \beta_k t)e^{i\Gamma_k(t,x)} \quad \text{and} \quad \Gamma_k(t,x) = \frac{1}{2}(\beta_k \cdot x) - \frac{1}{4}|\beta_k|^2 t + c_k t + \gamma_k.$$

Then, there exist $T_0 \in \mathbb{R}$, $\kappa > 0$ and an H^1 solution u of (4) such that, for all $t \leq T_0$, $\|u(t) - \mathbf{R}(t)\|_{H^1} \lesssim e^{-\kappa|t|}$.

Uniqueness (for critical and sub-critical nonlinearities) or classification (for super-critical nonlinearity) is an open problem. See Combet [2014] for multi-existence in the 1D super-critical case.

Note that the construction of multi-solitons and the study of the stability of the sums of multi-soliton has been extended to several other models, see *e.g.* Côte and Martel [2016] and Côte and Muñoz [2014] for the case of the nonlinear Klein-Gordon equation, and Ming, Rousset, and Tzvetkov [2015] for the water wave model.

For the 5D energy critical wave equation, the following existence result is proved in Martel and Merle [2016].

Theorem 9 (Existence of NLW multi-solitary waves, Martel and Merle [ibid.])**.** *Let $d = 5$. Let $K \geq 2$, and for any $k \in \{1, \dots, K\}$, let $\lambda_k > 0$, $\boldsymbol{\sigma}_k \in \mathbb{R}^5$, $\iota_k \in \{-1, +1\}$ and $\boldsymbol{\ell}_k \in \mathbb{R}^5$ with $|\boldsymbol{\ell}_k| < 1$. Assume that, for any $k' \neq k$, $\boldsymbol{\ell}_k \neq \boldsymbol{\ell}_{k'}$. Let $\mathbf{R} = \sum_{k=1}^{K} R_k$ where*

$$R_k(t,x) = \frac{\iota_k}{\lambda_k^{\frac{3}{2}}} W_{\boldsymbol{\ell}_k}\left(\frac{x - \boldsymbol{\ell}_k t - \boldsymbol{\sigma}_k}{\lambda_k}\right).$$

Assume that one of the following assumptions holds

(1) Two-solitons: $K = 2$

(2) Collinear speeds: *For all $k \in \{1, \dots, K\}$, $\boldsymbol{\ell}_k = \ell_k \mathbf{e}_1$ where $\ell_k \in (-1, 1)$.*

Then, there exist $T_0 \in \mathbb{R}$ and a solution u of (6) on $(-\infty, T_0]$ in the energy space such that

$$\lim_{t \to -\infty} \|\nabla_{x,t}(u(t) - \mathbf{R}(t))\|_{L^2} = 0.$$

5.2 Multi-solitons with strong interactions. We state a typical result where the strong interactions between the traveling waves indeed affect their trajectories.

Theorem 10 (Two-solitary waves with logarithmic distance, Nguyen [2016]). *Let $d \geq 1$. Let*

$$1 < p < \frac{d+2}{d-2} \ (p > 1 \text{ for } d = 1,2) \quad \textit{and} \quad p \neq 1 + \frac{4}{d}.$$

There exists a solution u of (4) such that $|z_1(t) - z_2(t)| \sim 2 \log t$ as $t \to -\infty$ and

$$\lim_{t \to -\infty} \left\| u(t) - e^{-i\gamma(t)} \sum_{k=1,2} Q(\cdot - z_k(t)) \right\|_{H^1} = 0.$$

As discussed in Section 2.3, such solutions were already known in the integrable case by the inverse scattering theory. The above result means that this behavior is universal for general NLS equations, under the same restriction that the traveling waves have equal scaling. The mass critical case $p = 1 + \frac{4}{d}$ is excluded since it displays a special behavior related to blow up and where the above behavior is visible only in rescaled variables, as previously described in Martel and Raphael [2015]. For the gKdV equation, a result similar to Theorem 10 is given in Nguyen [2017].

We mention a few other previous results of strong interactions: for the Hartree equation Krieger, Martel, and Raphaël [2009], for the energy critical wave equation Jendrej [2016b,a] and Jendrej and Lawrie [2017], for the mass critical gKdV equation Combet and Martel [2017b,a], and for the half-wave equation Gérard, Lenzmann, Pocovnicu, and Raphaël [2018].

5.3 Soliton interaction with the background. Several papers deal with the question of the interaction of a soliton with a changing background or an impurity. See Holmer and Zworski [2007, 2008], Holmer, Marzuola, and Zworski [2007a], and Holmer, Marzuola, and Zworski [2007b] for the interaction of a soliton of NLS with a Dirac mass or a slowly varying potential, and Muñoz [2012b,a] for the interaction of a gKdV soliton with a slowing variable bottom.

6 Decomposition into solitons for the energy critical wave equation

Here, we recall the few existing results of decomposition in solitons in non-integrable cases. First, a complete result of decomposition into solitons for equation (6) was proved in Duyckaerts, Kenig, and Merle [2013] for the radial 3D case.

Theorem 11 (Soliton resolution for the 3D radial critical wave equation, Duyckaerts, Kenig, and Merle [ibid.]). *Let $d = 3$. Let u be a global radial solution of* (6). *Then, there exist a solution v_{L} of the linear wave equation, $K \in \mathbb{N}$, $\iota_k \in \{-1, 1\}$, $\lambda_k > 0$, such that*

$$\lim_{t \to +\infty} \left\| (u(t), \partial_t u(t)) - \left(v_{\mathrm{L}}(t) + \sum_{k=1}^{K} \frac{\iota_k}{\lambda_k^{\frac{1}{2}}(t)} W\left(\frac{\cdot}{\lambda_k(t)}\right), \partial_t v_{\mathrm{L}}(t) \right) \right\|_{\dot{H}^1 \times L^2} = 0,$$

and $\lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_K(t) \ll t$, as $t \to +\infty$.

Note that the above result is in some sense more complete than for gKdV (Section 2.1), since the residue is proved to scatter. A similar result holds for blow up solutions, provided they exhibit type II blow up. The soliton resolution conjecture was later proved in the non-radial case for a subsequence of time for the $3, 4$ and 5D energy critical wave equation in Duyckaerts, Kenig, and Merle [2016] and Duyckaerts, Jia, Kenig, and Merle [2017]. Note that a fundamental idea in the approach of Duyckaerts, Kenig, and Merle [2013] is the introduction of the *method of channels of energy* for the linear wave equation (see Theorem 16 for a typical result in 5D).

See similar results for the wave map problem in Côte [2015] and Côte, Kenig, Lawrie, and Schlag [2015a,b].

7 Collision problem

Concerning the collision problem, we recall the discussion in Craig, Guyenne, Hammack, Henderson, and Sulem [2006] on inelastic collisions. To study the collision problem, it is natural to study the behavior as $t \to +\infty$ of the solutions constructed in Theorems 7, 8, 9. See Craig, Guyenne, Hammack, Henderson, and Sulem [ibid.], page 057106-4 for suggesting this approach which seems more canonical than to study initial data with the sum of two solitons initially distant.

7.1 Collision for the quartic gKdV equation I. We consider the quartic gKdV equation

$$\partial_t u + \partial_x(\partial_x^2 u + u^4) = 0 \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{8}$$

The article Martel and Merle [2011a] (see also Muñoz [2010] for generalization to any gKdV equation) gives the first rigorous results concerning collision of solitons for a non-integrable equation, and in particular the first proof of non-existence of pure two-soliton solutions, in the case where one soliton is much smaller than the other one.

Theorem 12 (Collision of solitons with very different size, Martel and Merle [2011a]). *Assume* $0 < c \ll 1$. *Let* u *be the solution of* (8) *such that*

$$\lim_{t\to-\infty} \left\| u(t) - Q(\cdot - t) - Q_c(\cdot - ct) \right\|_{H^1} \to 0.$$

(i) Global stability of the 2-soliton behavior. *There exist* $c_1^+ \underset{c\to 0}{\sim} 1$, $c_2^+ \underset{c\to 0}{\sim} c$, ρ_1, ρ_2 *such that the function* w^+ *defined by*

$$w^+(t,x) = u(t,x) - Q_{c_1^+}(x - \rho_1(t)) - Q_{c_2^+}(x - \rho_2(t))$$

satisfies

$$\lim_{t\to+\infty} \|w^+(t)\|_{H^1(x \geq \frac{c}{10}t)} = 0 \quad \textit{and} \quad \sup_{t\in\mathbb{R}} \|w^+(t)\|_{H^1} \lesssim c^{\frac{1}{3}}.$$

(ii) Inelasticity of the collision. *Moreover, for* $t \gg 1$,

$$c_1^+ - 1 \gtrsim c^{\frac{17}{6}}, \quad 1 - \frac{c_2^+}{c} \gtrsim c^{\frac{8}{3}}, \quad c^{\frac{17}{12}} \lesssim \|w_x^+(t)\|_{L^2} + c^{\frac{1}{2}} \|w^+(t)\|_{L^2} \lesssim c^{\frac{11}{12}}.$$

The first part of the theorem means that the two solitons are preserved through the collision, even the smallest one. Indeed, for c small, $\sup_t \|w^+(t)\|_{H^1} \lesssim c^{\frac{1}{3}} \ll \|Q_c\|_{H^1} \sim c^{\frac{1}{12}}$.

The second part of the theorem says that the sizes of the final solitons as $t \to +\infty$ are slightly changed with respect to their original sizes as $t \to -\infty$, and that the residue does not converge to zero. In particular, the solution is not a pure 2-soliton as $t \to +\infty$ in this regime. Thus, the collision is *not elastic.*

7.2 Collision for the quartic gKdV equation II. A first intuition on the general problem of two solitons with *almost same sizes* comes from the explicit multi-solitons of the integrable case. From LeVeque [1987], we have a sharp description of the behavior of the multi-soliton of (1) satisfying

$$\lim_{t\to\pm\infty} \left\| u(t) - Q_{c_1}(\cdot - c_1 t - \sigma_1^{\pm}) - Q_{c_2}(\cdot - c_2 t - \sigma_2^{\pm}) \right\|_{H^1} = 0,$$

in the special asymptotics where $0 < \mu_0 = \frac{c_2 - c_1}{c_1 + c_2} \ll 1$, *i.e.* for nearly equal solitons. Indeed, the following global in time estimate is proved for some explicit functions $c_k(t)$, $\sigma_k(t)$:

$$\sup_{t,x\in\mathbb{R}} \left| u(t,x) - Q_{c_1(t)}(x - \sigma_1(t)) - Q_{c_2(t)}(x - \sigma_2(t)) \right| \lesssim \mu_0^2.$$

Moreover, it is proved that $\min_{t\in\mathbb{R}}(\sigma_1(t)-\sigma_2(t)) \sim 2|\ln\mu_0|$. This means that the minimum separation between the two solitons is large when $\mu_0 \ll 1$. In particular, the two solitons never cross and the solution has two maximum points for all time. The interaction is repulsive, the solitons exchange their sizes and speeds at large distance and consequently avoid the collision.

We now recall results from Mizumachi [2003] for the quartic gKdV equation. Let u be a solution of (8) for which the initial data is close to the sum $Q(x)+Q_c(x+Y_0)$, where $Y_0>0$ is large and $0 \le c-1 \le \exp(-\frac{1}{2}Y_0)$, so that the quicker soliton can be initially *on the left* of the other soliton. It follows from Mizumachi [ibid.] that the interaction of the two solitons is repulsive: the two solitons remain separated for all positive time and eventually $u(t)$ behaves as

$$u(t,x) = Q_{c_1^+}(x-c_1^+t-\sigma_1^+) + Q_{c_2^+}(x-c_2^+t-\sigma_2^+) + w(t,x),$$

for large time, for some $c_1^+ > c_2^+$ close to 1 and w small in some sense. The situation for almost equal solitons of the quartic (gKdV) is thus at the main order similar to the one described in the integrable case in LeVeque [1987]. The analysis part in Mizumachi [2003] relies on techniques from Hayashi and Naumkin [1998, 2001] and on the use of spaces introduced in this context in Pego and Weinstein [1992].

Before presenting the main result from Martel and Merle [2011b], for simplicity, we change variables. For $c_2-c_1>0$ small, and any σ_1,σ_2, let $u(t)$ be the unique solution of (8) such that

$$\lim_{t\to-\infty} \|u(t)-Q_{c_1}(\cdot-c_1t-\sigma_1)-Q_{c_2}(\cdot-c_2t-\sigma_2)\|_{H^1}=0.$$

Let

$$c_0=\frac{c_1+c_2}{2},\quad \mu_0=\frac{c_2-c_1}{c_1+c_2},\quad y_1=\sigma_1\sqrt{c_0},\quad y_2=\sigma_2\sqrt{c_0}.$$

Then the function $U(t,x)=c_0^{-1/3}u(c_0^{-3/2}t, c_0^{-1/2}(x+t))$ solves

$$\partial_t U+\partial_x(\partial_x^2 U - U + U^4)=0,\quad (t,x)\in\mathbb{R}\times\mathbb{R}, \tag{9}$$

and is the unique solution of (9) satisfying

$$\lim_{t\to-\infty}\left\|U(t)-Q_{1-\mu_0}(\cdot+\mu_0t-y_1)-Q_{1+\mu_0}(\cdot-\mu_0t-y_2)\right\|_{H^1}=0.$$

The next result concerns the asymptotics $\mu_0>0$ small.

Theorem 13 (Inelastic interaction of two nearly equal solitons, Martel and Merle [ibid.]). *Assume*
$0<\mu_0\ll 1$. *Let U be the unique solution of* (9) *such that*

$$\lim_{t\to-\infty}\left\|U(t)-Q_{1-\mu_0}(\cdot+\mu_0t+Y_0+\ln 2)-Q_{1+\mu_0}(\cdot-\mu_0t-Y_0-\ln 2)\right\|_{H^1}=0,$$

where $Y_0 = \frac{1}{2}|\ln(\mu_0^2/\alpha)|$ *and* $\alpha = 12(10)^{2/3}(\int Q^2)^{-1}$. *Then the following holds.*

(i) Global stability of the 2-soliton behavior. *There exist* μ_1, μ_2, y_1, y_2 *such that*

$$w(t,x) = U(t) - Q_{1+\mu_1(t)}(x - y_1(t)) - Q_{1+\mu_2(t)}(x - y_2(t))$$

satisfies $|\min_{t\in\mathbb{R}}(y_1(t) - y_2(t)) - 2Y_0| \lesssim \mu_0$ *and*

$$\lim_{t\to+\infty} \|w(t)\|_{H^1(x>-\frac{9}{10}t)} = 0, \quad \sup_{t\in\mathbb{R}} |w(t)\|_{H^1} \lesssim \mu_0^{\frac{3}{2}}.$$

(ii) Inelasticity of the interaction.

$$\lim_{+\infty} \mu_1 - \mu_0 \gtrsim \mu_0^5, \quad \mu_0 - \lim_{+\infty} \mu_2 \gtrsim \mu_0^5, \quad \liminf_{t\to+\infty} \|w(t)\|_{H^1} \gtrsim \mu_0^3.$$

It follows that *no pure* 2*-soliton exists* also in this regime. The proofs of Theorems 12 and 13 are based on the construction of a refined approximate solution of the two-soliton problem for all t and x.

7.3 Collision for the quartic gKdV equation III. Still concerning the collision of two solitons for the quartic gKdV equation, we recall from Martel and Merle [2015] the following negative result.

Theorem 14 (Inelasticity of collision for gKdV, Martel and Merle [ibid.])**.** *Let* $K \geq 2$, $0 < c_K < \cdots < c_1 = 1$ *and* $\sigma_1, \ldots, \sigma_K \in \mathbb{R}$. *Let* u *be the solution of* (8) *satisfying*

$$\lim_{t\to-\infty} \left\| u(t) - \sum_{k=1}^{K} Q_{c_k}(\cdot - c_k t - \sigma_k) \right\|_{H^1} = 0.$$

Assume that $\sum_{k=2}^{K} (1 - c_k)^2 < \frac{1}{16}$. *Then,* $u(t)$ *is not an asymptotic multi-soliton as* $t \to +\infty$. *In particular, there exists no pure multi-soliton of* (1) *with the speeds* $c_1, c_2, \ldots, c_K$.

In the case of two solitons, the condition on the speeds reduces to $\frac{3}{4}c_1 < c_2 < c_1$. In contrast with Theorems 12 and 13, the result in Theorem 14 is *not perturbative* and the explicit condition on the speeds seems technical. The strategy of the proof of Theorem 14 is to study the asymptotic behavior of $u(t,x)$ for any t and for any large x (*i.e.* far from the collision region, which seems impossible to describe in the general case) and to find a contradiction with the fact that u is an asymptotic two-soliton in the two directions of time. Being a proof by contradiction, it does not give further information on the collision.

7.4 Collision for the perturbed integrable NLS equation. Let $\beta > 0$ and $0 < c \ll 1$. Under the following assumptions for the perturbation $|f(u)| \lesssim_0 u^2$, $f(u) \lesssim |u|^q$ ($q < 2$), it is proved in Perelman [2011], that there exists a solution u of

$$i\partial_t u + \partial_x^2 u + |u|^2 u + f(|u|^2)u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

satisfying $\lim_{t\to-\infty} \left\| u(t) - e^{it} Q - e^{i\Gamma_\beta(t,\cdot)} Q_c(\cdot - \beta t) \right\|_{H^1} \to 0$, and for which the small soliton splits in two after the collision, in the following sense

$$u(t,x) \sim e^{i\Gamma(t,x)} Q(x - \sigma(t)) + \psi^+(t,x) + \psi^-(t,x),$$

where $\psi^\pm$ are solutions of (2) corresponding to the *transmitted part* and the *reflected part* of the small soliton. The above estimate holds on large time intervals after the collision depending on $1/c$. The splitting means some strong form of inelasticity.

7.5 Collision for the 5D energy critical wave equation. In view of the results described so far, it was natural to search a situation where a non-perturbative approach would allow to treat *all two-soliton collisions*. In the case of the 5D energy critical wave equation, we now recall from Martel and Merle [2017] a result showing the inelastic nature of the collision of any two solitons, except the special case of same scaling and opposite signs.

Theorem 15 (Inelasticity of soliton collisions for NLW, Martel and Merle [ibid.]). *Let $d = 5$. For $k \in \{1,2\}$, let $\lambda_k^\infty > 0$, $\mathbf{y}_k^\infty \in \mathbb{R}^5$, $\epsilon_k \in \{\pm 1\}$, $\boldsymbol{\ell}_k \in \mathbb{R}^5$ with $|\boldsymbol{\ell}_k| < 1$, and*

$$W_k^\infty(t,x) = \frac{\epsilon_k}{(\lambda_k^\infty)^{\frac{3}{2}}} W_{\boldsymbol{\ell}_k}\left(\frac{x - \boldsymbol{\ell}_k t - \mathbf{y}_k^\infty}{\lambda_k^\infty}\right).$$

Assume that $\boldsymbol{\ell}_1 \neq \boldsymbol{\ell}_2$ and $\epsilon_1\lambda_1^\infty + \epsilon_2\lambda_2^\infty \neq 0$. Then, there exists a solution u of (6) *in the energy space such that*

(i) Two-soliton as $t \to -\infty$

$$\lim_{t\to-\infty} \|\nabla_{t,x} u(t) - \nabla_{t,x}(W_1^\infty(t) + W_2^\infty(t))\|_{L^2} = 0.$$

(ii) Dispersion as $t \to +\infty$. *For all $A > 0$ large enough,*

$$\liminf_{t\to+\infty} \|\nabla u(t)\|_{L^2(|x|>t+A)} \gtrsim A^{-\frac{5}{2}}. \tag{10}$$

Note first that the solution constructed in Theorem 15 is a two-soliton asymptotically as $t \to -\infty$ and that it does not necessarily exist for all $t \in \mathbb{R}$. However, by finite speed of propagation and small data Cauchy theory, it is straightforward to justify that it can be extended uniquely as a solution of (6) for all $t \in \mathbb{R}$ in the region $|x| > |t| + A$, provided that A is large enough. Thus, the limit in (10) makes sense. Since the estimate (10) gives an explicit lower bound on the loss of energy as dispersion as $t \to +\infty$, the solution u is not a two-soliton asymptotically as $t \to +\infty$ and *the collision is inelastic*. Note that the two-soliton could have any global behavior, like dislocation of the solitons and dispersion, blow-up or a different multi-soliton plus radiation, but the property obtained is universal.

The only case left open by Theorem 15 corresponds to the *dipole* case. It is the first result proving inelasticity rigorously without restriction on the relative sizes or speeds of the solitons except an exceptional case.

The strategy of the proof is to construct a refined approximate solution of the two-soliton problem for large negative times that displays an explicit dispersive radial part at the leading order and then to propagate the dispersion for any positive time at the exterior of large cones by finite speed of propagation and the method of *channels of energy* from Duyckaerts, Kenig, and Merle [2013] and Kenig, Lawrie, and Schlag [2014]. To finish, we recall such a typical result of channel of energy for the radial linear wave equation in 5D from Kenig, Lawrie, and Schlag [2014], Proposition 4.1 (see also Duyckaerts, Kenig, and Merle [2012, 2013] and Kenig, Lawrie, B. Liu, and Schlag [2015]).

Theorem 16 (Exterior energy estimates for the 5D linear wave equation, Kenig, Lawrie, and Schlag [2014])**.** *Any radial energy solution U_{L} of the* 5D *linear wave equation*

$$\begin{cases} \partial_t^2 U_{\mathrm{L}} - \Delta U_{\mathrm{L}} = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^5, \\ U_{\mathrm{L}|t=0} = U_0 \in \dot{H}^1, & \partial_t U_{\mathrm{L}|t=0} = U_1 \in L^2, \end{cases}$$

satisfies, for any $R > 0$,

$$\sum_{\pm} \left\{ \limsup_{t \to \pm\infty} \int_{|x|>|t|+R} \left(|\partial_t U_{\mathrm{L}}(t,x)|^2 + |\nabla U_{\mathrm{L}}(t,x)|^2 \right) dx \right\} \gtrsim$$

$$\gtrsim \|\pi_R^{\perp}(U_0, U_1)\|^2_{(\dot{H}^1 \times L^2)(|x|>R)}$$

where $\pi_R^{\perp}(U_0,U_1)$ denotes the orthogonal projection of (U_0,U_1) onto the complement of the plane $\mathrm{span}\{(|x|^{-3},0),(0,|x|^{-3})\}$ *in $(\dot{H}^1 \times L^2)(|x| > R)$.*

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