Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 2 (43–62)

# TOWARDS A THEORY OF DEFINABLE SETS

Stephen Jackson

**Abstract**

The subject of descriptive set theory is traditionally concerned with the theory of definable subsets of Polish spaces. By introducing large cardinal/determinacy axioms, a theory of definable subsets of Polish spaces and their associated ordinals has been developed over the last several decades which extends far up in the definability hierarchy. Recently, much interest has been focused on trying to extend the theory of definable objects to more general types of sets, not necessarily subsets of a Polish space or an ordinal. A large class of these objects are represented by equivalence relations on Polish spaces. Even for some of the simpler of these relations, an interesting combinatorial theory is emerging. We consider both problems of extending further the theory of definable subsets of Polish spaces, and that of determining the structure of these new types of definable sets.

## 1 Introduction and background

The field of descriptive set theory traditionally in concerned with the theory of definable sets in Polish spaces (complete, separable, metric spaces). As all uncountable Polish spaces are isomorphic by a Borel function, it is customary to refer to the elements of any of several standard Polish spaces as "reals." Aside from $\mathbb{R}$, familiar examples include the Baire space $\omega^\omega$ (homeomorphic to the space of irrationals in $\mathbb{R}$; here $\omega$ denotes the set of natural numbers), and $2^\omega$ (homeomorphic to the Cantor set in $\mathbb{R}$; here $2 = \{0, 1\}$). In the latter two cases, $\omega$ is endowed with the discrete topology, and $\omega^\omega$ or $2^\omega$ with the product topology. Note that if $G$ is any countable discrete group, then $2^G$ is likewise homeomorphic to the Cantor set, so it is naturally a compact Polish space.

Using the axiom of choice, $\mathsf{AC}$, "pathological" sets with a variety of properties can be constructed. Examples include Vitali sets (non-measurable sets), Bernstein sets (a set such that neither the set nor its complement contains a closed uncountable set), Lusin sets (a set of reals which meets every meager set in a countable set), and Sierpinski sets (a set

*MSC2010:* primary 03E15; secondary 03E60.

which meets every measure 0 set in a countable set). A theme of descriptive set theory is that if we restrict out attention to "definable" sets, then these pathologies disappear and a reasonable structure theory emerges. The notion of definable is made precise through hierarchies of collections of sets of increasing complexity. A *pointclass* $\boldsymbol{\Gamma}$ is a collection of subsets of Polish spaces which is closed under inverse images by continuous functions, that is, if $f : X \to Y$ is continuous and $A \subseteq Y$ is in $\boldsymbol{\Gamma}$, then $f^{-1}(A)$ is also in $\boldsymbol{\Gamma}$. A basic example is the pointclass of Borel sets, the smallest collection containing the open and closed sets and closed under countable unions and intersections. The Borel sets are stratified into the *Borel hierarchy*, the pointclasses $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, and $\boldsymbol{\Delta}^0_\alpha$, for $\alpha < \omega_1$. Here $\boldsymbol{\Sigma}^0_1 \restriction X$ is the collection of open sets in the Polish space $X$, $\boldsymbol{\Pi}^0_1 \restriction X$ the closed sets in $X$, $\boldsymbol{\Delta}^0_1 \restriction X = \boldsymbol{\Sigma}^0_1 \restriction X \cap \boldsymbol{\Pi}^0_1 \restriction X$, and in general $A \in \boldsymbol{\Sigma}^0_\alpha$ if $A = \bigcup_n A_n$ where each $A_n \in \boldsymbol{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$. Likewise, $A \in \boldsymbol{\Pi}^0_\alpha$ if $A = \bigcap_n A_n$ with each $A_n \in \boldsymbol{\Sigma}^0_{\beta_n}$ for some $\beta_n < \alpha$. Also, we define $\boldsymbol{\Delta}^0_\alpha = \boldsymbol{\Sigma}^0_\alpha \cap \boldsymbol{\Pi}^0_\alpha$. It is a classical fact that the Borel sets in any Polish space have the perfect set property (if they are uncountable then they contain a perfect set, or equivalently an uncountable closed set), and are Lebesgue measurable and have the Baire property. Thus, they cannot be any of the above types of pathological sets. Another example of a "regularity property" for sets is the *Ramsey property* for the set $A \subseteq [\omega]^\omega$ (here $[H]^\omega$ denotes the set of infinite subsets of $H$, which we can identify with the set of increasing functions from $\omega$ to $\omega$) which asserts that there is an infinite set $H \subseteq \omega$ such that either $[H]^\omega \subseteq A$ or $[H]^\omega \subseteq \omega^\omega - A$. Again, all Borel sets have this regularity property, this being a theorem of Galvin and Prikry (in fact the Borel sets are *completely Ramsey*, a somewhat stronger version of the Ramsey property).

The hierarchy of definable sets extends far beyond the Borel sets. The next hierarchy after the Borel sets is the *projective hierarchy*, so called because the main operation used in generating the hierarchy is projection from a product $X \times Y$ of Polish spaces to $X$. The *analytic*, or $\boldsymbol{\Sigma}^1_1$ sets, are defined by projecting closed (or equivalently Borel) sets: $A \subseteq X$ is $\boldsymbol{\Sigma}^1_1$ iff there is a closed set $F \subseteq X \times \omega^\omega$ such that $x \in A$ iff $\exists y\ (x, y) \in F$. In more succinct notation, we write $\boldsymbol{\Sigma}^1_1 \restriction X = \exists^{\omega^\omega} \boldsymbol{\Pi}^0_1 \restriction (X \times \omega^\omega)$, where $\exists^Y$ denotes the operation of applying existential quantification over $Y$. A set $A \subseteq X$ is *co-analytic*, or $\boldsymbol{\Pi}^1_1$, if it is the complement of an analytic set. That is, $\boldsymbol{\Pi}^1_1$ is the *dual pointclass* of $\boldsymbol{\Sigma}^1_1$, which we write as $\boldsymbol{\Pi}^1_1 = \check{\boldsymbol{\Sigma}}^1_1$. We set $\boldsymbol{\Delta}^1_1 = \boldsymbol{\Sigma}^1_1 \cap \boldsymbol{\Pi}^1_1$ to be the sets that are both $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$. A classical theorem of Suslin states that $\boldsymbol{\Delta}^1_1$ is the collection of Borel sets, so the projective hierarchy begins where the Borel hierarchy ends. We continue to define the $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, and $\boldsymbol{\Delta}^1_n$ sets for all $n \in \omega$ by setting $\boldsymbol{\Sigma}^1_{n+1} = \exists^{\omega^\omega} \boldsymbol{\Pi}^1_n$, $\boldsymbol{\Pi}^1_{n+1} = \check{\boldsymbol{\Sigma}}^1_{n+1}$ (or equivalently, $\boldsymbol{\Pi}^1_{n+1} = \forall^{\omega^\omega} \boldsymbol{\Sigma}^1_n$), and $\boldsymbol{\Delta}^1_{n+1} = \boldsymbol{\Sigma}^1_{n+1} \cap \boldsymbol{\Pi}^1_{n+1}$. The projective hierarchy is important because it includes all of the sets of conventional analysis. In fact, the sets of analysis generally occur at the first or second levels of this hierarchy. In any uncountable Polish space, all of the levels of the Borel and projective hierarchies are distinct, that is,

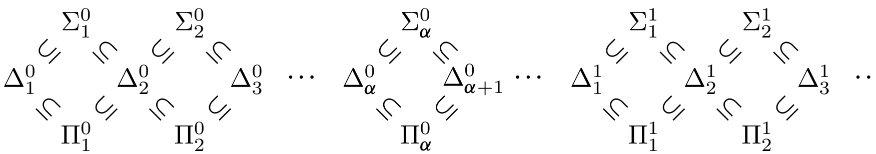


Figure 1: The Borel and projective hierarchies.

there is no collapsing in either of these hierarchies. The inclusions of pointclasses within these hierarchies is shown in Figure 1.

Beginning with the fundamental work of Gödel, it was realized that there were strong limits to how much further far one could extend the regularity results for Borel sets working just in ZFC set theory (the set theory of "ordinary mathematics"). For example, the $\boldsymbol{\Sigma}^1_1$ sets have the perfect set property, but it is consistent with ZFC that the $\boldsymbol{\Pi}^1_1$ sets do not. Likewise, while the $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ sets are all Lebesgue measurable and have the Baire property, it is consistent with ZFC that the collection of $\boldsymbol{\Delta}^1_2$ sets does not. A theorem of Silver asserts that the $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ sets are all (completely) Ramsey, but it is again consistent that there are $\boldsymbol{\Delta}^1_2$ sets which are not. Thus, in order to extend the theory further, one must assume additional axioms which go beyond the ZFC axioms. There are currently two main axiom schemes for doing this: large cardinal axioms and determinacy axioms. Large cardinal axioms, which are generally meant to be added to the ZFC axioms, assert that cardinals $\kappa$ with certain properties exist which cannot be shown to exist just from ZFC. Determinacy axioms, on the other hand, assert that certain two-player games are determined. If $A \subseteq \omega^\omega$, then we associate a two-player integer game $G(A)$ to $A$ in a natural way: the players I and II alternate picking integers $x(n) \in \omega$ as shown in Figure 2. They thereby jointly build an $x = (x(0), x(1), \dots) \in \omega^\omega$. Player I wins the game iff $x \in A$.

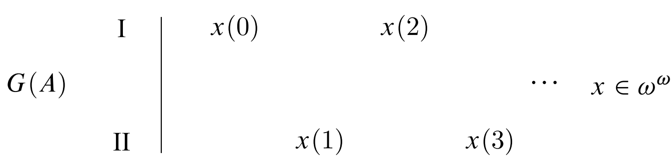


Figure 2: The basic game $G(A)$.

The notions of a winning strategy for one of the players, and of the game being determined (i.e., one of the players has a winning strategy) are defined in the natural manner. The axiom of determinacy, AD, is the assertion that $G(A)$ is determined for all $A \subseteq \omega^\omega$.

This axiom contradicts the axiom of choice, $\mathsf{AC}$, but is meant to be an axiom for certain inner models of the full universe of sets $V$ for which the sets are, in some sense, definable. If $\mathbf{\Gamma}$ is a pointclass, then $\det(\mathbf{\Gamma})$ is the statement that $G(A)$ is determined for all $A \subseteq \omega^\omega$ in $\mathbf{\Gamma}$. A celebrated theorem of Martin says all Borel games are determined. More generally, if $X$ is any set and $A \subseteq X^\omega$ is Borel in the product topology on $X^\omega$, where $X$ is given the discrete topology, then (in $\mathsf{ZFC}$) the game $G(A)$ is determined. In fact, a version of this result holds in $\mathsf{ZF}$ (without choice), where now every every Borel game $G(A)$ for $A \subseteq X^\omega$ is *quasi-determined* (see Moschovakis [1980]). Here a quasi-strategy is like a strategy except it is multi-valued. Results of Martin and Harrington show that $\mathsf{ZFC} + \det(\mathbf{\Sigma}^1_1)$ is strictly stronger than $\mathsf{ZFC}$. The assertions $\det(\mathbf{\Sigma}^1_n)$ are strictly increasing in strength, and projective determinacy, $\mathsf{PD}$, is the statement that all projective games are determined. The model $L(\mathbb{R})$ is the smallest model of set theory containing all of the reals and ordinals. Every set of reals in this model is definable in this model by a formula using only ordinal and real parameters. Because of this, the axiom $\mathsf{AD}^{L(\mathbb{R})}$ that all $A \in L(\mathbb{R})$ are determined is a plausible axiom. Since strategies in integer games are essentially reals, if $\mathsf{AD}^{L(\mathbb{R})}$ holds then in fact $L(\mathbb{R})$ satisfies the axiom $\mathsf{AD}$. The model $L(\mathbb{R})$ is thus the smallest candidate inner model (containing all the ordinals) which satisfies $\mathsf{AD}$. More generally, if $M$ is any inner model satisfying $\mathsf{AD}$, then we may consider the sets in $M$ as being, in some abstract sense, definable.

Work of Martin, Steel, and Woodin in the 80's established the precise connection between determinacy axioms and large cardinal axioms. It was shown, for example, that $\mathsf{ZFC} + \exists n$ Woodin cardinals $+\exists$ a larger measurable cardinal implies $\det(\mathbf{\Sigma}^1_{n+1})$. Also, $\mathsf{AD}$ is equiconsistent with the existence of $\omega$ many Woodin cardinals, and $\mathsf{AD}^{L(\mathbb{R})}$ is implied by the existence of $\omega$ many Woodin cardinals and a measurable cardinal above them. These fundamental results lend respectability to the determinacy axioms and show that not only is $\mathsf{AD}^{L(\mathbb{R})}$ an intuitively appealing axiom, but that it is actually implied by large cardinal axioms out in the $\mathsf{ZFC}$ universe.

Moving past $\mathsf{AD}$, we many consider the axiom $\mathsf{AD}_X$ which asserts that every game played on the set $X$, that is, where $A \subseteq X^\omega$, is determined. This axiom is inconsistent for $X = \omega_1$ or $X = \mathcal{P}(\mathbb{R})$, but for $X = \mathbb{R}$ the axiom of real game determinacy $\mathsf{AD}_\mathbb{R}$ is reasonable. This axiom is significantly stronger than $\mathsf{AD}$, and cannot hold in the minimal model $L(\mathbb{R})$ of $\mathsf{AD}$, or in any model of the form $L(T, \mathbb{R})$, for $T \subseteq \mathrm{On}$. Woodin has identified the exact consistency strength of $\mathsf{AD}_\mathbb{R}$ in the large cardinal hierarchy as well. Thus we have a progression of the determinacy axioms starting with $\det(\mathbf{\Delta}^1_1)$, which is a theorem of $\mathsf{ZFC}$, to $\det(\mathbf{\Sigma}^1_1)$, $\det(\mathbf{\Sigma}^1_n)$, $\mathsf{PD}$, $\mathsf{AD}$, and $\mathsf{AD}_\mathbb{R}$. There are also stronger determinacy axioms than $\mathsf{AD}_\mathbb{R}$ involving "long games," but we will not need to consider these here. We note that in any model of $\mathsf{AD}$ the sets of reals fall into a single hierarchy, called the Wadge hierarchy, which gives a far-reaching generalization of the Borel and projective hierarchies. In particular, in these models we can define the $\mathbf{\Sigma}^1_\alpha$, $\mathbf{\Pi}^1_\alpha$ classes for all $\alpha < \Theta$, where $\Theta$ is the length of the Wadge hierarchy in the model. Thus, these higher level analogs of the projective sets are defined and extend throughout the entire Wadge hierarchy of sets of reals.

Beginning in the 60's, and continuing to the present, it was realized that determinacy axioms were a powerful tool which allowed the classical results for Borel and analytic sets to be extended to larger classes of sets. Work of Kechris, Martin, Moschovakis, Solovay, Steel, Woodin, and others showed that assuming determinacy axioms, and in particular assuming AD, one could propagate a structural theory similar to the ZFC theory of Borel and analytic sets. This theory is largely presented in terms of *scales* and *Suslin cardinals*, and gives a tight connection between the theory of the sets of reals in a pointclass $\mathbf{\Gamma}$ and the properties of an ordinal $\delta(\mathbf{\Gamma})$ associated to the pointclass. The notion of a scale was isolated by Moschovakis, and has origins in the Novikov-Kondo proof of $\mathbf{\Pi}^1_1$ uniformization. We recall the following definition. By a tree $T$ on a set $X$ we mean a $T \subseteq X^{<\omega}$ which is closed under subsequence, that is, if $s \in T$ and $m$ is less than the length of $s$, then $s \restriction m \in T$. We let $[T] = \{x \in X^\omega \colon \forall n\; x \restriction n \in T\}$ be the set of infinite branches (or body) of $T$.

**Definition 1.** We say a set $A \subseteq \omega^\omega$ is $\kappa$-Suslin, for $\kappa \in \mathrm{On}$, if there is a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ such that $A = p[T] = \{x \in \omega^\omega \colon \exists f \in \kappa^\omega\; (x, f) \in [T]\}$.

We say $\kappa$ is a Suslin cardinal if there is a set $A$ which is $\kappa$-Suslin but not $\lambda$-Suslin for any $\lambda < \kappa$. The notions of semi-scale and scale are a more algebraic reformulation of having Suslin representations, presented in terms of norms $\varphi_n \colon A \to \kappa$. In fact, being $\kappa$-Suslin is equivalent to having a semi-scale with norms to $\kappa$, and also equivalent to having a scale with norms to $\kappa$. We refer the reader to Moschovakis [ibid.] for the precise definitions of semi-scales, scales, and the scale property for a pointclass.

Assuming AD, we can propagate the scale/Suslin cardinal analysis past the $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$ levels to the entire projective hierarchy and beyond. In Jackson [2010] one can find a presentation of the complete scale and Suslin cardinal analysis from AD. This analysis, though it extends throughout the full extent of the Suslin cardinals, presents the theory in terms of the ordinals $\delta(\mathbf{\Gamma})$. A much more detailed inductive analysis is necessary to analyze these ordinals and describe the cardinal structure below them. In Jackson [1999] and Jackson [1988] this analysis is described through the projective hierarchy. The extent of this analysis is currently far short of the extent of scales, and so much about the general cardinal structure of determinacy models remains unknown.

To give one example of the consequences of this analysis, we first recall that it is classical fact (proved in ZFC) that every $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ set is $\omega_2$-Borel, that is, is in the smallest collection containing the open and closed sets and closed under unions and intersections of length $< \omega_2$. In fact, every $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ set is an $\omega_1$ union of Borel sets (a proof can be found in Moschovakis [1980], A. S. Kechris [1978], or Jackson [2010]). From the above mentioned inductive analysis we get the following extension of this result, assuming determinacy holds for the sets in $L(\mathbb{R})$ (see Jackson [1989]).

**Theorem 2.** *Assume* $\mathsf{ZFC} + \mathsf{AD}^{L(\mathbb{R})}$*. Then every projective set is* $\omega_\omega$*-Borel.*

Moving forward, in trying to develop the theory of definable objects from stronger set-theoretic axioms, there are two main directions to pursue. The first is to extend this theory of sets of reals and their associated ordinals further, and to to attempt to describe the entire cardinal structure of determinacy models. We might refer to this as extending the theory of "reals and ordinals." A second direction is to study more general types of objects, moving past those that be identified with sets in a Polish space or wellordered sets. Of course, the study of these more general definable objects encompasses the first direction, but the point is that we can advance the study of these more general objects without having the complete theory of the cardinals structure in hand.

In Section 2 we describe in a little more detail some of the progress in developing the theory of "reals and ordinals" and problems that are reasonably aligned with this program. We describe some of the recent progress various researchers have made, in particular using new techniques from inner model theory. This emerging area of "descriptive inner model theory" holds much promise for future progress in this area. We also mention some of the old questions and conjectures which are still around and which may serve as a benchmark for further progress. In Section 3 we consider some questions related to more general types of objects. Here we see an interesting and fascinating combinatorial structure beginning to emerge. The focus here is not so much on extending the theory to higher and higher pointclasses, but to understand how the new nature of these objects affects their combinatorial structure. Thus, we frequently consider problems at the Borel level, where the sets and functions used in the definitions of the objects are Borel, or even continuous/clopen. Recent years have seen a growing interest in this study of "Borel combinatorics" and its connections with other areas such as ergodic theory, geometric group theory, and descriptive set theory.

## 2 The theory of reals and ordinals

A well-known consequence of $\mathsf{AD}$ is that all sets of reals have the perfect set property, are measurable (with respect to any Borel measure), and have the Baire property. It follows that we have the Fubini theorem and its analog for category, the Kuratowski-Ulam theorem, for arbitrary sets $A \subseteq X \times Y$ in products of Polish spaces. We then also have full additivity of measure and category, that is, an arbitrary well-ordered union of meager (or measure 0) sets is meager (measure 0). In particular, from the perfect set property we have that there are only two possibilities for the cardinality of a set in reals in a determinacy model: countable and the size of the continuum. We note that one must be careful with the term "cardinality" in a model without AC as, for example a map from a set $X$ onto a set $Y$ does not necessarily yield a map from $Y$ into $X$ (in a model of AD there are maps from $\mathbb{R}$ onto any ordinal $\alpha < \Theta$, which is very large in the $\aleph_\beta$ hierarchy, but there is only an injection from $\alpha$ to $\mathbb{R}$ if $\alpha$ is countable). Nevertheless, if a set of reals contains a perfect set, then it is in bijection with $\mathbb{R}$.

The cardinal structure inside a model of determinacy is interesting and non-trivial. As we indicated before, the cardinal structure is closely connected with certain associated pointclasses. At the projective level, the ordinals associated to these classes are called the *projective ordinals*. More precisely, let

$$\delta^1_n = \delta(\mathbf{\Pi}^1_n) = \sup\{|\preceq| \colon \preceq \text{ is a prewellordering of } \omega^\omega \text{ in } \mathbf{\Delta}^1_n\}$$

where a prewellordering $\preceq$ is a reflexive, transitive, connected relation whose strict part $(x \prec y \leftrightarrow (x \preceq y) \wedge \neg(y \preceq x))$ is wellfounded and $|\preceq|$ denotes its length. The work of Kechris, Kunen, Martin, Moschovakis, and Solovay established the basic properties of the $\boldsymbol{\delta}^1_n$, and computed their values for $n \leq 4$ (these results can be found in A. S. Kechris [1978] of Jackson [2010]). The author computed their values for all $n$ and described the structure of the cardinals below their supremeum (c.f. Jackson [1999] and Jackson [1988]). The Suslin cardinals below their supremum are the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ and their cardinal predecessors $\lambda^1_{2n+1} = (\boldsymbol{\delta}^1_{2n+1})^-$. The $\boldsymbol{\delta}^1_{2n+1}$-Suslin sets are the $\mathbf{\Sigma}^1_{2n+2}$ sets, and the $\lambda^1_{2n+1}$-Suslin sets are the $\mathbf{\Sigma}^1_{2n+1}$ sets. The cardinal structure below the supremum of the projective ordinals reveals some interesting and delicate patterns. The projective ordinals are all regular cardinals and the even ones are the successors of the odd ones, $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ (this was known from early work), and there are exactly $2^{n+1}-1$ many regular cardinals strictly between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$. The other cardinals between these two odd projective ordinals are all singular of cofinality one of these $2^{n+1}-1$ cardinals. The values of the $\boldsymbol{\delta}^1_n$ can be computed exactly. The result is that $\boldsymbol{\delta}^1_{2n+1} = \omega_{e(2n-1)+1}$, where $e(0) = 1$ and $e(n+1) = \omega^{e(n)}$ (ordinal exponentiation). Also, the exact cofinalities of the cardinals below the projective ordinals can be computed (see Jackson and Khafizov [2016] and Jackson and Löwe [2013]). Figure 3 shows some of the cardinal structure below the projective ordinals. Note that the three regular cardinals between $\boldsymbol{\delta}^1_3$ and $\boldsymbol{\delta}^1_5$ are $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}, \omega_{\omega\cdot 2+1}$, and $\omega_{\omega^\omega+1}$.

The detailed inductive analysis which provides the above mentioned analysis of cardinal cofinalities does not currently generalize to arbitrary levels of the Wadge hierarchy. While it does extend past the projective sets, likely to the first weakly inaccessible cardinal, it is known that the methods do not extend to the first "inductive like" pointclass (a non-selfdual pointclass closed under real quantification). Thus, some questions about cardinal cofinalities are still open past the projective hierarchy. To take an example, within

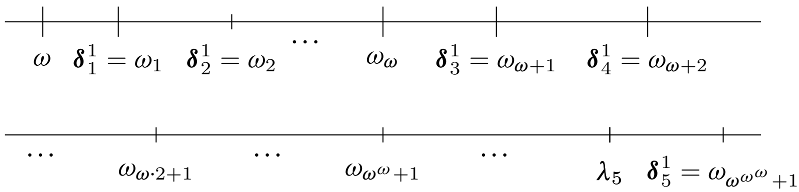


Figure 3: The cardinal structure below the projective ordinals.

the projective hierarchy there are never more than two regular cardinals in a row. Past the projective hierarchy, it is known that there can be three regular cardinals in a row (in Apter, Jackson, and B. Löwe [2013] this and stronger results are shown), however the evidence seems to suggest that there cannot be four in a row. This seems like a reasonable benchmark for understanding the cardinal structure, so we state:

**Conjecture 3.** Assuming $\mathsf{AD}$, there does not exists a cardinal $\kappa < \Theta$ such that $\kappa$, $\kappa^+$, $\kappa^{++}$, and $\kappa^{+++}$ are all regular.

Aside from the cofinalities of the cardinals, there are other interesting combinatorial properties of the cardinals which we may consider. One class of these concerns partition properties. In the Erdös-Rado partition notation we write $\kappa \to (\kappa)^\lambda$ if for all partitions $\mathcal{P} \colon \kappa^\lambda \to \{0,1\}$ of the increasing functions from $\lambda$ to $\kappa$ into two pieces, there is a set $H \subseteq \kappa$ of size $\kappa$ and an $i \in \{0,1\}$ such that $\mathcal{P} \restriction H^\lambda = i$. The statement that all sets $A \subseteq \omega^\omega$ are Ramsey is the strong partition property for $\kappa = \omega$. This follows from $\mathsf{AD}^+$, a technical strengthening of $\mathsf{AD}$ introduced by Woodin which has found many applications in determinacy theory (it is not known whether $\mathsf{AD}$ suffices for this result). Assuming $\mathsf{AD}$, the cardinal analysis shows that all regular Suslin cardinals below the projective ordinals, which are just the $\boldsymbol{\delta}^1_{2n+1}$, have the strong partition property. This is also not known to extend to arbitrary levels, so we ask:

**Conjecture 4.** Assuming $\mathsf{AD}$, every regular Suslin cardinal has the strong partition property.

It is shown in Jackson [2011] that for the (finitely many) regular cardinals $\kappa$ between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$ we have $\kappa \to (\kappa)^{\boldsymbol{\delta}^1_{2n+1}}$, but $\kappa \not\to (\kappa)^{\boldsymbol{\delta}^1_{2n+2}}$. This leads to the following general problem.

**Problem 5.** Assume $\mathsf{AD}$. Determine for each regular $\kappa < \Theta$ the $\lambda$ such that $\kappa \to (\kappa)^\lambda$.

We close this section by results concerning three large cardinal notions interpreted in models of determinacy. Namely, we consider the notions of Jónsson cardinal, measurable cardinal, and supercompact cardinal. For $\kappa < \lambda < \theta$, we say that $\kappa$ is $\lambda$-supercompact if there is a fine, normal measure $\nu$ on $\wp_\kappa(\lambda)$ (the subsets of $\lambda$ of size less than $\kappa$). Here fine means that if $A \in \wp_\kappa(\lambda)$ then $\{B : B \supseteq A\}$ has $\nu$ measure one. Normal means that if $f : \wp_\kappa(\lambda) \to \lambda$ is such that $f(A) \in A$ for $\nu$ almost all $A$, then $f$ is constant $\nu$ almost everywhere. Building on H. S. Becker and Jackson [2001] it is shown in Jackson [2001] that assuming $\mathsf{AD}^+$ that every regular cardinal $\kappa$ which is either a Suslin cardinal or the successor of a Suslin cardinal is $\lambda$-supercompact for all $\lambda < \Theta$. Again, this leads to a general problem:

**Problem 6.** Assume $\mathsf{AD}$. For which regular $\kappa$ and $\lambda > \kappa$ is $\kappa$ $\lambda$-supercompact?

We say a cardinal $\kappa$ is measurable if there is a non-principal, $\kappa$-complete ultrafilter $\mu$ of $\kappa$. We recall that assuming $\mathsf{AD}$, every ultrafilter on a set is countably additive, that is, is a measure. It is not difficult to show that the regular Suslin cardinals are measurable, but for general regular cardinals the problem seems to require new methods. Specifically, methods of inner model theory have begun to play an important role in determinacy theory. In Steel [1995] Steel made an important breakthrough by using progress in inner model theory to analyze the inner model $\mathrm{HOD}^{L(\mathbb{R})}$ of $L(\mathbb{R})$ assuming a large cardinal/determinacy hypothesis, which Woodin improved to just assuming $\mathsf{AD}^{L(\mathbb{R})}$. A consequence of this analysis is:

**Theorem 7** (Steel)**.** *Assume* $\mathsf{AD} + V = L(\mathbb{R})$. *Then every regular cardinal* $\kappa < \Theta$ *is measurable.*

As part of the HOD analysis, it is also shown that $\mathrm{HOD}^{L(\mathbb{R})}$ satisfies the GCH. Although this is a result about the model HOD, by "relativizing" it (i.e., using the fact that every set in $L(\mathbb{R})$ is definable from an ordinal and a real) we get:

**Theorem 8** (Steel)**.** *Assume* $\mathsf{AD} + V = L(\mathbb{R})$. *Then for any* $\kappa < \Theta$*, any wellordered sequence of subsets of* $\kappa$ *has length* $< \kappa^+$.

Again, the previous result was known to hold previously for Suslin cardinals $\kappa$. It is not known how to obtain either of the two previous theorems by direct determinacy arguments.

In a similar vein one can use the HOD analysis to prove a result concerning Jónsson cardinals. A cardinal $\kappa$ is said to be Jonsson if for all $f : \kappa^{<\omega} \to \kappa$ there an $A \subseteq \kappa$ with $|A| = \kappa$ such that $f(A^{<\omega}) \neq \kappa$. Thus, the Jonsson property is a weak form of the partition property, weaker than being measurable. From the HOD analysis we get Jackson, Ketchersid, Schlutzenberg, and Woodin [2014]:

**Theorem 9** (J, Ketchersid, Schlutzenberg, Woodin)**.** *Assume* $\mathsf{AD}^{L(\mathbb{R})}$. *Then every cardinal* $\kappa < \Theta$ *is Jónsson.*

Using additional arguments, Woodin has extended the last two theorems to models of $\mathsf{AD}^+$. It remains open if these techniques can be extended to answer Conjectures 3, 4.

## 3 More general sets

In Section 2 we considered the problem of developing the theory of definable sets of reals and ordinals. The theory at the lower levels of the definability hierarchy seems fairly well established, though many interesting problems remain in extending this theory to higher levels. As we described, this theory is developed assuming stronger axioms than ZFC. In this section we consider the problem of developing the theory of more general types of sets. To motivate some of the basic objects of study, consider the model $L(\mathbb{R})$. In this model, every set $A \subseteq L_\Theta(\mathbb{R})$ is the surjective image of $\mathbb{R}$. Say $\pi : \mathbb{R} \to A$ is an onto map. This defines naturally an equivalence relation $E$ on $\omega^\omega$, namely, $x\ E\ y$ iff $\pi(x) = \pi(y)$. It follows that $A$ is in bijection with the quotient space $\mathbb{R}/\sim$. So, all of the sets in this model, at least those of rank $< \Theta$, can be identified with the set of equivalence classes (quotient space) of an equivalence relation on the Polish space $\mathbb{R}$. Thus, the collection of sets which can be represented as quotient spaces by equivalence relations on Polish spaces is a quite large collection, greatly extending the collection of sets which can be identified with a subset of a Polish space, or which can be wellordered (identified with an ordinal).

The simplest equivalence relations on Polish spaces are the *smooth* ones. We say $(X, E)$ is smooth if there is a Borel map $f : X \to Y$, for some Polish space $Y$ such that $x\ E\ y$ iff $f(x) = f(y)$. We can, of course, replace "Borel" with more liberal notions of definable, but in most cases this is a good stand-in for the more general case. In this case, the quotient space $X/E$ can be identified with a subset of $Y$, namely the range of $f$. Conversely, any subset $A$ of a Polish space $X$ can be identified with the quotient space of a smooth equivalence relation on $X$. So, if $(X, E)$ is smooth, or if the classes can be wellordered, then we are in the case of Section 2. So, from the point of view of introducing new types of definable objects, we consider these to be "trivial" equivalence relations.

One of the simplest non-trivial equivalence relations is the equivalence relation of eventual agreement on $2^\omega$ known as $E_0$: $x\ E_0\ y$ iff $\exists n\ \forall m \geq n\ (x(m) = y(m))$. Note that the $E_0$ relation is a simple ($\boldsymbol{\Sigma}^0_2$) Borel equivalence relation. There is a natural action of the group $\mathbb{Z}$ on $2^\omega$ called the odometer action which is defined by $1 \cdot x$ (1 being the generator $\mathbb{Z}$) is obtained by adding 1 to $x$ viewed as an infinite binary expansion (with $x(0)$ being the least significant digit). This $\mathbb{Z}$ action is defined on all classes except $[\bar{0}]$ and $[\bar{1}]$, which are the constant 0 and 1 reals. The natural definition of the odometer map on the classes $[\bar{0}]$ and $[\bar{1}]$ amalgamates these two classes, but we can redefine the map on these two classes so that the $\mathbb{Z}$ action generates the $E_0$ equivalence relation, that is, $x\ E_0\ y$ iff $\exists n \in \mathbb{Z}\ (n \cdot x = y)$. The natural Bernoulli measure on $2^\omega$ is invariant under this action (the redefinition on the two distinguished classes doesn't affect anything). It follows that there cannot be a Borel, or even measurable, *selector* for $E_0$, that is, a set $S$ which meets every $E_0$ class in exactly one point. So, $E_0$ is not smooth. Of course, with AC one can form a selector by simply picking an element for each $E_0$ class, but this does not result in a definable set. This simple argument is just the standard Vitali argument for the construction of (non-definable) non-measurable set. From our current point of view, focusing on definable objects, the quotient space of $E_0$ is a new type of object, not given by a subset of a Polish space or an ordinal. This immediately raises a general question: what can we say about the structure of these definable equivalence relations on Polish spaces? As the above example shows, restricting the notion of definable to Borel still captures the main essence of the new phenomenon, and thus we led to the study of Borel equivalence relations on Polish spaces.

The motivation expressed in the above arguments for studying Borel equivalence relations is only one of many such possible. For example, classical dynamic can be viewed as the study of Borel actions of the group $\mathbb{Z}$ on Polish spaces, frequently equipped with other structure such an an invariant probability measure on $X$. From the point of view of "descriptive dynamics" however (a term likely coined by Kechris), we are not just interested in the structure up to measure zero sets, but rather what can be done everywhere in a definable (say Borel) manner. It is also of interest to restrict from Borel to continuous in many questions, that is, asking what can be done in continuous manner leads to interesting questions as well.

In the rest of this section we first give a brief (and selective) background on some results concerning Borel equivalence relations, and then describe some recent work on some problems in this area. We are particularly interested in problems concerning the combinatorial structure of these quotient spaces. We also mention some questions which arise when going past Borel equivalence relations to consider general equivalence relations in determinacy models.

If $G$ is a group acting on the Polish space $X$, then there is an equivalence relation $E_G$, the *orbit equivalence relation*, associated to the action: $x\ E_G\ y$ iff $\exists g \in G\ (g \cdot x = y)$. The case of interest is when $G$ is a Polish group (a topological group which is a Polish space in the group topology), and $G$ acts in a Borel way on $X$ (that is, the relation $R(g, x, y) \leftrightarrow g \cdot x = y$ is Borel). An important special of this is when $G$ is a countable discrete group, in which case $E_G$ is a *countable* Borel equivalence relation, that is, all of the $E_G$ classes are countable. In the case of a general Polish group, the relation $E_G$ need only be $\boldsymbol{\Sigma}^1_1$, though it is a fact that all of the individual orbits $[x]_{E_G}$ are Borel. When $G$ is countable, $E_G$ is Borel. Given any countable group $G$, a natural action is the (left) *shift action* of $G$ on $2^G$ defined by $g \cdot x(h) = x(g^{-1}h)$. The is a natural action and is also important as it is essentially a universal action of $G$ (we refer the reader to Dougherty, Jackson, and A. S. Kechris [1994] for details).

The theory largely splits in two directions: the case of general (uncountable) Polish groups, and the case where $G$ is countable. Both directions are interesting. For example, the Polish group $S_\infty$ of permutation of $\omega$ has a natural action, the *logic action* on the space of countable models of first-order theories (which can viewed as a Polish space). Various important questions in model theory/logic can be phrased as question about this Polish group action. One such question is the well-known Vaught conjecture on the number of models of a first-order theory (that is, is either countable or of size $c$), which can rephrased as a question about this action (we refer the reader to A. S. Kechris and H. Becker [1996] for details).

For the rest of this section we will focus on the case of countable $G$, which is illustrative of the general case and includes many cases of interest, particularly in relation to dynamics, ergodic theory, and some aspects of descriptive set theory (we note here that the degree notions of descriptive set theory such as Turing degree, arithmetical degrees, $\Delta^1_1$ degrees, etc., all give countable equivalence relations). We refer the reader to Dougherty, Jackson, and A. S. Kechris [1994], Jackson, A. Kechris, and Louveau [2002], and A. S. Kechris and Miller [2004] for more general background.

The Feldman-Moore theorem Feldman and Moore [1977] is fundamental to the study of countable Borel equivalence relations.

**Theorem 10.** *Let $E$ be a countable Borel equivalence relation on the Polish space $X$. then there is countable group $G$ and a Borel action $G \curvearrowright X$ of $G$ on $X$ such that $E = E_G$.*

Thus, we may approach the study of countable Borel equivalence relations "group by group," starting with the algebraically simplest groups and progressing through groups of increasing complexity. Finite groups only generate finite equivalence relations, and these are smooth since there is a Borel linear order on $X$ which we can use to select the least element from each class. The simplest infinite group is $\mathbb{Z}$. Since $E_0$ is given by a Borel action of $\mathbb{Z}$, these relations need not be smooth. A basic result of Slaman-Steel identifies these as the *hyperfinite* equivalence relations.

**Definition 11.** A countable Borel equivalence relation $E$ is hyperfinite if $E = \bigcup_n E_n$ is the increasing union of finite equivalence relations (that is, each $E_n$ class is finite).

The Slaman-Steel theorem (see Dougherty, Jackson, and A. S. Kechris [1994]) says that a countable Borel equivalence relation is hyperfinite iff there is a Borel ordering $<_X$ on $X$ such restricted to each class, $<_X \restriction [x]$ is either finite or order-isomorphic to $\mathbb{Z}$. That is, we have in a uniform Borel manner put the structure of a $\mathbb{Z}$ ordering onto each equivalence class.

The fundamental notion in the theory of Borel equivalence relations is the notion of a reduction: we say $(X, E) \leq (Y, F)$ if there is a Borel $f : X \to Y$ such that for all $x, y \in X$, $x \mathrel{E} y \leftrightarrow f(x) \mathrel{F} f(y)$. This is saying that have in a definable way (in this case a Borel way) an injection from the quotient space $X/E$ to $Y/F$. In other words, this corresponds to saying that $X/E$ has a definable cardinality no larger than that of $Y/F$. Again, "Borel" can be viewed as a stand-in for other notions of definability; we could consider models of determinacy and allow arbitrary functions $f$. The Cantor-Schroeder-Bernstein theorem applies here, so if $(X, E) \leq (Y, F)$ and $(Y, F) \leq (X, E)$, then the quotient spaces are in bijection. A result of Dougherty, Jackson, and A. S. Kechris [ibid.] says that all non-smooth hyperfinite equivalence relations are Borel bi-reducible, so they all the same definable cardinality. A central result in the subject is the *Harrington-Kechris-Louveau* dichotomy theorem (see Harrington, A. S. Kechris, and Louveau [1990]). This theorem states that if the Borel equivalence relation $(X, E)$ is not smooth, then $E_0 \leq (X, E)$. That is, there is nothing between the trivial (smooth) relations and the hyperfinite relations. In other words, $E_0$ is the smallest definable cardinal past those given as subsets of a Polish space (among those representable as Borel equivalence relations).

Two general questions are immediately suggested. The first involves understanding the definable cardinalities of these quotient spaces. That is, determine the structure of the reducibility relation among the family of Borel equivalence relations (or within the countable Borel equivalence relations). The second questions concerns the hyperfinite equivalence relations: which countable groups $G$ generate hyperfinite equivalence relations. That is, which groups $G$ have the property that if $G \curvearrowright X$ is a Borel action of $G$ on the Polish space $X$, then the orbit equivalence relation $E_G$ is hyperfinite? This hyperfiniteness question was first raised explicitly by Kechris and Weiss. The Connes-Feldman-Weiss theorem answers this question in the ergodic theory/dynamics perspective, that is, up to measure 0 sets with respect to an invariant probability measure $\mu$ on $X$. Their theorem says that if $G$ is *amenable* then, up to a measure 0 set, the action is hyperfinite, and conversely, if all the Borel actions of $G$ are hyperfinite up to a measure 0 set for some such measure, then $G$ is amenable. So, if $G$ is non-amenable then there are Borel actions of $G$ which are not (everywhere) hyperfinite. The other direction is far from clear, and is an important open problem in the area.

Concerning the first problem, a result of Dougherty, Jackson, and A. S. Kechris [1994] shows there is a "largest" countable Borel equivalence relation in the sense that every countable Borel equivalence relations reduces to it. This is given by the shift action of the group $F_2$ on $2^{F_2}$ ($F_2$ here is the free group on 2 generators). While it is not too difficult to show that there are incomparable Borel equivalence relations, the corresponding result for countable Borel equivalence relations was open for a significant time. Finally, A. S. Kechris and Adams [2000] resolved this problem using techniques from Zimmer's superrigidity theory in ergodic theory. They showed that there is a large family (of size continuum) of pairwise incomparable countable Borel equivalence relations. This result was strengthened by Hjorth. In Miller [n.d.] an elegant simplified presentation of some of these results can be found.

We mention the best currently known results on the hyperfiniteness problem. First, Weiss showed (unpublished, but see Jackson, A. Kechris, and Louveau [2002]) the following.

**Theorem 12** (Weiss)**.** *All equivalence relations generated by a Borel action of $\mathbb{Z}^n$ are hyperfinite.*

Next, Gao and the author extended the result to general abelian groups:

**Theorem 13** (Gao and Jackson [2015])**.** *All equivalence relations generated by a Borel action of a countable abelian group are hyperfinite.*

The method used to prove Theorem 13 is quite different from that of Theorem 12. Both proofs employ heavily the use of certain *marker structures* on the equivalence relation. By a marker structure we mean a Borel set $M \subseteq X$ which is complete (meets every equivalence class) and co-complete (its complement is complete). For the proofs, it is necessary to create marker structures with certain delicate geometric properties. Thus, some of the fundamental questions in this area are closely connected with the question of what types of marker structures we can put (in a Borel manner) on the equivalence relation. For the proof of Theorem 13, the notion of an *orthogonal* marker structure was introduced. This roughly says that the marker points $M$ give a decomposition of the points in an equivalence class into rectangular regions such that any two parallel faces of nearby regions are separated by a certain fixed positive fraction of the side lengths. The technology used in this proof has other applications. For example, it allows us to show that there is a *continuous* embedding from $2^{\mathbb{Z}^n}$ (with the shift action) into $E_0$ (the fact that there is a Borel action follows from the shift action on $2^{\mathbb{Z}^n}$ being hyperfinite). It also allows us to show that the Borel chromatic number of $F(2^{\mathbb{Z}^n})$ is 3 (we discuss this more below), and answer other combinatorial structuring questions.

Theorem 13 was extended further by Schneider and Seward who extended the result to nilpotent groups, and in fact showed the following.

**Theorem 14** (Schneider and Seward [n.d.])**.** *All equivalence relations generated by the action of a locally nilpotent group are hyperfinite.*

By a an important result of Gromov in geometric group theory, the class of finitely generated groups which have a nilpotent subgroup of finite index (the virtually nilpotent groups) coincides with the class of finitely generated groups of polynomial growth. We note that Theorem 14 for the case of finitely generated nilpotent groups (or finitely generated virtually nilpotent groups) was known previously, a result of Jackson, A. Kechris, and Louveau [2002]. This suggested the possibility that polynomial growth was the barrier to extending these hyperfiniteness results. However, in recent as yet unpublished work, Conley, Marks, Seward, Tucker-Drob, and the author have shown that there are finitely generated solvable, non-nilpotent (so not of polynomial growth) groups all of whose Borel actions are hyperfinite. Whether these arguments can be made to extend to all elementary amenable groups, or even all amenable groups, is not yet known.

Aside from the above questions concerning the cardinalities of the quotient spaces $X/E$, we are interested in questions about the combinatorial structure of these sets. We can ask these types of question at either the definable level, where we usually use "Borel" as a representative case, or at the topological level. Roughly speaking, in the latter case, we require the types of structures we are considering to be given in a continuous manner. As we said above, the hyperfiniteness arguments require certain types of marker structures, but there are many other kinds of structuring questions we can ask.

The notion of a continuous or Borel "structuring" of the countable Borel equivalence relation $E$ can be made precise in a natural manner. If $\mathfrak{L} = (c_i, R_i, f_i)$ is a language of first-order logic, by an $\mathfrak{L}$-structuring of $E$ we mean an assignment $[x] \mapsto \mathfrak{A}_x$ of $\mathfrak{L}$-structures $\mathfrak{A}_x$ to the equivalence classes of $E$, where the domain of the structure $\mathfrak{A}_x$ is the equivalence class $[x]$. If $E = E_G$ for some action of the group $G$, then we frequently also assume that there are unary function symbols $f_g$ in the language $\mathfrak{L}$ for each group element $g \in G$ (these are intended to represent the function $f_g(x) = g \cdot x$). The notion of the structuring being Borel (or continuous) is defined in a natural manner (e.g., for each $n$-ary relation symbol $R_i$ of $\mathfrak{L}$, the relation

$$R(x_1, \dots, x_n) \leftrightarrow x_1\, E\, x_2 \cdots E\, x_n \wedge \mathfrak{A}_{[x_1]}(x_1, \dots, x_n)$$

is a Borel (or clopen) relation on $X^n$. We can then, for example, ask if Borel or continuous structurings of $E$ exists with the structures $\mathfrak{A}_x$ satisfying certain properties (for example, if they satisfy a certain formula of first-order, or higher-order, logic).

Many types of interesting combinatorial questions can be phrased as instances of structuring questions. Consider a fixed countable group $G$. Given actions $G \curvearrowright X$ and $G \curvearrowright Y$ generating equivalence relations $E_X$ and $E_Y$, we say $f : X \to Y$ is *equivariant* if $f$ commutes with the actions, that is, $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$. A one-to-one equivariant map is necessarily a reduction from $E_X$ to $E_Y$. By a subshift of finite type we mean a closed, invariant (under the shift action) $Y \subseteq k^G$ for some $k \in \mathbb{N}$ which is defined by a finite set $p_1, \dots, p_t$ of "forbidden" patterns. Here a pattern is partial function $p : G \to k$ with finite domain. Then $y \in k^G$ is in the subshift $Y$ determined by the $p_i$ (with say $D_i = \mathrm{dom}(p_i)$) if for all $g \in G$, the function $p_g : D_i \to k$ given by $p_g(h) = g \cdot y(h)$ is not equal to any of the $p_i$. Asking if there is an equivariant map from $E_X$ to the subshift $Y$ is an instance of a structuring question. Subshift questions of this form are themselves quite general and include several interesting types of questions. We consider a few of these types of questions and some recent results concerning them.

If $G$ is a marked group, that is, comes with a distinguished set of generators $S$ (which does include the identity $e$), then there is a graphing $\Gamma(E_G)$ of the orbit equivalence relation $E_G$ for any action $G \curvearrowright X$ given by $x \Gamma(E_G) y$ iff $\exists s \in S\ (s \cdot x = y \vee s \cdot y = x)$. If the action is free, then on each equivalence class this graphing is isomorphic to the

Cayley graph associated to $(G, S)$. The *Borel chromatic number*, $\chi_b(E_G)$ of the equivalence relation is the least cardinal $k$ such that there is a Borel map $c: X \to k$ which is a chromatic coloring of the graph $\Gamma(E_G)$. We likewise define the *continuous chromatic number* $\chi_c(E_G)$, using continuous functions $f$. The study of definable chromatic numbers was initiated by Kechris, Solecki, and Todorcevic in A. S. Kechris, Solecki, and Todorcevic [1999]. One of their basic results is that the Borel chromatic number satisfies $\chi_b(E_G) \leq d+1$, where $d$ is the vertex degree of the Cayley graph, for any $E_G$ generated by a free action of $G$. We refer to the determination of $\chi_c(E_G)$ and $\chi_b(E_G)$ as the chromatic number problem. This is an instance of the more general *subshift problem*, which is determine for which subshifts $Y \subseteq k^G$ (determined by $k$ and the patterns $p_1, \ldots, p_t$) there is a continuous or Borel equivariant map from $E_G$ to $Y$. Another instance of the subshift problem is the *graph homomorphism problem*. Given a countable graph $\Gamma$, this problem is to determine whether there is a continuous or Borel graph homomorphism from $\Gamma(E_G)$ to $\Gamma$. Finally, we mention the *tiling problem*. By a tile we mean a finite set $T \subseteq G$. Given a finite set $T_1, \ldots, T_\ell$ of tiles, the tiling problem asks whether there is a continuous or Borel tiling of $E_G$. By this we mean Borel sets $A_i \subseteq X$ such that the sets $\{T_i \cdot g : g \in A_i\}$ partition $X$ (a "continuous" tiling means that the $A_i$ are clopen sets in $X$). There are many other types of structuring questions one can ask, but these serve as test questions for the type of definable structures we can put on the equivalence classes. While these questions are of interest for general countable groups, let us now restrict our attention to simpler groups.

Consider the groups $G = \mathbb{Z}^n$. As we said above, all of these groups induce only hyperfinite actions. Nevertheless, structuring questions about the equivalence relations generated by actions of these groups are non-trivial. Perhaps even more surprising, given the fact that all of these shift spaces $2^{\mathbb{Z}^n}$ continuously embed into $E_0$, is that some continuous structuring questions have answers that depend on $n$.

We mentioned above the method of orthogonal markers, which has been used in recent hyperfiniteness proofs. This method is normally used in a "positive" sense, that is, to produce Borel structurings on various types in the equivalence relations $E_G$. Another method which has been used to obtain negative results in the continuous setting is the method of *hyperaperiodic* points. The notion of hyperaperiodic point was introduced by Gao, Seward, and the author in Gao, Seward, and Jackson [2009] and also independently by Glasner and Uspenskij. Consider the shift space $2^G$. We say $x \in 2^G$ is a hyperaperiodic point if $\overline{[x]} \subseteq F(2^G)$, that is, the closure of the orbit of $x$ lies entirely in the free part of $2^G$. This definition can be reformulated as a purely combinatorial property of $x$. Namely, $x \in 2^G$ is hyperaperiodic iff it satisfies the following: for any $s \in G$ with $s \neq e$, there is a finite $T \subseteq G$ such that

$$\forall g \in G\ \exists t \in T\ (x(gt) \neq x(gst)).$$

This combinatorial property is sometimes referred to as $x$ being a "2-coloring." Hyperaperiodic elements are easy to construct for simple groups such as $\mathbb{Z}^n$, however the following result of Gao, Seward, and Jackson [ibid.] states that they exist for any countable group.

**Theorem 15** (Gao, Seward, and Jackson [ibid.]). *For every countable group $G$ there is an $x \in 2^G$ which is a hyperaperiodic point.*

Hyperaperiodic points are useful since $\overline{[x]}$ is a compact set contained within the free part of $2^G$, and this permits certain compactness arguments. However, to prove some more delicate results it is necessary to construct hyperiodic points with various addition properties. To illustrate the use of the orthogonal marker and hyperaperiodic element arguments, consider the Borel and continuous chromatic number problems for $F(2^{\mathbb{Z}^n})$. An easy category argument shows that $\chi_b(F(2^{\mathbb{Z}^n}) > 2$ for all $n$. Also, any easy argument using the existence of clopen marker sets (see Gao and Jackson [2015]) which are roughly $d$-spaced (for any $d > 1$) shows that $\chi_c(F(2^{\mathbb{Z}})) \leq 3$. It follows that in the $n = 1$ case we have $\chi_b(F(2^{\mathbb{Z}})) = \chi_c(F(2^{\mathbb{Z}})) = 3$. For $n \geq 2$, the arguments require the new methods. The result, from Gao, Jackson, Krohne, and Seward [n.d.], is the following.

**Theorem 16** (Gao, J, Krohne,Seward). *For any $n \geq 2$ we have:*

$$3 = \chi_b(F(2^{\mathbb{Z}^n})) < \chi_c(F(2^{\mathbb{Z}^n})) = 4.$$

This result has two points of interest. First, it shows a difference between the dimension $n = 1$ and $n \geq 2$ cases, even though both equivalence relations are hyperfinite. Second, it shows a difference between the continuous and Borel versions of the question.

The proof of Theorem 16 was first first accomplished by the construction of a particular hyperaperiodic point. The basic idea was to construct $x \in 2^{\mathbb{Z}^n}$ with certain periodicity requirements in one direction, but yet keeping the point hyperaperiodic. This is possible as $n \geq 2$. Later, Gao, Krohne, Seward, and the author proved a general theorem applicable to general subshift questions. The theorem (see Gao, Jackson, Krohne, and Seward [ibid.]) reduces the subshift question for $F(2^{\mathbb{Z}^n})$ down to a question about a family of finite graphs. Consider the case $n = 2$. For each $1 \leq n < p, q$ we define a finite graph $\Gamma_{n,p,q}$. The graph is obtained by starting with 12 individual "grid-graphs." by a grid-graph we mean a graph which is isomorphic to a finite rectangular region of $\mathbb{Z}^2$ with edges inherited from the Cayley graphing of $\mathbb{Z}^2$. Certain vertices are identified among the vertices in these grid-graph, and the quotient graph is $\Gamma_{n,p,q}$. To give the reader a feel for the construction we show three of the grid graphs in Figure 4 (each of the other graphs comprising $\Gamma_{n,p,q}$ is similar to one of these). In each of these grid-graphs, certain rectangular subregions are marked with labels such as $R_\times$, $R_a$, etc. In the graph $\Gamma_{n,p,q}$ the corresponding points within regions with the same label are identified. For example, the upper-left points of each $R_\times$ region are identified in forming $\Gamma_{n,p,q}$.

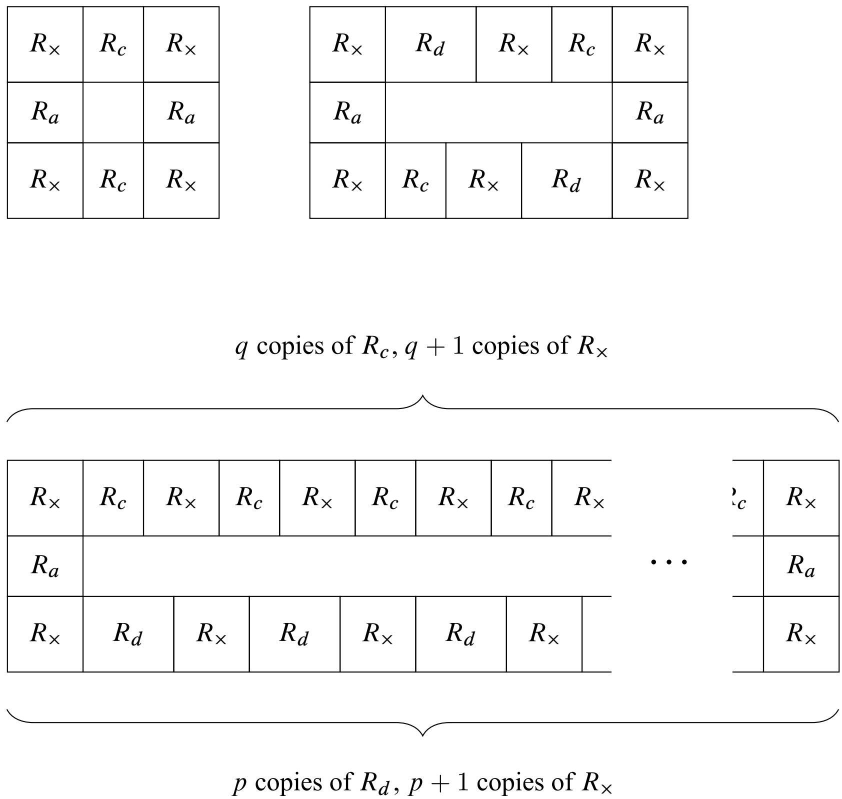


Figure 4: The grid-graphs in $\Gamma_{n,p,q}$.

The following result of Gao, Jackson, Krohne, and Seward [n.d.] shows that a subshift question for $F(2^{\mathbb{Z}^n})$ reduces to question about the graph $\Gamma_{n,p,q}$.

**Theorem 17.** *Let $Y \subseteq k^{\mathbb{Z}^2}$ be a subshift of finite type described by $(k; p_1, \ldots, p_t)$. Then the following are equivalent.*

1. *There is a continuous, equivariant map $f : F(2^{\mathbb{Z}^2}) \to Y$.*
2. *There are positive integers $n, p, q$ with $n < p, q$, $(p, q) = 1$, and $n \geq \max\{a_i, b_i : dom(p_i) = [0, a_i) \times [0, b_i)\} - 1$ and a $g : \Gamma_{n,p,q} \to k$ which respects $Y$.*
3. *For all $n \geq \max\{a_i, b_i : dom(p_i) = [0, a_i) \times [0, b_i)\} - 1$, for all sufficiently large $p, q$ with $(p, q) = 1$ there is a $g : \Gamma_{n,p,q} \to k$ which respects $Y$.*

In this theorem, when we say $g : \Gamma_{n,p,q} \to k$ respects the subshift $y$ we mean that in any $a_i \times b_i$ rectangular subregions $R$ of one of the grid-graphs forming $\Gamma_{n,p,q}$, $g \restriction R$ is not equal to $p_i$. In other words, we can find a continuous equivariant map from $F(2^{\mathbb{Z}^2})$ into the subshift $Y \subseteq k^{\mathbb{Z}^2}$ iff we can find such a map from $\Gamma_{n,p,q} \to k$ for some $p, q$ with $(p,q) = 1$ (equivalently, if we can such maps $g$ for all sufficiently large $p, q$ with $(p,q) = 1$).

Using this result, a number of subshift questions can be answered for $F(2^{\mathbb{Z}^n})$. Moreover, some general results about the decidability of the subshift problem in general can be shown which highlight a key difference between the dimension $n = 1$ and $n \geq 2$ cases. A subshift $Y$ is coded by a finite sequence $(k; p_1, \dots, p_t)$, which can be viewed as an integer. Let $Y_m$ be the subshift coded by the integer $m$. Consider the set $S(n)$ of $m \in \omega$ such that there is a continuous, equivariant map from $F(2^{\mathbb{Z}^n})$ to $Y_m$. From Theorem 17 it follows that for each $n$ the set $S(n)$ is computably enumerable, that is, is a $\Sigma^0_1$ set. The question we consider is whether this set is actually computable (i.e., a $\Delta^0_1$ set). We have the following result of Gao, Jackson, Krohne, and Seward [ibid.].

**Theorem 18.** *For $n = 1$, the subshift problem is decidable, that is, $S(1)$ is computable. For $n \geq 2$ the subshift problem is not computable.*

Theorem 18 shows a remarkable difference between the shift actions of $\mathbb{Z}$ and $\mathbb{Z}^n$ for $n \geq 2$. In Gao, Jackson, Krohne, and Seward [ibid.] it is further shown that even the specific graph homomorphism problem for $F(2^{\mathbb{Z}^n})$ is not computable for $n \geq 2$.

The above results are for the shift actions of the groups $\mathbb{Z}^n$. Let us mention a result of a similar flavor but for a completely different class of groups. this result, obtained by Marks [2016] concerns the free product of groups. The result is:

**Theorem 19** (Marks)**.** *If $G$, $H$ are finitely generated marked groups, then*

$$\chi_b(F(2^{G*H})) \geq \chi_b(F(2^G) + \chi_b(F(2^H)) - 1.$$

where $F(2^G)$ denotes the free part of the shift action of $G$ on $2^G$ and $G * H$ denotes the free product of the groups $G$ and $H$ (the statement of Theorem 19 above actually incorporates an improvement of the result due to Seward and Tucker-Drob). What is interesting is that Marks' method in proving this result involves games and Borel determinacy (a result due to Martin). This surprising result introduces yet another new technique into the subject.

In this section we have been mainly concerned with objects given by Borel equivalence relations on Polish spaces. Although Borel is frequently taken as a representative of definable, let us finally return to considering general sets in a model determinacy. Woodin [2006] has shown an interesting result about cardinalities in determinacy models which shows that the exact determinacy hypotheses assumed may be important. Woodin shows that assuming $\mathsf{AD}_{\mathbb{R}}$, the axiom of real game determinacy (which is considerably stronger than AD or $\mathsf{AD}^+$), there are exactly 5 cardinals below the set $\omega_1^\omega$. He also shows that there are more than 5 cardinals below this set if one assumes $\mathsf{AD} + \neg\mathsf{AD}_\mathbb{R}$. This surprising result shows that for some sets, questions about their definable structure may depend on the background axioms assumed.

The results we have discussed show that a very rich theory of definable sets is emerging, and is connected with many other areas of mathematics. We believe this will continue to be an interesting and fruitful line of investigation.

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Received 2018-03-02.

Stephen Jackson
Department of Mathematics
University of North Texas
Denton, TX, 76203
Stephen.Jackson@unt.edu