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# CONFORMAL FIELD THEORY, VERTEX OPERATOR ALGEBRAS AND OPERATOR ALGEBRAS

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**Abstract**

We present recent progress in theory of local conformal nets which is an operator algebraic approach to study chiral conformal field theory. We emphasize representation theoretic aspects and relations to theory of vertex operator algebras which gives a different and algebraic formulation of chiral conformal field theory.

## 1 Introduction

Quantum field theory is a vast area in physics and two-dimensional conformal field theory has caught much attention recently. A two-dimensional conformal field theory decomposes into two chiral conformal field theories, and here we present mathematical studies of a chiral conformal field theory based on operator algebras. It is within a scope of what is called algebraic quantum field theory and our mathematical object is called a local conformal net.

The key idea in algebraic quantum field theory is to work on operator algebras generated by observables in a spacetime region rather than quantum fields. In chiral conformal field theory, the spacetime becomes a one-dimensional circle and a spacetime region is an interval in it, which is a nonempty, nondense, open and connected set in the circle, so we deal with a continuous family of operator algebras parameterized by intervals. This is what a local conformal net is.

Each operator algebra of a local conformal net acts on the same Hilbert space from the beginning, but we also consider its representation theory on another Hilbert space. Such a representation corresponds to a notion of a charge, and a unitary equivalence class of a representation is called a superselection sector. In the Doplicher-Haag-Roberts theory Doplicher, Haag, and Roberts [1971], a representation is realized with a DHR endomorphism of one operator algebra, and such an endomorphism produces a subfactor in the sense of the Jones theory Jones [1983], Jones [1985]. Subfactor theory plays an important

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role in this approach. It has revolutionized theory of operator algebras and revealed its surprising deep relations to 3-dimensional topology, quantum groups and solvable lattice models. Its connection to quantum field theory was clarified by Longo [1989] and it has been an important tool also in conformal field theory since then.

Representation theory of a local conformal net gives a powerful tool to study chiral conformal field theory. We present $\alpha$-induction, a certain induction procedure for representation theory of a local conformal net, and its use for classification theory.

A vertex operator algebra gives another axiomatization of a chiral conformal field theory and it has started with the famous Moonshine conjecture Conway and Norton [1979]. The axiomatic framework has been established in Frenkel, Lepowsky, and Meurman [1988] and we have had many research papers on this topic. This is an algebraic axiomatization of Fourier expansions of a family of operator-valued distributions on the one-dimensional circle. Since a local conformal net and a vertex operator algebra give different axiomatizations of the same physical theory, it is natural to expect that they have many common features. There have been many parallel results in the two theories, but a precise relation between the two were not known until recently. We have established that if a vertex operator algebra satisfies unitarity and an extra mild assumption called strong locality, then we can construct the corresponding local conformal net and also recover the original vertex operator algebra from the local conformal net. Strong locality is known to be satisfied for most examples and we do not know any example of a vertex operator algebra which does not have strong locality.

There are many open problems to study in the operator algebraic approach to chiral conformal field theory. We present some of them in this article.

We refer a reader to lecture notes Kawahigashi [2015b] for more details with an extensive bibliography.

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## 2 Algebraic quantum field theory and local conformal nets

In a common approach to quantum field theory such as the Wightman axioms, we deal with quantum fields which are a certain kind of operator-valued distributions on the spacetime acting on the same Hilbert space together with a spacetime symmetry group. An operator-valued distribution $T$ applied to a test function $f$ gives $\langle T, f \rangle$ which is an (often unbounded) operator. Handling distributions, rather than functions, and unbounded operators causes technical difficulties, so an idea of algebraic quantum field theory of Haag-Kastler is to study operator algebras generated by observables in a spacetime region. Let $T$ be an operator-valued distribution and $f$ be a test function supported in $O$ which is a spacetime region. Then $\langle T, f\rangle$ gives an observable in $O$ (if it is self-adjoint). Let $A(O)$ be the von Neumann algebra generated by these observables. (A von Neumann algebra is an algebra of bounded linear operators on a Hilbert space containing the identity operator which is closed under the $*$-operation and the strong operator topology. This topology is given by pointwise convergence on the Hilbert space.) We have a family $\{A(O)\}$ of von Neumann algebras. We impose physically natural axioms on such a family and make a mathematical study of these axioms.

We apply the above general idea to 2-dimensional conformal field theory. We first consider the 2-dimensional Minkowski space with space coordinate $x$ and time coordinate $t$. We have a certain restriction procedure of a conformal field theory on the Minkowski space to the two light rays $\{x = \pm t\}$. In this way, we can regard one light ray as a kind of spacetime though it has only one dimension. Then conformal symmetry can move the point at infinity of this light ray, so our space should be now $S^1$, the one-point compactification of a light ray. A spacetime region is now a nonempty nondense open connected subset of $S^1$ and such a set is called an interval. Our mathematical object is a family of von Neumann algebras $\{A(I)\}$ parametrized by an interval $I \subset S^1$. We impose the following axioms on this family.

1. (Isotony) For two intervals $I_1 \subset I_2$, we have $A(I_1) \subset A(I_2)$.

2. (Locality) When two intervals $I_1, I_2$ are disjoint, we have $[A(I_1), A(I_2)] = 0$.

3. (Möbius covariance) We have a unitary representation $U$ of $PSL(2,\mathbb{R})$ on the Hilbert space such that we have $U(g)A(I)U(g)^* = A(gI)$ for all $g \in PSL(2,\mathbb{R})$, where $g$ acts on $S^1$ as a fractional linear transformation on $\mathbb{R} \cup \{\infty\}$ and $S^1 \setminus \{-1\}$ is identified with $\mathbb{R}$ through the Cayley transform $C(z) = -i(z-1)/(z+1)$.

4. (Conformal covariance) We have a projective unitary representation, still denoted by $U$, of Diff($S^1$), the group of orientation preserving diffeomorphisms of $S^1$, extending the unitary representation $U$ of $PSL(2,\mathbb{R})$ such that

$$U(g)\mathfrak{A}(I)U(g)^* = A(gI), \quad g \in \mathrm{Diff}(S^1),$$
$$U(g)xU(g)^* = x, \quad x \in A(I),\ g \in \mathrm{Diff}(I'),$$

   where $I'$ is the interior of the complement of $I$ and Diff($I'$) is the set of diffeomorphisms of $S^1$ which are the identity map on $I$.

5. (Positive energy condition) The generator of the restriction of $U$ to the rotation subgroup of $S^1$, the conformal Hamiltonian, is positive.

6. (Existence of the vacuum vector) We have a unit vector $\Omega$, called the vacuum vector, such that $\Omega$ is fixed by the representation $U$ of $PSL(2,\mathbb{R})$ and $(\bigvee_{I\subset S^1} A(I))\Omega$ is

dense in the Hilbert space, where $\bigvee_{I\subset S^1} A(I)$ is the von Neumann algebra generated by $A(I)$'s.

7. (Irreducibility) The von Neumann algebra $\bigvee_{I\subset S^1} A(I)$ is the algebra of all the bounded linear operators on the Hilbert space.

Isotony is natural because a larger spacetime domain should have more observables. Locality comes from the Einstein causality in the 2-dimensional Minkowski space that observables in spacelike separated regions should commute with each other. Note that we have a simple condition of disjointness instead of spacelike separation. Conformal covariance represents an infinite dimensional symmetry. This gives a reason for the name "conformal" field theory. The positive energy condition expresses positivity of the eigenvalues of the conformal Hamiltonian. The vacuum state is a physically distinguished state of the Hilbert space. Irreducibility means that our Hilbert space is irreducible.

It is non-trivial to construct an example of a local conformal net. Basic sources of constructions are as follows. These are also sources of constructions of vertex operator algebras as we see below.

1. Affine Kac-Moody algebras Gabbiani and Fröhlich [1993], Wassermann [1998], Toledano Laredo [1999]

2. Virasoro algebra Xu [2000a], Kawahigashi and Longo [2004a]

3. Even lattices Kawahigashi and Longo [2006], Dong and Xu [2006]

When we have some examples of local conformal nets, we have the following methods to construct new ones.

1. Tensor product

2. Simple current extension Böckenhauer and Evans [1998]

3. Orbifold construction Xu [2000b]

4. Coset construction Xu [2000a]

5. Extension by a $Q$-system Longo and Rehren [1995], Kawahigashi and Longo [2004a], Xu [2007]

The first four constructions were first studied for vertex operator algebras. The last one was first studied for a local conformal net and later for vertex operator algebras Huang, Kirillov, and Lepowsky [2015]. The Moonshine net, the operator algebraic counterpart of the famous Moonshine vertex operator algebra, is constructed from the Leech lattice, an even lattice of rank 24, with a combination of the orbifold construction and a simple current extension Kawahigashi and Longo [2006], for example. This is actually given by a 2-step simple current extension as in Kawahigashi and Suthichitranont [2014]. The $Q$-system in the last construction was first introduced in Longo [1994]. It is also known under the name of a Frobenius algebra in algebraic literature.

Irreducibility implies that each $A(I)$ has a trivial center. Such an algebra is called a factor. It turns out that each algebra $A(I)$ is isomorphic to the Araki-Woods factor of type $\mathrm{III}_1$ because the split property automatically holds by Morinelli, Tanimoto, and Weiner [2018] and it implies hyperfiniteness of $A(I)$. This shows that each single algebra $A(I)$ contains no information about a local conformal net and what is important is relative relation among $A(I)$'s.

## 3 Representation theory and superselection sectors

We now present representation theory of a local conformal net. Each algebra $A(I)$ of a local conformal net $\{A(I)\}$ acts on the same Hilbert space having the vacuum vector from the beginning, but we also consider a representation of an algebra $A(I)$ on another common Hilbert space (without a vacuum vector).

The Haag duality $A(I') = A(I)'$ automatically holds from the axioms, where the prime on the right hand side denotes the commutant, and this implies that each representation is represented with an endomorphism $\lambda$ of $A(I)$ for some fixed $I$. This is a standard Doplicher-Haag-Roberts theory adapted to a local conformal net Fredenhagen, Rehren, and Schroer [1989]. An endomorphism $\lambda$ produces $\lambda(A(I))$ which is subalgebra of $A(I)$ and a factor, so it is called a subfactor. It is an object in the Jones theory of subfactors Jones [1983]. The relative size of the subfactor $\lambda(A(I))$ with respect to $A(I)$ is called the Jones index $[A(I) : \lambda(A(I))]$. It turns out that the square root $[A(I) : \lambda(A(I))]^{1/2}$ of the Jones index gives a proper notion of the dimension of the corresponding representation of $\{A(I)\}$ Longo [1989]. The dimension $\dim(\lambda)$ takes its value in the interval $[1, \infty]$.

It is important to have a notion of a tensor product of representations of a local conformal net. Note that while it is easy to define a tensor product of representations of a group, we have no notion of a tensor product of representations of an algebra. It turns out that a composition of endomorphisms of $A(I)$ for a fixed $I$ gives a right notion of a tensor product of representations Doplicher, Haag, and Roberts [1971]. In this way, we have a tensor category of finite dimensional representations of $\{A(I)\}$. The original action of $A(I)$ on the Hilbert space is called the vacuum representation and has dimension 1. It plays a role of a trivial representation. In the original setting of the Doplicher-Haag-Roberts theory on the higher dimensional Minkowski space, the tensor product operation is commutative in a natural sense and we have a symmetric tensor category. Now in the setting of chiral conformal field theory, the commutativity is more subtle, and we have a structure of braiding Fredenhagen, Rehren, and Schroer [1989]. We thus have a braided tensor category of finite dimensional representations.

We are often interested in a situation where we have only finitely many irreducible representations and such finiteness is usually called rational. (This rationality is well-studied in a context of representation theory of quantum groups at roots of unity in connection to quantum invariants in 3-dimensional topology.) We have defined complete rationality for a local conformal net, which means we have only finitely many irreducible representations up to unitary equivalence and all of them have finite dimensions, and given its operator algebraic characterization in terms of finiteness of the Jones index of a certain subfactor in Kawahigashi, Longo, and Müger [2001]. (We originally assumed two more properties for complete rationality, but they have been shown to be automatic by Longo and Xu [2004], Morinelli, Tanimoto, and Weiner [2018], respectively.) This characterization is given by only studying the vacuum representation. We have further proved that complete rationality implies that the braiding of the representations is non-degenerate, that is, we have the following theorem in Kawahigashi, Longo, and Müger [2001].

**Theorem 3.1.** *The tensor category of finite dimensional representations of a completely rational local conformal net is modular.*

It is an important open problem to decide which modular tensor category arises as the representation category of a completely rational local conformal net. The history of classification theory of factors, group actions and subfactors in theory of von Neumann algebras due to Connes, Haagerup, Jones, Ocneanu and Popa culminating in Popa [1994] tells us that as long as we have an analytic condition, generally called amenability, we have no nontrivial obstruction to realization of algebraic invariants. This strongly suggests that any modular tensor category is realized as the representation category of some local conformal net, because we now have amenability automatically. This conjecture has caught much attention these days because of recent work of Jones. We turn to this problem again in the next section.

## 4 Subfactors and tensor categories

In the Jones theory of subfactors, we study an inclusion $N \subset M$ of factors. In the original setting of Jones [1983], one considers type $\mathrm{II}_1$ factors, but one has to deal with type III factors in conformal field theory. The Jones theory has been extended to type III factors by Pimsner-Popa and Kosaki, and many algebraic arguments are now more or less parallel in the type $\mathrm{II}_1$ and type III cases. For simplicity, we assume factors are of type $\mathrm{II}_1$ in this section. We refer reader to Evans and Kawahigashi [1998] for details of subfactor theory.

We start with a subfactor $N \subset M$. The Jones index $[M : N]$ is a number in the interval $[1, \infty]$. In this section, we assume that the index is finite. On the algebra $M$, we have the left and right actions of $M$ itself. We restrict the left action to the subalgebra $N$, and we have a bimodule ${}_N M_M$. We make the completion of $M$ with respect to the inner product arising from the trace functional and obtain the Hilbert space $L^2(M)$. For simplicity, we still write ${}_N M_M$ for this Hilbert space with the left action of $N$ and the right action of $M$. We make relative tensor powers such as ${}_N M \otimes_M M \otimes_N M \otimes_M \cdots$ and their irreducible decomposition gives four kinds of bimodules, $N$-$N$, $N$-$M$, $M$-$N$ and $M$-$M$. If we have only finitely many irreducible bimodules in this way, we say that the subfactor $N \subset M$ is of finite depth. In this case, finite direct sums of these irreducible $N$-$N$ (and $M$-$M$) bimodules (up to isomorphism) give a fusion category. Note that the relative tensor product is not commutative in general and we have no braiding structure.

If we have a free action of a finite group $G$ on a factor $M$, we have a subfactor $N = M^G \subset M$. The index is the order of $G$ and the fusion category of $N$-$N$ bimodules is the representation category of $G$. There are other constructions of subfactors from actions of finite groups and their quantum group versions give many interesting examples of subfactors. If the index is less than 4, the set of all the possible values is $\{4 \cos^2 \pi/n \mid n = 3, 4, 5, \ldots\}$ Jones [1983]. Classification of subfactors with index less than 4 has been given in Ocneanu [1988] and this is well-understood today in terms of quantum groups or conformal field theory. Such classification of subfactors has been extended to index value 5 Jones, Morrison, and Snyder [2014] recently.

There are some exceptional subfactors which do not seem to arise from such constructions involving (quantum) groups. The most notable examples are the Haagerup subfactor Asaeda and Haagerup [1999], the Asaeda-Haagerup subfactor Asaeda and Haagerup [ibid.] and the extended Haagerup subfactor Bigelow, Peters, Morrison, and Snyder [2012] in the index range $(4, 5)$. (The first two were constructed along an extension of the line of Ocneanu [1988] and the last one is based on the planar algebra of Jones.) Such a subfactor produces an exceptional fusion category and then it produces an exceptional modular tensor category through the Drinfeld center construction. (See Izumi [2000] for an operator algebraic treatment of this.) Such a modular tensor category does not seem to arise from a combination of other known constructions applied to the Wess-Zumino-Witten models. The above three subfactors were found through a combinatorial search for a very narrow range of index values. This strongly suggests that there is a huge variety of exceptional fusion categories and modular tensor categories beyond what is known today. History of classification theory of subfactors even strongly suggests that there is a huge variety of exceptional modular tensor categories even up to Witt equivalence ignoring Drinfeld centers, because it seems impossible to exhaust all examples by prescribing construction methods.

As explained in the previous section, we strongly believe that all such exceptional modular tensor categories do arise from local conformal nets. This would mean that there is a huge variety of chiral conformal field theories beyond what is known today. For the Haagerup subfactor, a partial evidence for this conjecture is given in Evans and Gannon [2011].

## 5 $\alpha$-induction, modular invariants and classification theory

We next present an important tool to study representation of a local conformal net. For a subgroup $H$ of another group $G$ and a representation of $H$, we have a notion of an induced representation of $G$. We have some similar notion for a representation of a local conformal net. Let $\{A(I) \subset B(I)\}$ be an inclusion of local conformal nets and assume the index $[B(I) : A(I)]$ is finite. For a a representation of $\{A(I)\}$ which is given by an endomorphism $\lambda$ of a factor $A(I)$ for some fixed interval $I$, we extend $\lambda$ to an endomorphism of $B(I)$. This extension depends on a choice of positive and negative crossings in the braiding structure of representations of $\{A(I)\}$ and we denote it with $\alpha_\lambda^\pm$ where $\pm$ stands for the choice of positive and negative crossings. This gives an "almost" representation of $\{B(I)\}$ and it is called a soliton endomorphism. This induction machinery is called $\alpha$-induction. It was first introduced in Longo and Rehren [1995] and studied in detail in Xu [1998], Böckenhauer and Evans [1998]. Ocneanu had a graphical calculus in a very different context involving the $A$-$D$-$E$ Dynkin diagrams and the two methods were unified in Böckenhauer, Evans, and Kawahigashi [1999], Böckenhauer, Evans, and Kawahigashi [2000]. It turns out that the intersection of irreducible endomorphisms of $B(I)$ arising from $\alpha^+$-induction and $\alpha^-$-induction exactly gives those corresponding the representations of $\{B(I)\}$ by Kawahigashi, Longo, and Müger [2001], Böckenhauer, Evans, and Kawahigashi [1999], Böckenhauer, Evans, and Kawahigashi [2000].

Let $\{A(I)\}$ be completely rational in the above setting. Then $\{B(I)\}$ is automatically also completely rational. (The converse also holds.) The modular tensor category of $\{A(I)\}$ gives a (finite dimensional) unitary representation of $SL(2,\mathbb{Z})$ from its braiding. (The dimension of the representation is the number of irreducible representations of $\{A(I)\}$ up to unitary equivalence.) Define the matrix $(Z_{\lambda\mu})$ by $Z_{\lambda\mu} = \dim \mathrm{Hom}(\alpha_\lambda^+, \alpha_\mu^-)$ where $\lambda, \mu$ denote endomorphisms of $A(I)$ corresponding to irreducible representations of $\{A(I)\}$. Then we have the following in Böckenhauer, Evans, and Kawahigashi [1999].

**Theorem 5.1.** *The matrix $Z$ commutes with the above unitary representation of $SL(2,\mathbb{Z})$.*

Such $Z$ also satisfies $Z_{\lambda\mu} \in \{0,1,2,\dots\}$ and $Z_{00} = 1$ where $0$ denotes the vacuum representation of $\{A(I)\}$. Such a matrix is called a modular invariant of the representation of $SL(2,\mathbb{Z})$. The number of modular invariants for a given local conformal net $\{A(I)\}$ is always finite and often quite limited. This gives the following classification method of all possible irreducible extensions $\{B(I)\}$ for a given local conformal net $\{A(I)\}$. (Any irreducible extension automatically has a finite index by Izumi, Longo, and Popa [1998].)

1. Find all possible modular invariants $(Z_{\lambda\mu})$ for the modular tensor category arising from representations of $\{A(I)\}$.
2. For each $(Z_{\lambda\mu})$, determine all possible $Q$-systems corresponding to $\bigoplus Z_{0\lambda}\lambda$.
3. Pick up only local $Q$-systems.

Consider a local conformal net $\{A(I)\}$. The projective unitary representation of $\mathrm{Diff}(S^1)$ gives a representation of the Virasoro algebra and it gives a positive real number $c$ called the central charge. This is a numerical invariant of a local conformal net and the value of $c$ is in the set $\{1 - 6/n(n+1) \mid n = 3, 4, 5, \dots\} \cup [1, \infty)$. We now restrict ourselves to the case $c < 1$. Let $\mathrm{Vir}_c(I)$ be the von Neumann algebra generated by $U(g)$ where $g \in \mathrm{Diff}(S^1)$ acts trivially on $I'$. This gives an extension $\{\mathrm{Vir}_c(I) \subset A(I)\}$. It turns out $\{\mathrm{Vir}_c(I)\}$ is completely rational and we can apply the above method to classify all possible $\{A(I)\}$. The modular invariants have been classified in Cappelli, Itzykson, and Zuber [1987], and locality and a certain 2-cohomology argument imply that the extensions exactly correspond to so-called type I modular invariants. We thus have a complete classification of local conformal nets with $c < 1$ as follows Kawahigashi and Longo [2004a].

**Theorem 5.2.** *Any local conformal net with $c < 1$ is one of the following.*

1. *The Virasoro nets $\{\mathrm{Vir}_c(I)\}$ with $c < 1$.*
2. *Their simple current extensions with index 2.*
3. *Four exceptionals at $c = 21/22, 25/26, 144/145, 154/155$.*

The four exceptionals correspond to the Dynkin diagrams $E_6$ and $E_8$. Three of them are identified with certain coset constructions, but the remaining one with $c = 144/145$ does not seem to be related to any other known constructions so far. All these four are given by an extension by a $Q$-system. Note that this appearance of modular invariants is different from its original context in 2-dimensional conformal field theory.

## 6 Vertex operator algebras

A vertex operator algebra gives another mathematical axiomatization of a chiral conformal field theory. It deals with Fourier expansions of operator-valued distributions, vertex operators, on $S^1$ in an algebraic manner.

Recall that we have a complete list of finite simple groups today as follows Frenkel, Lepowsky, and Meurman [1988].

1. Cyclic groups of prime order.
2. Alternating groups of degree 5 or higher.
3. 16 series of groups of Lie type over finite fields.
4. 26 sporadic finite simple groups.

The largest group among the 26 groups in the fourth in terms of the order is called the Monster group, and its order is approximately $8 \times 10^{53}$. This group was first constructed by Griess. It has been known that the smallest dimension of a non-trivial irreducible representation of the Monster group is 196883.

The next topic in this section is the $j$-function. This is a function of a complex number $\tau$ with $\mathrm{Im}\,\tau > 0$ with the following expansion.

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots,$$

where we set $q = \exp(2\pi i \tau)$.

This function has modular invariance property

$$j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right),$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

and this property and the condition that the top term of the Laurent series of $q$ start with $q^{-1}$ determine the $j$-function uniquely except for the constant term.

McKay noticed that the first non-trivial coefficient of the Laurent expansion of the $j$-function except for the constant term is 196884 which is "almost" 196883. Extending this idea, Conway and Norton [1979] formulated the Moonshine conjecture as follows.

**Conjecture 6.1.**
1. *We have some graded infinite dimensional $\mathbb{C}$-vector space $V = \bigoplus_{n=0}^{\infty} V_n$ $(\dim V_n < \infty)$ with some natural algebraic structure and its automorphism group is the Monster group.*
2. *Each element $g$ of the Monster group acts on each $V_n$ linearly. The Laurent series*

$$\sum_{n=0}^{\infty} (\mathrm{Tr}\, g|_{V_n}) q^{n-1}$$

*arising from the trace value of the $g$-action on $V_n$ is a classical function called a Hauptmodul corresponding to a genus 0 subgroup of $SL(2, \mathbb{R})$. (The case $g$ is the identity element is the $j$-function without the constant term.)*

"Some natural algebraic structure" in the above conjecture has been formulated as a vertex operator algebra in Frenkel, Lepowsky, and Meurman [1988] and the full Moonshine conjecture has been proved by Borcherds [1992]. The axioms of a vertex operator algebra are given as follows.

Let $V$ be a $\mathbb{C}$-vector space. We say that a formal series $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ with coefficients $a_{(n)} \in \mathrm{End}(V)$ is a field on $V$, if for any $b \in V$, we have $a_{(n)}b = 0$ for all sufficiently large $n$.

A $\mathbb{C}$-vector space $V$ is called a vertex algebra if we have the following properties.

1. (State-field correspondence) For each $a \in V$, we have a field $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ on $V$.

2. (Translation covariance) We have a linear map $T \in \mathrm{End}(V)$ such that we have $[T, Y(a, z)] = \frac{d}{dz} Y(a, z)$ for all $a \in V$.

3. (Existence of the vacuum vector) We have a vector $\Omega \in V$ with $T\Omega = 0$, $Y(\Omega, z) = \mathrm{id}_V$, $a_{(-1)}\Omega = a$.

4. (Locality) For all $a, b \in V$, we have $(z - w)^N [Y(a, z), Y(b, w)] = 0$ for a sufficiently large integer $N$.

We then call $Y(a, z)$ a vertex operator. (The locality axiom is one representation of the idea that $Y(a, z)$ and $Y(b, w)$ should commute for $z \neq w$.)

Let $V$ be a $\mathbb{C}$-vector space and $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ be a field on $V$. If the endomorphisms $L_n$ satisfy the Virasoro algebra relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{(m^3 - m)\delta_{m+n,0}}{12} c,$$

with central charge $c \in \mathbb{C}$, then we say $L(z)$ is a Virasoro field. If $V$ is a vertex algebra and $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is a Virasoro field, then we say $\omega \in V$ is a Virasoro vector. A Virasoro vector $\omega$ is called a conformal vector if $L_{-1} = T$ and $L_0$ is diagonalizable on $V$, that is, $V$ is an algebraic direct sum of the eigenspaces of $L_0$. Then the corresponding vertex operator $Y(\omega, z)$ is called the energy-momentum field and $L_0$ the conformal Hamiltonian. A vertex algebra with a conformal vector is called a conformal vertex algebra. We then say $V$ has central charge $c \in \mathbb{C}$.

A nonzero element $a$ of a conformal vertex algebra in $\mathrm{Ker}(L_0 - \alpha)$ is said to be a homogeneous element of conformal weight $d_a = \alpha$. We then set $a_n = a_{(n+d_a-1)}$ for $n \in \mathbb{Z} - d_a$. For a sum $a$ of homogeneous elements, we extend $a_n$ by linearity.

A homogeneous element $a$ in a conformal vertex algebra $V$ and the corresponding field $Y(a, z)$ are called quasi-primary if $L_1 a = 0$ and primary if $L_n a = 0$ for all $n > 0$.

We say that a conformal vertex algebra $V$ is of CFT type if we have $\mathrm{Ker}(L_0-\alpha)\neq 0$ only for $\alpha\in\{0,1,2,3,\dots\}$ and $V_0=\mathbb{C}\Omega$.

We say that a conformal vertex algebra $V$ is a vertex operator algebra if we have the following.

1. We have $V=\bigoplus_{n\in\mathbb{Z}} V_n$, where $V_n=\mathrm{Ker}(L_0-n)$.
2. We have $V_n=0$ for all sufficiently small $n$.
3. We have $\dim(V_n)<\infty$ for $n\in\mathbb{Z}$.

Basic sources of constructing vertex operator algebras are affine Kac-Moody and Virasoro algebras due to Frenkel-Zhu and even lattices due to Frenkel-Lepowsky-Meurman. Methods to construct new examples from known examples are a tensor product, a simple current extension due Schellekens-Yankielowicz, orbifold construction due to Dijkgraaf-Vafa-Verlinde-Verlinde, coset construction due to Frenkel-Zhu, and an extension by a Q-system due to Huang-Kirillov-Lepowsky. These are parallel to constructions of local conformal nets, but constructions of vertex operator algebras are earlier except for the extension by a Q-system.

A representation theory of a vertex operator algebra is known as a theory of modules. It has been shown by Huang that we have a modular tensor category for a well-behaved vertex operator algebra. (The well-behavedness condition is basically the so-called $C_2$-cofiniteness.)

## 7 From a vertex operator algebra to a local conformal net and back

We now would like to construct a local conformal net from a vertex operator algebra $V$. First of all, we need a Hilbert space of states, and it should be the completion of $V$ with respect to some natural inner product. A vertex operator algebra with such an inner product is called unitary. Many vertex operator algebras are unitary, but also many others are non-unitary. In order to have the corresponding local conformal net, we definitely have to assume that $V$ is unitary. We now give a precise definition of a unitary vertex operator algebra.

An invariant bilinear form on a vertex operator algebra $V$ is a bilinear form $(\cdot,\cdot)$ on $V$ satisfying

$$(Y(a,z)b,c)=(b,Y(e^{zL_1}(-z^{-2})^{L_0}a,z^{-1})c)$$

for all $a,b,c\in V$.

For a vertex operator algebra $V$ with a conformal vector $\omega$, an automorphism $g$ as a vertex algebra is called a VOA automorphism if we have $g(\omega)=\omega$.

Let $V$ be a vertex operator algebra and suppose we have a positive definite inner product $(\cdot \mid \cdot)$, where we assume this is antilinear in the first variable. We say the inner product is normalized if we have $(\Omega \mid \Omega) = 1$. We say that the inner product is invariant if there exists a VOA antilinear automorphism $\theta$ of $V$ such that $(\theta\cdot \mid \cdot)$ is an invariant bilinear form on $V$. We say that $\theta$ is a PCT operator associated with the inner product.

If we have an invariant inner product, we automatically have $(L_n a \mid b) = (a \mid L_{-n} b)$ for $a, b \in V$ and also $V_n = 0$ for $n < 0$. The PCT operator $\theta$ is unique and we have $\theta^2 = 1$ and $(\theta a \mid \theta b) = (b \mid a)$ for all $a, b \in V$. (See Carpi, Kawahigashi, Longo, and Weiner [n.d., Section 5.1] for details.)

A unitary vertex operator algebra $V$ is a pair of a vertex operator algebra and a normalized invariant inner product. It is simple if we have $V_0 = \mathbb{C}\Omega$.

Now suppose $V$ is a unitary vertex operator algebra. A vertex operator $Y(a, z)$ should mean a Fourier expansion of an operator-valued distribution on $S^1$. For a test function $f$ with Fourier coefficients $\hat{f}_n$, the action of the distribution $Y(a, z)$ applied to the test function $f$ on $b \in V$ should be $\sum_{n \in \mathbb{Z}} \hat{f}_n a_n b$. In order to make sense out of this, we need convergence of this infinite sum. To insure such convergence, we introduce the following notion of energy-bounds.

Let $(V, (\cdot \mid \cdot))$ be a unitary vertex operator algebra. We say that $a \in V$ (or $Y(a, z)$) satisfies energy-bounds if we have positive integers $s, k$ and a constant $M > 0$ such that we have

$$\|a_n b\| \le M(|n| + 1)^s \|(L_0 + 1)^k b\|,$$

for all $b \in V$ and $n \in \mathbb{Z}$. If every $a \in V$ satisfies energy-bounds, we say $V$ is energy-bounded.

We have the following Proposition in Carpi, Kawahigashi, Longo, and Weiner [ibid.].

**Proposition 7.1.** *If $V$ is a simple unitary vertex operator algebra generated by $V_1$ and $F \subset V_2$ where $F$ is a family of quasi-primary $\theta$-invariant Virasoro vectors, then $V$ is energy-bounded.*

We now assume $V$ is energy-bounded. Let $H$ be the completion of $V$ with respect to the inner product. For any $a \in V$ and $n \in \mathbb{Z}$, we regard $a_{(n)}$ as a densely defined operator on $H$. This turns out to be closable. Let $f(z)$ be a smooth function on $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with Fourier coefficients

$$\hat{f}_n = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$$

for $n \in \mathbb{Z}$. For every $a \in V$, we define the operator $Y_0(a, f)$ with domain $V$ by

$$Y_0(a, f)b = \sum_{n \in \mathbb{Z}} \hat{f}_n a_n b$$

for $b \in V$. The convergence follows from the energy-bounds and $Y_0(a, f)$ is a densely defined operator. This is again closable. We denote by $Y(a, f)$ the closure of $Y_0(a, f)$ and call it a smeared vertex operator.

We define $\mathcal{A}_{(V,(\cdot|\cdot))}(I)$ to be the von Neumann algebra generated by the (possibly unbounded) operators $Y(a, f)$ with $a \in V$, $f \in C^\infty(S^1)$ and supp $f \subset I$. The family $\{\mathcal{A}_{(V,(\cdot|\cdot))}(I)\}$ clearly satisfies isotony. We can verify that $(\bigvee_I \mathcal{A}_{(V,(\cdot|\cdot))}(I))\Omega$ is dense in $H$. A proof of conformal covariance is nontrivial, but can be done as in Toledano Laredo [1999] by studying the representations of the Virasoro algebra and Diff$(S^1)$. We also have the vacuum vector $\Omega$ and the positive energy condition. However, locality is not clear at all from our construction, so we make the following definition.

We say that a unitary vertex operator algebra $(V, (\cdot \mid \cdot))$ is strongly local if it is energy-bounded and we have $\mathcal{A}_{(V,(\cdot|\cdot))}(I) \subset \mathcal{A}_{(V,(\cdot|\cdot))}(I')'$ for all intervals $I \subset S^1$.

A strongly local unitary vertex operator algebra produces a local conformal net through the above procedure by definition, but the definition of strong locality looks like we assume what we want to prove, and it would be useless unless we have a good criterion for strong locality. The following theorem gives such a criterion Carpi, Kawahigashi, Longo, and Weiner [n.d.].

**Theorem 7.2.** *Let $V$ be a simple unitary vertex operator algebra generated by $V_1 \cup F$ where $F \subset V_2$ is a family of quasi-primary $\theta$-invariant Virasoro vectors, then $V$ is strongly local.*

The above criteria applies to vertex operator algebras arising from the affine Kac-Moody and Virasoro algebras. We also have the following theorem which we can apply to many examples Carpi, Kawahigashi, Longo, and Weiner [ibid.].

**Theorem 7.3.** *(1) Let $V_1, V_2$ be simple unitary strongly local vertex operator algebras. Then $V_1 \otimes V_2$ is also strongly local.*

*(2) Let $V$ be a simple unitary strongly local vertex operator algebra and $W$ its subalgebra. Then $W$ is also strongly local.*

The second statement of the above theorem shows that strong locality passes to orbifold and coset constructions, in particular.

For a unitary vertex operator algebra $V$, we write Aut$(V)$ for the automorphism group of $V$. For a local conformal net $\{A(I)\}$, we have a notion of the automorphism group and we write Aut$(A)$ for this. We have the following in Carpi, Kawahigashi, Longo, and Weiner [ibid.].

**Theorem 7.4.** *Let $V$ be a strongly local unitary vertex operator algebra and $\{A_{(V,(\cdot|\cdot))}(I)\}$ the corresponding local conformal net. Suppose* Aut$(V)$ *is finite. Then we have* Aut$(A_{(V,(\cdot|\cdot))}) =$ Aut$(V)$.

The Moonshine vertex operator algebra $V^\natural$ is strongly local and unitary, so we can apply the above result to this to obtain the Moonshine net. It was first constructed in Kawahigashi and Longo [2006] with a more ad-hoc method.

For the converse direction, we have the following Carpi, Kawahigashi, Longo, and Weiner [n.d.].

**Theorem 7.5.** *Let $V$ be a simple unitary strongly local vertex operator algebra and $\{\mathcal{A}_{(V,(\cdot|\cdot))}(I)\}$ be the corresponding local conformal net. Then one can recover the vertex operator algebra structure on $V$, which is an algebraic direct sum of the eigenspaces of the conformal Hamiltonian, from the local conformal net $\{\mathcal{A}_{(V,(\cdot|\cdot))}(I)\}$.*

This is proved by using the Tomita-Takesaki theory and extending the methods in Fredenhagen and Jörß [1996]. Establishing correspondence between the representation theories of a vertex operator algebra and a local conformal net is more difficult, though we have some recent progress due to Carpi, Weiner and Xu. The method of Tener [2017] may be more useful for this. We list the following conjecture on this. (For a representation of a local conformal net, we define the character as $\mathrm{Tr}(q^{L_0-c/24})$ when it converges for some small values of $q$. We have a similar definition for a module of a vertex operator algebra.)

**Conjecture 7.6.** *We have a bijective correspondence between completely rational local conformal nets and simple unitary $C_2$-cofinite vertex operator algebras. We also have equivalence of tensor categories for finite dimensional representations of a completely rational local conformal net and modules of the corresponding vertex operator algebra. We further have coincidence of the corresponding characters of the irreducible representations of a completely rational local conformal net and irreducible modules of the corresponding vertex operator algebra.*

Recall that we have a classical correspondence between Lie algebras and Lie groups. The correspondence between affine Kac-Moody algebras and loop groups is similar to this, but "one step higher". Our correspondence between vertex operator algebras and local conformal nets is something even one more step higher.

Finally we discuss the meaning of strong locality. We have no example of a unitary vertex operator algebra which is known to be not strongly local. If there should exist such an example, it would not correspond to a chiral conformal field theory in a physical sense. This means that one of the following holds: any simple unitary vertex operator algebra is strongly local or the axioms of unitary vertex operator algebras are too weak to exclude non-physical examples.

## 8 Other types of conformal field theories

Here we list operator algebraic treatments of conformal field theories other than chiral ones.

Full conformal field theory is a theory on the 2-dimensional Minkowski space. We axiomatize a net of von Neumann algebras $\{B(I \times J)\}$ parameterized by double cones (rectangles) in the Minkowski space in a similar way to the case of local conformal nets. From this, a restriction procedure produces two local conformal nets $\{A_L(I)\}$ and $\{A_R(I)\}$. We assume both are completely rational. Then we have a subfactor $A_L(I) \otimes A_R(J) \subset B(I \times J)$ which automatically has a finite index, and the study of $\{B(I \times J)\}$ is reduced to studies of $\{A_L(I)\}$, $\{A_R(I)\}$ and this subfactor. A modular invariant again naturally appears here and we have a general classification theory. For the case of central charge less than 1, we obtain a complete and concrete classification result as in Kawahigashi and Longo [2004b].

A boundary conformal field theory is a quantum field theory on the half-Minkowski space $\{(t, x) \in \mathfrak{M} \mid x > 0\}$. The first general theory to deal with this setting was given in Longo and Rehren [2004]. We have more results in Carpi, Kawahigashi, and Longo [2013] and Bischoff, Kawahigashi, and Longo [2015]. In this case, a restriction procedure gives one local conformal net. We assume that this is completely rational. Then we have a non-local, but relatively local extension of this completely rational local conformal net which automatically has a finite index. The study of a boundary conformal field theory is reduced to studies of this local conformal net and a non-local extension. For the case of central charge less than 1, we obtain a complete and concrete classification result as in Kawahigashi, Longo, Pennig, and Rehren [2007] along the line of this general theory.

We also have results on the phase boundaries and topological defects in the operator algebraic framework in Bischoff, Kawahigashi, Longo, and Rehren [2016], Bischoff, Kawahigashi, Longo, and Rehren [2015]. See Fuchs, Runkel, and Schweigert [2004] for earlier works on topological defects.

A superconformal field theory is a version of $\mathbb{Z}_2$-graded conformal field theory having extra supersymmetry. We have operator algebraic versions of $N = 1$ and $N = 2$ superconformal field theories as in Carpi, Kawahigashi, and Longo [2008] and Carpi, Hillier, Kawahigashi, Longo, and Xu [2015] based on $N = 1$ and $N = 2$ super Virasoro algebras, and there we have superconformal nets rather than local conformal nets. We also have relations of this theory to noncommutative geometry in Kawahigashi and Longo [2005], Carpi, Hillier, Kawahigashi, and Longo [2010], Carpi, Hillier, Kawahigashi, Longo, and Xu [2015].

## 9 Future directions

We list some problems and conjectures for the future studies at the end of this article.

**Conjecture 9.1.** *For a completely rational local conformal net, we have convergent characters for all irreducible representations and they are closed under modular transformations of $SL(2, \mathbb{Z})$. Furthermore, the $S$-matrix defined with braiding gives transformation rules of the characters under the transformation $\tau \mapsto -1/\tau$.*

This conjecture was made in Gabbiani and Fröhlich [1993, page 625] and follows from Conjecture 7.6.

We say that a local conformal net is holomorphic if its only irreducible representation is the vacuum representation. The following is Xu [2009, Conjecture 3.4] which is the operator algebraic counterpart of the famous uniqueness conjecture of the Moonshine vertex operator algebra.

**Conjecture 9.2.** *A holomorphic local conformal net with $c = 24$ and the eigenspace of $L_0$ with eigenvalue $1$ being $0$ is unique up to isomorphism.*

A reason to expect such uniqueness from an operator algebraic viewpoint is that a set of simple algebraic invariants should be a complete invariant as long as we have some kind of amenability, which is automatic in the above case.

The following is an operator algebraic counterpart of Höhn [2003, Conjecture 3.5].

**Conjecture 9.3.** *Fix a modular tensor category $\mathcal{C}$ and a central charge $c$. Then we have only finitely many local conformal nets with representation category $\mathcal{C}$ and central charge $c$.*

From an operator algebraic viewpoint, the following problem is also natural.

**Problem 9.4.** *Suppose a finite group $G$ is given. Construct a local conformal net whose automorphism group is $G$ in some canonical way.*

This "canonical" method should produce the Moonshine net if $G$ is the Monster group. We may have to consider some superconformal nets rather than local conformal nets to get a nice solution.

Conformal field theory on Riemann surfaces has been widely studied and conformal blocks play a important role there. It is not clear at all how to formulate this in our operator algebraic approach to conformal field theory, so we have the following problem.

**Problem 9.5.** *Formulate a conformal field theory on a Riemann surface in the operator algebraic approach.*

It is expected that the $N = 2$ full superconformal field theory is related to Calabi-Yau manifolds, so we also list the following problem.

**Problem 9.6.** *Construct an operator algebraic object corresponding to a Calabi-Yau manifold in the setting of $N = 2$ full superconformal field theory and study the mirror symmetry in this context.*

The structure of a modular tensor category naturally appears also in the context of topological phases of matters and anyon condensation as in Kawahigashi [2015a], Kawahigashi [2017], Kong [2014]. (The results in Böckenhauer, Evans, and Kawahigashi [2001] can be also seen in this context.) We list the following problem.

**Problem 9.7.** *Relate local conformal nets directly with topological phases of matters and anyon condensation.*

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