Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 2 (1281–1300)

REPRESENTATION THEORY OF W-ALGEBRAS AND HIGGS BRANCH CONJECTURE

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Abstract

We survey a number of results regarding the representation theory of W-algebras and their connection with the resent development of the four dimensional $N = 2$ superconformal field theories.

1 Introduction

(Affine) W-algebras appeared in 80's in the study of the two-dimensional conformal field theory in physics. They can be regarded as a generalization of infinite-dimensional Lie algebras such as affine Kac-Moody algebras and the Virasoro algebra, although W-algebras are not Lie algebras but vertex algebras in general. W-algebras may be also considered as an affinizaton of finite W-algebras introduced by Premet [2002] as a natural quantization of Slodowy slices. W-algebras play important roles not only in conformal field theories but also in integrable systems (e.g. V. G. Drinfeld and Sokolov [1984], De Sole, V. G. Kac, and Valeri [2013], and Bakalov and Milanov [2013]), the geometric Langlands program (e.g. E. Frenkel [2007], Gaitsgory [2016], Tan [2016], and Aganagic, E. Frenkel, and Okounkov [2017]) and four-dimensional gauge theories (e.g. Alday, Gaiotto, and Tachikawa [2010], Schiffmann and Vasserot [2013], Maulik and Okounkov [2012], and Braverman, Finkelberg, and Nakajima [2016a]).

In this note we survey the resent development of the representation theory of W-algebras. One of the fundamental problems in W-algebras was the Frenkel-Kac-Wakimoto conjecture E. Frenkel, V. Kac, and Wakimoto [1992] that stated the existence and construction of rational W-algebras, which generalizes the integrable representations of affine Kac-Moody algebras and the minimal series representations of the Virasoro algebra. The notion of the associated varieties of vertex algebras played a crucial role in the proof of the Frenkel-Kac-Wakimoto conjecture, and has revealed an unexpected connection of vertex algebras with the geometric invariants called the Higgs branches in the four dimensional $N = 2$ superconformal field theories.

MSC2010: primary 81R10; secondary 17B67, 17B67.

Acknowledgments. The author benefited greatly from discussion with Christopher Beem, Davide Gaiotto, Madalena Lemos, Victor Kac, Anne Moreau, Hiraku Nakajima, Takahiro Nishinaka, Wolfger Peelaers, Leonardo Rastelli, Shu-Heng Shao, Yuji Tachikawa, and Dan Xie. The author is partially supported in part by JSPS KAKENHI Grant Numbers 17H01086, 17K18724.

2 Vertex algebras

A *vertex algebra* consists of a vector space V with a distinguished vacuum vector $|0\rangle \in V$ and a vertex operation, which is a linear map $V \otimes V \to V((z))$, written $a \otimes b \mapsto a(z)b = (\sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1})b$, such that the following are satisfied:

- (Unit axioms) $(|0\rangle)(z) = 1_V$ and $a(z)|0\rangle \in a + zV[[z]]$ for all $a \in V$.
- (Locality) $(z-w)^n[a(z), b(w)] = 0$ for a sufficiently large n for all $a, b \in V$.

The operator $T : a \mapsto a_{(-2)}|0\rangle$ is called the translation operator and it satisfies $(Ta)(z) = [T, a(z)] = \partial_z a(z)$. The operators $a_{(n)}$ are called *modes*.

For elements a, b of a vertex algebra V we have the following *Borcherds identity* for any $m, n \in \mathbb{Z}$:

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}, \tag{1}$$

$$(a_{(m)}b)_{(n)} = \sum_{j\geq 0} (-1)^j \binom{m}{j} (a_{(m-j)}b_{(n+j)} - (-1)^m b_{(m+n-j)}a_{(j)}). \tag{2}$$

By regarding the Borcherds identity as fundamental relations, representations of a vertex algebra are naturally defined (see V. Kac [1998] and E. Frenkel and Ben-Zvi [2004] for the details).

One of the basic examples of vertex algebras are universal affine vertex algebras. Let G be a simply connected simple algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Kac-Moody algebra associated with $\mathfrak{g}$. The commutation relations of $\widehat{\mathfrak{g}}$ are given by

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K, \quad [K, \widehat{\mathfrak{g}}] = 0 \quad (x, y \in \mathfrak{g},\ m, n \in \mathbb{Z}), \tag{3}$$

where $(\ |\ )$ is the normalized invariant inner product of $\mathfrak{g}$, that is, $(\ |\ ) = 1/2h^\vee \times$Killing form and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. For $k \in \mathbb{C}$, let

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as multiplication by k. There is a unique vertex algebra structure on $V^k(\mathfrak{g})$ such that $|0\rangle = 1 \otimes 1$ is the vacuum vector and

$$x(z) = \sum_{n \in \mathbb{Z}} (xt^n) z^{-n-1} \quad (x \in \mathfrak{g}).$$

Here on the left-hand-side $\mathfrak{g}$ is considered as a subspace of $V^k(\mathfrak{g})$ by the embedding $\mathfrak{g} \hookrightarrow V^k(\mathfrak{g})$, $x \mapsto (xt^{-1})|0\rangle$. $V^k(\mathfrak{g})$ is called the *universal affine vertex algebra associated with* $\mathfrak{g}$ *at a level* k. The Borcherds identity (1) for $x, y \in \mathfrak{g} \subset V^k(\mathfrak{g})$ is identical to the commutation relation (3) with $K = k$ id, and hence, any $V^k(\mathfrak{g})$-module is a $\widehat{\mathfrak{g}}$-module of level k. Conversely, any smooth $\widehat{\mathfrak{g}}$-module of level k is naturally a $V^k(\mathfrak{g})$-module, and therefore, the category $V^k(\mathfrak{g})$ -Mod of $V^k(\mathfrak{g})$-modules is the same as the category of smooth $\widehat{\mathfrak{g}}$-modules of level k. Let $L_k(\mathfrak{g})$ be the unique simple graded quotient of $V^k(\mathfrak{g})$, which is isomorphic to the irreducible highest weight representation $L(k\Lambda_0)$ with highest weight $k\Lambda_0$ as a $\widehat{\mathfrak{g}}$-module. The vertex algebra $L_k(\mathfrak{g})$ is called the *simple affine vertex algebra* associated with $\mathfrak{g}$ at level k, and $L_k(\mathfrak{g})$ -Mod forms a full subcategory of $V^k(\mathfrak{g})$ -Mod, the category of smooth $\widehat{\mathfrak{g}}$-modules of level k.

A vertex algebra V is called *commutative* if both sides of (1) are zero for all $a, b \in V$, $m, n \in \mathbb{Z}$. If this is the case, V can be regarded as a *differential algebra* (=a unital commutative algebra with a derivation) by the multiplication $a.b = a_{(-1)}b$ and the derivation T. Conversely, any differential algebra can be naturally equipped with the structure of a commutative vertex algebra. Hence, commutative vertex algebras are the same[1] as differential algebras (Borcherds [1986]).

Let X be an affine scheme, $J_\infty X$ the *arc space* of X that is defined by the functor of points $\mathrm{Hom}(\mathrm{Spec}\, R, J_\infty X) = \mathrm{Hom}(\mathrm{Spec}\, R[[t]], X)$. The ring $\mathbb{C}[J_\infty X]$ is naturally a differential algebra, and hence is a commutative vertex algebra. In the case that X is a Poisson scheme $\mathbb{C}[J_\infty X]$ has the structure of *Poisson vertex algebra* (Arakawa [2012b]), which is a vertex algebra analogue of Poisson algebra (see E. Frenkel and Ben-Zvi [2004] and V. Kac [2015] for the precise definition).

It is known by Haisheng Li [2005] that any vertex algebra V is canonically filtered, and hence can be regarded[2] as a quantization of the associated graded Poisson vertex algebra $\mathrm{gr}\, V = \bigoplus_p F^p V / F^{p+1} V$, where $F^\bullet V$ is the canonical filtration of V. By definition,

$$F^p V = \mathrm{span}_{\mathbb{C}}\{(a_1)_{(-n_1-1)} \dots (a_r)_{(-n_r-1)}|0\rangle \mid a_i \in V,\ n_i \geq 0,\ \sum_i n_i \geq p\}.$$

[1]However, the modules of a commutative vertex algebra are not the same as the modules as a differential algebra.

[2]This filtration is separated if V is non-negatively graded, which we assume.

The subspace

$$R_V := V/F^1V = F^0V/F^1V \subset \operatorname{gr} V$$

is called *Zhu's C_2-algebra of V*. The Poisson vertex algebra structure of $\operatorname{gr} V$ restricts to the Poisson algebra structure of R_V, which is given by

$$\bar{a}.\bar{b} = \overline{a_{(-1)}b}, \quad \{\bar{a}, \bar{b}\} = \overline{a_{(0)}b}.$$

The Poisson variety

$$X_V = \operatorname{Specm}(R_V)$$

called the *associated variety* of V (Arakawa [2012b]). We have Li [2005] the inclusion

(4) $$\operatorname{Specm}(\operatorname{gr} V) \subset J_\infty X_V.$$

A vertex algebra V is called *finitely strongly generated* if R_V is finitely generated. In this note all vertex algebras are assumed to be finitely strongly generated. V is called *lisse* (or *C_2-cofinite*) if $\dim X_V = 0$. By (4), it follows that V is lisse if and only if $\dim \operatorname{Spec}(\operatorname{gr} V) = 0$ (Arakawa [2012b]). Hence lisse vertex algebras can be regarded as an analogue of finite-dimensional algebras.

For instance, consider the case $V = V^k(\mathfrak{g})$. We have $F^1V^k(\mathfrak{g})) = \mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g})$, and there is an isomorphism of Poisson algebras

$$\mathbb{C}[\mathfrak{g}^*] \xrightarrow{\sim} R_V, \quad x_1 \dots x_r \mapsto \overline{(x_1t^{-1}) \dots (x_rt^{-1})|0\rangle} \quad (x_i \in \mathfrak{g}).$$

Hence

(5) $$X_{V^k(\mathfrak{g})} \cong \mathfrak{g}^*.$$

Also, we have the isomorphism $\operatorname{Spec}(\operatorname{gr} V^k(\mathfrak{g})) \cong J_\infty\mathfrak{g}^*$. By (5), we have $X_{L_k(\mathfrak{g})} \subset \mathfrak{g}^*$, which is G-invariant and conic. It is known Dong and Mason [2006] that

(6) $$L_k(\mathfrak{g}) \text{ is lisse} \iff L_k(\mathfrak{g}) \text{ is integrable as a } \widehat{\mathfrak{g}}\text{-module } (\iff k \in \mathbb{Z}_{\geq 0}).$$

Hence the lisse condition may be regarded as a generalization of the integrability condition to an arbitrary vertex algebra.

A vertex algebra is called *conformal* if there exists a vector ω, called the *conformal vector*, such that the corresponding field $\omega(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ satisfies the following conditions. (1) $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c \operatorname{id}_V$, where c is a constant called the central charge of V; (2) L_0 acts semisimply on V; (3) $L_{-1} = T$. For a conformal vertex algebra V we set $V_\Delta = \{v \in V \mid L_0v = \Delta V\}$, so that $V = \bigoplus_\Delta V_\Delta$. The universal affine vertex algebra $V^k(\mathfrak{g})$ is conformal by the Sugawara construction provided that $k \neq -h^\vee$. A *positive energy representation* M of a conformal vertex algebra V is a V-module M on which L_0 acts diagonally and the L_0-eivenvalues on M is bounded from below. An *ordinary representation* is a positive energy representation such that each L_0-eivenspaces are finite-dimensional. For a finitely generated ordinary representation M, the normalized character

$$\chi_V(q) = \mathrm{tr}_V(q^{L_0 - c/24})$$

is well-defined.

For a conformal vertex algebra $V = \bigoplus_\Delta V_\Delta$, one defines *Zhu's algebra* I. B. Frenkel and Zhu [1992] $\mathrm{Zhu}(V)$ of V by

$$\mathrm{Zhu}(V) = V/V \circ V, \quad V \circ V = \mathrm{span}_{\mathbb{C}}\{a \circ b \mid a, b \in V\},$$

where $a \circ b = \sum_{i \geq 0} \binom{\Delta_a}{i} a_{(i-2)} b$ for $a \in \Delta_a$. $\mathrm{Zhu}(V)$ is a unital associative algebra by the multiplication $a * b = \sum_{i \geq 0} \binom{\Delta_a}{i} a_{(i-1)} b$. There is a bijection between the isomorphism classes $\mathrm{Irrep}(V)$ of simple positive energy representation of V and that of simple $\mathrm{Zhu}(V)$-modules (I. B. Frenkel and Zhu [1992] and Zhu [1996]). The grading of V gives a filtration on $\mathrm{Zhu}(V)$ which makes it quasi-commutative, and there is a surjective map

$$R_V \twoheadrightarrow \mathrm{gr}\, \mathrm{Zhu}(V) \tag{7}$$

of Poisson algebras. Hence, if V is lisse then $\mathrm{Zhu}(V)$ is finite-dimensional, so there are only finitely many irreducible positive energy representations of V. Moreover, the lisse condition implies that any simple V-module is a positive energy representation (Abe, Buhl, and Dong [2004]).

A conformal vertex algebra is called *rational* if any positive energy representation of V is completely reducible. For instance, the simple affine vertex algebra $L_k(\mathfrak{g})$ is rational if and only if $L_k(\mathfrak{g})$ is integrable, and if this is the case $L_k(\mathfrak{g})$ -Mod is exactly the category of integrable representations of $\widehat{\mathfrak{g}}$ at level k. A theorem of Y. Zhu [1996] states that if V is a rational, lisse, $\mathbb{Z}_{\geq 0}$-graded conformal vertex algebra such that $V_0 = \mathbb{C}|0\rangle$, then the character $\chi_M(e^{2\pi i \tau})$ converges to a holomorphic functor on the upper half plane for any $M \in \mathrm{Irrep}(V)$. Moreover, the space spanned by the characters $\chi_M(e^{2\pi i \tau})$, $M \in \mathrm{Irrep}(V)$, is invariant under the natural action of $SL_2(\mathbb{Z})$. This theorem was strengthened in Dong, Lin, and Ng [2015] to the fact that $\{\chi_M(e^{2\pi i \tau}) \mid M \in \mathrm{Irrep}(V)\}$ forms a vector valued modular function by showing the congruence property. Furthermore, it has been shown in Huang [2008] that the category of V-modules form a modular tensor category.

3 W-algebras

W-algebras are defined by the method of the *quantized Drinfeld-Sokolov reduction* that was discovered by Feigin and E. Frenkel [1990]. In the most general definition of W-algebras given by V. Kac, Roan, and Wakimoto [2003], W-algebras are associated with the pair $(\mathfrak{g}, f)$ of a simple Lie algebra $\mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$. The corresponding W-algebra is a one-parameter family of vertex algebra denoted by $\mathcal{W}^k(\mathfrak{g}, f)$, $k \in \mathbb{C}$. By definition,

$$\mathcal{W}^k(\mathfrak{g}, f) := H^0_{DS,f}(V^k(\mathfrak{g})),$$

where $H^{\bullet}_{DS,f}(M)$ denotes the BRST cohomology of the quantized Drinfeld-Sokolov reduction associated with $(\mathfrak{g}, f)$ with coefficient in a $V^k(\mathfrak{g})$-module M, which is defined as follows. Let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple associated with f, $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [h, x] = 2j\}$, so that $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$. Set $\mathfrak{g}_{\geq 1} = \bigoplus_{j \geq 1} \mathfrak{g}_j$, $\mathfrak{g}_{>0} = \bigoplus_{j \geq 1/2} \mathfrak{g}_j$. Then $\chi : \mathfrak{g}_{\geq 1}[t, t^{-1}] \to \mathbb{C}$, $xt^n \mapsto \delta_{n,-1}(f|x)$, defines a character. Let $F_\chi = U(\mathfrak{g}_{>0}[t, t^{-1}]) \otimes_{U(\mathfrak{g}_{>0}[t] + \mathfrak{g}_{\geq 1}[t,t^{-1}])} \mathbb{C}_\chi$, where $\mathbb{C}_\chi$ is the one-dimensional representation of $\mathfrak{g}_{>0}[t] + \mathfrak{g}_{\geq 1}[t, t^{-1}]$ on which $\mathfrak{g}_{\geq 1}[t, t^{-1}]$ acts by the character χ and $\mathfrak{g}_{>0}[t]$ acts triviality. Then, for a $V^k(\mathfrak{g})$-module M,

$$H^{\bullet}_{DS,f}(M) = H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}_{>0}[t, t^{-1}], M \otimes F_\chi),$$

where $H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}_{>0}[t, t^{-1}], N)$ is the semi-infinite $\mathfrak{g}_{>0}[t, t^{-1}]$-cohomology Feĭgin [1984] with coefficient in a $\mathfrak{g}_{>0}[t, t^{-1}]$-module N. Since it is defined by a BRST cohomology, $\mathcal{W}^k(\mathfrak{g}, f)$ is naturally a vertex algebra, which is called *W-algebra associated with* $(\mathfrak{g}, f)$ *at level* k. By E. Frenkel and Ben-Zvi [2004] and V. G. Kac and Wakimoto [2004], we know that $H^i_{DS,f}(V^k(\mathfrak{g})) = 0$ for $i \neq 0$. If $f = 0$ we have by definition $\mathcal{W}^k(\mathfrak{g}, f) = V^k(\mathfrak{g})$. The W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is conformal provided that $k \neq -h^\vee$.

Let $\mathcal{S}_f = f + \mathfrak{g}^e \subset \mathfrak{g} \cong \mathfrak{g}^*$, the *Slodowy slice* at f, where $\mathfrak{g}^e$ denotes the centralizer of e in $\mathfrak{g}$. The affine variety $\mathcal{S}_f$ has a Poisson structure obtained from that of $\mathfrak{g}^*$ by Hamiltonian reduction (Gan and Ginzburg [2002]). We have

$$X_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathcal{S}_f, \quad \operatorname{Spec}(\operatorname{gr} \mathcal{W}^k(\mathfrak{g}, f)) \cong J_\infty \mathcal{S}_f \tag{8}$$

(De Sole and V. G. Kac [2006] and Arakawa [2015a]). Also, we have

$$\operatorname{Zhu}(\mathcal{W}^k(\mathfrak{g}, f)) \cong U(\mathfrak{g}, f)$$

(Arakawa [2007] and De Sole and V. G. Kac [2006]), where $U(\mathfrak{g}, f)$ is the *finite W-algebra* associated with $(\mathfrak{g}, f)$ (Premet [2002]). Therefore, the W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ can be regarded as an affinization of the finite W-algebra $U(\mathfrak{g}, f)$. The map (7) for $\mathcal{W}^k(\mathfrak{g}, f)$

is an isomorphism, which recovers the fact Premet [2002] and Gan and Ginzburg [2002] that $U(\mathfrak{g}, f)$ is a quantization of the Slodowy slice $\mathcal{S}_f$. The definition of $\mathcal{W}^k(\mathfrak{g}, f)$ naturally extends V. Kac, Roan, and Wakimoto [2003] to the case that $\mathfrak{g}$ is a (basic classical) Lie superalgebra and f is a nilpotent element in the even part of $\mathfrak{g}$.

We have $\mathcal{W}^k(\mathfrak{g}, f) \cong \mathcal{W}^k(\mathfrak{g}, f')$ if f and f' belong to the same nilpotent orbit of $\mathfrak{g}$. The W-algebra associated with a minimal nilpotent element f_{mim} and a principal nilpotent element f_{prin} are called a minimal W-algebra and a principal W-algebra, respectively. For $\mathfrak{g} = \mathfrak{sl}_2$, these two coincide and are isomorphic to the Virasoro vertex algebra of central charge $1-6(k-1)^2/(k+2)$ provided that $k \neq -2$. In V. Kac, Roan, and Wakimoto [ibid.] it was shown that almost every superconformal algebra appears as the minimal W-algebra $\mathcal{W}^k(\mathfrak{g}, f_{min})$ for some Lie superalgebra $\mathfrak{g}$, by describing the generators and the relations (OPEs) of minimal W-algebras. Except for some special cases, the presentation of $\mathcal{W}^k(\mathfrak{g}, f)$ by generators and relations is not known for other nilpotent elements.

Historically, the principal W-algebras were first extensively studied (see Bouwknegt and Schoutens [1995]). In the case that $\mathfrak{g} = \mathfrak{sl}_n$, the non-critical principal W-algebras is isomorphic to the Fateev-Lukyanov's W_n-algebra Fateev and Lykyanov [1988] (Feigin and E. Frenkel [1990] and E. Frenkel and Ben-Zvi [2004]). The critical principal W-algebra $\mathcal{W}^{-h^\vee}(\mathfrak{g}, f_{prin})$ is isomorphic to the Feigin-Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $\widehat{\mathfrak{g}}$, that is the center of the critical affine vertex algebra $V^{-h^\vee}(\mathfrak{g})$ (Feigin and E. Frenkel [1992]). For a general f, we have the isomorphism

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \cong Z(\mathcal{W}^{-h^\vee}(\mathfrak{g}, f)),$$

(Arakawa [2012a] and Arakawa and Moreau [2018a]), where $Z(\mathcal{W}^{-h^\vee}(\mathfrak{g}, f))$ denotes the center of $\mathcal{W}^{-h^\vee}(\mathfrak{g}, f)$. This fact has an application to *Vinberg's Problem* for the centralizer $\mathfrak{g}^e$ of e in $\mathfrak{g}$ Arakawa and Premet [2017].

4 Representation theory of W-algebras

The definition of $\mathcal{W}^k(\mathfrak{g}, f)$ by the quantized Drinfeld-Sokolov reduction gives rise to a functor

$$\begin{aligned} V^k(\mathfrak{g})\text{-Mod} &\to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod} \\ M &\mapsto H^0_{DS,f}(M). \end{aligned}$$

Let $\mathcal{O}_k$ be the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ at level k. Then $\mathcal{O}_k$ is naturally considered as a full subcategory of $V^k(\mathfrak{g})$-Mod. For a weight λ of $\widehat{\mathfrak{g}}$ of level k, let $L(\lambda)$ be the irreducible highest weight representations of $\widehat{\mathfrak{g}}$ with highest weight λ.

Theorem 1 (Arakawa [2005]). *Let $f_{min} \in \mathbb{O}_{min}$ and let k be an arbitrary complex number.*

i. *We have $H^i_{DS,f_{min}}(M) = 0$ for any $M \in \mathcal{O}_k$ and $i \in \mathbb{Z}\backslash\{0\}$. Therefore, the functor*

$$\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f_{min})\text{-Mod}, \quad M \mapsto H^0_{DS,f_{min}}(M)$$

is exact.

ii. *For a weight λ of $\widehat{\mathfrak{g}}$ of level k, $H^0_{DS,f_{min}}(L(\lambda))$ is zero or isomorphic to an irreducible highest weight representation of $\mathcal{W}^k(\mathfrak{g}, f_{min})$. Moreover, any irreducible highest weight representation the minimal W-algebra $\mathcal{W}^k(\mathfrak{g}, f_{min})$ arises in this way.*

By Theorem 1 and the Euler-Poincaré principal, the character $\operatorname{ch} H^0_{DS,f_\theta}(L(\lambda))$ is expressed in terms of $\operatorname{ch} L(\lambda)$. Since $\operatorname{ch} L(\lambda)$ is known Kashiwara and Tanisaki [2000] for all non-critical weight λ, Theorem 1 determines the character of all non-critical irreducible highest weight representation of $\mathcal{W}^k(\mathfrak{g}, f_{min})$. In the case that k is critical the character of irreducible highest weight representation of $\mathcal{W}^k(\mathfrak{g}, f_{min})$ is determined by the Lusztig-Feigin-Frenkel conjecture (Lusztig [1991], Arakawa and Fiebig [2012], and E. Frenkel and Gaitsgory [2009]).

Remark 2. Theorem 1 holds in the case that $\mathfrak{g}$ is a basic classical Lie superalgebra as well. In particular one obtains the character of irreducible highest weight representations of superconformal algebras that appear as $\mathcal{W}^k(\mathfrak{g}, f_{min})$ once the character of irreducible highest weight representations of $\widehat{\mathfrak{g}}$ is given.

Let KL_k be the full subcategory of $\mathcal{O}_k$ consisting of objects on which $\mathfrak{g}[t]$ acts locally finitely. Altough the functor

$$\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H^0_{DS,f}(M) \tag{9}$$

is not exact for a general nilpotent element f, we have the following result.

Theorem 3 (Arakawa [2015a]). *Let f, k be arbitrary. We have $H^i_{DS,f_{min}}(M) = 0$ for any $M \in \mathrm{KL}_k$ and $i \neq 0$. Therefore, the functor*

$$\mathrm{KL}_k \to \mathcal{W}^k(\mathfrak{g}, f_{min})\text{-Mod}, \quad M \mapsto H^0_{DS,f_{min}}(M)$$

is exact.

In the case that f is a principal nilpotent element, Theorem 3 has been proved in E. Frenkel and Gaitsgory [2010] using Theorem 4 below.

The restriction of the quantized Drinfeld-Sokolv reduction functor to KL_k does not produce all the irreducible highest weight representations of $\mathcal{W}^k(\mathfrak{g}, f)$. However, one can modify the functor (9) to the "$-$"-reduction functor $H^0_{-,f}(?)$ (defined in E. Frenkel, V. Kac, and Wakimoto [1992]) to obtain the following result for the principal W-algebras.

Theorem 4 (Arakawa [2007]). *Let f be a principal nilpotent element, and let k be an arbitrary complex number.*

i. *We have $H^i_{-,f_{prin}}(M) = 0$ for any $M \in \mathcal{O}_k$ and $i \in \mathbb{Z}\backslash\{0\}$. Therefore, the functor*

$$\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f_{prin})\text{-Mod}, \quad M \mapsto H^0_{-,f_{prin}}(M)$$

is exact.

ii. *For a weight λ of $\widehat{\mathfrak{g}}$ of level k, $H^0_{-,f_{prin}}(L(\lambda))$ is zero or isomorphic to an irreducible highest weight representation of $\mathcal{W}^k(\mathfrak{g}, f_{prin})$. Moreover, any irreducible highest weight representation the principal W-algebra $\mathcal{W}^k(\mathfrak{g}, f_{prin})$ arises in this way.*

In type A we can derive the similar result as Theorem 4 for any nilpotent element f using the work of Brundan and Kleshchev [2008] on the representation theory of finite W-algebras (Arakawa [2011]). In particular the character of all ordinary irreducible representations of $\mathcal{W}^k(\mathfrak{sl}_n, f)$ has been determined for a non-critical k.

5 BRST reduction of associated varieties

Let $\mathcal{W}_k(\mathfrak{g}, f)$ be the unique simple graded quotient of $\mathcal{W}^k(\mathfrak{g}, f)$. The associated variety $X_{\mathcal{W}_k(\mathfrak{g},f)}$ is a subvariety of $X_{\mathcal{W}^k(\mathfrak{g},f)} = \mathcal{S}_f$, which is invariant under the natural $\mathbb{C}^*$-action on $\mathcal{S}_f$ that contracts to the point $f \in \mathcal{S}_f$. Therefore $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse if and only if $X_{\mathcal{W}_k(\mathfrak{g},f)} = \{f\}$.

By Theorem 3, $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient of the vertex algebra $H^0_{DS,f}(L_k(\mathfrak{g}))$, provided that it is nonzero.

Theorem 5 (Arakawa [2015a]). *For any $f \in \mathfrak{g}$ and $k \in \mathbb{C}$ we have*

$$X_{H^0_{DS,f}(L_k(\mathfrak{g}))} \cong X_{L_k(\mathfrak{g})} \cap \mathcal{S}_f.$$

Therefore,

i. *$H^0_{DS,f}(L_k(\mathfrak{g})) \neq 0$ if and only if $\overline{G.f} \subset X_{L_k(\mathfrak{g})}$;*

ii. *If* $X_{L_k(\mathfrak{g})} = \overline{G.f}$ *then* $X_{H^0_{DS,f}(L_k(\mathfrak{g}))} = \{f\}$. *Hence* $H^0_{DS,f}(L_k(\mathfrak{g}))$ *is lisse, and thus, so is its quotient* $\mathcal{W}_k(\mathfrak{g}, f)$.

Theorem 5 can be regarded as a vertex algebra analogue of the corresponding result Losev [2011] and Ginzburg [2009] for finite W-algebras.

Note that if $L_k(\mathfrak{g})$ is integrable we have $H^0_{DS,f}(L_k(\mathfrak{g})) = 0$ by (6). Therefore we need to study more general representations of $\widehat{\mathfrak{g}}$ to obtain lisse W-algebras using Theorem 5.

Recall that the irreducible highest weight representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$ is called *admissible* V. G. Kac and Wakimoto [1989] (1) if λ is regular dominant, that is, $\langle \lambda + \rho, \alpha^\vee \rangle \notin -\mathbb{Z}_{\geq 0}$ for any $\alpha \in \Delta^{re}_+$, and (2) $\mathbb{Q}\Delta(\lambda) = \mathbb{Q}\Delta^{re}$. Here Δ^{re} is the set of real roots of $\widehat{\mathfrak{g}}$, Δ^{re}_+ the set of positive real roots of $\widehat{\mathfrak{g}}$, and $\Delta(\lambda) = \{\alpha \in \Delta^{re} \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$, the set of integral roots of λ. Admissible representations are (conjecturally all) modular invariant representations of $\widehat{\mathfrak{g}}$, that is, the characters of admissible representations are invariant under the natural action of $SL_2(\mathbb{Z})$ (V. G. Kac and Wakimoto [1988]). The simple affine vertex algebra $L_k(\mathfrak{g})$ is admissible as a $\widehat{\mathfrak{g}}$-module if and only if

$$(10) \qquad k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{N},\ (p,q) = 1,\ p \geq \begin{cases} h^\vee & \text{if } (q, r^\vee) = 1, \\ h & \text{if } (q, r^\vee) = r^\vee \end{cases}$$

(V. G. Kac and Wakimoto [2008]). Here h is the Coxeter number of $\mathfrak{g}$ and $r^\vee$ is the lacity of $\mathfrak{g}$. If this is the case k is called an *admissible number* for $\widehat{\mathfrak{g}}$ and $L_k(\mathfrak{g})$ is called an *admissible affine vertex algebra*.

Theorem 6 (Arakawa [2015a])**.** *Let* $L_k(\mathfrak{g})$ *be an admissible affine vertex algebra.*

i. *(Feigin-Frenkel conjectrue) We have* $X_{L_k(\mathfrak{g})} \subset \mathfrak{N}$, *the nilpotent cone of* $\mathfrak{g}$.

ii. *The variety* $X_{L_k(\mathfrak{g})}$ *is irreducible. That is, there exists a nilpotent orbit* $\mathbb{O}_k$ *of* $\mathfrak{g}$ *such that*

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k}.$$

iii. *More precisely, let* k *be an admissible number of the form* (10)*. Then*

$$X_{L_k(\mathfrak{g})} = \begin{cases} \{x \in \mathfrak{g} \mid (\operatorname{ad} x)^{2q} = 0\} & \text{if } (q, r^\vee) = 1, \\ \{x \in \mathfrak{g} \mid \pi_{\theta_s}(x)^{2q/r^\vee} = 0\} & \text{if } (q, r^\vee) = r^\vee, \end{cases}$$

where θ_s *is the highest short root of* $\mathfrak{g}$ *and* π_{θ_s} *is the irreducible finite-dimensional representation of* $\mathfrak{g}$ *with highest weight* θ_s.

From Theorem 5 and Theorem 6 we immediately obtain the following assertion, which was (essentially) conjectured by V. G. Kac and Wakimoto [2008].

Theorem 7 (Arakawa [2015a]). *Let $L_k(\mathfrak{g})$ be an admissible affine vertex algebra, and let $f \in \mathbb{O}_k$. Then the simple affine W-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse.*

In the case that $X_{L_k(\mathfrak{g})} = \overline{G.f_{prin}}$, the lisse W-algebras obtained in Theorem 7 is the *minimal series* principal W-algebras studied in E. Frenkel, V. Kac, and Wakimoto [1992]. In the case that $\mathfrak{g} = \mathfrak{sl}_2$, these are exactly the minimal series Virasoro vertex algebras (Feĭgin and Fuchs [1984], Beilinson, Belavin, V. Drinfeld, and et al. [2004], and Wang [1993]). The Frenkel-Kac-Wakimoto conjecture states that these minimal series principal W-algebras are rational. More generally, all the lisse W-algebras $\mathcal{W}_k(\mathfrak{g}, f)$ that appear in Theorem 7 are conjectured to be rational (V. G. Kac and Wakimoto [2008] and Arakawa [2015a]).

6 The rationality of minimal series principal W-algebras

An admissible affine vertex algebra $L_k(\mathfrak{g})$ is called *non-degenearte* (E. Frenkel, V. Kac, and Wakimoto [1992]) if

$$X_{L_k(\mathfrak{g})} = \mathfrak{N} = \overline{G.f_{prin}}.$$

If this is the case k is called a *non-degenerate admissible number* for $\widehat{\mathfrak{g}}$. By Theorem 6 (iii), "most" admissible affine vertex algebras are non-degenerate. More precisely, an admissible number k of the form (10) is non-degenerate if and only if

$$q \geq \begin{cases} h & \text{if } (q, r^\vee) = 1, \\ r^\vee {}^L h^\vee & \text{if } (q, r^\vee) = r^\vee \end{cases}$$

where ${}^L h^\vee$ is the dual Coxeter number of the Langlands dual Lie algebra ${}^L\mathfrak{g}$. For a non-degenerate admissible number k, the simple principal W-algebra $\mathcal{W}_k(\mathfrak{g}, f_{prin})$ is lisse by Theorem 7.

The following assertion settles the Frenkel-Kac-Waimoto conjecture E. Frenkel, V. Kac, and Wakimoto [ibid.] in full generality.

Theorem 8 (Arakawa [2015b]). *Let k be a non-degenerate admissible number. Then the simple principal W-algebra $\mathcal{W}_k(\mathfrak{g}, f_{prin})$ is rational.*

The proof of Theorem 8 based on Theorem 4, Theorem 7, and the following assertion on admissible affine vertex algebras, which was conjectured by Adamović and Milas [1995].

Theorem 9 (Arakawa [2016]). *Let $L_k(\mathfrak{g})$ be an admissible affine vertex algebra. Then $L_k(\mathfrak{g})$ is rational in the category $\mathcal{O}$, that is, any $L_k(\mathfrak{g})$-module that belongs to $\mathcal{O}$ is completely reducible.*

The following assertion, which has been widely believed since V. G. Kac and Wakimoto [1990] and E. Frenkel, V. Kac, and Wakimoto [1992], gives a yet another realization of minimal series principal W-algebras.

Theorem 10 (Arakawa, Creutzig, and Linshaw [2018]). *Let $\mathfrak{g}$ be simply laced. For an admissible affine vertex algebra $L_k(\mathfrak{g})$, the vertex algebra $(L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]}$ is isomorphic to a minimal series principal W-algebra. Conversely, any minimal series principal W-algebra associated with $\mathfrak{g}$ appears in this way.*

In the case that $\mathfrak{g} = \mathfrak{sl}_2$ and k is a non-negative integer, the statement of Theorem 10 is well-known as the GKO construction of the discrete series of the Virasoro vertex algebras Goddard, Kent, and Olive [1986]. Some partial results have been obtained previously in Arakawa, Lam, and Yamada [2017] and Arakawa and Jiang [2018]. From Theorem 10, it follows that the minimal series principal W-algebra $\mathcal{W}_{p/q-h^\vee}(\mathfrak{g}, f_{prin})$ of ADE type is unitary, that is, any simple $\mathcal{W}_{p/q-h^\vee}(\mathfrak{g}, f_{prin})$-module is unitary in the sense of Dong and Lin [2014], if and only if $|p-q| = 1$.

7 Four-dimensional $N = 2$ superconformal algebras, Higgs branch conjecture and the class $\mathcal{S}$ chiral algebras

In the study of four-dimensional $N = 2$ superconformal field theories in physics, Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees [2015] have constructed a remarkable map

$$\Phi : \{\text{4d } N = 2 \text{ SCFTs}\} \to \{\text{vertex algebras}\} \tag{11}$$

such that, among other things, the character of the vertex algebra $\Phi(\mathcal{T})$ coincides with the *Schur index* of the corresponding 4d $N = 2$ SCFT $\mathcal{T}$, which is an important invariant of the theory $\mathcal{T}$.

How do vertex algebras coming from 4d $N = 2$ SCFTs look like? According to Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees [ibid.], we have

$$c_{2d} = -12c_{4d},$$

where c_{4d} and c_{2d} are central charges of the 4d $N = 2$ SCFT and the corresponding vertex algebra, respectively. Since the central charge is positive for a unitary theory, this implies that the vertex algebras obtained by Φ are never unitary. In particular integrable affine vertex algebras never appear by this correspondence.

The main examples of vertex algebras considered in Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees [ibid.] are the affine vertex algebras $L_k(\mathfrak{g})$ of types D_4, F_4, E_6, E_7,

E_8 at level $k = -h^\vee/6-1$, which are non-rational, non-admissible affine vertex algebras studies in Arakawa and Moreau [2016]. One can find more examples in the literature, see e.g. Beem, Peelaers, Rastelli, and van Rees [2015], Xie, Yan, and Yau [2016], Córdova and Shao [2016], and Song, Xie, and Yan [2017].

Now, there is another important invariant of a 4d $N = 2$ SCFT $\mathcal{T}$, called the *Higgs branch*, which we denote by $Higgs_{\mathcal{T}}$. The Higgs branch $Higgs_{\mathcal{T}}$ is an affine algebraic variety that has the hyperKähler structure in its smooth part. In particular, $Higgs_{\mathcal{T}}$ is a (possibly singular) symplectic variety.

Let $\mathcal{T}$ be one of the 4d $N = 2$ SCFTs studied in Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees [2015] such that that $\Phi(\mathcal{T}) = L_k(\mathfrak{g})$ with $k = h^\vee/6-1$ for types D_4, F_4, E_6, E_7, E_8 as above. It is known that $Higgs_{\mathcal{T}} = \overline{\mathbb{O}_{min}}$, and this equals Arakawa and Moreau [2016] to the the associated variety $X_{\Phi(\mathcal{T})}$. It is expected that this is not just a coincidence.

Conjecture 11 (Beem and Rastelli [2017])**.** For a 4d $N = 2$ SCFT $\mathcal{T}$, we have

$$Higgs_{\mathcal{T}} = X_{\Phi(\mathcal{T})}.$$

So we are expected to recover the Higgs branch of a 4d $N = 2$ SCFT from the corresponding vertex algebra, which is a purely algebraic object.

We note that Conjecture 11 is a physical conjecture since the Higgs branch is not a mathematical defined object at the moment. The Schur index is not a mathematical defined object either. However, in view of (11) and Conjecture 11, one can try to define both Higgs branches and Schur indices of 4d $N = 2$ SCFTs using vertex algebras. We note that there is a close relationship between Higgs branches of 4d $N = 2$ SCFTs and *Coulomb branches* of three-dimensional $N = 4$ gauge theories whose mathematical definition has been given by Braverman, Finkelberg, and Nakajima [2016b].

Although Higgs branches are symplectic varieties, the associated variety X_V of a vertex algebra V is only a Poisson variety in general. A vertex algebra V is called *quasi-lisse* (Arakawa and Kawasetsu [n.d.]) if X_V has only finitely many symplectic leaves. If this is the case symplectic leaves in X_V are algebraic (Brown and Gordon [2003]). Clearly, lisse vertex algebras are quasi-lisse. The simple affine vertex algebra $L_k(\mathfrak{g})$ is quasi-lisse if and only if $X_{L_k(\mathfrak{g})} \subset \mathfrak{N}$. In particular, admissible affine vertex algebras are quasi-lisse. See Arakawa and Moreau [2016, 2017, 2018b] for more examples of quasi-lisse vertex algebras. Physical intuition expects that vertex algebras that come from 4d $N = 2$ SCFTs via the map Φ are quasi-lisse.

By extending Zhu's argument Zhu [1996] using a theorem of Etingof and Schelder Etingof and Schedler [2010], we obtain the following assertion.

Theorem 12 (Arakawa and Kawasetsu [n.d.])**.** *Let V be a quasi-lisse $\mathbb{Z}_{\geq 0}$-graded conformal vertex algebra such that $V_0 = \mathbb{C}$. Then there only finitely many simple ordinary*

V-modules. Moreover, for an ordinary V-module M, the character $\chi_M(q)$ satisfies a modular linear differential equation.

Since the space of solutions of a modular linear differential equation is invariant under the action of $SL_2(\mathbb{Z})$, Theorem 12 implies that a quasi-lisse vertex algebra possesses a certain modular invariance property, although we do not claim that the normalized characters of ordinary V-modules span the space of the solutions. Note that Theorem 12 implies that the Schur indices of 4d $N = 2$ SCFTs have some modular invariance property. This is something that has been conjectured by physicists (Beem and Rastelli [2017]).

There is a distinct class of four-dimensional $N = 2$ superconformal field theories called the *theory of class* $\mathcal{S}$ (Gaiotto [2012] and Gaiotto, Moore, and Neitzke [2013]), where $\mathcal{S}$ stands for 6. The vertex algebras obtained from the theory of class $\mathcal{S}$ is called the *chiral algebras of class* $\mathcal{S}$ (Beem, Peelaers, Rastelli, and van Rees [2015]). The Moore-Tachikawa conjecture Moore and Tachikawa [2012], which was recently proved in Braverman, Finkelberg, and Nakajima [2017], describes the Higgs branches of the theory of class $\mathcal{S}$ in terms of two-dimensional topological quantum field theories.

Let $\mathbb{V}$ be the category of vertex algebras, whose objects are semisimple groups, and $\mathrm{Hom}(G_1, G_2)$ is the isomorphism classes of conformal vertex algebras V with a vertex algebra homomorphism

$$V^{-h_1^\vee}(\mathfrak{g}_1) \otimes V^{-h_2^\vee}(\mathfrak{g}_2) \to V$$

such that the action of $\mathfrak{g}_1[t] \oplus \mathfrak{g}_2[t]$ on V is locally finite. Here $\mathfrak{g}_i = \mathrm{Lie}(G_i)$ and $h_i^\vee$ is the dual Coxeter number of $\mathfrak{g}_i$ in the case that $\mathfrak{g}_i$ is simple. If $\mathfrak{g}_i$ is not simple we understand $V^{-h_i^\vee}(\mathfrak{g}_i)$ to be the tensor product of the critical level universal affine vertex algebras corresponding to all simple components of $\mathfrak{g}_i$. The composition $V_1 \circ V_2$ of $V_1 \in \mathrm{Hom}(G_1, G_2)$ and $V_2 \in \mathrm{Hom}(G_1, G_2)$ is given by the relative semi-infinite cohomology

$$V_1 \circ V_2 = H^{\frac{\infty}{2}+\bullet}(\widehat{\mathfrak{g}}_2, \mathfrak{g}_2, V_1 \otimes V_2),$$

where $\widehat{\mathfrak{g}}_2$ denotes the direct sum of the affine Kac-Moody algebra associated with the simple components of $\mathfrak{g}_2$. By a result of Arkhipov and Gaitsgory [2002], one finds that the identity morphism id_G is the algebra $\mathfrak{D}_G^{ch}$ of *chiral differential operators* on G (Malikov, Schechtman, and Vaintrob [1999] and Beilinson and V. Drinfeld [2004]) at the critical level, whose associated variety is canonically isomorphic to T^*G.

The following theorem, which was conjectured in Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees [2015] (see Tachikawa [n.d.(a),(b)]) for mathematical expositions), describes the chiral algebras of class $\mathcal{S}$.

Theorem 13 (Arakawa [n.d.]). *Let $\mathbb{B}_2$ the category of 2-bordisms. There exists a unique monoidal functor $\eta_G : \mathbb{B}_2 \to \mathbb{V}$ which sends (1) the object S^1 to G, (2) the cylinder,* *which is the identity morphism* id_{S^1}*, to the identity morphism* $\mathrm{id}_G = \mathfrak{D}^{ch}_G$*, and (3) the cap to* $H^0_{DS,f_{prin}}(\mathfrak{D}^{ch}_G)$*. Moreover, we have* $X_{\eta_G(B)} \cong \eta^{BFN}_G(B)$ *for any 2-bordism* B*, where* η^{BFN}_G *is the functor form* $\mathbb{B}_2$ *to the category of symplectic variaties constructed in Braverman, Finkelberg, and Nakajima [2017].*

The last assertion of the above theorem confirms the Higgs branch conjecture for the theory of class $\mathcal{S}$.

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Received 2017-12-06.

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