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SPATIAL REFINEMENTS AND KHOVANOV HOMOLOGY

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Abstract

We review the construction and context of a stable homotopy refinement of Khovanov homology.

1 Introduction

While studying critical points and geodesics, Morse [1925, 1930, 1996] introduced what is now called *Morse theory*—using functions for which the second derivative test does not fail (*Morse functions*) to decompose manifolds into simpler pieces. The finite-dimensional case was further developed by many authors (see Bott [1980] for a survey of the history), and an infinite-dimensional analogue introduced by Palais and Smale [1964], Palais [1963], and Smale [1964]. In both cases, a Morse function f on M leads to a chain complex $C_*(f)$ generated by the critical points of f. This chain complex satisfies the *fundamental theorem of Morse homology*: its homology $H_*(f)$ is isomorphic to the singular homology of M. This is both a feature and a drawback: it means that one can use information about the topology of M to deduce the existence of critical points of f, but also implies that $C_*(f)$ does not see the smooth topology of M. (See Milnor [1963, 1965] for an elegant account of the subject's foundations and some of its applications.)

Much later, Floer [1988c,a,b] introduced some new examples of infinite-dimensional, Morse-like theories. Unlike Palais-Smale's Morse theory, in which the descending manifolds of critical points are finite-dimensional, in Floer's setting both ascending and descending manifolds are infinite-dimensional. Also unlike Palais-Smale's setting, Floer's homology groups are not isomorphic to singular homology of the ambient space (though the singular homology acts on them). Indeed, most Floer (co)homology theories seem to have no intrinsic cup product operation, and so are unlikely to be the homology of any natural space.

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Cohen, J. Jones, and Segal [1995] proposed that although Floer homology is not the homology of a space, it could be the homology of some associated spectrum (or pro-spectrum), and outlined a construction, under restrictive hypotheses, of such an object. While they suggest that these spectra might be determined by the ambient, infinite-dimensional manifold together with its polarization (a structure which seems ubiquitous in Floer theory), their construction builds a CW complex cell-by-cell, using the moduli spaces appearing in Floer theory. (We review their construction in Section 2.4. Steps towards describing Floer homology in terms of a polarized manifold have been taken by Lipyanskiy [n.d.].) Although the Cohen-Jones-Segal approach has been stymied by analytic difficulties, it has inspired other constructions of stable homotopy refinements of various Floer homologies and related invariants; see Furuta [2001], Bauer and Furuta [2004], Bauer [2004], Manolescu [2003], Kronheimer and Manolescu [n.d.], Douglas [n.d.], Cohen [2010, 2009], Kragh [n.d., 2013], Abouzaid and Kragh [2016], Khandhawit [2015a,b], Sasahira [n.d.], and Khandhawit, Lin, and Sasahira [n.d.].

From the beginning, Floer homologies have been used to define invariants of objects in low-dimensional topology—3-manifolds, knots, and so on. In a slightly different direction, Khovanov [2000] defined another knot invariant, which he calls $\mathfrak{sl}_2$ *homology* and everyone else calls *Khovanov homology*, whose graded Euler characteristic is the Jones polynomial from V. Jones [1985]. (See Bar-Natan [2002] for a friendly introduction.) While it looks formally similar to Floer-type invariants, Khovanov homology is defined combinatorially. No obvious infinite-dimensional manifold or functional is present. Still, Seidel and Smith [2006] (inspired by earlier work of Khovanov and Seidel [2002] and others) gave a conjectural reformulation of Khovanov homology via Floer homology. Over $\mathbb{Q}$, the isomorphism between Seidel-Smith's and Khovanov's invariants was recently proved by Abouzaid and Smith [n.d.]. Manolescu [2007] gave an extension of the reformulation to $\mathfrak{sl}_n$ homology constructed by Khovanov and Rozansky [2008].

Inspired by this history, Lipshitz and Sarkar [2014a,c,b] gave a combinatorial definition of a spectrum refining Khovanov homology, and studied some of its properties. This circle of ideas was further developed in Lipshitz, Ng, and Sarkar [2015] and in Lawson, Lipshitz, and Sarkar [n.d.(a),(b)], and extended in many directions by other authors (see Section 3.3). Another approach to a homotopy refinement was given by Everitt and Turner [2014], though it turns out their invariant is determined by Khovanov homology, cf. Everitt, Lipshitz, Sarkar, and Turner [2016]. Inspired by a different line of inquiry, Hu, D. Kriz, and I. Kriz [2016] also gave a construction of a Khovanov stable homotopy type. Lawson, Lipshitz, and Sarkar [n.d.(a)] show that the two constructions give homotopy equivalent spectra, perhaps suggesting some kind of uniqueness.

Most of this note is an outline of a construction of a Khovanov homotopy type, following Lawson, Lipshitz, and Sarkar [ibid.] and Hu, D. Kriz, and I. Kriz [2016], with an emphasis on the general question of stable homotopy refinements of chain complexes. In the last two sections, we briefly outline some of the structure and uses of the homotopy type (Section 3.3) and some questions and speculation (Section 3.4). Another exposition of some of this material can be found in Lawson, Lipshitz, and Sarkar [2017].

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2 Spatial refinements

The spatial refinement problem can be summarized as follows.

Start with a chain complex C_* with a distinguished, finite basis, arising in some interesting setting. Incorporating more information from the setting, construct a based CW complex (or spectrum) whose reduced cellular chain complex, after a shift, is isomorphic to C_* with cells corresponding to the given basis.

A result of Carlsson [1981] implies that there is no universal solution to the spatial refinement problem, i.e., no functor S from chain complexes (supported in large gradings, say) to CW complexes so that the composition of S and the reduced cellular chain complex functor is the identity (cf. Prasma et al. [n.d.]). Specifically, for $G = \mathbb{Z}/2\times\mathbb{Z}/2$ he defines a module P over $\mathbb{Z}[G]$ so that there is no G-equivariant Moore space $M(P,n)$ for any n. If C_* is a free resolution of P over $\mathbb{Z}[G]$ then $S(C_*)$ would be such a Moore space, a contradiction.

Thus, spatially refining C_* requires context-specific work. This section gives general frameworks for such spatial refinements, and the next section has an interesting example of one.

2.1 Linear and cubical diagrams. Let C_* be a freely and finitely generated chain complex with a given basis. After shifting we may assume C_* is supported in gradings $0,\dots,n$. Let $[n+1]$ be the category with objects $0,1,\dots,n$ and a unique morphism $i\to j$ if $i\geq j$. Let $\mathcal{B}(\mathbb{Z})$ denote the category of finitely generated free abelian groups, with objects finite sets and $\mathrm{Hom}_{\mathcal{B}(\mathbb{Z})}(S,T)$ the set of linear maps $\mathbb{Z}\langle S\rangle\to\mathbb{Z}\langle T\rangle$ or, equivalently, $T\times S$ matrices of integers. Then C_* may be viewed as a functor F from $[n+1]$ to $\mathcal{B}(\mathbb{Z})$ subject to the condition that F sends any length two arrow (that is, a morphism $i\to j$ with $i-j=2$) to the zero map. Given such a functor $F\colon [n+1]\to\mathcal{B}(\mathbb{Z})$, that is, a *linear diagram*

$$\mathbb{Z}\langle F(n)\rangle\to\mathbb{Z}\langle F(n-1)\rangle\to\cdots\to\mathbb{Z}\langle F(1)\rangle\to\mathbb{Z}\langle F(0)\rangle \tag{2-1}$$

with every composition the zero map, we obtain a chain complex C_* by shifting the gradings, $C_i = \mathbb{Z}\langle F(i)\rangle[i]$, and letting $\partial_i = F(i \to i-1)$. This construction is functorial. That is, if $\mathscr{B}(\mathbb{Z})^{[n+1]}_\bullet$ denotes the full subcategory of the functor category $\mathscr{B}(\mathbb{Z})^{[n+1]}$ generated by those functors which send every length two arrow to the zero map, and if Kom denotes the category of chain complexes, then the above construction is a functor $\mathsf{ch}\colon \mathscr{B}(\mathbb{Z})^{[n+1]}_\bullet \to \mathsf{Kom}$. Indeed, it would be reasonable to call an element of $\mathscr{B}(\mathbb{Z})^{[n+1]}_\bullet$ a *chain complex in* $\mathscr{B}(\mathbb{Z})$.

A linear diagram $F \in \mathscr{B}(\mathbb{Z})^{[n+1]}_\bullet$ may also be viewed as a *cubical diagram* $G\colon [2]^n \to \mathscr{B}(\mathbb{Z})$ by setting

$$G(v) = \begin{cases} F(i) & \text{if } v = (\underbrace{0,\dots,0}_{n-i},\underbrace{1,\dots,1}_{i}) \\ \varnothing & \text{otherwise.} \end{cases} \tag{2-2}$$

On morphisms, G is either zero or induced from F, as appropriate. Conversely, a cubical diagram $G \in \mathscr{B}(\mathbb{Z})^{[2]^n}$ gives a linear diagram $F \in \mathscr{B}(\mathbb{Z})^{[n+1]}_\bullet$ by setting $F(i) = \coprod_{|v|=i} G(v)$, where $|v|$ denotes the number of 1's in v. The component of $F(i+1 \to i)$ from $\mathbb{Z}\langle G(u)\rangle \subset \mathbb{Z}\langle F(i+1)\rangle$ to $\mathbb{Z}\langle G(v)\rangle \subset \mathbb{Z}\langle F(i)\rangle$ is

$$\begin{cases} (-1)^{u_1+\cdots+u_{k-1}} G(u \to v) & \text{if } u - v = \widehat{e}_k \text{, the } k^{\text{th}} \text{ unit vector,} \\ 0 & \text{if } u - v \text{ is not a unit vector.} \end{cases} \tag{2-3}$$

These give functors $\mathscr{B}(\mathbb{Z})^{[n+1]}_\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathscr{B}(\mathbb{Z})^{[2]^n}$ with $\beta \circ \alpha = \mathrm{Id}$.

The composition $\mathsf{ch} \circ \beta\colon \mathscr{B}(\mathbb{Z})^{[2]^n} \to \mathsf{Kom}$ is the *totalization* Tot, and may be viewed as an iterated mapping cone. Up to chain homotopy equivalence, one can also construct Tot using homotopy colimits. Define a category $[2]_+$ by adjoining a single object $*$ to $[2]$ and a single morphism $1 \to *$; let $[2]^n_+ = ([2]_+)^n$. Given $G \in \mathscr{B}(\mathbb{Z})^{[2]^n}$, by treating abelian groups as chain complexes supported in homological grading zero, we get an associated cubical diagram $A\colon [2]^n \to \mathsf{Kom}$. Extend A to a diagram $A_+\colon [2]^n_+ \to \mathsf{Kom}$ by setting

$$A_+(v) = \begin{cases} A(v) & \text{if } v \in [2]^n \\ 0 & \text{otherwise.} \end{cases} \tag{2-4}$$

Then the totalization of G is the *homotopy colimit* of A_+. (See Segal [1974], Bousfield and Kan [1972], and Vogt [1973].)

2.2 Spatial refinements of diagrams of abelian groups. As a next step, given a finitely generated chain complex represented by a functor $F\colon [n+1] \to \mathscr{B}(\mathbb{Z})$ we wish to construct a based cell complex with cells in dimensions $N, \dots, N+n$ whose reduced cellular complex—with distinguished basis given by the cells—is isomorphic to the given complex shifted up by N.

Let $\mathcal{T}(S^N)$ be the category with objects finite sets and morphisms $\mathrm{Hom}_{\mathcal{T}(S^N)}(S,T)$ the set of all based maps $\bigvee_S S^N \to \bigvee_T S^N$ between wedges of N-dimensional spheres; applying reduced N^{th} homology to the morphisms produces a functor, also denoted $\widetilde{H}_N$, from $\mathcal{T}(S^N)$ to $\mathscr{B}(\mathbb{Z})$. A *strict N-dimensional spatial lift* of F is a functor $P\colon [n+1] \to \mathcal{T}(S^N)$ satisfying $\widetilde{H}_N \circ P = F$ and P is the constant map on any length two arrow in $[n+1]$, i.e., a *strict chain complex in $\mathcal{T}(S^N)$ lifting F*. Just as morphisms in $\mathscr{B}(\mathbb{Z})$ are matrices, if we replace S^N by the sphere spectrum $\mathbb{S}$, we may view a morphism in $\mathcal{T}(\mathbb{S})$ as a matrix of maps $\mathbb{S} \to \mathbb{S}$ by viewing $\bigvee_S \mathbb{S}$ as a coproduct and $\bigvee_T \mathbb{S}$ as a product.

Given such a linear diagram P, we can construct a based cell complex by taking mapping cones and suspending sequentially, cf. Cohen, J. Jones, and Segal [1995, §5]. If CW denotes the category of based cell complexes, then P induces a diagram $X\colon [n+1] \to \mathsf{CW}$,

$$X(n) \xrightarrow{f_n} X(n-1) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X(1) \xrightarrow{f_1} X(0) \tag{2-5}$$

with every composition the constant map. Since $f_1 \circ f_2$ is the constant map, there is an induced map $g_1\colon \Sigma X(2) \to \mathrm{Cone}(f_1)$ from the reduced suspension to the reduced cone. Then we get a diagram $Y\colon [n] \to \mathsf{CW}$,

$$Y(n-1) = \Sigma X(n) \xrightarrow{\Sigma f_n} \cdots \xrightarrow{\Sigma f_3} Y(1) = \Sigma X(2) \xrightarrow{g_1} Y(0) = \mathrm{Cone}(f_n). \tag{2-6}$$

Take the mapping cone of g_1 and suspend to get a diagram $Z\colon [n-1] \to \mathsf{CW}$ and so on. The reduced cellular chain complex of the final CW complex is the original chain complex, shifted up by N. This construction is also functorial: if $\mathcal{T}(S^N)_\bullet^{[n+1]}$ is the full subcategory generated by the functors which send every length two arrow to the constant map, then the construction is a functor $\mathcal{T}(S^N)_\bullet^{[n+1]} \to \mathsf{CW}$. The construction can also be carried out in a single step. Construct a category $[n+1]_+$ by adjoining a single object $*$ and a unique morphism $i \to *$ for all $i \neq 0$. Extend $X\colon [n+1] \to \mathsf{CW}$ to $X_+\colon [n+1]_+ \to \mathsf{CW}$ by sending $*$ to a point and take the homotopy colimit of X_+.

A linear diagram $P \in \mathcal{T}(S^N)_\bullet^{[n+1]}$ produces a cubical diagram $Q\colon [2]^n \to \mathcal{T}(S^N)$ by the analogue of Equation (2-2). There is a *totalization* functor $\mathsf{Tot}\colon \mathcal{T}(S^N)^{[2]^n} \to \mathsf{CW}$

extending the functor $\mathscr{T}(S^N)_\bullet^{[n+1]} \to \mathsf{CW}$ so that

$$\begin{array}{ccccc} \mathscr{T}(S^N)_\bullet^{[n+1]} & \longrightarrow & \mathscr{T}(S^N)^{[2]^n} & \longrightarrow & \mathsf{CW} \\ \downarrow & & \downarrow & & \downarrow \widetilde{C}_*^{\mathrm{cell}}[-N] \\ \mathscr{B}(\mathbb{Z})_\bullet^{[n+1]} & \longrightarrow & \mathscr{B}(\mathbb{Z})^{[2]^n} & \longrightarrow & \mathsf{Kom}. \end{array} \tag{2-7}$$

commutes. The totalization functor is defined as an iterated mapping cone or as a homotopy colimit of an extension of Q analogous to Equation (2-4).

2.3 Lax spatial refinements. Instead of working with strict functors as in the previous section, sometimes is it more convenient to work with lax functors. A *lax* or *homotopy coherent* or $(\infty, 1)$ *functor* $F : \mathscr{C} \to \mathsf{Top}$ is a diagram that commutes up to homotopies which are specified, and the homotopies themselves commute up to higher homotopies which are also specified, and so on; for details see Vogt [1973], Cordier [1982], and Lurie [2009a]. More precisely, F consists of based topological spaces $F(x)$ for $x \in \mathscr{C}$, and higher homotopy maps $F(f_n, \dots, f_1) : [0,1]^{n-1} \times F(x_0) \to F(x_n)$ for composable morphisms $x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n$ with certain boundary conditions and restrictions involving basepoints and identity morphisms; the case $n = 1$ is maps corresponding to the arrows in the diagram, $n = 2$ is homotopies corresponding to pairs of composable arrows, etc. A strict functor may be viewed as a lax functor. Let $\mathfrak{h}\mathsf{Top}^{\mathscr{C}}$ denote the category of lax functors $\mathscr{C} \to \mathsf{Top}$, with morphisms given by lax functors $\mathscr{C} \times [2] \to \mathsf{Top}$. (There are also higher morphisms corresponding to lax functors $\mathscr{C} \times [n] \to \mathsf{Top}$.)

There is a notion of a lax functor to $\mathscr{T}(S^N)$ induced from the notion of lax functors to Top. Let $\mathfrak{h}\mathscr{T}(S^N)_\bullet^{[n+1]}$ be the subcategory of $\mathfrak{h}\mathscr{T}(S^N)^{[n+1]}$ consisting of those objects (respectively, morphisms) F such that $F\big(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n\big)$ is the constant map to the basepoint for any string of morphisms in $[n+1]$ (respectively, $[n+1] \times [2]$) with some f_i of length ≥ 2 in $[n+1]$ (respectively, $[n+1] \times \{0,1\}$). We call such functors *chain complexes in* $\mathscr{T}(S^N)$. (In Cohen-Jones-Segal's language, chain complexes in $\mathscr{T}(S^N)$ are functors $\mathscr{J}_0^n \to \mathscr{T}_*$.)

If one starts with a chain complex $F \in \mathscr{B}(\mathbb{Z})_\bullet^{[n+1]}$ and wishes to refine it to a based cell complex, instead of constructing a strict N-dimensional spatial lift in $\mathscr{T}(S^N)_\bullet^{[n+1]}$, it is enough to construct a *lax* N*-dimensional spatial lift*, that is, a functor $P \in \mathfrak{h}\mathscr{T}(S^N)_\bullet^{[n+1]}$ with $\widetilde{H}_N \circ P = F$. Such a P produces a cell complex by adjoining a basepoint to get a lax diagram $[n+1]_+ \to \mathsf{CW}$ and then taking homotopy colimits. Alternatively, we may convert P to a lax cubical diagram $Q \in \mathfrak{h}\mathscr{T}(S^N)^{[2]^n}$ and proceed as before. The iterated mapping cone construction becomes intricate since the associated diagram $X : [2]^n \to \mathsf{CW}$ is lax. So, extend to a lax diagram $X_+ : [2]^n_+ \to \mathsf{CW}$ as before and then take its homotopy colimit. This generalization to the lax set-up remains functorial and the analogue of Diagram (2-7) still commutes.

2.4 Framed flow categories. Cohen, J. Jones, and Segal [1995] first proposed lax spatial refinements of diagrams $F\colon [n+1] \to \mathscr{B}(\mathbb{Z})$ via framed flow categories, using the Pontryagin-Thom construction. A *framed flow category* is an abstraction of the gradient flows of a Morse-Smale function. Concretely, a framed flow category $\mathscr{C}$ consists of:

1. A finite set of *objects* $\mathrm{Ob}(\mathscr{C})$ and a *grading* $\mathrm{gr}\colon \mathrm{Ob}(\mathscr{C}) \to \mathbb{Z}$. After translating, we may assume the gradings lie in $[0, n]$.
2. For $x, y \in \mathscr{C}$ with $\mathrm{gr}(x) - \mathrm{gr}(y) - 1 = k$, a *morphism set* $\mathfrak{M}(x, y)$ which is a k-dimensional $\langle k\rangle$-manifold. A *$\langle k\rangle$-manifold M* is a smooth manifold with corners so that each codimension-c corner point lies in exactly c facets (closure of a codimension-1 component), equipped with a decomposition of its boundary $\partial M = \cup_{i=1}^{k} \partial_i M$ so that each $\partial_i M$ is a *multifacet* of M (union of disjoint facets), and $\partial_i M \cap \partial_j M$ is a multifacet of $\partial_i M$ and $\partial_j M$, cf. Jänich [1968] and Laures [2000].
3. An associative *composition* map $\mathfrak{M}(y, z) \times \mathfrak{M}(x, y) \hookrightarrow \partial_{\mathrm{gr}(y)-\mathrm{gr}(z)} \mathfrak{M}(x, z) \subset \mathfrak{M}(x, z)$. Setting

   (2-8) $$\mathfrak{M}(i, j) = \coprod_{\substack{x,y \\ \mathrm{gr}(x)=i, \mathrm{gr}(y)=j}} \mathfrak{M}(x, y),$$

   the composition is required to induce an isomorphism of $\langle i - j - 2\rangle$-manifolds

   (2-9) $$\partial_{j-k} \mathfrak{M}(i, k) \cong \mathfrak{M}(j, k) \times \mathfrak{M}(i, j).$$

4. *Neat embeddings* $\iota_{i,j}\colon \mathfrak{M}(i, j) \hookrightarrow [0, 1)^{i-j-1} \times (-1, 1)^{D(i-j)}$ for some large $D \in \mathbb{N}$, namely, smooth embeddings satisfying

   (2-10) $$\iota_{i,k}^{-1}([0, 1)^{j-k-1} \times \{0\} \times [0, 1)^{i-j-1} \times (-1, 1)^{D(i-k)}) = \partial_{j-k} \mathfrak{M}(i, k)$$

   and certain orthogonality conditions near boundaries. These embeddings are required to be coherent with respect to composition. The space of such collections of neat embeddings is $(D - 2)$-connected.
5. *Framings* of the normal bundles of $\iota_{i,j}$, also coherent with respect to composition, which give extensions $\bar{\iota}_{i,j}\colon \mathfrak{M}(i, j) \times [-1, 1]^{D(i-j)} \hookrightarrow [0, 1)^{i-j-1} \times (-1, 1)^{D(i-j)}$.

A framed flow category produces a lax linear diagram $P \in \mathfrak{h}\mathscr{T}(S^N)_{\bullet}^{[n+1]}$ with $N = nD$. On objects, set $P(i) = \{x \in \mathscr{C} \mid \mathrm{gr}(x) = i\}$. On morphisms, define the map

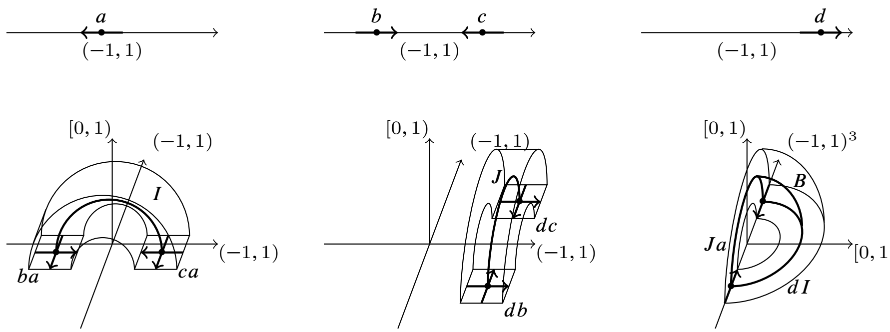

Figure 2.1: **A framed flow category $\mathscr{C}$.** $\mathrm{Ob}(\mathscr{C}) = \{x, y, z, w\}$ *in gradings* $3, 2, 1, 0$*, respectively. The* 0*-dimensional morphism spaces are:* $\mathfrak{M}(x, y) = \{a\}$, $\mathfrak{M}(y, z) = \{b, c\}$*, and* $\mathfrak{M}(z, w) = \{d\}$*, each embedded in* $(-1, 1)$*. The* 1*-dimensional morphism spaces are:* $\mathfrak{M}(x, z) = I$ *(resp.* $\mathfrak{M}(y, w) = J$*) an interval, embedded in* $[0, 1) \times (-1, 1)^2$*, with endpoints* $\{ba, ca\}$ *(resp.* $\{db, dc\}$*) embedded in* $\{0\} \times (-1, 1)^2$ *by the product embedding. The* 2*-dimensional morphism space is a disk* B*, embedded in* $[0, 1)^2 \times (-1, 1)^3$*; it is a* $\langle 2 \rangle$*-manifold with boundary decomposed as a union of two arcs,* $\partial_1 B = dI \subset \{0\} \times [0, 1) \times (-1, 1)^3$ *and* $\partial_2 B = Ja \subset [0, 1) \times \{0\} \times (-1, 1)^3$*. Coherent framings of all the normal bundles are represented by the tubular neighborhoods* $\bar{\iota}_{i,j}$*. In the last subfigure,* $(-1, 1)^3$ *is drawn as an interval by projecting to the middle* $(-1, 1)$*.*

associated to the sequence $m_0 \to m_1 \to \cdots \to m_k$ to be the constant map unless all the arrows are length one. To a sequence of length one arrows, $i \to i-1 \to \cdots \to j$, associate a map

$$(2\text{-}11)\qquad \begin{aligned} [0,1]^{i-j-1} \times \bigvee_{x \in P(i)} S^N &= [0,1]^{i-j-1} \times \coprod_{x \in P(i)} [-1,1]^{nD}/\partial \\ &\to \bigvee_{y \in P(j)} S^N = \coprod_{y \in P(j)} [-1,1]^{nD}/\partial \end{aligned}$$

using $\bar{\iota}_{i,j}$ and the Pontryagin-Thom construction.

We can then apply the totalization functor to P to get a cell complex with cells in dimensions $N, N+1, \ldots, N+n$. As Cohen, J. Jones, and Segal [1995] sketch, for a flow category coming from a generic gradient flow of a Morse function, the cell complex produced by the totalization functor is the N^{th} reduced suspension of the Morse cell complex built from the unstable disks of the critical points.

Much of the above data can also be reformulated in the language of S-modules from Pardon [n.d., §4.3]. Let $S[n+1]$ be the (non-symmetric) multicategory with objects pairs

(j,i) of integers with $0 \leq j \leq i \leq n$, unique multimorphisms $(i_0,i_1),(i_1,i_2),\ldots,(i_{k-1},i_k) \to (i_0,i_k)$ when $k \geq 1$, and no other multimorphisms (cf. *shape multicategories* from Lawson, Lipshitz, and Sarkar [n.d.(b)]). Let $\underline{\mathsf{Top}}$ be the multicategory of based topological spaces whose multimorphisms $X_1,\ldots,X_k \to Y$ are maps $X_1 \wedge \cdots \wedge X_k \to Y$. An *$S$-module* is a multifunctor $S[n+1] \to \underline{\mathsf{Top}}$.

Given $\mathscr{C}$, define S-modules $\mathbf{S},\mathbf{J}$ by setting $\mathbf{S}(j,i) = \vee_{y\in P(j)} S^{D(i-j)}$ and $\mathbf{J}(j,i) = (\vee_{x\in P(i)} S^{D(i-j)}) \wedge \mathcal{J}(i,j)$ where $\mathcal{J}$ is the category with objects integers and morphisms $\mathcal{J}(i,j)$ the one-point compactification of $[0,1)^{i-j-1}$ if $i \geq j$ (which is a point if $i = j$) and composition $\mathcal{J}(j,k) \wedge \mathcal{J}(i,j) \to \mathcal{J}(i,k)$ induced by the inclusion map $[0,1)^{j-k-1} \times \{0\} \times [0,1)^{i-j-1} \hookrightarrow \partial[0,1)^{i-k-1}$, cf. Cohen, J. Jones, and Segal [1995, §5]. On a multimorphism $(i_0,i_1),\ldots,(i_{k-1},i_k) \to (i_0,i_k)$, $\mathbf{S}$ sends the $(y_0,\ldots,y_{k-1}) \in P(i_0) \times \cdots \times P(i_{k-1})$ summand $S^{D(i_1-i_0)} \wedge \cdots \wedge S^{D(i_k-i_{k-1})}$ homeomorphically to the y_0 summand $S^{D(i_k-i_0)}$, and $\mathbf{J}$ sends the $(x_1,\ldots,x_k) \in P(i_1)\times\cdots\times P(i_k)$ summand $(S^{D(i_1-i_0)} \wedge \cdots \wedge S^{D(i_k-i_{k-1})}) \wedge (\mathcal{J}(i_1,i_0) \wedge \cdots \wedge \mathcal{J}(i_k,i_{k-1}))$ to the x_k summand $S^{D(i_k-i_0)} \wedge \mathcal{J}(i_k,i_0)$, homeomorphically on the first factor, and using the composition in $\mathcal{J}$ on the second factor. Given a neat embedding of $\mathscr{C}$, we can define another S-module $\mathbf{N}$ by setting $\mathbf{N}(j,i) = \iota^{\nu}_{i,j}$, the Thom space of the normal bundle of $\iota_{i,j}$, if $i > j$. (When $i = j$, $\mathbf{N}(j,j)$ is a point.) On multimorphisms, $\mathbf{N}$ is induced from the inclusion maps $\mathrm{im}(\bar{\iota}_{i_1,i_0})\times\cdots\times\mathrm{im}(\bar{\iota}_{i_k,i_{k-1}}) \to \mathrm{im}(\bar{\iota}_{i_k,i_0})$. The Pontryagin-Thom collapse map is a natural transformation—an *S-module map*—from $\mathbf{J}$ to $\mathbf{N}$, which sends the $x \in P(i)$ summand of $\mathbf{J}(j,i)$ to the Thom-space summand $\cup_{y\in P(j)} \iota^{\nu}_{x,y}$ in $\mathbf{N}(j,i)$. A framing of $\mathscr{C}$ produces another S-module map $\mathbf{N} \to \mathbf{S}$ which sends the summand $\iota^{\nu}_{x,y}$ in $\mathbf{N}(j,i)$ to the y summand of $\mathbf{S}(j,i)$. Composing we get an S-module map $\mathbf{J} \to \mathbf{S}$, which is precisely the data needed to recover a lax diagram in $\mathfrak{h}\mathcal{T}(S^N)^{[n+1]}_{\bullet}$.

Finally, as popularized by Abouzaid, note that since the (smooth) framings of $\iota_{i,j}$ were only used to construct maps $\iota^{\nu}_{i,j} \to \vee_{y\in P(j)} S^{D(i-j)}$, a weaker structure on the flow category—namely, coherent trivializations of the Thom spaces $\iota^{\nu}_{i,j}$ as spherical fibrations—might suffice.

2.5 Speculative digression: matrices of framed cobordisms. Perhaps it would be tidy to reformulate the notion of stably framed flow categories as chain complexes in some category $\mathcal{B}(\mathsf{Cob})$ equipped with a functor $\mathcal{B}(\mathsf{Cob}) \to \mathcal{T}(\mathbb{S})$. It is clear how such a definition would start. Objects in $\mathcal{B}(\mathsf{Cob})$ should be finite sets. By the Pontryagin-Thom construction, a map $\mathbb{S} \to \mathbb{S}$ is determined by a framed 0-manifold; therefore, a morphism in $\mathcal{B}(\mathsf{Cob})$ should be a matrix of framed 0-manifolds. To account for the homotopies in $\mathcal{T}(\mathbb{S})$, $\mathcal{B}(\mathsf{Cob})$ should have higher morphisms. For instance, given two $(T \times S)$-matrices A,B of framed 0-manifolds, a 2-morphism from A to B should be a $(T \times S)$-matrix of framed 1-dimensional cobordisms. Given two such $(T \times S)$-matrices

M, N of framed 1-dimensional cobordisms, a 3-morphism from M to N should be a $(T \times S)$-matrix of framed 2-dimensional cobordisms with corners, and so on. That is, the target category Cob seems to be the extended cobordism category, an (∞,∞)-category studied, for instance, by Lurie [2009b].

Since matrix multiplication requires only addition and multiplication, the construction $\mathscr{B}(\mathscr{C})$ makes sense for any *rig* or *symmetric bimonoidal* category $\mathscr{C}$ and, presumably, for a rig (∞,∞)-category, for some suitable definition; and perhaps the framed cobordism category Cob is an example of a rig (∞,∞)-category. Maybe the Pontryagin-Thom construction gives a functor $\mathscr{B}(\mathsf{Cob}) \to \mathscr{T}(\mathbb{S})$, and that a stably framed flow category is just a functor $[n+1] \to \mathscr{B}(\mathsf{Cob})$.

Rather than pursuing this, we will focus on a tiny piece of Cob, in which all 0-manifolds are framed positively, all 1-dimensional cobordisms are trivially-framed intervals and, more generally, all higher cobordisms are trivially-framed disks. In this case, all of the information is contained in the objects, 1-morphisms, and 2-morphisms, and this tiny piece equals $\mathscr{B}(\mathsf{Sets})$ with Sets being viewed as a rig category via disjoint union and Cartesian product.

2.6 The cube and the Burnside category. The *Burnside category* $\mathscr{B}$ (associated to the trivial group) is the following weak 2-category. The objects are finite sets. The 1-morphisms $\mathrm{Hom}(S,T)$ are $T \times S$ matrices of finite sets; composition is matrix multiplication, using the disjoint union and product of sets in place of $+$ and $\times$ of real numbers. The 2-morphisms are matrices of entrywise bijections between matrices of sets.

(The category $\mathscr{B}$ is denoted $\mathcal{S}_2$ by Hu, D. Kriz, and I. Kriz [2016], and is an example of what they call a $\star$-*category*. The realization procedure below is a concrete analogue of the Elmendorf and Mandell [2006] infinite loop space machine; see also Lawson, Lipshitz, and Sarkar [n.d.(a), §8].)

There is an abelianization functor $\mathsf{Ab}\colon \mathscr{B} \to \mathscr{B}(\mathbb{Z})$ which is the identity on objects and sends a morphism $(A_{t,s})_{s\in S,t\in T}$ to the matrix $(\#A_{t,s})_{s\in S,t\in T} \in \mathbb{Z}^{T\times S}$. We are given a diagram $G \in \mathscr{B}(\mathbb{Z})^{[2]^n}$ which we wish to lift to a diagram $Q \in \mathfrak{h}\mathscr{T}(S^N)^{[2]^n}$. As we will see, it suffices to lift G to a diagram $D\colon [2]^n \to \mathscr{B}$.

Since $\mathscr{B}$ is a weak 2-category, we should first clarify what we mean by a diagram in $\mathscr{B}$. A *strictly unital lax 2-functor*—henceforth just called a lax functor—$D\colon [2]^n \to \mathscr{B}$ consists of the following data:

1. A finite set $F(x) \in \mathscr{B}$ for each $v \in [2]^n$.

2. An $F(v) \times F(u)$-matrix of finite sets $F(u \to v) \in \mathrm{Hom}_{\mathscr{B}}(F(u), F(v))$ for each $u > v \in [2]^n$.

3. A 2-isomorphism $F_{u,v,w}\colon F(v \to w) \circ_1 F(u \to v) \to F(u \to w)$ for each $u > v > w \in [2]^n$ so that for each $u > v > w > z$, $F_{u,w,z} \circ_2 (\mathrm{Id} \circ_1 F_{u,v,w}) = (F_{v,w,z} \circ_1 \mathrm{Id}) \circ_2 F_{u,v,z}$, where $\circ_i$ denotes composition of i-morphisms ($i = 1, 2$).

Next we turn such a lax diagram $D\colon [2]^n \to \mathscr{B}$ into a lax diagram $Q \in \mathfrak{h}\mathcal{T}(S^N)^{[2]^n}$, $N \geq n+1$, satisfying $\widetilde{H}_N \circ Q = \mathsf{Ab} \circ D$. Associate a *box* $B_x = [-1,1]^N$ to each $x \in D(v)$, $v \in [2]^n$. For each $u > v$, let $D(u \to v) = (A_{y,x})_{x \in D(u), y \in D(v)}$ and let $E(u \to v)$ be the space of embeddings $\iota_{u,v} = \{\iota_{u,v,x}\}_{x \in D(u)}$ where

$$\iota_{u,v,x}\colon \coprod_y A_{y,x} \times B_y \hookrightarrow B_x \tag{2-12}$$

whose restriction to each copy of B_y is a sub-box inclusion, i.e., composition of a translation and dilation. The space $E(A)$ is $(N-2)$-connected.

For any such data $\iota_{u,v}$, a collapse map and a fold map give a *box map*

$$\widehat{\iota}_{u,v}\colon \bigvee_{x \in D(u)} S^N = \coprod_{x \in D(u)} B_x/\partial \to \coprod_{\substack{x \in D(u)\\ y \in D(v)\\ a \in A_{y,x}}} B_y/\partial \to \coprod_{y \in D(v)} B_y/\partial = \bigvee_{y \in D(v)} S^N. \tag{2-13}$$

Given $u > v > w$ and data $\iota_{u,v}$, $\iota_{v,w}$, the composition $\widehat{\iota}_{v,w} \circ \widehat{\iota}_{u,v}$ is also a box map corresponding to some induced embedding data.

The construction of the lax diagram $Q \in \mathfrak{h}\mathcal{T}(S^N)^{[2]^n}$ is inductive. On objects, Q agrees with D. For (non-identity) morphisms $u \to v$, choose a box map $Q(u \to v) = \widehat{\iota}_{u,v}\colon \bigvee_{x \in D(u)} S^N \to \bigvee_{y \in D(v)} S^N$ refining $D(u \to v)$. Staying in the space of box maps, the required homotopies exist and are unique up to homotopy because each $E(A)$ is $N-2 \geq n-1$ connected, and there are no sequences of composable morphisms of length $> n-1$. (See Lawson, Lipshitz, and Sarkar [ibid.] for details.)

The above construction closely follows the Pontryagin-Thom procedure from Section 2.4. Indeed, functors from the cube to the Burnside category correspond to certain kinds of flow categories (*cubical* ones), and the realizations in terms of box maps and cubical flow categories agree.

3 Khovanov homology

3.1 The Khovanov cube. Khovanov homology was defined by Khovanov [2000] using the Frobenius algebra $V = H^*(S^2)$. Let $x_- \in H^0(S^2)$ and $x_+ \in H^2(S^2)$ be the positive generators. (Our labeling is opposite Khovanov's convention, as the maps in our cube go from 1 to 0.) Via the equivalence of Frobenius algebras and $(1+1)$-dimensional topological field theories (cf. Abrams [1996]), we can reinterpret V as a functor from the $(1+1)$-dimensional bordism category Cob^{1+1} to $\mathscr{B}(\mathbb{Z})$ that assigns $\{x_+, x_-\}$ to circle, and hence $\prod_{\pi_0(C)}\{x_+, x_-\}$ to a one-manifold C. For $x \in V(C)$, let $\|x\|_+$ (respectively, $\|x\|_-$) denote the number of circles in C labeled x_+ (respectively, x_-) by x. For a cobordism $\Sigma\colon C_1 \to C_0$, the map $V(\Sigma)\colon \mathbb{Z}\langle V(C_1)\rangle = \otimes_{\pi_0(C_1)}\mathbb{Z}\langle x_+, x_-\rangle \to \mathbb{Z}\langle V(C_0)\rangle = \otimes_{\pi_0(C_0)}\mathbb{Z}\langle x_+, x_-\rangle$ is the tensor product of the maps induced by the connected components of Σ; and if $\Sigma\colon C_1 \to C_0$ is a connected, genus-g cobordism, then the (y, x)-entry of the matrix representing the map, $x \in V(C_1)$, $y \in V(C_0)$, is

$$\begin{cases} 1 & \text{if } g = 0,\ \|x\|_+ + \|y\|_- = 1, \\ 2 & \text{if } g = 1,\ \|x\|_+ = \|y\|_- = 0, \\ 0 & \text{otherwise} \end{cases} \tag{3-1}$$

(cf. Bar-Natan [2005] and Hu, D. Kriz, and I. Kriz [2016]).

Now, given a link diagram L with n crossings numbered $c_1, \dots, c_n$, Khovanov [2000] constructs a cubical diagram $G_{Kh} = V \circ \mathbb{L} \in \mathscr{B}(\mathbb{Z})^{[2]^n}$ where $\mathbb{L}\colon [2]^n \to \mathsf{Cob}^{1+1}$ is the *cube of resolutions* (extending Kauffman [1987]) defined as follows. For $v \in [2]^n$, let $\mathbb{L}(v)$ be the complete resolution of the link diagram L formed by resolving the i^{th} crossing c_i  by the 0-*resolution*  if $v_i = 0$ and by the 1-*resolution*  if $v_i = 1$. For a morphism $u \to v$, $\mathbb{L}(u \to v)$ is the cobordism which is an elementary saddle from the 1-resolution to the 0-resolution near crossings c_i for each i with $u_i > v_i$, and is a product cobordism elsewhere.

The dual of the resulting total complex, shifted by n_-, the number of negatives crossings  in L, is usually called the Khovanov complex

$$\mathcal{C}^*_{Kh}(L) = \mathsf{Dual}(\mathsf{Tot}(G_{Kh}))[n_-], \tag{3-2}$$

and its cohomology $Kh^*(L)$ the Khovanov homology, which is a link invariant. There is an internal grading, the *quantum grading*, that comes from placing the two symbols x_+ and x_- in two different quantum gradings, and the entire complex decomposes along this grading, so Khovanov homology inherits a second grading $Kh^i(L) = \oplus_j Kh^{i,j}(L)$, and its quantum-graded Euler characteristic

$$\sum_{i,j} (-1)^i q^j \operatorname{rank}(Kh^{i,j}(L)) \tag{3-3}$$

recovers the unnormalized Jones polynomial of L. The quantum grading persists in the space-level refinement but, for brevity, we suppress it.

3.2 The stable homotopy type. Following Section 2.6, to give a space-level refinement of Khovanov homology it suffices to lift G_{Kh} to a lax functor $[2]^n \to \mathscr{B}$.

Hu, D. Kriz, and I. Kriz [2016, §3.2] shows that the TQFT $V\colon \mathsf{Cob}^{1+1} \to \mathscr{B}(\mathbb{Z})$ does not lift to a functor $\mathsf{Cob}^{1+1} \to \mathscr{B}$. However, we may instead work with the *embedded* cobordism category Cob_e^{1+1}, which is a weak 2-category whose objects are closed 1-manifolds embedded in S^2, morphisms are compact cobordisms embedded in $S^2 \times [0,1]$, and 2-morphisms are isotopy classes of isotopies in $S^2 \times [0,1]$ rel boundary. The cube $\mathbb{L}\colon [2]^n \to \mathsf{Cob}^{1+1}$ factors through a functor $\mathbb{L}_e\colon [2]^n \to \mathsf{Cob}_e^{1+1}$. (This functor $\mathbb{L}_e$ is lax, similar to what we had for functors to the Burnside category except without strict unitarity.) So it remains to lift V to a (lax) functor $V_e\colon \mathsf{Cob}_e^{1+1} \to \mathscr{B}$

$$\begin{array}{ccccc} & & \mathsf{Cob}_e^{1+1} & \xrightarrow{V_e} & \mathscr{B}. \\ & \overset{\mathbb{L}_e}{\nearrow} & \downarrow & & \downarrow \\ [2]^n & \underset{\mathbb{L}}{\searrow} & & & \\ & & \mathsf{Cob}^{1+1} & \xrightarrow{V} & \mathscr{B}(\mathbb{Z}) \end{array} \tag{3-4}$$

On an embedded one-manifold C, we must set

$$V_e(C) = \prod_{\pi_0(C)} \{x_+, x_-\}. \tag{3-5}$$

For an embedded cobordism $\Sigma\colon C_1 \to C_0$ with C_i embedded in $S^2 \times \{i\}$, the matrix $V_e(\Sigma)$ is a tensor product over the connected components of Σ, i.e., if $\Sigma = \amalg_{j=1}^m (\Sigma_j\colon C_{1,j} \to C_{0,j})$ and $(y^j, x^j) \in V_e(C_{0,j}) \times V_e(C_{1,j})$, then the $(\Pi_{j=1}^m y^j, \Pi_{j=1}^m x^j)$ entry of $V_e(\Sigma)$ equals

$$V_e(\Sigma_1)_{y^1,x^1} \times \cdots \times V_e(\Sigma_m)_{y^m,x^m}. \tag{3-6}$$

And finally, if $\Sigma\colon C_1 \to C_0$ is a connected genus-g cobordism, then for $x \in V_e(C_1)$ and $y \in V_e(C_0)$, the (y,x)-entry of $V_e(\Sigma)$ must be a

$$\begin{cases} \text{1-element set} & \text{if } g = 0,\ \|x\|_+ + \|y\|_- = 1, \\ \text{2-element set} & \text{if } g = 1,\ \|x\|_+ = \|y\|_- = 0, \\ \varnothing & \text{otherwise.} \end{cases} \tag{3-7}$$

One-element sets do not have any non-trivial automorphisms, so we may set all the one-element sets to $\{\mathrm{pt}\}$. The two-element sets must be chosen carefully: they have to behave naturally under isotopy of cobordisms (the 2-morphisms in Cob_e^{1+1}) and must admit natural isomorphisms $V_e(\Sigma \circ \Sigma') \cong V_e(\Sigma) \circ V_e(\Sigma')$ when composing cobordisms $\Sigma'\colon C_2 \to C_1$ and $\Sigma\colon C_1 \to C_0$.

Decompose $S^2 \times [0,1]$ as a union of two compact 3-manifolds glued along Σ, $A \cup_\Sigma B$. Set $V_e(\Sigma)$ to be the (cardinality two) set of unordered bases $\{\alpha, \beta\}$ for $\ker(H^1(\Sigma) \to$

$H^1(\partial\Sigma)) \cong \mathbb{Z}^2$ so that α (respectively, β) is the restriction of a generator of $\ker(H^1(A) \to H^1(A \cap (S^2 \times \{0,1\}))) \cong \mathbb{Z}$ (respectively, $\ker(H^1(B) \to H^1(B \cap (S^2 \times \{0,1\}))) \cong \mathbb{Z}$), and so that, if we orient Σ as the boundary of A then $\langle \alpha \cup \beta, [\Sigma] \rangle = 1$ (or equivalently, if we orient Σ as the boundary of B then $\langle \beta \cup \alpha, [\Sigma] \rangle = 1$). This assignment is clearly natural.

Given cobordisms $\Sigma' \colon C_2 \to C_1$ and $\Sigma \colon C_1 \to C_0$, we need to construct a natural 2-isomorphism $V_e(\Sigma) \circ V_e(\Sigma') \to V_e(\Sigma \circ \Sigma')$. The only non-trivial case is when Σ and Σ' are genus-0 cobordisms gluing to form a connected, genus-1 cobordism. In that case, letting $x \in V_e(C_2)$ (respectively, $y \in V_e(C_0)$) denote the generator that labels all circles of C_2 by x_- (respectively, all circles of C_0 by x_+), we need to construct a bijection between the (y,x)-entry $M_{y,x}$ of $V_e(\Sigma) \circ V_e(\Sigma')$ and the (y,x)-entry $N_{y,x}$ of $V_e(\Sigma \circ \Sigma')$. Consider an element Z of $M_{y,x}$; Z specifies an element $z \in V(C_1)$. There is a unique circle C in C_1 that is non-separating in $\Sigma \circ \Sigma'$ and is labeled x_+ by z. Choose an orientation o of $\Sigma \circ \Sigma'$, orient C as the boundary of Σ, and let $[C]$ denote the image of C in $H_1(\Sigma \circ \Sigma', \partial(\Sigma \circ \Sigma'))$. Assign to Z the unique basis in $N_{y,x}$ that contains the Poincaré dual of $[C]$. It is easy to check that this map is well-defined, independent of the choice of o, natural, and a bijection.

This concludes the definition of the functor $V_e \colon \mathsf{Cob}_e^{1+1} \to \mathscr{B}$. The spatial lift $Q_{Kh} \in \mathfrak{h}\mathcal{T}(S^N)^{[2]^n}$ is then induced from the composition $V_e \circ \mathbb{L}_e \colon [2]^n \to \mathscr{B}$. Totalization produces a cell complex $\mathsf{Tot}(Q_{Kh})$ with

$$\widetilde{C}^*_{\text{cell}}(\mathsf{Tot}(Q_{Kh}))[N+n_-] = \mathcal{C}^*_{Kh}(L). \tag{3-8}$$

We define the Khovanov spectrum $\mathcal{X}_{Kh}(L)$ to be the formal $(N+n_-)^{\text{th}}$ desuspension of $\mathsf{Tot}(Q_{Kh})$. The stable homotopy type of $\mathcal{X}_{Kh}(L)$ is a link invariant; see Lipshitz and Sarkar [2014a], Hu, D. Kriz, and I. Kriz [2016], and Lawson, Lipshitz, and Sarkar [n.d.(a)]. The spectrum decomposes as a wedge sum over quantum gradings, $\mathcal{X}_{Kh}(L) = \bigvee_j \mathcal{X}^j_{Kh}(L)$. There is also a reduced version of the Khovanov stable homotopy type, $\widetilde{\mathcal{X}}_{Kh}(L)$, refining the reduced Khovanov chain complex.

3.3 Properties and applications. In order to apply the Khovanov homotopy type to knot theory, one needs to extract some concrete information from it beyond Khovanov homology. Doing so, one encounters three difficulties:

☹1. The number of vertices of the Khovanov cube is 2^n, where n is the number of crossings of L, so the number of cells in the CW complex $\mathcal{X}_{Kh}(L)$ grows at least that fast. So, direct computation must be by computer, and for relatively low crossing number links.

☹2. For low crossing number links, $Kh^{i,j}(L)$ is supported near the diagonal $2i - j = \sigma(L)$, so each $\mathcal{X}^j_{Kh}(L)$ has nontrivial homology only in a small number of adjacent gradings, and these $Kh^{i,j}(L)$ have no p-torsion for $p > 2$. If X is a spectrum so that $\widetilde{H}^i(X)$ is nontrivial only for $i \in \{k, k+1\}$ then the homotopy type of X is determined by $\widetilde{H}^*(X)$, while if $\widetilde{H}^*(X)$ is nontrivial in only three adjacent gradings and has no p-torsion ($p > 2$) then the homotopy type of X is determined by $\widetilde{H}^*(X)$ and the Steenrod operations Sq^1 and Sq^2 (see Baues [1995, Theorems 11.2, 11.7]).

☹3. There are no known formulas for most algebro-topological invariants of a CW complex. (The situation is a bit better for simplicial complexes.)

Lipshitz and Sarkar [2014c] found an explicit formula for the operation $\mathrm{Sq}^2 \colon Kh^{i,j}(L; \mathbb{F}_2) \to Kh^{i+2,j}(L; \mathbb{F}_2)$. The operation Sq^1 is the Bockstein, and hence easy to compute. Using these, one can determine the spectra $\mathcal{X}^j_{Kh}(L)$ for all prime links up to 11 crossings. All these spectra are wedge sums of (de)suspensions of 6 basic pieces (cf. ☹2), and all possible basic pieces except $\mathbb{C}P^2$ occur (see ?1). The first knot for which $\mathcal{X}^{i,j}_{Kh}(K)$ is not a Moore space is also the first non-alternating knot: $T(3,4)$. Extending these computations:

Theorem 1 (Seed [n.d.]). *There are pairs of knots with isomorphic Khovanov cohomologies but non-homotopy equivalent Khovanov spectra.*

The first such pair is 11^n_{70} and 13^n_{2566}. D. Jones, Lobb, and Schütz [n.d.(b)] introduced moves and simplifications allowing them to give a by-hand computation of Sq^2 for $T(3,4)$ and some other knots.

Theorem 2 (Lawson, Lipshitz, and Sarkar [n.d.(a)]). *Given links L, L', $\mathcal{X}_{Kh}(L \amalg L') \simeq \mathcal{X}_{Kh}(L) \wedge \mathcal{X}_{Kh}(L')$ and, if L and L' are based, $\widetilde{\mathcal{X}}_{Kh}(L \# L') \simeq \widetilde{\mathcal{X}}_{Kh}(L) \wedge \widetilde{\mathcal{X}}_{Kh}(L')$ and $\mathcal{X}_{Kh}(L \# L') \simeq \mathcal{X}_{Kh}(L) \wedge_{\mathcal{X}_{Kh}(U)} \mathcal{X}_{Kh}(L')$. Finally, if $m(L)$ is the mirror of L then $\mathcal{X}_{Kh}(m(L))$ is the Spanier-Whitehead dual to $\mathcal{X}_{Kh}(L)$.*

Corollary 3.1. *For any integer k there is a knot K so that the operation $\mathrm{Sq}^k \colon Kh^{*,*}(K) \to Kh^{*+k,*}(K)$ is nontrivial. (Compare ?3.)*

Proof. Choose a knot K_0 so that in some quantum grading, $\widetilde{Kh}(K_0)$ has 2-torsion but $\widetilde{Kh}(K_0; \mathbb{F}_2)$ has vanishing Sq^i for $i > 1$. (For instance, $K_0 = 13^n_{3663}$ works, Shumakovitch [2014].) Let $K = \overbrace{K_0 \# \cdots \# K_0}^{k}$. By the Cartan formula, $\mathrm{Sq}^k(\alpha) \neq 0$ for some $\alpha \in \widetilde{Kh}(K; \mathbb{F}_2)$. The short exact sequence

$$0 \to \widetilde{Kh}(K; \mathbb{F}_2) \to Kh(K; \mathbb{F}_2) \to \widetilde{Kh}(K; \mathbb{F}_2) \to 0$$

from Rasmussen [2005, §4.3] is induced by a cofiber sequence of Khovanov spectra from Lipshitz and Sarkar [2014a, §8], so if $\beta \in Kh(K; \mathbb{F}_2)$ is any preimage of α then by naturality, $\mathrm{Sq}^k(\beta) \neq 0$, as well. □

Plamenevskaya [2006] defined an invariant of links L in S^3 transverse to the standard contact structure, as an element of the Khovanov homology of L.

Theorem 3 (Lipshitz, Ng, and Sarkar [2015]). *Given a transverse link L in S^3 there is a well-defined cohomotopy class of $\mathcal{X}_{Kh}(L)$ lifting Plamenevskaya's invariant.*

While Lipshitz, Ng, and Sarkar [ibid.] show that Plamenevskaya's class is known to be invariant under flypes, the homotopical refinement is not presently known to be. It remains open whether either invariant is effective (i.e., stronger than the self-linking number).

The Steenrod squares on Khovanov homology was used by Lipshitz and Sarkar [2014b] to tweak the concordance invariant and slice-genus bound s by Rasmussen [2010] to give potentially new concordance invariants and slice genus bounds. In the simplest case, Sq^2, these concordance invariants are, indeed, different from Rasmussen's invariants. They can be used to give some new results on the 4-ball genus for certain families of knots, see Lawson, Lipshitz, and Sarkar [n.d.(a)]. More striking, Feller, Lewark, and Lobb [n.d.] used these operations to resolve whether certain knots are *squeezed*, i.e., occur in a minimal-genus cobordism between positive and negative torus knots.

In a different direction, the Khovanov homotopy type admits a number of extensions. Lobb, Orson, and Schütz [2017] and, independently, Willis [n.d.] proved that the Khovanov homotopy type stabilizes under adding twists, and used this to extend it to a colored Khovanov stable homotopy type; further stabilization results were proved by Willis [ibid.] and Islambouli and Willis [n.d.]. D. Jones, Lobb, and Schütz [n.d.(a)] proposed a homotopical refinement of the $\mathfrak{sl}_n$ Khovanov-Rozansky homology for a large class of knots and there is also work in progress in this direction by Hu, I. Kriz, and Somberg [n.d.]. Sarkar, Scaduto, and Stoffregen [n.d.] gave a homotopical refinement of the odd Khovanov homology of Ozsváth, Rasmussen, and Szabó [2013].

The construction of the functor V_e is natural enough that it was used by Lawson, Lipshitz, and Sarkar [n.d.(b)] to give a space-level refinement of the arc algebras and tangle invariants from Khovanov [2002]. In the refinement, the arc algebras are replaced by ring spectra (or, if one prefers, spectral categories), and the tangle invariants by module spectra.

3.4 Speculation.

We conclude with some open questions:

?1. Does $\mathbb{C}P^2$ occur as a wedge summand of the Khovanov spectrum associated to some link? (Cf. Section 3.3.) More generally, are there non-obvious restrictions on the spectra which occur in the Khovanov homotopy types?

?2. Is the obstruction to amphichirality coming from the Khovanov spectrum stronger than the obstruction coming from Khovanov homology? Presumably the answer is "yes," but verifying this might require interesting new computational techniques.

ⓘ3. Are there prime knots with arbitrarily high Steenrod squares? Other power operations? Again, we expect that the answer is "yes."

ⓘ4. How can one compute Steenrod operations, or stable homotopy invariants beyond homology, from a flow category? (Compare Lipshitz and Sarkar [2014c].)

ⓘ5. Is the refined Plamenevskaya invariant from Lipshitz, Ng, and Sarkar [2015] effective? Alternatively, is it invariant under negative flypes / SZ moves?

ⓘ6. Is there a well-defined homotopy class of maps of Khovanov spectra associated to an isotopy class of link cobordisms $\Sigma \subset [0,1] \times \mathbb{R}^3$? Given such a cobordism Σ in general position with respect to projection to $[0,1]$, there is an associated map, but it is not known if this map is an isotopy invariant. More generally, one could hope to associate an $(\infty, 1)$-functor from a quasicategory of links and embedded cobordisms to a quasicategory of spectra, allowing one to study families of cobordisms. If not, this is a sense in which Khovanov homotopy, or perhaps homology, is *un*natural. Applications of these cobordism maps would also be interesting (cf. Swann [2010]).

ⓘ7. If analytic difficulties are resolved, applying the Cohen-Jones-Segal construction to the symplectic Khovanov homology of Seidel and Smith [2006] should also give a Khovanov spectrum. Is that symplectic Khovanov spectrum homotopy equivalent to the combinatorial Khovanov spectrum? (Cf. Abouzaid and Smith [n.d.].)

ⓘ8. The (symplectic) Khovanov complex admits, in some sense, an $O(2)$-action, cf. Manolescu [2006], Seidel and Smith [2010], Hendricks, Lipshitz, and Sarkar [n.d.], and Sarkar, Seed, and Szabó [2017]. Does the Khovanov stable homotopy type?

ⓘ9. Is there a homotopical refinement of the Lee [2005] or Bar-Natan [2005] deformation of Khovanov homology? Perhaps no genuine spectrum exists, but one can hope to find a lift of the theory to a module over *ku* or *ko* or another ring spectrum (cf. Cohen [2009]). Exactly how far one can lift the complex might be predicted by the polarization class of a partial compactification of the symplectic Khovanov setting from Seidel and Smith [2006].

ⓘ10. Can one make the discussion in Section 2.5 precise? Are there other rig (or ∞-rig) categories, beyond Sets, useful in refining chain complexes in categorification or Floer theory to get modules over appropriate ring spectra?

ⓘ11. Is there an intrinsic, diagram-free description of $\mathcal{X}_{Kh}(K)$ or, for that matter, for Khovanov homology or the Jones polynomial?

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