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LITTLE DISKS OPERADS AND FEYNMAN DIAGRAMS

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Abstract

The little disks operads are classical objects in algebraic topology which have seen a wide range of applications in the past. For example they appear prominently in the Goodwillie-Weiss embedding calculus, which is a program to understand embedding spaces through algebraic properties of the little disks operads, and their action on the spaces of configurations of points (or disks) on manifolds. In this talk we review the recent understanding of the rational homotopy theory of the little disks operads, and how the resulting knowledge can be used to fulfil the promise of the Goodwillie-Weiss calculus, at least in the "simple" setting of long knot spaces and over the rationals. The derivations prominently use and are connected to graph complexes, introduced by Kontsevich and other authors.

1 Introduction

The little disks operads are collections of spaces $\mathrm{D}_n(r)$ of rectilinear embeddings of r little disks in the n-dimensional unit disk

$$\mathrm{D}_n(r) = \mathrm{Emb}^{rl}(\mathbb{D}_n^{\sqcup r}, \mathbb{D}_n).$$

Here rectilinear means that the embedding may rescale and translate the little disks, but not rotate or otherwise deform them. The operadic compositions are defined through the gluing of configurations of disks, with one configuration being

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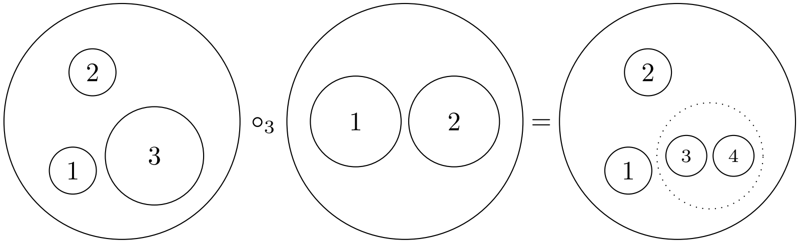
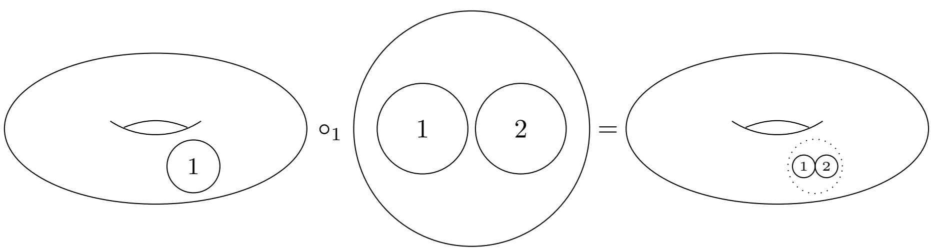
inserted in place of small disk as the following example illustrates.

(1)



The framed little n-disks operad D_n^{fr} is a variant in which one allows the embeddings to also rotate the little disks.

The little disks operads have seen various applications in algebra, topology, and even some branches of mathematical physics. We will focus on one particular relatively recent application here, the manifold calculus of Weiss and Goodwillie [1999] and Weiss [1999]. To this end let M be a manifold of dimension m. We may consider the spaces of embeddings of m-dimensional disks in M, $\mathrm{conf}_M(r) = \mathrm{Emb}(\mathbb{D}_m^r, M)$. Again by composition of embeddings (i.e., gluing of disks) the operad D_m (and likewise D_m^{fr}) naturally acts on the the collection of spaces conf_M.



The idea of the Goodwillie-Weiss manifold calculus is then that properties of the space M, and spaces derived from it, may be accessed using the spaces $\mathrm{conf}_M(r)$ and the action of $\mathrm{D}_n^{(fr)}$ upon them. In particular, for N another manifold of dimension n, and under the technical condition that $n \geq m+3$, one can express the space of embeddings from M into N as a derived mapping space between the right D_m^{fr}-modules conf_M and conf_N, see the work of Weiss and Boavida de Brito [2013],

$$\mathrm{Emb}(M,N) \simeq \mathrm{Map}^h_{mod-\mathrm{D}_m^{fr}}(\mathrm{conf}_M, \mathrm{conf}_N). \tag{2}$$

In other words, the manifold calculus replaces the complicated topological space of knottings of M in N on the left by a (potentially) accessible algebraic object on the right. One problem of this approach had been that the algebraic object on the right is still relatively complicated and hard to understand. However, due to recent progress in understanding the rational (or real) homotopy theory of the little disks operads and configuration spaces Campos and Willwacher [2016] and Idrissi [2016], information about the right-hand side may be obtained.

For this exposition we will in fact restrict to the simplest setting, when $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. In this case one studies the spaces of long knots $\mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n)$, i.e., of embeddings of $\mathbb{R}^m$ into $\mathbb{R}^n$ which agree with the standard embedding outside of a compact. For technical reasons one furthermore reduces to the homotopy fiber over immersions

$$\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) = \mathrm{hofiber}\left(\mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n) \to \mathrm{Imm}_\partial(\mathbb{R}^m, \mathbb{R}^n)\right).$$

The appropriate version of the embedding calculus for this situation then states that for $n - m \geq 3$ there is a weak equivalence (cf. Weiss and Boavida de Brito [2015], Ducoulombier and Turchin [2017], and Dwyer and Hess [2012])

$$\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \simeq \Omega^{m+1}\mathrm{Map}^h_{op}(\mathrm{D}_m, \mathrm{D}_n), \tag{3}$$

where on the right-hand side we have the $m+1$-fold loop space of the derived mapping space of operads between D_m and D_n. In particular, note that the Goodwillie-Weiss calculus states that the homotopy type of the space of (codimension ≥ 3-)knots is already fully encoded in the homological algebra of the relatively "simple" insertion operations (1).

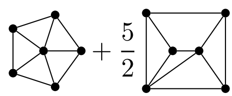
The final result we want to study is that the rational homotopy type of the right-hand side of (3) may be computed and expressed fully in terms of combinatorial data, through graph complexes and algebraic structures on graph complexes. The results we review here are mostly taken from joint works of the author Fresse and Willwacher [2015] and Fresse, Turchin, and Willwacher [2017b]. We also refer to these works for more technical details, which often have to be omitted from our exposition for reasons of brevity.

Notation and conventions. We generally work over the ground field $\mathbb{Q}$ unless otherwise stated, i.e., all vector spaces, commutative algebras etc. are considered over $\mathbb{Q}$. As usual we abbreviate the phrase "differential graded" by dg, and "differential graded commutative algebra" by dgca. We omit the prefix dg if clear from the context. For example "vector space" will typically mean dg vector space. We generally work in cohomological conventions, so that all of our differentials have degree +1. For a (dg) vector space V we denote by $V[k]$ the degree shifted vecor space. If $v \in V$ has degree d then the corresponding object in $V[k]$ has degree

$d-k$. For an introduction to the language of operads we refer the reader to the textbooks Loday and Vallette [2012] or Fresse [n.d.], whose notation we shall essentially follow. A standard reference for the little disks operads is May [1972, section 4].

2 Graph complexes

Graph complexes are differential graded vector spaces of linear combinations or series of combinatorial graphs. There are various versions depending on the type of graphs considered, for example complexes of undirected graphs, directed acyclic graphs, ribbon graphs etc. Here we will consider only the simplest version, as introduced by M. Kontsevich. We define GC_n to be the $\mathbb{Q}$-vector space of series of isomorphism classes of admissible graphs. Here an admissible graph is a connected undirected graph with an orientation, all of whose vertices have valence ≥ 2, and which does not have odd symmetries, a condition we shall elucidate shortly.

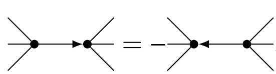


The definition depends on an integer n, which determines the cohomological degree of graphs, with a graph in GC_n being assigned degree

$$(\#\text{vertices}-1)n-(\#\text{edges})(n-1).$$

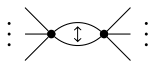
In other words, we consider the vertices as carrying degree n, and the edges as carrying degree $1-n$. An orientation on a graph Γ is the following data, depending on the parity of n:

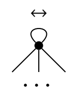
- For n even, an orientation of Γ is an ordering or of the set of edges of Γ. If two such orderings or, or$'$ differ by a permutation σ, we identify the oriented graphs up to sign
$$(\Gamma,\mathrm{or})=\mathrm{sgn}(\sigma)(\Gamma,\mathrm{or}').$$
- For n odd an orientation consists of an ordering of the set of vertices and half-edges. Again we identify two such orderings up to sign. Note that providing an ordering of the set of half-edges up to sign is equivalent to providing a direction on edges, identifying directions up to sign



The presence of the orientation implies that graphs with orientation reversing (odd) symmetries yield zero vectors in the graph complex. More concretely, for n even and odd symmetry of a graph is an automorphism inducing an odd permutation on the set of edges. Similarly, for n odd an odd symmetry is an automorphism inducing an odd permutation on the set of half-edges and vertices.

Note that this in particular implies that for n even any graph with a double edge is considered zero, for it has an odd symmetry by swapping the edges in the double edge. Likewise, for n odd any graph with a tadpole (or short cycle) is zero due to the symmetry reversing the cycle.

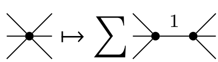


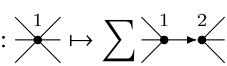


We define on GC_n a differential d, splitting vertices of graphs. More concretely

$$d\Gamma = \sum_v \mathit{split}(\Gamma, v)$$

with the operation $\mathit{split}(\Gamma, v)$ replacing the vertex v by two vertices connected by an edge and summing over all ways of reconnecting the edges incident at v to the two new vertices. Here the orientation on graphs in $\mathit{split}(\Gamma, v)$ is chosen so that for n even the newly created edge becomes the first in the ordering of edges. For n odd assume without loss of generality that v is the first vertex in the ordering. The orientation is chosen such that the newly created vertices are the first two in the ordering, with the newly created edge pointing from the first to the second.

n even : 

n odd : 

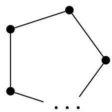
It is an easy exercise to check that with these conventions on the orientation $d^2 = 0$, so that we can consider the graph cohomology $H(\mathrm{GC}_n) = \ker d / \operatorname{im} d$. This cohomology is a somewhat mysterious object that can at present only partially be computed. Let us recall a few known facts.

First note that the differential cannot alter the loop order of a graph, and hence the graph complex decomposes into a direct product of subcomplexes of fixed loop order $\mathrm{GC}_n^{k\text{-loop}}$. Furthermore, GC_n depends on n essentially only up to parity, and hence one can see that

$$H^j(\mathrm{GC}_{n+2}^{k\text{-loop}}) \cong H^{j+2k}(\mathrm{GC}_n^{k\text{-loop}}).$$

In particular, knowing $H(\mathrm{GC}_n)$ for one even and one odd n suffices to determine $H(\mathrm{GC}_n)$ for all n. On the other hand, despite considerable effort, the author has not found any relation between $H(\mathrm{GC}_n)$ and $H(\mathrm{GC}_{n+1})$ that would allow for the computation of one through the other.

In low loop orders the graph cohomology can be computed explicitly by hand or with the help of computers. For example, in loop order 1 the most general graph has the form

(4) $L_k =$  (k vertices and k edges, deg$=n-k$).

These graphs have odd symmetries, and hence vanish as elements of the graph complex, unless $k \equiv 2n+1 \mod 4$, so that

$$H(\mathrm{GC}_n^{1\text{-loop}}) \cong \prod_{\substack{k\geq 1 \\ k\equiv 2n+1 \mod 4}} \mathbb{Q}W_k, \tag{5}$$

with the class W_k living in cohomological degree $k-n$.

For loop order $k \geq 2$ one can show that the inclusion of the subcomplex

$$\mathrm{GC}_n^{k\text{-loop},\geq 3} \subset \mathrm{GC}_n^{k\text{-loop}}$$

spanned by graphs all of whose vertices are at least trivalent is a quasi-isomorphism, cf. Kontsevich [1993] and Willwacher [2015b]. In this subcomplex, a graph with v vertices must then have at least $\frac{3}{2}v$ edges, and hence one can derive the simple upper degree bound for $k \geq 2$

$$H^{>-k(n-3)-3}(\mathrm{GC}_n^{k\text{-loop}}) = 0. \tag{6}$$

By somewhat different methods one can also derive a lower degree bound (cf. Willwacher [2015b])

$$H^{<-k(n-2)-1}(\mathrm{GC}_n^{k\text{-loop}}) = 0.$$

Next, we shall note that GC_n carries the structure of a dg Lie algebra. The Lie bracket is defined combinatorially by inserting a graph in vertices of another.

$$[\gamma, \nu] = \gamma \bullet \nu - (-1)^{|\gamma||\nu|}\nu \bullet \gamma$$

with

$$\gamma \bullet \nu := \sum_{x\in V\gamma} \gamma(\text{insert } \nu \text{ in place of } x)$$

In the case of $n = 2$ the author showed the following.

Theorem 1 (Willwacher [ibid.]). The zeroth cohomology $H^0(\mathrm{GC}_2)$ can be identified with the (completed) Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$. Furthermore $H^1(\mathrm{GC}_2) \cong \mathbb{K}$ and $H^{\leq 1}(\mathrm{GC}_2) = 0$.

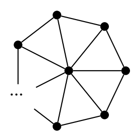
We will not recall the somewhat technical definition of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ defined by Drinfeld [1990], but rather recall the following important result of Francis Brown.

Theorem 2 (Brown [2012]). There is an injective Lie algebra morphism

$$F_{Lie}(\sigma_3, \sigma_5, \sigma_7, \dots) \to \mathfrak{grt}_1$$

from the complete free Lie algebra in generators $\sigma_3, \sigma_5, \dots$.

Both results together yield an infinite family of nontrivial graph cohomology classes. One can provide explicit integral formulas for the graph cocycles representing σ_{2j+1} as in "P. Etingof's conjecture about Drinfeld associators" [2014]. Concretely, σ_{2k+1} is represented by a linear combination of diagrams of loop order $2k+1$

$\sigma_{2k+1} \simeq$  $+ (\cdots),$

where the first "wheel" graph has $2k+2$ vertices, and the terms (...) on the right which are not explicitly written are linear combinations of graphs all of whose vertices have valence $\leq 2k$.

We shall not recall here in detail the known results for the graph cohomology $H(\mathrm{GC}_3)$, or equivalently $H(\mathrm{GC}_n)$ for n odd. Let us just mention that a large family of non-trivial cohomology classes in top degree is known through Chern-Simons theory. Furthermore there are conjectures regarding the precise shape of the top degree cohomology, see Vogel [2011] and Kneissler [2000, 2001a,b] for details.

Computer generated tables of the numbers $\dim H^j(\mathrm{GC}_n^{k\text{-loop}})$ can be found in Figure 1. The red lines depict the degree bounds beyond which the cohomology is zero. The cohomology classes giving rise to the numbers appearing in the (computer accessible portion of) the tables can be "explained", see Khoroshkin, Willwacher, and Živković [2017] for more details. However, at large we still do not know what $H(\mathrm{GC}_n)$ is, and in particular which entries of the table are zero. To this end, let us just note a famous vanishing conjecture which goes back to Kontsevich, and in a similar form to Drinfeld.

Conjecture 3. $H^1(\mathrm{GC}_2) = 0$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
7	1	0	0	0	0	0	0	0	0	1				
6	0	0	0	0	0	0	0	0	0	0				
5	0	0	0	0	0	0	0	0	0	0				
4	0	0	0	0	0	0	0	0	0	0				
3	1	0	0	0	0	1	0	1	1	2				
2	0	0	0	0	0	0	0	0	0	0				
1	0	0	0	0	0	0	0	0	0	0	0			
0	0	0	1	0	1	0	1	1	1	1	2	2	3	
-1	1	0	0	0	0	0	0	0	0	0	0	0	0	

	1	2	3	4	5	6	7	8	9	10	11	12
8	1											
4	1											
0	1											
-2	0	0	0	0	0	0	0	0	0	0	0	0
-3	0	1	1	1	2	2	3	4	5	6	8	9
-4	0	0	0	0	0	0	0	0				
-5	0	0	0	0	0	0	0	0				
-6	0	0	0	0	0	1	1	2				
-7	0	0	0	0	0	0	0	0				
-8	0	0	0	0	0	0	0	0	0			

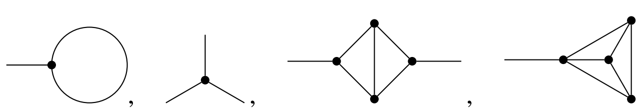
Figure 1: Computer generated tables of $\dim H^j(\mathrm{GC}_2^{k\text{-loop}})$ (top) and $\dim H^j(\mathrm{GC}_3^{k\text{-loop}})$ (bottom), with j being the row index and the loop order k the column index. All entries above the upper and below the lower red line are zero. Entries between the red lines which are not shown have not been computed.

	Even	Odd		Even	Odd
loop order	$\tilde{\chi}_b^{even}$	$\tilde{\chi}_b^{odd}$	loop order	$\tilde{\chi}_b^{even}$	$\tilde{\chi}_b^{odd}$
1	0	1	16	-3	6
2	1	1	17	-1	4
3	0	1	18	8	-5
4	1	2	19	12	-14
5	-1	1	20	27	-21
6	1	2	21	14	-11
7	0	2	22	-25	21
8	0	2	23	-39	44
9	-2	1	24	-496	504
10	1	3	25	-2979	2969
11	0	1	26	-412	413
12	0	3	27	38725	-38717
13	-2	4	28	10583	-10578
14	0	2	29	-667610	667596
15	-4	2	30	28305	-28290

Figure 2: Table of the Euler characteristics of the graph complexes GC_n for even and odd n from Willwacher and Živković [2015]. Note that for high loop orders the Euler characteristics for the even and odd complexes are astonishingly similar, with the sign difference being due to conventions.

As a final remark we shall mention that while the above tables suggest that the cohomology of the graph complexes $H(\mathrm{GC}_n)$ for even and odd n is very different. However, the Euler characteristic computations of GC_n from Willwacher and Živković [2015], which we reproduce in Figure 2, show that at least the Euler characteristics of both complexes in high loop orders are strikingly similar. The author can currently not explain this fact.

2.1 A variant with external legs. We will also need a slight variant of the above graph complexes. We may consider a complex of graphs $\mathrm{HGC}_{m,n}$ built using graphs with "external legs" or hairs, as shown in the following pictures



We require that all non-hair vertices are at least trivalent. The cohomological degree of a graph is determined by the formula

$$n(\#\text{internal vertices}) - (n-1)(\#\text{edges}) + m(\#\text{internal vertices} - 1).$$

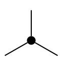
Here we count the edges being part of a hair as edges as well. One also equips these hairy graphs with an orientation. Graphs that possess odd symmetries are hence considered zero in the graph complex. The differential is again given by splitting (non-hair) vertices. This operation cannot change the number of hairs, nor the first Betti number (i.e„ the number of loops) of a graph, Hence the complex $\mathrm{HGC}_{m,n}$ splits into a direct product of subcomplexes of fixed number of hairs and loops

$$\mathrm{HGC}_{m,n}^{k\text{-loop},h\text{-hair}} \subset \mathrm{HGC}_{m,n}.$$

In general, the cohomology of these complexes is not known, but at least one has partial information. In low loop orders, or for low numbers of hairs one can explicitly compute the cohomology. For example, in loop order zero the cohomology is precisely one dimensional, and represented by the following graph cocycles:

(7)

 for $n-m$ even, in cohomological degree $m-n+1$

 for $n-m$ odd, in cohomological degree $2(m-n)+3$

The computation of the loop order one cohomology we leave to the reader. The computation in loop order two can be found in Conant, Costello, Turchin, and Weed [2014].

Next, using that non-hair vertices can be required to be at least trivalent one can easily derive the upper degree bounds

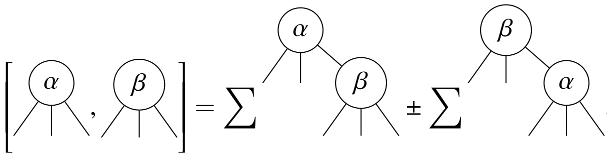
$$H^{>-k(n-3)-(h-1)(n-m-2)-1}(\mathrm{HGC}_{m,n}^{k\text{-loop},h\text{-hair}}) = 0. \tag{8}$$

For example, we will use below that for $n-3 \geq m$ the only non-trivial classes in degree $-m$ live in loop order $k=0$ and are hence found in the list (7).

By other methods (see Arone and Turchin [2014] and Willwacher [2015b]) one can also obtain the lower degree bound

$$H^{<-k(n-2)-(h-1)(n-m-1)}(\mathrm{HGC}_{m,n}^{k\text{-loop},h\text{-hair}}) = 0.$$

Furthermore, the complexes $\mathrm{HGC}_{m,n}$ are also dg Lie algebras. The Lie bracket of two hairy graphs is computed by connecting a hair of one graph to vertices of the other as indicated in the following picture:



For further information about the hairy graph cohomology $H(\mathrm{HGC}_{m,n})$, we refer the reader to Arone and Turchin [2015] containing a computation of the Euler characteristic, to Khoroshkin, Willwacher, and Živković [2015] containing a construction of infinite series of nontrivial classes and numerical results, or to Fresse, Turchin, and Willwacher [2017b] for general information.

3 The little disks operads

3.1 Cohomology of the little disks operad. The cohomology of the little disks operads D_n has been computed by Arnold (for $n = 2$) and F. Cohen (for all n).

Theorem 4 (Arnold [1969] and Cohen [1976]). *For $n \geq 2$ and $r \geq 1$ the cohomology algebra of the space $\mathrm{D}_n(r)$ has the presentation*

$$H(\mathrm{D}_n(r)) = \mathbb{Q}[\omega_{ij} \mid 1 \leq i \neq j \leq n]/\langle R \rangle,$$

where ω_{ij} are generators of degree $n-1$ and the relations R read

$$\omega_{ij} = (-1)^n \omega_{ji} \qquad \omega_{ij}^2 = 0 \qquad \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$$

From the operad structure on D_n the collection of graded commutative algebras $H(\mathrm{D}_n)$ receives a cooperad structure. Generally, we will call a cooperad in the category of dg commutative algebras a Hopf cooperad, so that in particular $H(\mathrm{D}_n)$ is a Hopf cooperad.

To understand the operad structure it is slightly easier to consider the dual operad $H_\bullet(\mathrm{D}_n)$. For $n = 1$ this is just the associative operad. For $n \geq 2$ it can be identified with the n-Poisson operad Pois_n, which generated by a commutative product operation $\wedge$ of degree 0 and a Lie bracket $[,]$ of degree $1-n$ satisfying the compatibility relation

$$[x_1, x_2 \wedge x_3] = [x_1, x_2] \wedge x_3 + x_2 \wedge [x_1, x_3].$$

3.2 Rational homotopy theory of operads. Rational homotopy theory is the study of rational homotopy types of spaces. In Sullivan's approach, the main ingredient is a Quillen adjunction between the categories of simplicial sets (which we shall think of as topological spaces) and of dg commutative algebras

$$\Omega : \mathrm{sSet} \leftrightarrows \mathrm{dgca}^{op} : G.$$

Sullivan's functor Ω sends a simplicial set X to the dg commutative algebra of piecewise polynomial differential forms

$$\Omega(X) := \mathrm{Hom}_{\mathrm{sSet}}(X, \Omega_{poly}(\Delta^\bullet)).$$

We readily extend the definition to topological spaces instead of simplicial sets via the singular simplicial complexes functor $\mathrm{Sing}_\bullet$. We shall quietly abuse the notation and write, for a topological space X,

$$\Omega(X) := \Omega(\mathrm{Sing}_\bullet X).$$

When X is a manifold $\Omega(X) \otimes_{\mathbb{Q}} \mathbb{R}$ is weakly equivalent to the dg commutative algebra of de Rham differential forms on X. For our purposes, rational homotopy theory can be seen as the study of the quasi-isomorphism type of the dg commutative algebra $\Omega(X)$.

Let us next consider a topological operad $\mathcal{T}$. To study its rational homotopy type we would like to apply the functor Ω and study the resulting cooperad object in dg commutative algebras. Unfortunately, due to incompatible monoidality properties of the functor Ω the collection $\Omega(\mathcal{T})$ is not naturally a cooperad. More concretely, the problem here is that one has a natural quasi-isomorphism $\Omega(X) \otimes \Omega(Y) \to \Omega(X \times Y)$, but no natural morphism in the other direction. There are essentially three known approaches to work around this technical problem, by (i) using operads up to homotopy or (ii) changing the functor Ω or (iii) to use completed tensor products in the smooth setting.

While all three approaches have been used in the literature, we use here approach (ii). We shall follow Fresse's rational homotopy theory for operads, see Fresse [n.d.] and Fresse and Willwacher [2015, section 0], which we briefly outline. For brevity we call a cooperad in dg commutative algebras a Hopf cooperad. Fresse constructs a Quillen adjunction

$$\Omega_\sharp : \mathrm{sSet\text{-}Op} \leftrightarrows \mathrm{Hopf\text{-}Op}^c : G$$

between the model categories of reduced operads in simplicial sets, and that of dg Hopf cooperads. The functor $\Omega_\sharp$ here is defined as left adjoint of the realization functor G and shall be seen as an operadic upgrade of Sullivan's functor Ω. In each arity, $\Omega_\sharp$ is weakly equivalent to Ω.

For our purposes, studying the rational homotopy type of a topological operad $\mathcal{T}$ amounts to studying the quasi-isomorphism class of the dg Hopf cooperad $\Omega_\sharp(\mathcal{T})$, where we again quietly extend $\Omega_\sharp$ to topological spaces instead of simplicial sets, taking singular simplices. Furthermore, if $\mathcal{S}$ and $\mathcal{T}$ are simplicial (or topological) operads, then one can use the Quillen adjunction above to compute

$$\mathrm{Map}_{\mathrm{sSet\text{-}Op}}(\mathcal{S}, \mathcal{T}^{\mathbb{Q}}) = \mathrm{Map}_{\mathrm{sSet\text{-}Op}}(\mathcal{S}, G(\Omega_\sharp(\mathcal{T}))) \simeq \mathrm{Map}_{HopfOp^c}(\Omega_\sharp(\mathcal{T}), \Omega_\sharp(\mathcal{S})).$$

We finally note that Fresse's framework has the downside that it requires our operads to be reduced, i.e., that $\mathcal{T}(0) = \mathcal{T}(1) = *$ is a point. Unfortunately the little disks operads introduced above are not reduced, since $\mathrm{D}_n(1)$ is not a point, only contractible. However, there are homotopy equivalent variants of D_n, for example the Fulton-MacPherson operad FM_n Getzler and Jones [1994], which are reduced. Generally, an E_n operad is a topological operad weakly equivalent to D_n. In the following we will abuse the notation a bit and denote by E_n some chosen reduced operad weakly equivalent to D_n. Furthermore, for technical reasons the arity zero operations in $\mathcal{T}(0) = *$ are encoded in a Λ-structure instead of considering them as operations in the operad. A Λ-structure is the collection of all possible composition maps with nullary operations $\mathcal{T}(r+s) \to \mathcal{T}(r)$, which are required to satisfy natural compatibility relations. For simplicity of notation we will hide this further technical complication and do not mark the presence of the Λ-structure in our notation. We refer to Fresse's book Fresse [n.d.] for details. It is shown in Fresse, Turchin, and Willwacher [2017a] that the mapping spaces computed in the category of reduced operads are weakly equivalent to those computed in the full category of operads, thus justifying our restriction to the reduced setting.

3.3 Formality and intrinsic formality of E_n operads. Today the rational homotopy types of the little disks operads are fully understood through the following Theorem.

Theorem 5 (Formality Theorem for the E_n operads). The dg Hopf cooperads $\Omega_\sharp(E_n)$ are formal, i.e., connected by a chain of weak equivalences to the cohomology cooperad $\mathrm{e}_n^c = H(E_n)$.

The Theorem has the following history. The formality of E_2 was first shown by Tamarkin [2003]. The statement for higher n was first established by Kontsevich [1999], albeit over the ground ring $\mathbb{R}$. It has then been noted in Guillén Santos, Navarro, Pascual, and Roig [2005] that Kontsevich's statement can be improved to yield formality over $\mathbb{Q}$, provided one disregards the arity zero operations in the cooperads. Finally, the remaining statement of formality over $\mathbb{Q}$ with zero-ary operations was shown by Fresse and the author in Fresse and Willwacher [2015]. Also, surprisingly, we were able to show a significantly stronger result. A dg Hopf cooperad $\mathfrak{C}$ is called intrinsically formal if any dg Hopf cooperad $\mathfrak{D}$ with $H(\mathfrak{D}) \simeq H(\mathfrak{C})$ is weakly equivalent to $\mathfrak{C}$. One then has:

Theorem 6 (Intrinsic formality for the little diks operadsFresse and Willwacher [ibid.]). The dg Hopf cooperad $\Omega_\sharp(E_n)$ is intrinsically formal for $n \geq 3$ and n not divisible by 4. If $n \geq 3$ is divisible by 4, one does not have intrinsic formality, but the following statement is retained: Suppose that $\mathfrak{D}$ is another dg Hopf cooperad such that $H(D) \cong H(E_n)$. Suppose further that $I : \mathfrak{D} \to \mathfrak{D}$ is an involution which agrees on cohomology with the canonical involution of E_n by mirror reflection along a coordinate axis. Then $\mathfrak{D} \simeq H(E_n)$.

For $n = 2$ the analogous statement is an open conjecture.

Conjecture 7. The little 2-disks operad is rationally intrinsically formal.

In fact, this conjecture would follow from the vanishing Conjecture 3 on the graph cohomology above.

To conclude this subsection let us also remark on several closely connected formality questions for the little disks operads. First we note that the little disks operads come with natural maps $\mathrm{D}_m \to \mathrm{D}_n$. One may ask what the rational (or real) homotopy type of these maps are. The question has been answered in works of Turchin and the author Turchin and Willwacher [2014] and Fresse and the author Fresse and Willwacher [2015], improving upon earlier results by Lambrechts and Volić Loday and Vallette [2012].

Theorem 8 (Fresse and Willwacher [2015], Turchin and Willwacher [2014], and Loday and Vallette [2012]). Let $n \geq m \geq 1$. Then the map $E_m \to E_n$ is rationally formal for $n - m \neq 1$, and not formal (even over $\mathbb{R}$) for $n - m = 1$.

Furthermore the real homotopy type of the map $E_{n-1} \to E_n$ can be fully described.

In a different direction, note that the group $O(n)$ naturally acts on the little n-disks operad $\mathrm{D}(n)$. One can hence ask whether the operad $\mathrm{D}(n)$ is $O(n)$-equivariantly formal or not. This is equivalent to asking whether the framed little disks operads are formal or not. This latter question has also been answered by now: The framed little 2-disks operad D_2^{fr} was shown to be formal by Giansiracusa and Salvatore [2010] (over $\mathbb{R}$) and independently by Ševera [2010] (over $\mathbb{Q}$).

Furthermore, the formality (over $\mathbb{R}$) of D_n^{fr} for n is even is shown in Khoroshkin and Willwacher [2017]. For $n \geq 3$ odd the operad D_n^{fr} is not formal, but explicit "small" models capturing the real homotopy type can be found Khoroshkin and Willwacher [2017] and Moriya [2017]

At present, the (arguably) most important remaining open problem regarding the formality of the little disks operads is the following.

Open Problem 9. Determine the $O(m) \times O(n-m)$-equivariant rational homotopy type of the Map $D_m \to D_n$.

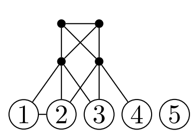
This homotopy type appears for example in connection to the Goodwillie-Weiss manifold calculus, see Section 5 below.

3.4 Kontsevich's graphical model for E_n.

The intrinsic formality statement (Theorem 6 above) can be shown by an analysis of the graph complex GC_n discussed in Section 2. The object that builds the bridge are the graphical models Graphs_n for the E_n opeads introduced by Kontsevich [1999]. More concretely, one defines a collection of dg commutative algebras $\mathrm{Graphs}_n(r)$ as follows. The space $\mathrm{Graphs}_n(r)$ is the space of linear combinations of isomorphism classes of graphs of the following type:

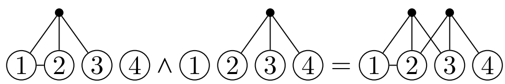
- The graph is an undirected graph with r numbered "external" vertices, and an arbitrary (but finite) number of "internal" vertices.
- All internal vertices have valence at least 2.
- Every connected component contains at least one external vertex.
- Graphs are equipped with an orientation, as in Section 2, and we identify orientations up to sign. More concretely, for n even an orientation is an ordering of the edges, while for n odd an orientation is an odering of the set of half-edges and vertices. As discussed before, the presence of the orientation renders graphs with odd symmetries zero

The following is an examples of a graph in $\mathrm{Graphs}_n(5)$.



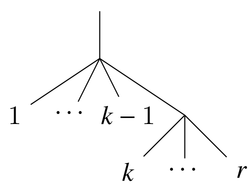
The space $\mathrm{Graphs}_n(r)$ is equipped with a differential by edge contraction, schematically:

Furthermore, we have a commutative product by gluing two graphs along the external vertices.



It is clear that $\mathrm{Graphs}_n(r)$ is free as a graded commutative algebra, generated by graphs that are internally connected, i.e., connected after removal of the external vertices.

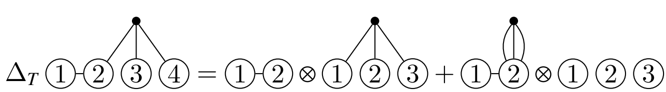
Furthermore there is cooperad structure on the collection Graphs_n, with the coproduct being the contraction of subgraphs. More precisely, for the generating cocomposition Δ_T corresponding to a tree



we have that

$$\Delta_T(\Gamma) = \sum_\gamma \Gamma/\gamma \otimes \gamma, \tag{9}$$

where the sum is over all subgraphs $\gamma \subset \Gamma$ containing the external vertices $k, \dots, r$ and no other external vertices, and the graph Γ/γ is obtained by contracting γ to one external vertex. Here is an example for $r = 4$ and $k = 2$:



In this formulas, and more generally (9) there is a natural way to define an orientation on the graphs on the right-hand side of the equation, given an orientation on the left-hand side, thus fixing the signs.

Finally, we note that for $n \geq 2$ we have a map of dg Hopf cooperads

$$\mathrm{Graphs}_n \to H(\mathrm{D}_n) = \mathrm{e}_n^c,$$

which is defined by sending any graph with internal vertices to zero, and sending a graph Γ without internal vertices and edges $(i_1, j_1), \dots, (i_k, j_k)$ to

$$\omega_{i_1 j_1} \cdots \omega_{i_k j_k},$$

cf. Theorem 4. Kontsevich and Lambrechts-Volić have then shown:

Proposition 10 (Kontsevich [1999] and Loday and Vallette [2012]). The map of dg Hopf cooperads $\mathrm{Graphs}_n \to \mathrm{e}_n^c$ above is a quasi-isomorphism.

The advantage of the dg Hopf cooperad Graphs_n over the smaller weakly equivalent cooperad e_n^c is that it has a large group of automorphism. More precisely we may consider the dg Lie algebra

$$\mathbb{Q}L \ltimes \mathrm{GC}_n, \tag{10}$$

where the generator L on the left acts on a graph by multiplication with the loop order.

$$[L, \gamma] = (\#\mathrm{loops}) \cdot \gamma$$

The statement is now that the dg Lie algebra (10) acts on the Hopf cooperad Graphs_n by biderivations, i.e., compatibly with both the cooperad and the dg commutative algebra structures. More concretely, the action of the generator L on a graph $\Gamma \in \mathrm{Graphs}_n$ is by

$$L \cdot \Gamma = (\#(\mathrm{edges}) - \#(\mathrm{internal\ vertices}))\Gamma.$$

The action of $\gamma \in \mathrm{GC}_n$ on $\Gamma \in \mathrm{Graphs}_n$ can be defined combinatorially as the contraction of subgraphs of shape γ in Γ, see Willwacher [2015b] for more details and the explicit formula.

3.5 A sketch of a proof of Theorem 6. As an illustration of the connection between the graph complexes GC_n of Section 2 we sketch here a proof of our intrinsic formality theorem (Theorem 6 above). We deviate slightly from the original reference Fresse and Willwacher [2015], where a different approach was used, using Bousfield's obstruction theory. The proof proceeds along the following sequence of steps.

1. As for most algebraic objects, we can define a deformation complex $\mathrm{Def}(\mathfrak{C},\mathfrak{D})$ for Hopf cooperads $\mathfrak{C}$ and $\mathfrak{D}$, governing maps between $\mathfrak{C}$ and $\mathfrak{D}$. More concretely, $\mathrm{Def}(\mathfrak{C},\mathfrak{D})$ is a dg Lie or L_∞-algebra, whose Maurer-Cartan elements correspond to maps from a cofibrant replacement of $\mathfrak{C}$ to a fibrant replacement of $\mathfrak{D}$. In particular $H^2(\mathrm{Def}(\mathfrak{C},\mathfrak{D}))$ is a space of potential obstructions to constructing such maps. Furthermore, given a Hopf operad map $f:\mathfrak{C}\to\mathfrak{D}$ and the corresponding Maurer-Cartan element α_f, the twisted dg Lie algebra $\mathrm{Def}(\mathfrak{C}\xrightarrow{f}\mathfrak{D}) := \mathrm{Def}(\mathfrak{C},\mathfrak{D})^{\alpha_f}$ governs deformations of the map f. We refer to Fresse, Turchin, and Willwacher [2017b, sections 3, 5] for details.

   We are interested in particular in $\mathrm{Def}(\mathrm{e}_n^c) := \mathrm{Def}(\mathrm{e}_n^c \xrightarrow{\mathrm{id}} \mathrm{e}_n^c)$, governing automorphisms of $\mathrm{e}_n^c = H(\mathrm{D}_n)$. If $\mathfrak{D}$ is a Hopf cooperad with $H(\mathfrak{D}) = \mathrm{e}_n^c$, then $H^2(\mathrm{Def}(\mathrm{e}_n^c))$ appears as a space of potential obstructions of lifting the cohomology map $\mathrm{e}_n^c \xrightarrow{=} H(\mathfrak{C})$ to a weak equivalence of Hopf cooperads between e_n^c and $\mathfrak{C}$. Our goal henceforth is to understand the space $H^2(\mathrm{Def}(\mathrm{e}_n^c))$, obstructing the intrinsic formality of the little n-disks operad.

2. If $\mathfrak{g}$ is a dg Lie algebra acting on e_n^c, or a quasi-isomorphic object, we obtain a map $H(\mathfrak{g}) \to H(\mathrm{Def}(\mathrm{e}_n^c))[1]$. It turns out that in the case of $\mathfrak{g} = \mathbb{Q}L \ltimes \mathrm{GC}_n$ acting on $\mathrm{Graphs}_n \simeq \mathrm{e}_n^c$ as described above the resulting map

$$\mathbb{Q}L \oplus H(\mathrm{GC}_n) \to H(\mathrm{Def}(\mathrm{e}_n^c))[1]$$

   is an isomorphism. This means in particular that the space of (potential) obstructions to intrinsic formality is given precisely by $H^1(\mathrm{GC}_n)$.

3. By the degree counting result (6) we hence see that for $n \geq 3$ the only possible obstructions are given by the graph cohomology classes represented by multiples of the loop graphs (4) appearing in (5). These graphs live in degree 1 only if n is divisible by 4. The intrinsic formality statment hence follows for $n \geq 3$ not divisible by 4.

4. Suppose next that $n \geq 3$ is divisible by 4. One can check that the $O(n)$ action on E_n is such that conjugation with the involution $S \in O(n)$ flipping the sign of one of the coordinates amounts to a multiplication of graphs in GC_n by $(-1)^{\#\mathrm{loops}}$. One can hence conclude that under the additional requirement of the presence of an involution on the operad as in Theorem 6, the relevant obstructions to intrinsic formality lie in the even loop order part $H(\mathrm{GC}_n)^{\mathbb{Z}_2}$. Hence the one-loop graphs cannot contribute.

By a similar analysis one can also make statements about the homotopy automorphisms of e_n^c for $n \geq 3$. Infinitesimally, they are governed by $\mathbb{Q}L \ltimes H^0(\mathrm{GC}_n)$. The piece $\mathbb{Q}L$ corresponds to the "trivial" automorphisms $S_\lambda : e_n^c \to e_n^c$ which just rescale the Lie cobracket by the factor $\lambda \in \mathbb{Q}^\times$. By invoking again the degree counting result 6 we see that (for $n \geq 3$) $H^0(\mathrm{GC}_n) = 0$ if $n \neq 3 \mod 4$ and otherwise $H^0(\mathrm{GC}_n)$ is one-dimensional, spanned by L_n. Hence one can say that for $n \geq 3$ the rationalization of the little n-disks operad has non-trivial homotopy automorphisms only if $n = 4k + 3$, and then only a 1-parameter family of such. Note that this is in striking contrast to the case of $n = 2$, where the homotopy automorphisms form the infinite dimensional Grothendieck-Teichmüller group, cf. Fresse [n.d.] or Theorem 1.

4 Mapping spaces and long knots

Let us turn again to the computation of the rational homotopy type of the space of long knots. Using the result (3) from the Goodwillie-Weiss embedding calculus we see that the quantity to evaluate is the mapping space $\mathrm{Map}^h_{op}(E_m, E_n)$. We are interested in the rationalization of this space. By Fresse, Turchin, and Willwacher [2017b, Theorem 15 and Proposition 6.1] the rationalization is weakly equivalent to the space

$$\mathrm{Map}^h_{op}(E_m, E_n^{\mathbb{Q}}) \simeq \mathrm{Map}^h_{\text{Hopf-Op}^c}(\Omega_\sharp E_n, \Omega_\sharp E_m) \simeq \mathrm{Map}^h_{\text{Hopf-Op}^c}(e_n^c, e_m^c)$$

if $n - m \geq 3$. For the last equivalence one uses the formality result of the previous section.

The space on the right-hand side of the above equation can be expressed through purely combinatorial data, and shown to be weakly equivalent to the nerve (Maurer-Cartan space) of the Lie algebra of hairy graphs $\mathrm{HGC}_{m,n}$.

Theorem 11 (Fresse, Turchin, and Willwacher [ibid.]). For $n \geq m \geq 2$ we have that

$$\mathrm{Map}^h_{\text{Hopf-Op}^c}(e_n^c, e_m^c) \simeq MC_\bullet(\mathrm{HGC}_{m,n}) := MC(\mathrm{HGC}\hat{\otimes}\Omega_{poly}(\Delta^\bullet))$$

where the hairy graph complex $\mathrm{HGC}_{m,n}$ is equipped with the Lie algebra structure of Section 2.1.

In fact, the same statement continues to hold for $m = 1$, if one modifies the Lie algebra structure on $\mathrm{HGC}_{1,n}$ to an L_∞-algebra structure described in Willwacher [2015a].

Theorem 11 states in particular that the real homotopy type of the spaces of long knots in codimension $n - m \geq 3$ is fully expressed through combinatorial data encoded in the graph complexes. In particular one finds that for $k \geq 0$ and $n-m \geq 3$ (see also Arone and Turchin [2015] for an earlier but weaker result)

$$\mathbb{Q} \otimes \pi_k(\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)) \cong H^{-k-m}(\mathrm{HGC}_{m,n}).$$

Since the cohomology of the graph complex $\mathrm{HGC}_{m,n}$ can be explicitly computed in low degrees as described in Section 2.1, this result allows us to compute rational homotopy groups for low degrees k explicitly. As one application, let us consider here the case $k = 0$, detecting nontrivial knottings of $\mathbb{R}^m$ in $\mathbb{R}^n$. From the degree bound (8) we conclude that the only possible contribution can come from graphs in loop order 0, and those graph cohomology classes are listed in (7). Concretely the “line” graph of degree $n-m-2$ occurring for $n-m$ even produces nontrivial knottings if $n = 4j-1$ and $m = 2j$ for some j. Similarly the tripod graph of degree $2(n-m)-3$ occurring for $n-m$ odd yields nontrivial zeroth homotopy $n = 6j$ and $m = 4j-1$ for some j, see Fresse, Turchin, and Willwacher [2017b, Corollary 20]. In particular one hence recovers Haefliger's classical result Haefliger [1965] on the existence of non-trivial knots of dimension $4j-1$ in $6j$-space.

5 Outlook, extensions and open problems

Above we have seen that a rational version of the Goodwillie-Weiss manifold calculus can be used to compute the rational homotopy type of the space of higher dimensional long knots through the combinatorial structure of the graph complexes $\mathrm{HGC}_{m,n}$. Evidently, one can similarly hope to attack arbitrary embedding spaces $\mathrm{Emb}(M,N)$ via the rationalized Goodwillie-Weiss calculus. To complete this program one in particular needs to understand the rational homotopy types of all spaces and operads involved. As of today, at least over the reals, we understand the real homotopy types of (unframed) configuration spaces of points on compact orientable manifolds through the works Campos and Willwacher [2016] and Idrissi [2016]. Also, we understand the real homotopy type of the framed little disks operads Khoroshkin and Willwacher [2017]. The main open problem is the following.

Open Problem 12. Determine the real or rational homotopy type of the n-framed configuration spaces of points on a manifold M as a right E_n^{fr}-module, where $n \leq \dim M$.

This problem is also closely related to Open Open Problem 9 above.

After the real or rational homotopy types of the aforementioned objects are understood there should be no principle obstacle for generalizing the mapping space

space computations we reviewed here to arbitrary M and N. I can even describe the expected form of the graph complexes replacing the hairy graph complexes $\mathrm{HGC}_{m,n}$ in this setting. The relevant graphs should be hairy graphs as before, except that the hairs are additionaly decorated by a dgca model for the source space M, and the internal vertices may be decorated by the homology of the target N as in Campos and Willwacher [2016].

Unfortunately, the embedding calculus is fully applicable only in codimensions $\dim N - \dim M \geq 3$. In lower codimension one still has the map (2) (from left to right), but one can not assert it to be a weak equivalence. Nevertheless I expect that valuable information about the left-hand side can be obtained from the right-hand side. For example, in codimension 0, we have a map into the homotopy automorphisms of right framed-little disks (co)modules

$$\mathrm{Diff}(M) \to \mathrm{Aut}^h_{\Omega(E_n^{fr})\text{-comod}}(\Omega(\mathrm{conf}_M)). \tag{11}$$

I claim that the right-hand side can be computed and expressed through graph complexes. This then provides an arena in which one can study diffeomorphism groups, although the map (11) is generally not (expected to be) a rational homotopy equivalence.

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