Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 2 (1339–1362)

A VIEW ON INVARIANT RANDOM SUBGROUPS AND LATTICES

Tsachik Gelander

Abstract

For more than half a century lattices in Lie groups played an important role in geometry, number theory and group theory. Recently the notion of Invariant Random Subgroups (IRS) emerged as a natural generalization of lattices. It is thus intriguing to extend results from the theory of lattices to the context of IRS, and to study lattices by analyzing the compact space of all IRS of a given group. This article focuses on the interplay between lattices and IRS, mainly in the classical case of semisimple analytic groups over local fields.

Let G be a locally compact group. We denote by $\mathrm{Sub}(G)$ the space of closed subgroups of G equipped with the Chabauty topology. The compact space $\mathrm{Sub}(G)$ is usually too complicated to work with directly. However, considering a random point in $\mathrm{Sub}(G)$ is often much more effective. Note that G acts on $\mathrm{Sub}(G)$ by conjugation. An invariant random subgroup (or shortly IRS) is a G-invariant probability measure on $\mathrm{Sub}(G)$. We denote by $\mathrm{IRS}(G)$ the space of all IRSs of G equipped with the w^*-topology. By Riesz' representation theorem and Alaoglu's theorem, $\mathrm{IRS}(G)$ is compact.

The Dirac measures in $\mathrm{IRS}(G)$ correspond to normal subgroups. Any lattice Γ in G induces an IRS μ_Γ which is defined as the push forward of the G-invariant probability measure from G/Γ to $\mathrm{Sub}(G)$ via the map $g\Gamma \mapsto g\Gamma g^{-1}$.

More generally consider a probability measure preserving action $G \curvearrowright (X, m)$. By a result of Varadarajan, the stabilizer of almost every point in X is closed in G. Moreover, the stabilizer map $X \to \mathrm{Sub}_G,\ x \mapsto G_x$ is measurable, and hence one can push the measure m to an IRS on G. In other words the random subgroup is the stabilizer of a random point in X. In a sense, the study of pmp G-spaces can be divided to the study of stabilizers (i.e. IRSs), the study of orbit spaces and the interplay between the two. Vice versa, every IRS arises (non-uniquely) in this way (see Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a, Theorem 2.6]).

MSC2010: primary 22D05; secondary 22E40, 22B99, 20G15, 37A05, 53C35.

Since its rebirth in the beginning of the current decade (see Section 10 for a short summery of the history of IRS), the topic of IRS played an important role in various parts of group theory, geometry and dynamics, and attracted the attention of many mathematicians for various different reasons. Not aiming to give an overview, I will try to highlight here several aspects of the evolving theory.

1 IRS and Lattices

IRSs can be considered as a generalisation of lattices, and one is tempted to extend results from the theory of lattices to IRS. In the other direction, as often happens in mathematics where one considers random objects to prove result about deterministic ones, the notion of IRS turns out to yield an extremely powerful tool to study lattices. In this section I will try to give a taste of the interplay between IRSs and Lattices focusing mainly on the second point of view. Attempting to expose the phenomenon in a rather clear way, avoiding technicality, I will assume throughout most of this section that G is a noncompact simple Lie group, although most of the results can be formulated in the much wider setup of semisimple analytic groups over arbitrary local fields, see Section 1.5.

1.1 Borel Density. Let $\mathrm{PSub}(G)$ denote the space of proper closed subgroups. Since G is an isolated point in $\mathrm{Sub}(G)$ (see Toyama [1949] and Kuranishi [1951]) we deduce that $\mathrm{PSub}(G)$ is compact. Letting $\mathrm{PIRS}(G)$ denote the subspace of $\mathrm{IRS}(G)$ consisting of the measures supported on $\mathrm{PSub}(G)$, we deduce:

Lemma 1.1. *The space of proper IRSs, PIRS(G) is compact.*

Let us say that that an IRS μ is discrete if a random subgroup is μ almost surely discrete, and denote by $\mathrm{DIRS}(G)$ the subspace of $\mathrm{IRS}(G)$ consisting of discrete IRSs. The following is a generalization of the classical Borel Density Theorem:

Theorem 1.2. *(Borel Density Theorem for IRS, Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a, Theorem 2.9]) Every proper IRS in G is discrete, i.e. PIRS$(G) =$ DIRS(G). Moreover, for every $\mu \in$ DIRS(G), μ-almost every subgroup is either trivial or Zariski dense.*

In order to prove Theorem 1.2 one first observes that there are only countably many conjugacy classes of non-trivial finite subgroups in G, hence the measure of their union is zero with respect to any non-atomic IRS. Then one can apply the same idea as in Furstenberg's proof of the classical Borel density theorem Furstenberg [1976]. Indeed, taking the Lie algebra of $H \in \mathrm{Sub}(G)$ as well as of its Zariski closure induce measurable maps (see

Gelander and Levit [2017, §4])

$$H \mapsto \mathrm{Lie}(H), \;\; H \mapsto \mathrm{Lie}(\overline{H}^Z).$$

As G is noncompact, Furstenberg's argument implies that the Grassman variety of non-trivial subspaces of $\mathrm{Lie}(G)$ does not carry an $\mathrm{Ad}(G)$-invariant measure. It follows that $\mathrm{Lie}(H) = 0$ and $\mathrm{Lie}(\overline{H}^Z) \in \{\mathrm{Lie}(G), 0\}$ almost surely, and the two statements of the theorem follow.

1.2 Weak Uniform Discreteness. Let U be an identity neighbourhood in G. A family of subgroups $\mathfrak{F} \subset \mathrm{Sub}(G)$ is called *U-uniformly discrete* if $\Gamma \cap U = \{1\}$ for all $\Gamma \in \mathfrak{F}$.

Definition 1.3. A family $\mathfrak{F} \subset \mathrm{DIRS}(G)$ of invariant random subgroups is said to be *weakly uniformly discrete* if for every $\epsilon > 0$ there is an identity neighbourhood $U_\epsilon \subset G$ such that

$$\mu(\{\Gamma \in \mathrm{Sub}_G : \Gamma \cap U_\epsilon \neq \{1\}\}) < \epsilon$$

for every $\mu \in \mathfrak{F}$.

A justification for this definition is given by the following result which is proved by an elementary argument and yet provides a valuable information:

Theorem 1.4. *Let G be a connected non-compact simple Lie group. Then DIRS(G) is weakly uniformly discrete.*

Let $U_n, n \in \mathbb{N}$ be a descending sequence of compact sets in G which form a base of identity neighbourhoods, and set

$$K_n = \{\Gamma \in \mathrm{Sub}_G : \Gamma \cap U_n = \{1\}\}.$$

Since G has NSS (no small subgroups), i.e. there is an identity neighbourhood which contains no non-trivial subgroups, we have:

Lemma 1.5. *The sets K_n are open in* $\mathrm{Sub}(G)$.

Proof. Fix n and let $V \subset U_n$ be an open identity neighbourhood which contains no non-trivial subgroups, such that $V^2 \subset U_n$. It follows that a subgroup Γ intersects U_n non-trivially iff it intersects $U_n \setminus V$. Since $U_n \setminus V$ is compact, the lemma is proved. □

In addition, observe that the ascending union $\bigcup_n K_n$ exhausts $\mathrm{Sub}_d(G)$, the set of all discrete subgroups of G. Therefore we have:

Claim 1.6. For every $\mu \in \mathrm{DIRS}(G)$ and $\epsilon > 0$ we have $\mu(K_n) > 1 - \epsilon$ for some n. □

Let

$$\mathcal{K}_{n,\epsilon} := \{\mu \in \mathrm{DIRS}(G) : \mu(K_n) > 1 - \epsilon\}.$$

Since $\mathrm{Sub}(G)$ is metrizable, it follows from Lemma 1.5 that $\mathcal{K}_{n,\epsilon}$ is open. By Claim 1.6, for any given $\epsilon > 0$, the sets $\mathcal{K}_{n,\epsilon}$, $n \in \mathbb{N}$ form an ascending cover of $\mathrm{DIRS}(G)$. Since the latter is compact, we have $\mathrm{DIRS}(G) \subset \mathcal{K}_{m,\epsilon}$ for some $m = m(\epsilon)$. It follows that

$$\mu\big(\{\Gamma \in \mathrm{Sub}(G) : \Gamma \text{ intersects } U_m \text{ trivially}\}\big) > 1 - \epsilon,$$

for every $\mu \in \mathrm{DIRS}(G)$. Thus Theorem 1.4 is proved. □

Picking $\epsilon < 1$ and applying the theorem for the IRS μ_Γ where $\Gamma \leq G$ is an arbitrary lattice, one deduces the Kazhdan–Margulis theorem Každan and Margulis [1968], and in particular that there is a positive lower bound on the volume of locally G/K-orbifolds:

Corollary 1.7 (Kazhdan–Margulis theorem)**.** *There is an identity neighbourhood $\Omega \subset G$ such that for every lattice $\Gamma \leq G$ there is $g \in G$ such that $g\Gamma g^{-1} \cap \Omega = \{1\}$.*

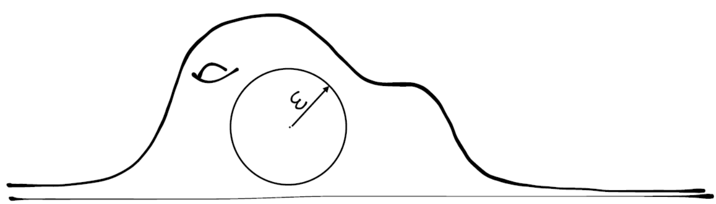

Figure 1: Every X-manifold has a thick part.

A famous conjecture of Margulis Margulis [1991, page 322] asserts that the set of all torsion-free anisotropic arithmetic lattices in G is U-uniformly discrete for some identity neighbourhood $U \subset G$. Theorem 1.4 can be regarded as a probabilistic variant of this conjecture as it implies that all lattices in G are jointly weakly uniformly discrete.

In the language of pmp actions Theorem 1.4 can be reformulated as follows:

Theorem 1.8 (p.m.p. actions are uniformly weakly locally free)**.** *For every $\epsilon > 0$ there is an identity neighbourhood $U_\epsilon \subset G$ such that the stabilizers of $1 - \epsilon$ of the points, in any non-atomic probability measure preserving G-space (X, m) are U_ϵ-uniformly discrete. I.e., there is a subset $Y \subset X$ with $m(Y) > 1 - \epsilon$ such that $U_\epsilon \cap G_y = \{1\}$, $\forall y \in Y$.*

1.3 Local Rigidity.

Observe that local rigidity implies Chabauty locally rigid:

Proposition 1.9. *Let G be a connected Lie group and $\Gamma \leq G$ a locally rigid lattice. Then Γ is Chabauty locally rigid, i.e. the conjugacy class of Γ is Chabauty open.*

Proof. Let $\Gamma \leq G$ a locally rigid lattice. Let U be a compact identity neighborhood in G satisfying:

- $U \cap \Gamma = \{1\}$,
- U contains no nontrivial groups,

and let V be an open symmetric identity neighborhood with $V^2 \subset U$. By the choice of V we for a subgroup $H \leq G$, that $H \cap U \neq \{1\}$ iff H meets the compact set $U \setminus V$.

Recall that Γ, being a lattice in a Lie group, is finitely presented and let $\langle \Sigma | R \rangle$ be a finite presentation of Γ. Denote $S = \{s_1, \ldots, s_k\}$. We can pick a sufficiently small identity neighborhood Ω so that for every choice of $g_i \in s_i\Omega$, $i = 1, \ldots, k$ and every $w \in R$ we have $w(g_1, \ldots, g_n) \in U$.

Now if $H \in \mathrm{Sub}(G)$ is sufficiently close to Γ in the Chabauty topology then $H \cap s_i\Omega \neq \emptyset$, $i = 1, \ldots, k$ and $H \cap (U \setminus V) = \emptyset$, i.e. $H \cap U = \{1\}$. Picking $h_i \in H \cap s_i\Omega$, $i = 1, \ldots, k$ one sees that the assignment $s_i \mapsto h_i$ induces a homomorphism from Γ into H. Since Γ is locally rigid it follows that if H is sufficiently close to Γ then it contains a conjugate of Γ. However there are only finitely many subgroups containing Γ and intersecting U trivially, hence if H is sufficiently close to Γ then it is a conjugate of Γ. □

Denote by $\mathrm{EIRS}(G)$ the space ergodic IRSs of G, i.e. the set of extreme points of $\mathrm{IRS}(G)$.

Corollary 1.10. *Let G be a connected Lie group and $\Gamma \leq G$ a locally rigid lattice. Then the IRS μ_Λ is isolated in EIRS(G).*

Proof. Let Γ be as above. If μ is an IRS of G sufficiently close to μ_Γ then with positive μ-probability a random subgroup is a conjugate of Γ. Thus if μ is ergodic then it must be μ_Γ. □

1.4 Farber property.

Definition 1.11. A sequence μ_n of invariant random subgroups of G is called *Farber*[1] if μ_n converge to the trivial IRS, $\delta_{\{1\}}$.

[1] Various authors use various variants of this notion.

One of the key results of Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a] is the following theorem:

Theorem 1.12. *Let G be a simple Lie group of real rank at least 2. Let Γ_n be a sequence of pairwise non-conjugate lattices, then μ_{Γ_n} is Farber.*

The proof relies on the following variant of Stuck–Zimmer theorem (see Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [ibid.]):

Theorem 1.13. *Assuming $\mathrm{rank}(G) \geq 2$, a proper non-trivial ergodic IRS of G is μ_Γ for some lattice Γ, i.e.*

$$EIRS(G) = \{\delta_G, \delta_{\{1\}}, \mu_\Gamma,\ \Gamma \text{ a lattice in } G\}.$$

Proof of Theorem 1.12. Since G is of rank ≥ 2 it has Kazhdan's property (T), Každan [1967]. By E. Glasner and Weiss [1997], $\mathrm{EIRS}(G)$ is compact. Since δ_G and δ_{μ_Γ} where Γ is a lattice in G are isolated points, it follows that $\delta_{\{1\}}$ is the unique accumulation point of $\mathrm{EIRS}(G)$. In particular $\mu_{\Gamma_n} \to \delta_{\{1\}}$. $\square$

1.5 Semisimple analytic groups. Above we have carried out the arguments under the assumption that G is a simple Lie group. In this section we state the results in the more general setup of analytic groups over local fields.

Definition 1. Let k be a local field and $\mathbb{G}$ a connected k-isotropic k-simple linear k-algebraic group.

- A *simple analytic group* is a group of the form $\mathbb{G}(k)$.
- A *semisimple analytic group* is an almost direct product of finitely many simple analytic groups, possibly over different local fields.

Note that if k is a local field and $\mathbb{G}$ is a connected semisimple linear k-algebraic group without k-anisotropic factors then $\mathbb{G}(k)$ is a semisimple analytic group. Such a group is indeed *analytic* in the sense of e.g. Serre [2006]. Associated to a semisimple analytic group G are its universal covering group $\widetilde{G}$ and adjoint group $\overline{G}$. There are central k-isogenies $\widetilde{G} \xrightarrow{\widetilde{p}} G \xrightarrow{\overline{p}} \overline{G}$ and this data is unique up to a k-isomorphism Margulis [1991, p. I.4.11]. For a semisimple analytic group G, denote by G^+ the subgroup of G generated by its unipotent elements Margulis [ibid., pp. I.1.5, I.2.3]. If G is simply connected then $G = G^+$. If G is Archimedean then G^+ is the connected component G_0 at the identity. In general G/G^+ is a compact abelian group. The group G^+ admits no proper finite index subgroups.

Definition 2. A simple analytic group G is *happy* if $\mathrm{char}(k)$ does not the divide $|Z|$ where Z is the kernel of the map $\widetilde{G} \to G$. A semisimple analytic group is *happy* if all of its almost direct factors are.

Note that a simply connected or a zero characteristic semisimple analytic group is automatically happy. From the work of Barnea and Larsen [2004] one obtains that a semisimple analytic group G is happy, iff G/G^+ is a finite abelian group, iff some (equivalently every) compact open subgroup in the non-Archimedean factor of G is finitely generated.

Self Chabauty isolation.

Definition 3. A l.c.s.c. group G is *self-Chabauty-isolated if the point G is isolated in* $\mathrm{Sub}(G)$ *with the Chabauty topology.*

Note that G is self-Chabauty-isolated if and only if there is a finite collection of open subsets $U_1, \dots, U_n \subset G$ so that the only closed subgroup intersecting every U_i nontrivially is G itself. The following result is proved in Gelander and Levit [2017, §6].

Theorem 4. Let G be a happy semisimple analytic group. Then G^+ is self-Chabauty-isolated.

As an immediate consequence we deduce the analog of Lemma 1.1, namely that the space $\mathrm{PSub}(G^+)$ is compact for every G as in Theorem 4.

Borel Density. The following generalization of Theorem 1.2 was obtained in Gelander and Levit [ibid., Theorem 1.9]:

Theorem 5 (Borel density theorem for IRS)**.** Let k be a local field and G a happy semisimple analytic group over k. Assume that G has no almost k-simple factors of type B_n, C_n or F_4 if $\mathrm{char}(k) = 2$ and of type G_2 if $\mathrm{char}(k) = 3$.

Let μ be an ergodic invariant random subgroup of G. Then there is a pair of normal subgroups $N, M \lhd G$ so that

$$N \le H \le M, \quad H/N \text{ is discrete in } G/N \quad \text{and} \quad \overline{H}^{Z} = M$$

for μ-almost every closed subgroup H in G. Here $\overline{H}^{Z}$ is the Zariski closure of H.

Weak Uniform Discreteness. As shown in Gelander [2018, Theorem 2.1] the analog of Theorem 1.4 holds for general semisimple Lie groups:

Theorem 1.14. *Let G be a connected center-free semisimple Lie group with no compact factors. Then $\mathrm{DIRS}(G)$ is weakly uniformly discrete.*

Consider now a general locally compact σ-compact group G. Since $\mathrm{Sub}_d(G) \subset \mathrm{Sub}(G)$ is a measurable subset, by restricting attention to it, one may replace Property NSS by the weaker Property NDSS (no discrete small subgroups), which means that there is an identity neighbourhood which contains no non-trivial discrete subgroups. In that generality, the analog of Lemma 1.5 would say that K_n are relatively open in $\mathrm{Sub}_d(G)$. Thus, the ingredients required for the argument above are:

1. $\mathrm{DIRS}(G)$ is compact,

2. G has NDSS.

In particular we have:

Theorem 1.15. *Let G be a locally compact σ-compact group which satisfies* (1) *and* (2). *Then DIRS(G) is weakly uniformly discrete.*

If G possesses the Borel density theorem and G is self-Chabauty-isolated then (1) holds. By the previous paragraphs happy semisimple analytic groups enjoy these two properties, and hence (1).

***p*-adic groups.** Note that a p-adic analytic group G has NDSS, and hence $\mathrm{DIRS}(G)$ is uniformly discrete (in the obvious sense). Moreover, if $G \leq \mathrm{GL}_n(\mathbb{Q}_p)$ is a rational algebraic subgroup, then the first principal congruence subgroup $G(p\mathbb{Z}_p)$ is a torsion-free open compact subgroup. In particular the space $\mathrm{DIRS}(G)$ is $G(p\mathbb{Z}_p)$-uniformly discrete. Supposing further that G is simple, then in view of the Borel density theorem we have:

Let (X, μ) be a probability G-space essentially with no global fixed points. Then the action of the congruence subgroup $G(p\mathbb{Z}_p)$ on X is essentially free.

Positive characteristic. Algebraic groups over local fields of positive characteristic do not posses property NDSS, and the above argument does not apply to them.

Conjecture 1.16. *Let k be a local field of positive characteristic, let $\mathbb{G}$ be simply connected absolutely almost simple k-group with positive k-rank and let $G = \mathbb{G}(k)$ be the group of k-rational points. Then DIRS(G) weakly uniformly discrete.*

The analog of Corollary 1.7 in positive characteristic was proved in Salehi Golsefidy [2013] and Raghunathan [1972]. A. Levit proved Conjecture 1.16 for k-rank one groups Levit [n.d.].

Local Rigidity. Combining Theorem 7.2 and Proposition 7.9 of Gelander and Levit [2017] we obtain the following extension of Proposition 1.9:

Theorem 6. (Chabauty local rigidity Gelander and Levit [ibid.]) Let G be a semisimple analytic group and Γ an irreducible lattice in G. If Γ is locally rigid then it is also Chabauty locally rigid.

Let us also mention the following generalization of the classical Weil local rigidity theorem:

Theorem 7. (Gelander and Levit [2016, Theorem 1.2] Let X be a proper geodesically complete CAT(0) space without Euclidean factors and with Isom$X$ acting cocompactly. Let Γ be a uniform lattice in Isom$X$. Assume that for every de Rham factor Y of X isometric to the hyperbolic plane the projection of Γ to Isom$Y$ is non-discrete. Then Γ is locally rigid.

Farber Property. The proof presented above for Theorem 1.12 follows the lines developed at Gelander and Levit [2017] and is simpler and applies to a more general setup than the original proof from Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a]. In particular, the following general version of Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [ibid., Theorem 4.4] is proved in Gelander and Levit [2017]:

Theorem 8. Gelander and Levit [ibid., Theorem 1.1] Let G be a semisimple analytic group. Assume that G is happy, has property (T) and rank$(G) \geq 2$. Let Γ_n be a sequence of pairwise non-conjugate irreducible lattices in G. Then Γ_n is Farber.

Farber property for congruence subgroups. Relying on intricate estimates involving the trace formula, Raimbault [2013] and Fraczyk [2016] were able to establish Benjamini–Schramm convergence for every sequence of congruence lattices in the rank one groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$.

Using property (τ) as a replacement for property (T), Levit [2017a] established Benjamini–Schramm convergence for every sequence of congruence lattices in any higher-rank semisimple group G over local fields. Whenever lattices in G are known to satisfy the congruence subgroup property this applies to all irreducible lattices in G.

1.6 The IRS compactification of moduli spaces. One may also use IRS(G) in order to obtain new compactifications of certain natural spaces.

Example 1.17. Let Σ be a closed surface of genus ≥ 2. Every hyperbolic structure on Σ corresponds to an IRS in $\mathrm{PSL}_2(\mathbb{R})$. Indeed one take a random point and a random tangent vector w.r.t the normalized Riemannian measure on the unit tangent bundle and consider the associated embedding of the fundamental $\pi_1(\Sigma)$ in $\mathrm{PSL}_2(\mathbb{R})$ via deck transformations.

Taking the closure in $\mathrm{IRS}(G)$ of the set of hyperbolic structures on Γ, one obtains an interesting compactification of the moduli space of Σ.

Problem 1.18. Analyse the IRS compactification of $\mathrm{Mod}(\Sigma)$.

Note that the resulting compactification is similar to (but is not exactly) the Deligne–Munford compactification.

2 The Stuck–Zimmer theorem

One of the most remarkable manifestations of rigidity for invariant random subgroups is the following celebrated result due to Stuck and Zimmer [1994].

Theorem 9. Let k be a local field and G be connected, simply connected semi-simple linear algebraic k-group. Assume that G has no k-anisotropic factors, has Kazhdan's property (T) and $\mathrm{rank}_k(G) \geq 2$. Then every properly ergodic and irreducible probability measure preserving Borel action of G is essentially free.

We recall that a probability measure preserving action is properly ergodic if it is ergodic and not essentially transitive, and is irreducible if every non-central normal subgroup is acting ergodically.

In the work of Stuck and Zimmer G is assumed to be a Lie group. The modifications necessary to deal with arbitrary local fields were carried on in Levit [2017b]. Much more generally, Bader and Shalom obtained a variant of the Stuck–Zimmer theorem for products of locally compact groups with property (T) in Bader and Shalom [2006].

The connection between invariant random subgroups and stabilizer structure for probability measure preserving actions allows one to derive the following:

Theorem 10. Let G be as in Theorem 9. Then any irreducible invariant random subgroup of G is either $\delta_{\{e\}}, \delta_G$ or μ_Γ for some irreducible lattice Γ in G.

We would like to point out that the Stuck–Zimmer theorem is a generalization of the following normal subgroup theorem of Margulis.

Theorem 11. Let G be as in Theorem 9 and Γ an irreducible lattice in G. Then any non-trivial normal subgroup $N \lhd \Gamma$ is either central or has finite index in Γ.

The Stuck–Zimmer theorem implies the normal subgroup theorem — indeed the ideas that go into its proof build upon the ideas of Margulis. One key ingredient is the intermediate factor theorem of Nevo and Zimmer, which in turn generalizes the factor theorem of Margulis. We point out that this aspect of the proof is entirely independent of property (T).

Question 2.1. Do Theorems 9 and 10 hold for all higher rank semisimple linear groups, regardless of property (T)?

Observe that the role played by Kazhdan's property (T) in the proof of Theorem 9 is in establishing the following fact.

Proposition 2.2. *Let G be a second countable locally compact group with Kazhdan's property (T). Then every properly ergodic probability measure preserving Borel action of G is not weakly amenable in the sense of Zimmer.*

The Nevo–Zimmer intermediate factor theorem is then used to show that a non-weakly amenable action is essentially free. On the other hand, the contrapositive of weak amenability follows quite readily from the fact that there are no non-trivial cocycles into amenable groups associated to probability measure preserving Borel actions of G.

To summarize, it is currently unknown if groups like $\mathrm{SL}_(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p)$ admit any non-trivial irreducible invariant random subgroups not coming from lattices.

We would like to point out that rank one semisimple linear groups, discrete hyperbolic or relatively hyperbolic groups as well as mapping class groups and $\mathrm{Out}(F_n)$ have a large supply of exotic invariant random subgroups Bowen [2015b], Dahmani, Guirardel, and Osin [2017], and Bowen, Grigorchuk, and Kravchenko [2015].

3 The Benjamini–Schramm topology

Let $\mathfrak{M}$ be the space of all (isometry classes of) pointed proper metric spaces equipped with the Gromov–Hausdorff topology. This is a huge space and for many applications it is enough to consider compact subspaces of it obtained by bounding the geometry. That is, let $f(\epsilon, r)$ be an integer valued function defined on $(0, 1) \times \mathbb{R}^{>0}$, and let $\mathfrak{M}_f$ consist of those spaces for which $\forall \epsilon, r$, the ϵ-entropy of the r-ball $B_X(r, p)$ around the special point is bounded by $f(\epsilon, r)$, i.e. no $f(\epsilon, r) + 1$ points in $B_X(r, p)$ form an ϵ-discrete set. Then $\mathfrak{M}_f$ is a compact subspace of $\mathfrak{M}$.

In many situations one prefers to consider some variants of $\mathfrak{M}$ which carry more information about the spaces. For instance when considering graphs, it may be useful to add colors and orientations to the edges. The Gromov–Hausdorff distance defined on these objects should take into account the coloring and orientation. Another example is smooth Riemannian manifolds, in which case it is better to consider framed manifolds, i.e. manifold with a chosen point and a chosen frame at the tangent space at that point. In that case, one replace the Gromov–Hausdorff topology by the ones determined by (ϵ, r) relations (see Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a, Section 3] for details), which remembers also the directions from the special point.

We define the *Benjamini–Schramm* space $\mathfrak{BS} = \mathrm{Prob}(\mathfrak{M})$ to be the space of all Borel probability measures on $\mathfrak{M}$ equipped with the weak-$*$ topology. Given f as above, we set $\mathfrak{BS}_f := \mathrm{Prob}(\mathfrak{M}_f)$. Note that $\mathfrak{BS}_f$ is compact.

The name of the space is chosen to hint that this is the same topology induced by 'local convergence', considered by Benjamini and Schramm in Benjamini and Schramm [2001], when restricting to measures on rooted graphs. Recall that a sequence of random rooted bounded degree graphs converges to a limiting distribution iff for every n the statistics of the n ball around the root (i.e. the probability vector corresponding to the finitely many possibilities for n-balls) converges to the limit.

The case of general proper metric spaces can be described similarly. A sequence $\mu_n \in \mathfrak{BS}_f$ converges to a limit μ iff for any compact pointed 'test-space' $M \in \mathfrak{M}$, any r and some arbitrarily small[2] $\epsilon > 0$, the μ_n probability that the r ball around the special point is 'ϵ-close' to M tends to the μ-probability of the same event.

Example 3.1. An example of a point in $\mathfrak{BS}$ is a measured metric space, i.e. a metric space with a Borel probability measure. A particular case is a finite volume Riemannian manifold — in which case we scale the Riemannian measure to be one, and then randomly choose a point and a frame.

Thus a finite volume locally symmetric space $M = \Gamma\backslash G/K$ produces both a point in the Benjamini–Schramm space and an IRS in G. This is a special case of a general analogy that I'll now describe. Given a symmetric space X, let us denote by $\mathfrak{M}(X)$ the space of all pointed (or framed) complete Riemannian orbifolds whose universal cover is X, and by $\mathfrak{BS}(X) = \mathrm{Prob}(\mathfrak{M}(X))$ the corresponding subspace of the Benjamini–Schramm space.

Let G be a non-compact simple Lie group with maximal compact subgroup $K \leq G$ and an associated Riemannian symmetric space $X = G/K$. There is a natural map

$$\{\text{discrete subgroups of } G\} \to \mathfrak{M}(X),\ \Gamma \mapsto \Gamma\backslash X.$$

It can be shown that this map is continuous, hence inducing a continuous map

$$\mathrm{DIRS}(G) \to \mathfrak{BS}(X).$$

[2]This doesn't mean that it happens for all ϵ.

It can be shown that the later map is one to one, and since $\mathrm{DIRS}(G)$ is compact, it is a homeomorphism to its image (see Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a, Corollary 3.4]). [3]

Remark 3.2 (Invariance under the geodesic flow). Given a tangent vector $\overline{v}$ at the origin (the point corresponding to K) of $X = G/K$, define a map $\mathfrak{F}_{\overline{v}}$ from $\mathfrak{M}(X)$ to itself by moving the special point using the exponent of $\overline{v}$ and applying parallel transport to the frame. This induces a homeomorphism of $\mathfrak{BS}(X)$. The image of $\mathrm{DIRS}(G)$ under the map above is exactly the set of $\mu \in \mathfrak{BS}(X)$ which are invariant under $\mathfrak{F}_{\overline{v}}$ for all $\overline{v} \in T_K(G/K)$.

Thus we can view geodesic-flow invariant probability measures on framed locally X-manifolds as IRS on G and vice versa, and the Benjamini–Schramm topology on the first coincides with the IRS-topology on the second.

Remark 3.3. The analogy above can be generalised, to some extent, to the context of general locally compact groups. Given a locally compact group G, fixing a right invariant metric on G, we obtain a map $\mathrm{Sub}_G \to \mathfrak{M}$, $H \mapsto G/H$, where the metric on G/H is the induced one. Moreover, this map is continuous hence defines a continuous map $\mathrm{IRS}(G) \to \mathfrak{BS}$.

For the sake of simplicity let us forget 'the frame' and consider pointed X-manifolds, and $\mathfrak{BS}(X)$ as probability measures on such. We note that while for general Riemannian manifolds there is a benefit for working with framed manifolds, for locally symmetric spaces of non-compact type, pointed manifolds, and measures on such, behave nicely enough.

In order to examine convergence in $\mathfrak{BS}(X)$ it is enough to use as 'test-space' balls in locally X-manifolds. Moreover, since X is non-positively curved, a ball in an X-manifold is isometric to a ball in X iff it is contractible.

Note that since X is a homogeneous space, all choices of a probability measure on X correspond to the same point in $\mathfrak{BS}(X)$. Abusing notations, we shall denote this point by X.

Definition 3.4. Let us say that a sequence in $\mathfrak{BS}(X)$ is *Farber* if it converges to X.

For an X-manifold M and $r > 0$, we denote by $M_{\geq r}$ the r-thick part in M:

$$M_{\geq r} := \{x \in M : \mathrm{InjRad}_M(x) \geq r\},$$

where $\mathrm{InjRad}_M(x) = \sup\{\epsilon : B_M(x, \epsilon) \text{ is contractible}\}$.

[3] It was Miklos Abert who pointed out to me, about eight years ago, the analogy between Benjamini–Schramm convergence (at that time 'local convergence') and convergence of Invariant Random Subgroups, which later played an important rule in the work Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a] and earlier in the work of Abért, Y. Glasner, and Virág [2016, 2014].

Proposition 3.5. *Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a, Corollary 3.8] A sequence M_n of finite volume X-manifolds is Farber iff*

$$\frac{\mathrm{vol}((M_n)_{\geq r})}{\mathrm{vol}(M_n)} \to 1,$$

for every $r > 0$.

Theorem 1.12 can be reformulated as:

Theorem 3.6. *Let X be an irreducible Riemannian symmetric space of non compact type of rank at least 2. For any r and ϵ there is V such that if M is an X-manifold of volume $v \geq V$ then $\frac{\mathrm{vol}(M_{\geq r})}{v} \geq 1 - \epsilon$ (see Figure 2).*

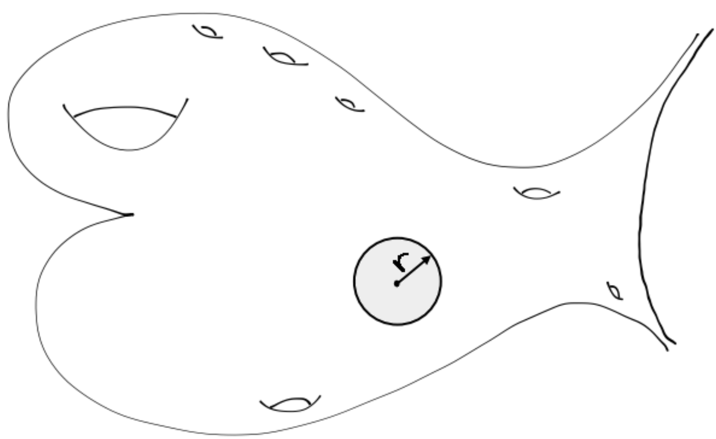

Figure 2: A large volume manifold is almost everywhere fat.

4 Applications to L_2-invariants

Let Γ be a uniform lattice in G. The right quasi-regular representation ρ_Γ of G in $L^2(\Gamma\backslash G, \mu_G)$ decomposes as a direct sum of irreducible representations. Every irreducible unitary representation π of G appears in ρ_Γ with finite multiplicity $m(\pi, \Gamma)$.

Definition 12. The *normalized relative Plancherel measure* of G with respect to Γ is an atomic measure on the unitary dual $\widehat{G}$ given by

$$\nu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi\in\widehat{G}} m(\pi,\Gamma)\delta_\pi.$$

The following result, extending earlier works of de George and Wallach [1978], Delorme [1986] and many others, was proved in Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a] for real Lie groups and then generalized to non-archimedean groups in Gelander and Levit [2017]:

Theorem 13. Let G be a semisimple analytic group in zero characteristic. Fix a Haar measure on G and let ν_G be the associated Plancherel measure on $\widehat{G}$. Let Γ_n be a uniformly discrete sequence of lattices in G with μ_{Γ_n} being weak-$*$ convergent to $\delta_{\{e\}}$. Then

$$\nu_{\Gamma_n}(E) \xrightarrow{n\to\infty} \nu_G(E)$$

for every relatively quasi-compact ν^G-regular subset $E \subset \widehat{G}$.

One of the consequence of Theorem 13 is the convergence of normalized Betti numbers (cf. Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2011]). Recently we were able to get rid of the co-compactness and the uniform discreteness assumptions and proved the following general version, making use of the Bowen–Elek Bowen [2015a, §4] simplicial approximation technique:

Theorem 4.1. *Abert, Bergeron, Biringer, and Gelander [n.d.] Let X be a symmetric space of non compact type of $dim(X) \neq 3$ and (M_n) is a weakly convergent sequence of finite volume X-manifolds. Then for all k, the normalized Betti numbers $b_k(M_n)/\mathrm{vol}(M_n)$ converge.*

Here, the only three-dimensional irreducible symmetric spaces of noncompact type are scales of $\mathbb{H}^3$. In fact, the conclusion of Theorem 4.1 is false when $X = \mathbb{H}^3$. As an example, let $K \subset S^3$ be a knot such that the complement $M = S^3 \setminus K$ admits a hyperbolic metric, e.g. the figure-8 knot. Using meridian–longitude coordinates, let M_n be obtained by Dehn filling M with slope $(1, n)$; then each M_n is a homology 3-sphere. The manifolds $M_n \to M$ geometrically, see Benedetti and Petronio [1992, Ch E.6], so the measures μ_{M_n} weakly converge to μ_M (c.f. Bader, Gelander, and Sauer [2016, Lemma 6.4]) and the volumes $\mathrm{vol}(M_n) \to \mathrm{vol}(M)$. However, $0 = b_1(M_n) \not\to b_1(M) = 1$, so the normalized Betti numbers of the sequence $M_1, M, M_2, M, \ldots$ do not converge.

Corollary 14. Suppose that (M_n) is a Farber sequence of finite volume X-manifolds. Then for all $k \in \mathbb{N}$, we have $b_k(M_n)/\mathrm{vol}(M_n) \to \beta_k^{(2)}(X)$.

In the thin case, we were able to push our analytic methods far enough to give a proof for $X = \mathbb{H}^d$, see Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017a, Theorem 1.8]. Hence, there is no problem in allowing $X = \mathbb{H}^3$ in Corollary 14. The analog of Corollary 14 for p-adic Bruhat Tits buildings is proved in Gelander and Levit [2017].

5 Measures on the space of Riemannian manifolds

When $X = G/K$ is a symmetric space of noncompact type, say, the quotient of a discrete, torsion-free IRS Γ of G is a random X-manifold M. Fixing a base point p in X, the projection of p to $\Gamma\backslash X$ is a natural base point for the quotient. So, we can regard the quotient of an IRS as a random pointed X-manifold. In fact, the conjugation invariance of Γ directly corresponds to a property called *unimodularity* of the random pointed X-manifold, just as IRSs of discrete groups correspond to unimodular random Schreier graphs.

In Biringer and Abert [2016], the Abert and Biringer study unimodular probability measures on the more general space $\mathfrak{M}^d$ of all pointed Riemannian d-manifolds, equipped with the smooth topology. One can construct such unimodular measures from finite volume d-manifolds, or from IRSs of continuous groups as above (see Biringer and Abert [ibid., Proposition 1.9]). Under certain geometric assumptions like pinched negative curvature or local symmetry, they show that sequences of unimodular probability measures are precompact, in parallel with the compactness of the space of IRSs of a Lie group, see Biringer and Abert [ibid., Theorems 1.10 and 1.11]. They also show that unimodular measures on $\mathfrak{M}^d$ are just those that are 'compatible' with its foliated structure. Namely, $\mathfrak{M}^d$ is almost a foliated space, where a leaf is obtained by fixing a manifold M and varying the basepoint. While this foliation may be highly singular, they show in Biringer and Abert [ibid., Theorem 1.6] that after passing to an (actually) foliated desingularization, unimodular measures are just those that are created by integrating the Riemannian measures on the leaves against some invariant transverse measure. This is a precise analogue of the hard-to-formalize statement that a unimodular random graph is a random pointed graph in which the vertices are 'distributed uniformly' across each fixed graph.

6 Soficity of IRS

Definition 15. An IRS μ is *co-sofic* if it is a weak-$*$ limit in $\mathrm{IRS}(G)$ of ones supported on lattices.

The following result justify the name (cf. Abert, Gelander, and Nikolov [2017, Lemma 16]):

Proposition 6.1. *Let F_n be the free group of rank n. A Dirac mass δ_N, $N \lhd F_n$ is co-sofic iff the corresponding group $G = F_n/N$ is sofic.*

Given a group G it is natural to ask:

Question 6.2. Is every IRS in G co-sofic?

In particular for $G = F_n$ this is equivalent to the Aldous–Lyons conjecture that every unimodular network (supported on rank n Schreier graphs) is a limit of ones corresponding to finite Schreier graphs Aldous and Lyons [2007].

Therefore it is particularly intriguing to study Question 6.2 for G, a locally compact group admitting F_n as a lattice. This is the case for $G = SL_2(\mathbf{R}), SL_2(\mathbf{Q}_p)$ and $\mathrm{Aut}(T)$.

7 Exotic IRS

In the lack of Margulis' normal subgroup theorem there are IRS supported on non-lattices. Indeed, from a lattice $\Gamma \leq G$ and a normal subgroup of infinite index $N \lhd \Gamma$ one can cook an IRS in G supported on the closure of the conjugacy class N^G.

A more interesting example in $\mathrm{SO}(n,1)$ (from Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2016]) is obtained by choosing two compact hyperbolic manifolds A, B with totally geodesic boundary, each with two components, and all four components are pairwise isometric and then glue random copies of A, B along an imaginary line to obtain a random hyperbolic manifold whose fundamental group is an IRS in $\mathrm{SO}(n,1)$. If A, B are chosen wisely, the random subgroup obtained is not contained in a lattice. However, all IRSs obtained that way are co-sofic. Other constructions of exotic IRS in $\mathrm{SO}(3,1)$ are given in Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [ibid.].

8 Existence

There are many well known examples of discrete groups without nontrivial IRS, for instance $\mathrm{PSL}_n(\mathbb{Q})$, and also the Tarski Monsters. In Gelander [2015, §8] I asked for non-discrete examples, and in particular weather the Neretin group (of almost adthomorphisms of a regular tree) admits non-trivial IRS. Recently Boudec and Bon [2017] constructed an example of a non-discrete locally compact group with no non-trivial IRS. (Note however that it is not compactly generated.)

9 Character Rigidity

Definition 16. Let Γ be a discrete group. A *character* on Γ is an irreducible positive definite complex-valued class function $\varphi : \Gamma \to \mathbb{C}$ satisfying $\varphi(e) = 1$.

The irreducibility of φ simply means that it cannot be written as a convex combination of two distinct characters. This notion was introduced by Thoma in Thoma [1964a,b]. In the abelian case Definition 16 reduces to the classical notion.

We will say that Γ has *Character rigidity* if only the obvious candidates occur as characters of Γ. The following theorem of Bekka [2007] is an outstanding example of such a result.

Theorem 17. Let φ be a character of the group $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$. Then either φ factors through an irreducible representation of some finite congruence quotient $\mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z})$ or φ vanishes outside the center of Γ.

The connection between invariant random subgroups and characters arises from the following construction. Let (X, μ) be a Borel probability space with an action of Γ preserving μ. Consider the following real-valued function $\varphi : \Gamma \to \mathbf{R}$ that is associated to the action of Γ. The function g is given by

$$\varphi(\gamma) = \mu(\mathrm{Fix}(\gamma))$$

for every $\gamma \in \Gamma$, where

$$\mathrm{Fix}(\gamma) = \{x \in X \ : \ \gamma x = x\}.$$

For instance $\varphi(\gamma) = 1$ if γ lies in the kernel of the action and $\varphi(\gamma) = 0$ if μ-almost every point of X is not fixed by γ. It turns out that φ is a positive define class function satisfying $\varphi(e) = 1$.

Let Γ be an irreducible lattice in a higher rank semisimple linear group G with property (T). It can be shown by means of induced actions that Theorem 10 holds for the lattice Γ as well, namely any properly ergodic action of Γ has central stabilizers.

We see that Theorem 17 in fact implies Theorem 9 in the special case of the particular arithmetic group $\Gamma = \mathrm{SL}_n(\mathbb{Z}), n \geq 3$, which in turn implies the normal subgroup theorem of Margulis[4]. A character rigidity result is in general much stronger than invariant random subgroups rigidity — indeed, not all characters arise in the above manner from probability measure preserving actions.

Recently Peterson [2014] has been able to vastly generalize Bekka's result, as follows.

[4]The normal subgroup theorem for $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$ is in fact a much older theorem, dating back to Mennicke's work on the congruence subgroup problem Mennicke [1965].

Theorem 18. Character rigidity in the sense of Theorem 17 holds for any irreducible lattice in a higher rank semisimple Lie group without compact factors and with property (T).

Let us survey a few other well-known classification results for characters of discrete groups. In his original papers Thoma studied characters of the infinite symmetric group Thoma [1964a]. Dudko and Medynets studied characters of the Higman—Thompson and related groups Dudko and Medynets [2014]. Peterson and Thom establish character rigidity for linear groups over infinite fields or localizations of orders in number fields Peterson and Thom [2016b], generalizing several previous results Kirillov [1965] and Ovčinnikov [1971].

10 History

The interplay between a group theoretic and geometric viewpoints characterises the theory of IRS from its beginning. Two groundbreaking papers, Stuck and Zimmer [1994] and Aldous and Lyons [2007] represent these two points of view. Zimmer's work, throughout, was deeply influenced by Mackey's virtual group philosophy which draws an analogy between the subgroups of G and its ergodic actions. When G is a center free, higher rank simple Lie group, it is proved in Stuck and Zimmer [1994] that every non-essentially-free ergodic action is in fact a transitive action on the cosets of a lattice subgroup. These results can be viewed as yet another implementation of higher rank rigidity, but they also show that Mackey's analogy becomes much tighter when one considers non-essentially-free actions.

The Aldous–Lyons paper is influenced by the geometric notion of *Benjamini–Schramm* convergence in graphs, sometimes also referred to as *weak convergence* or as *convergence in local statistics*, developed in Aldous and Steele [2004],Benjamini and Schramm [2001],Benjamini, Lyons, Peres, and Schramm [1999]. Any finite graph[5] gives rise to a random rooted graph, upon choosing the root uniformly at random. Thus the collection of finite graphs, embeds as a discrete set, into the space of Borel probability measures on the (compact) space of rooted graphs. Random rooted graph in the w^*-closure of this set are subject to the *mass transport principal* introduced by Benjamini and Schramm [2001]: For every integrable function on the space of bi-rooted graphs

$$\int \sum_{x \in V(G)} f(G, o, x) d\mu([G, o]) = \int \sum_{x \in V(G)} f(G, x, o) d\mu([G, o]).$$

[5] or more generally an infinite graph whose automorphism group contains a lattice.

Aldous and Lyons define random unimodular graphs to be random rooted graphs subject to the mass transport principal. In Aldous and Lyons [2007, Question 10.1] they ask whether every random unimodular graph is in the w^*-closure of the set of finite graphs. When one specialises this theory to Schreier graphs of a given finitely generated group Γ (more generally to the quotients of the Cayley-Abels graph of a given compactly generated group G) one obtains the theory of IRS in Γ or in G. For probability measures on the Chabauty space of subgroups Sub_Γ — the mass transport principal is equivalent to invariance under the adjoint action of the group. When $\Gamma = F_d$ is the free group and $N \lhd \Gamma$ is a normal subgroup the group Γ/N is *sofic* in the sense of Gromov and Weiss if and only if the IRS δ_N is a w^*-limit of IRS supported on finite index subgroups. Thus the Aldous-Lyons question in the setting of Schreier graphs of F_d specializes to Gromov's question whether every group is sofic.

In a pair of papers Abért, Y. Glasner, and Virág [2016, 2014], Abért, Glasner and Virág introduced the notion of IRS and used it to answer a long standing question in graph theory. A sequence $\{X_n\}$ of finite, distinct d-regular Ramanujan graphs Benjamini–Schramm converges to the universal covering tree T_d. They provided a quantitative estimate for this result, for a Ramanujan graph X,

$$\Pr\{x \in X \mid \mathrm{inj}_X(x) \leq \beta \log\log(|X|)\} = O\left(\log(|X|)^{-\beta}\right),$$

where $\beta = (30\log(d-1))^{-1}$, $\mathrm{Inj}_X(x) = \max\{R \in \mathbf{N} \mid B_X(x,R) \text{ is contractible}\}$ and the probability is the uniform over the vertices of X. The proof combines the geometric and group theoretic viewpoints in an essential way: They start with a sequence of Ramanujan (Schreier) graphs $\{X_n\}$. Passing if necessary to a subsequence they assume that $X_n \to \Delta\backslash F_{d/2}$, where Δ is an IRS in $F_{d/2}$. Now the main technical result of their paper shows that the Schreier graph of an IRS has to satisfy Kesten's spectral gap theorem $\rho(\mathrm{Cay}(\Gamma/\Delta, S)) \geq \rho(\mathrm{Cay}(\Gamma, S))$ with equality if and only if $\Delta = \langle e \rangle$ a.s. Thus the limiting object is indeed the tree.

More generally they develop the theory of Benjamini–Shcramm limits of unimodular random graphs, as well as for Γ-Schreier graphs for arbitrary finitely generated group Γ. In this case the IRS version of Kesten's theorem reads $\rho(\mathrm{Cay}(\Gamma/\Delta, S)) \geq \rho(\mathrm{Cay}(\Gamma, S))$, with an (a.s.) equality, iff Δ is (a.s.) amenable. In hope of reproducing this same beautiful picture for general finitely generated groups, Abért, Glasner and Virág phrased a fundamental question that was quickly answered by Bader, Duchesne, and Lécureux [2016] giving rise to the following theorem: Every amenable IRS in a group Γ is supported on the subgroups of the amenable radical of Γ.

Independently of all of the above, Lewis Bowen in Bowen [2014], introduced the notion of an IRS, and of the *Poisson boundary relative to an IRS*. He used these notions to solve a long standing question in dynamics — proving that the Furstenberg entropy spectrum of the free group is a closed interval. Let (G, μ) be a locally compact group with a Borel probability measure on it, and (X, ν) a (G, μ) space. This means that $G \curvearrowright X$ acts on X measurably and ν is a μ-stationary probability measure in the sense that $\nu = \mu * \nu$. The Furstenberg entropy of this space is

$$h_\mu(X, \nu) = \int \int -\log \frac{d\eta \circ g}{d\eta}(x) d\eta(x) d\mu(g).$$

$\mathrm{Spec}(G, \mu) := \{h_\mu(X, \nu) \mid (X, \nu) \text{ an ergodic}(G, \mu)\text{-space}\}$ is called *the Furstenberg entropy spectrum* and it is bounded in the interval $[0, h_{max}(\mu)]$. The value 0 is obtained when the action is measure preserving, and the maximal value is always attained by the Poisson boundary $B(G, \mu)$. The study of the entropy spectrum is tightly related to the study of factors of the Poisson boundary. Nevo and Zimmer, Nevo and Zimmer [2000], consider a restricted spectrum, that comes only form actions subject to certain mixing properties and show that this restricted spectrum $\mathrm{Spec}'(G)$ is finite for a centre free, higher rank semisimple Lie group G. This result was then used in their proof of the intermediate factor theorem, which in retrospect also validated the proof of the Stuck-Zimmer theorem Nevo and Zimmer [1999, 2002b,a]. Bowen's work on IRS filled in a gap in the other direction — providing as it did many examples of stationary actions.

Let $K \in \mathrm{Sub}(G)$. The Poisson boundary $B(K \backslash G, \mu)$ is the (Borel) quotient of the space of all μ-random walks on $K \backslash G$ under the shift $\sigma(Kg_0, Kg_1, Kg_2, \ldots) = (Kg_1, Kg_2, \ldots)$. If $\Delta = N$ happens to be normal then one retains the Kaimanovich–Vershik description of the Poisson boundary on G/N and clearly G acts on this space from the left giving it the structure of a (G, μ)-space. In the more general setting introduced by Bowen, G still acts on the natural bundle over $\mathrm{Sub}(G)$, where the fibre over $K \in \mathrm{Sub}(G)$ is $B(K \backslash G, \mu)$. The natural action of G on the space of all walks on all these coset spaces, given by $g(Kg_1, Kg_2, \ldots) = (gKg^{-1}gg_1, gKg^{-1}gg_2, \ldots)$ clearly commutes with the shift and gives rise to a well defined action of (G, μ) on this bundle. Any choice of an IRS $\theta \in \mathrm{IRS}(G)$ gives rise to a (G, μ)-stationary measure on this bundle. Now for $F_d = \langle s_1, \ldots, s_d \rangle$, $\mu = \frac{1}{2d}\left(\sum_{i=1}^d s_i + s_i^{-1}\right)$ the proof that $\mathrm{Spec}(F_d, \mu) = [0, h_{\max}(\mu)]$ is completed by finding a certain path $\alpha : I :\to \mathrm{IRS}(F_d)$ in the space of IRSs with the following properties: (i) the path starts at the trivial IRS (corresponding to the action on the Poisson boundary), (ii) it ends at an IRS giving rise to arbitrarily small entropy values and (iii) the entropy function is continuous on this path. Continuity of the entropy function is very special and so are the IRS that are chosen in order to allow for this continuity. The existence of paths on the other hand is actually general, in Bowen [2015b] Bowen proves that the collection of ergodic IRS on F_d that are not supported on finite index subgroups is path connected.

Vershik [2012, 2010], also indpendently, arrived at IRS from his study of the representation theory and especially the characters of S_f^∞ — the group of finitely supported permutations of a countable set. To an IRS $\mu \in \mathrm{IRS}(\Gamma)$ in a countable group define its *Vershik character* as follows

$$\phi_\mu : \Gamma \to \mathbf{R}_{\geq 0}, \qquad \phi_\mu(\gamma) = \mu\left(\{\Delta \in \mathrm{Sub}(\Gamma) \mid \gamma \in \Delta\}\right).$$

If the IRS is realized as the stabilizer Γ_x of a random point in a p.m.p. action $\Gamma \curvearrowright (X, \nu)$ (by Abért, Y. Glasner, and Virág [2014], every IRS can be realized in this fashion), the same IRS is given by $\phi_\nu(\gamma) = \nu(Fix(\gamma))$. Vershik also describes the GNS constuctions associated with this character. Let $R = \{(x,y) \in X \times X | y \in \Gamma x\}$ and let η be the infinite measure on R given by $\int f(x,y)\eta(x,y) = \int \sum_{y \in \Gamma x} f(x,y) d\mu(x)$. Γ acts on R via its action on the first coordinate $\gamma(x,y) = (\gamma x, y)$ and hence it acts on the Hilbert space $L_2(R,\eta)$. Let $\chi(x,y) = 1_{x=y} \in L_2(R,\eta)$ be the characteristic function of the diagonal. It is easy to verify that $phi_\mu(\gamma) = \langle \gamma\chi, \chi \rangle$. The defintion of the Vershik character clarified the deep connection between character rigidity in the snese of Connes and and the Stuck–Zimmer theorem.

Acknowledgments. Several people helped me in writing up parts of this paper: Ian Biringer with Section 5, Arie Levit with Section 9 and Section 2 and Yair Glasner with Section 10.

References

M. Abert, N. Bergeron, I. Biringer, and T. Gelander (n.d.). "Convergence of normalized Betti numbers in nonpositive curvature" (cit. on p. 1353).

Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet (2011). "On the growth of Betti numbers of locally symmetric spaces". *C. R. Math. Acad. Sci. Paris* 349.15-16, pp. 831–835. MR: 2835886 (cit. on p. 1353).

– (Dec. 2016). "On the growth of L2-invariants of locally symmetric spaces, II: exotic invariant random subgroups in rank one". arXiv: 1612.09510 (cit. on p. 1355).

– (2017a). "On the growth of L^2-invariants for sequences of lattices in Lie groups". *Ann. of Math. (2)* 185.3, pp. 711–790. arXiv: 1210.2961. MR: 3664810 (cit. on pp. 1339, 1340, 1344, 1347, 1350–1354).

– (2017b). "On the growth of L^2-invariants for sequences of lattices in Lie groups". *Ann. of Math. (2)* 185.3, pp. 711–790. MR: 3664810.

Miklos Abert, Tsachik Gelander, and Nikolay Nikolov (2017). "Rank, combinatorial cost, and homology torsion growth in higher rank lattices". *Duke Math. J.* 166.15, pp. 2925–2964. MR: 3712168 (cit. on p. 1354).

Miklós Abért, Yair Glasner, and Bálint Virág (2014). "Kesten's theorem for invariant random subgroups". *Duke Math. J.* 163.3, pp. 465–488. MR: 3165420 (cit. on pp. 1351, 1358, 1360).

– (2016). "The measurable Kesten theorem". *Ann. Probab.* 44.3, pp. 1601–1646. MR: 3502591 (cit. on pp. 1351, 1358).

David Aldous and Russell Lyons (2007). "Processes on unimodular random networks". *Electron. J. Probab.* 12, no. 54, 1454–1508. MR: 2354165 (cit. on pp. 1355, 1357, 1358).

David Aldous and J. Michael Steele (2004). "The objective method: probabilistic combinatorial optimization and local weak convergence". In: *Probability on discrete structures*. Vol. 110. Encyclopaedia Math. Sci. Springer, Berlin, pp. 1–72. MR: 2023650 (cit. on p. 1357).

Uri Bader, Pierre-Emmanuel Caprace, Tsachik Gelander, and Shahar Mozes (2012). "Simple groups without lattices". *Bull. Lond. Math. Soc.* 44.1, pp. 55–67. MR: 2881324.

Uri Bader, Bruno Duchesne, and Jean Lécureux (2016). "Amenable invariant random subgroups". *Israel J. Math.* 213.1. With an appendix by Phillip Wesolek, pp. 399–422. MR: 3509477 (cit. on p. 1358).

Uri Bader, Tsachik Gelander, and Roman Sauer (Dec. 2016). "Homology and homotopy complexity in negative curvature". arXiv: 1612.04871 (cit. on p. 1353).

Uri Bader and Yehuda Shalom (2006). "Factor and normal subgroup theorems for lattices in products of groups". *Invent. Math.* 163.2, pp. 415–454. MR: 2207022 (cit. on p. 1348).

Y. Barnea and M. Larsen (2004). "Random generation in semisimple algebraic groups over local fields". *J. Algebra* 271.1, pp. 1–10. MR: 2022476 (cit. on p. 1345).

Bachir Bekka (2007). "Operator-algebraic superridigity for $\mathrm{SL}_n(\mathbb{Z}), n \geq 3$". *Invent. Math.* 169.2, pp. 401–425. MR: 2318561 (cit. on p. 1356).

Riccardo Benedetti and Carlo Petronio (1992). *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, pp. xiv+330. MR: 1219310 (cit. on p. 1353).

I. Benjamini, R. Lyons, Y. Peres, and O. Schramm (1999). "Group-invariant percolation on graphs". *Geom. Funct. Anal.* 9.1, pp. 29–66. MR: 1675890 (cit. on p. 1357).

Itai Benjamini and Oded Schramm (2001). "Recurrence of distributional limits of finite planar graphs". *Electron. J. Probab.* 6, no. 23, 13. MR: 1873300 (cit. on pp. 1350, 1357).

Ian Biringer and Miklos Abert (June 2016). "Unimodular measures on the space of all Riemannian manifolds". arXiv: 1606.03360 (cit. on p. 1354).

Adrien Le Boudec and Nicolas Matte Bon (Sept. 2017). "Locally compact groups whose ergodic or minimal actions are all free". arXiv: 1709.06733 (cit. on p. 1355).

Lewis Bowen (2014). "Random walks on random coset spaces with applications to Furstenberg entropy". *Invent. Math.* 196.2, pp. 485–510. MR: 3193754 (cit. on p. 1358).

– (2015a). "Cheeger constants and L^2-Betti numbers". *Duke Math. J.* 164.3, pp. 569–615. MR: 3314481 (cit. on p. 1353).

– (2015b). "Invariant random subgroups of the free group". *Groups Geom. Dyn.* 9.3, pp. 891–916. MR: 3420547 (cit. on pp. 1349, 1359).

Lewis Bowen, Rostislav Grigorchuk, and Rostyslav Kravchenko (2015). "Invariant random subgroups of lamplighter groups". *Israel J. Math.* 207.2, pp. 763–782. MR: 3359717 (cit. on p. 1349).

F. Dahmani, V. Guirardel, and D. Osin (2017). "Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces". *Mem. Amer. Math. Soc.* 245.1156, pp. v+152. arXiv: 1111.7048. MR: 3589159 (cit. on p. 1349).

Patrick Delorme (1986). "Formules limites et formules asymptotiques pour les multiplicités dans $L^2(G/\Gamma)$". *Duke Math. J.* 53.3, pp. 691–731. MR: 860667 (cit. on p. 1353).

Artem Dudko and Konstantin Medynets (2014). "Finite factor representations of Higman-Thompson groups". *Groups Geom. Dyn.* 8.2, pp. 375–389. arXiv: 1212.1230. MR: 3231220 (cit. on p. 1357).

Mikolaj Fraczyk (Dec. 2016). "Strong Limit Multiplicity for arithmetic hyperbolic surfaces and 3-manifolds". arXiv: 1612.05354 (cit. on p. 1347).

Harry Furstenberg (1976). "A note on Borel's density theorem". *Proc. Amer. Math. Soc.* 55.1, pp. 209–212. MR: 0422497 (cit. on p. 1340).

Tsachik Gelander (2014). "Lectures on lattices and locally symmetric spaces". In: *Geometric group theory*. Vol. 21. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, pp. 249–282. MR: 3329730.

– (Mar. 2015). "A lecture on Invariant Random Subgroups". arXiv: 1503.08402 (cit. on p. 1355).

– (2018). "Kazhdan–Margulis theorem for invariant random subgroups". *Adv. Math.* 327, pp. 47–51. MR: 3761990 (cit. on p. 1345).

Tsachik Gelander and Arie Levit (May 2016). "Local Rigidity Of Uniform Lattices". arXiv: 1605.01693 (cit. on p. 1347).

– (July 2017). "Invariant random subgroups over non-Archimedean local fields". arXiv: 1707.03578 (cit. on pp. 1341, 1345, 1347, 1353, 1354).

David L. de George and Nolan R. Wallach (1978). "Limit formulas for multiplicities in $L^2(\Gamma\backslash G)$". *Ann. of Math. (2)* 107.1, pp. 133–150. MR: 0492077 (cit. on p. 1353).

E. Glasner and B. Weiss (1997). “Kazhdan’s property T and the geometry of the collection of invariant measures”. *Geom. Funct. Anal.* 7.5, pp. 917–935. MR: 1475550 (cit. on p. 1344).

Yair Glasner (2017). “Invariant random subgroups of linear groups”. *Israel J. Math.* 219.1. With an appendix by Tsachik Gelander and Glasner, pp. 215–270. arXiv: 1407.2872. MR: 3642021.

D. A. Každan (1967). “On the connection of the dual space of a group with the structure of its closed subgroups”. *Funkcional. Anal. i Priložen.* 1, pp. 71–74. MR: 0209390 (cit. on p. 1344).

D. A. Každan and G. A. Margulis (1968). “A proof of Selberg’s hypothesis”. *Mat. Sb. (N.S.)* 75 (117), pp. 163–168. MR: 0223487 (cit. on p. 1342).

A. A. Kirillov (1965). “Positive-definite functions on a group of matrices with elements from a discrete field”. *Dokl. Akad. Nauk SSSR* 162, pp. 503–505. MR: 0193183 (cit. on p. 1357).

Benoît Kloeckner (2009). “The space of closed subgroups of $\mathbb{R}^n$ is stratified and simply connected”. *J. Topol.* 2.3, pp. 570–588. MR: 2546586.

Masatake Kuranishi (1951). “On everywhere dense imbedding of free groups in Lie groups”. *Nagoya Math. J.* 2, pp. 63–71. MR: 0041145 (cit. on p. 1340).

A. Levit (n.d.). “Weak uniform discreteness of lattices in positive characteristic” (cit. on p. 1346).

Arie Levit (May 2017a). “On Benjamini–Schramm limits of congruence subgroups”. arXiv: 1705.04200 (cit. on p. 1347).

– (2017b). “The Nevo-Zimmer intermediate factor theorem over local fields”. *Geom. Dedicata* 186, pp. 149–171. MR: 3602889 (cit. on p. 1348).

G. A. Margulis (1991). *Discrete subgroups of semisimple Lie groups*. Vol. 17. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, pp. x+388. MR: 1090825 (cit. on pp. 1342, 1344).

Jens L. Mennicke (1965). “Finite factor groups of the unimodular group”. *Ann. of Math. (2)* 81, pp. 31–37. MR: 0171856 (cit. on p. 1356).

Amos Nevo and Robert J. Zimmer (1999). “Homogenous projective factors for actions of semi-simple Lie groups”. *Invent. Math.* 138.2, pp. 229–252. MR: 1720183 (cit. on p. 1359).

– (2000). “Rigidity of Furstenberg entropy for semisimple Lie group actions”. *Ann. Sci. École Norm. Sup. (4)* 33.3, pp. 321–343. MR: 1775184 (cit. on p. 1359).

– (2002a). “A generalization of the intermediate factors theorem”. *J. Anal. Math.* 86, pp. 93–104. MR: 1894478 (cit. on p. 1359).

– (2002b). “A structure theorem for actions of semisimple Lie groups”. *Ann. of Math. (2)* 156.2, pp. 565–594. MR: 1933077 (cit. on p. 1359).

S. V. Ovčinnikov (1971). "Positive definite functions on Chevalley groups". *Funkcional. Anal. i Priložen.* 5.1, pp. 91–92. MR: 0291371 (cit. on p. 1357).

Jesse Peterson (2014). "Character rigidity for lattices in higher-rank groups" (cit. on p. 1356).

Jesse Peterson and Andreas Thom (2016a). "Character rigidity for special linear groups". *J. Reine Angew. Math.* 716, pp. 207–228. MR: 3518376.

– (2016b). "Character rigidity for special linear groups". *J. Reine Angew. Math.* 716, pp. 207–228. MR: 3518376 (cit. on p. 1357).

Vladimir Platonov and Andrei Rapinchuk (1994). *Algebraic groups and number theory*. Vol. 139. Pure and Applied Mathematics. Translated from the 1991 Russian original by Rachel Rowen. Academic Press, Inc., Boston, MA, pp. xii+614. MR: 1278263.

M. S. Raghunathan (1972). *Discrete subgroups of Lie groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. Springer-Verlag, New York-Heidelberg, pp. ix+227. MR: 0507234 (cit. on p. 1346).

Jean Raimbault (Nov. 2013). "On the convergence of arithmetic orbifolds". arXiv: 1311.5375 (cit. on p. 1347).

Alireza Salehi Golsefidy (2013). "Lattices of minimum covolume are non-uniform". *Israel J. Math.* 196.1, pp. 363–373. MR: 3096596 (cit. on p. 1346).

Jean-Pierre Serre (2006). *Lie algebras and Lie groups*. Vol. 1500. Lecture Notes in Mathematics. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition. Springer-Verlag, Berlin, pp. viii+168. MR: 2179691 (cit. on p. 1344).

Garrett Stuck and Robert J. Zimmer (1994). "Stabilizers for ergodic actions of higher rank semisimple groups". *Ann. of Math. (2)* 139.3, pp. 723–747. MR: 1283875 (cit. on pp. 1348, 1357).

Elmar Thoma (1964a). "Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe". *Math. Z.* 85, pp. 40–61. MR: 0173169 (cit. on pp. 1356, 1357).

– (1964b). "Über unitäre Darstellungen abzählbarer, diskreter Gruppen". *Math. Ann.* 153, pp. 111–138. MR: 0160118 (cit. on p. 1356).

Hiraku Toyama (1949). "On discrete subgroups of a Lie group". *Kōdai Math. Sem. Rep.,* 1.2. Volume numbers not printed on issues until Vol. 7 (1955), pp. 36–37. MR: 0029918 (cit. on p. 1340).

A. M. Vershik (2010). "Nonfree actions of countable groups and their characters". *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 378.Teoriya Predstavleniĭ, Dinamicheskie Sistemy, Kombinatornye Metody. XVIII, pp. 5–16, 228. MR: 2749291 (cit. on p. 1360).

– (2012). "Totally nonfree actions and the infinite symmetric group". *Mosc. Math. J.* 12.1, pp. 193–212, 216. MR: 2952431 (cit. on p. 1360).

Robert J. Zimmer (1984). *Ergodic theory and semisimple groups*. Vol. 81. Monographs in Mathematics. Birkhäuser Verlag, Basel, pp. x+209. MR: 776417.

Received 2017-11-27.

Tsachik Gelander
tsachik.gelander@weizmann.ac.il