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HIGHER ORDER COMMUTATORS AND MULTI-PARAMETER BMO

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Abstract

In this article we highlight the interplay of multi-parameter BMO spaces and boundedness of corresponding commutators. In a variety of settings, we discuss two-sided norm estimates for commutators of classical singular operators with a symbol function. In its classical form, this concerns a theorem by Nehari, factorisation of Hardy space, Hankel and Toeplitz forms. We highlight recent results in which a characterization of L^p boundedness of iterated commutators of multiplication by a symbol function and tensor products of Riesz and Hilbert transforms is obtained, completing a theory on characterisation of BMO spaces begun by Cotlar, Ferguson and Sadosky. In the light of real analysis, we discuss results in a more intricate situation; commutators of multiplication by a symbol function and Calderón-Zygmund or Journé operators. We show that the boundedness of these commutators is also determined by the inclusion of their symbol function in the same multi-parameter BMO class. In this sense the Hilbert or Riesz transforms or their tensor products are a representative testing class for Calderón-Zygmund or Journé operators.

1 Introduction

A classical result of Nehari Nehari [1957] studies L^2 boundedness of Hankel operators with anti-analytic symbol b mapping analytic functions into the space of anti-analytic functions by

$$H_b : f \mapsto P_- bf.$$

A BMO condition on the symbol characterises boundedness. This theorem has an equivalent formulation in terms of the boundedness of the commutator of the multiplication operator with symbol function b and the Hilbert transform

$$[H, b] = Hb - bH.$$

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To see this correspondence one uses that up to a constant $H = P_+ - P_-$ and rewrites the commutator as a sum of Hankel operators with orthogonal ranges. One writes the two-sided inequality on the operator norm

$$\|b\|_{BMO} \lesssim \|[H,b]\|_{L^2\to L^2} \lesssim \|b\|_{BMO}.$$

This two-sided estimate uses the classical factorisation into inner and outer functions. Notably, the lower commutator estimate relies heavily on its corollary, a factorisation theorem of functions in the complex Hardy space H^1 into a product of two H^2 functions. Here is a sketch of the argument showing necessity and sufficiency of a BMO condition for the boundedness of H_b.

$$\begin{aligned}
\|H_b\| &= \sup_{\|g\|_{H^2_-}=1} \sup_{\|f\|_{H^2}=1} |(H_b f, g)| \\
&= \sup_{\|g\|_{H^2_-}=1} \sup_{\|f\|_{H^2}=1} |(P_-(((P_- + P_+)b)f), g)| \\
&= \sup_{\|g\|_{H^2_-}=1} \sup_{\|f\|_{H^2}=1} |(P_-((P_-b)f), g)| \\
&= \sup_{\|g\|_{H^2_-}=1} \sup_{\|f\|_{H^2_+}=1} |((P_-b)f, g)| \\
&= \sup_{\|g\|_{H^2}=1} \sup_{\|f\|_{H^2}=1} |(P_-b, \bar{f}g)| \\
&= \sup_{\|g\|_{H^2}=1} \sup_{\|f\|_{H^2}=1} |(b, \bar{f}g)|.
\end{aligned}$$

Using $H^1 - BMO$ duality and factorisation of Hardy space for the necessity, we get the characterisation of BMO.

Let $H^2(\mathbb{T}^2)$ denote the Banach space of analytic functions in $L^2(\mathbb{T}^2)$. In Ferguson and Sadosky [2000], Ferguson and Sadosky study the symbols of bounded 'big' and 'little' Hankel operators on the bidisk. Big Hankel operators are those which project on to a 'big' subspace of $L^2(\mathbb{T}^2)$ - the orthogonal complement of $H^2(\mathbb{T}^2)$; while little Hankel operators project onto the smaller subspace of complex conjugates of functions in $H^2(\mathbb{T}^2)$ - or anti-analytic functions. The corresponding commutators are

$$[H_1 H_2, b],$$

and

$$[H_1, [H_2, b]]$$

where $b = b(x_1, x_2)$ and H_k are the Hilbert transforms acting in the k^{th} variable. Ferguson and Sadosky show that the first commutator is bounded if and only if the symbol b belongs to the so called little BMO class, consisting of those functions that are uniformly in BMO in each variable separately. Their argument is based on a classical fact on Toeplitz operators. They also show that if b belongs to the product BMO space, as identified by Chang and Fefferman Chang and Fefferman [1985], Chang and Fefferman [1980] then the second commutator is bounded. The fact that boundedness of the second commutator implies that b is in product BMO was shown in the groundbreaking paper of Ferguson and Lacey Ferguson and Lacey [2002]. The absence of factorisation theorems in this multi-parameter setting lead the authors to study two-sided commutator estimates - a very difficult task, considering the complicated structure of the product BMO space. The set up still has Hankel operators at heart, but the techniques to tackle this question in several parameters are very different and have brought valuable new insight and use to existing theories, for example in the interpretation of Journé's lemma Journé [1986] in combination with Carleson's example Carleson [1974]. Lacey and Terwilliger extended this result to an arbitrary number of iterates in Lacey and Terwilleger [2009], requiring thus, among others, a refinement of Pipher's iterated multi-parameter version Pipher [1986] of Journé's lemma. One can then deduce a weak factorisation theorem on the bi-disk. Commutators of the mixed type whose base case is for example

$$[H_1, [H_2 H_3, b]]$$

were considered by Ou, Strouse and the author in Yumeng, Petermichl, and Strouse [2016]. One classifies boundedness of these commutators by a little product BMO class: those functions $b = b(x_1, x_2, x_3)$ so that $b(\cdot, x_2, \cdot)$ and $b(\cdot, \cdot, x_3)$ are uniformly in product BMO. Similar results can be obtained for any finite iteration of any finite tensor product of Hilbert transforms. The proof for this Hilbert transform case is a simple application of Toeplitz operators, if one admits the work by Ferguson, Lacey and Terwilliger.

The main focus in this note however, is in the setting of real analysis, where Hankel and Toeplitz operators cannot be used as a tool.

When leaving the notion of Hankel operators behind, their interpretation as commutators allow for natural generalizations. Through the use of completely different real variable methods, Coifman, Rochberg and Weiss Coifman, Rochberg, and Weiss [1976] extended Nehari's one-parameter theory to real analysis in the sense that the Hilbert transforms were replaced by Riesz transforms. The missing features of the Riesz transforms include analytic projection on one hand as well as strong factorisation theorems of analytic function spaces.

The authors in Coifman, Rochberg, and Weiss [1976] obtained sufficiency, i.e. that a BMO symbol b yields an $L^2(\mathbb{R}^d)$ bounded commutator for certain more general, convolution type singular integral operators. For necessity, they showed that the collection of Riesz transforms was representative enough:

$$\|b\|_{\mathrm{BMO}} \lesssim \sup_{1\le j\le d} \|[R_j, b]\|_{2\to 2}.$$

Notably this lower bound was obtained somewhat indirectly through use of spherical harmonics in combination with the mean oscillation characterisation of BMO in one parameter.

These one-parameter results in Coifman, Rochberg, and Weiss [ibid.] were extended to the multi-parameter setting in the work by Lacey, Pipher, Wick and the author Lacey, Petermichl, Pipher, and Wick [2009]. Both the upper and lower estimate have proofs very different from those in one parameter. For the lower estimate, the methods in Ferguson and Lacey [2002] or Lacey and Terwilleger [2009] find an extension to real variables through operators closer to the Hilbert transform than the Riesz transforms (cone operators) and an indirect passage on the Fourier transform side.

In a recent paper Dalenc and Ou [2014] it is shown that iterated commutators formed with any arbitrary Calderón-Zygmund operators are bounded if the symbol belongs to product BMO.

Ou, Strouse and the author considered in Yumeng, Petermichl, and Strouse [2016] all generalisations of the base case

$$[R_{1,j_1}, [R_{2,j_2} R_{3,j_3}, b]],$$

where R_{k,j_k} are Riesz transforms of direction j_k acting in the k^{th} variable. We show necessity and sufficiency of the little product BMO condition when the R_{k,j_k} are allowed to run through all Riesz transforms by means of a two-sided estimate. While in the Hilbert transform case, Toeplitz operators with operator symbol arise naturally, using Riesz transforms in $\mathbb{R}^d$ as a replacement, there is an absence of analytic structure and tools relying on analytic projection or orthogonal spaces are not readily available. We again overcome part of this difficulty through the use of Calderón-Zygmund operators whose Fourier multiplier symbols are adapted to cones. In this situation, the Toeplitz forms create an additional difficulty which is overcome through an intermediate passage and the construction of a multi-parameter cone operator not of tensor product type.

Further it was shown in work by Holmes, Ou, Strouse, Wick and the author Yumeng, Petermichl, and Strouse [ibid.], Holmes, Petermichl, and Wick [2018] that the tensor products of Riesz transforms in the upper estimate can be replaced by Journé operators, these are singular integral operators of the product type.

Much like discussed in the base cases of the results Coifman, Rochberg, and Weiss [1976], Lacey, Petermichl, Pipher, and Wick [2009], boundedness of commutators involving Hilbert or Riesz transforms are a testing condition. If these commutators are bounded, the symbol necessarily belongs to a BMO, little BMO, product BMO or little product BMO. Then, iterated commutators using a much more general class than that of tensor products of Riesz transforms are also bounded: commutators withCalderón-Zygmund or Journé operators.

2 Aspects of Multi-Parameter Theory

This section contains some review on Hardy spaces in several parameters as well as some definitions and lemmas relevant to us.

2.1 Chang-Fefferman BMO. We describe the elements of product Hardy space theory, as developed by Chang and Fefferman as well as Journé. By this we mean the Hardy spaces associated with domains like the poly-disk or $\mathbb{R}^{\vec{d}} := \bigotimes_{s=1}^{t} \mathbb{R}^{d_s}$ for $\vec{d} = (d_1, \ldots, d_t)$. While doing so, we typically do not distinguish whether we are working on $\mathbb{R}^d$ or $\mathbb{T}^d$. In higher dimensions, the Hilbert transform is usually replaced by the collections of Riesz transforms.

The (real) one-parameter Hardy space $H^1_{\mathrm{Re}}(\mathbb{R}^d)$ denotes the class of functions with the norm

$$\sum_{j=0}^{d} \|R_j f\|_1$$

where R_j denotes the j^{th} Riesz transform or the Hilbert transform if the dimension is one. Here and below we adopt the convention that R_0, the 0^{th} Riesz transform, is the identity. This space is invariant under the one-parameter family of isotropic dilations, while the product Hardy space $H^1_{\mathrm{Re}}(\mathbb{R}^{\vec{d}})$ is invariant under dilations of each coordinate separately. That is, it is invariant under a t parameter family of dilations, hence the terminology 'multi-parameter' theory. One way to define a norm on $H^1_{\mathrm{Re}}(\mathbb{R}^{\vec{d}})$ is

$$\|f\|_{H^1} \sim \sum_{0 \le j_l \le d_l} \|\prod_{l=1}^{t} \mathrm{R}_{l,j_l} f\|_1.$$

R_{l,j_l} is the Riesz transform in the j_l^{th} direction of the l^{th} variable, and the 0^{th} Riesz transform is the identity operator.

The dual of the real Hardy space $H^1_{\mathrm{Re}}(\mathbb{R}^{\vec{d}})^*$ is $\mathrm{BMO}(\mathbb{R}^{\vec{d}})$, the t-fold product BMO space. It is a theorem of S.-Y. Chang and R. Fefferman Chang and Fefferman [1985], Chang and Fefferman [1980] that this space has a characterization in terms of a product Carleson measure.

Define

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^{\vec{d}})} := \sup_{U \subset \mathbb{R}^{\vec{d}}} \left(|U|^{-1} \sum_{R \subset U} \sum_{\vec{\varepsilon} \in \mathrm{sig}_{\vec{d}}} |(b, w_R^{\vec{\varepsilon}})|^2 \right)^{1/2} . \tag{1}$$

Here the supremum is taken over all open subsets $U \subset \mathbb{R}^{\vec{d}}$ with finite measure, and we use a wavelet basis $w_R^{\vec{\varepsilon}}$ adapted to rectangles $R = Q_1 \times \cdots \times Q_t$, where each Q_l is a cube. The superscript $\vec{\varepsilon}$ reflects the fact that multiple wavelets are associated to any dyadic cube, see Lacey, Petermichl, Pipher, and Wick [2009] for details. In this note most often we use the well known Haar wavelet basis. The fact that the supremum admits all open sets of finite measure cannot be omitted, as Carleson's example shows Carleson [1974]. This fact is responsible for some of the difficulties encountered when working with this space.

Theorem 2.1 (Chang, Fefferman)**.** *We have the equivalence of norms*

$$\|b\|_{(H^1_{Re}(\mathbb{R}^{\vec{d}}))^*} \sim \|b\|_{BMO(\mathbb{R}^{\vec{d}})}.$$

That is, $BMO(\mathbb{R}^{\vec{d}})$ is the dual to $H^1_{Re}(\mathbb{R}^{\vec{d}})$.

This BMO norm is invariant under a t-parameter family of dilations. Here the dilations are isotropic in each parameter separately. See also Fefferman [1979] and Fefferman [1987].

2.2 Little BMO. Following Cotlar and Sadosky [1990] and Ferguson and Sadosky [2000], we review the space little BMO, often written as 'bmo'. A locally integrable function $b : \mathbb{R}^{\vec{d}} = \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_s} \to \mathbb{C}$ is in bmo if and only if

$$\|b\|_{\mathrm{bmo}} = \sup_{\vec{Q} = Q_1 \times \cdots \times Q_s} |\vec{Q}|^{-1} \int_{\vec{Q}} |b(\vec{x}) - b_{\vec{Q}}| < \infty$$

Here the Q_k are d_k-dimensional cubes and $b_{\vec{Q}}$ denotes the average of b over $\vec{Q}$.

It is easy to see that this space consists of all functions that are uniformly in BMO in each variable separately. Let $\vec{x}_{\hat{v}} = (x_1, \ldots, x_{v-1}, \cdot, x_{v+1}, \ldots, x_s)$. Then $b(\vec{x}_{\hat{v}})$ is a function in x_v only with the other variables fixed. Its BMO norm in x_v is

$$\|b(\vec{x}_{\hat{v}})\|_{\mathrm{BMO}} = \sup_{Q_v} |Q_v|^{-1} \int_{Q_v} |b(\vec{x}) - b(\vec{x}_{\hat{v}})_{Q_v}| dx_v$$

and the little BMO norm becomes

$$\|b\|_{\mathrm{bmo}} = \max_v \{\sup_{\vec{x}_{\hat{v}}} \|b(\vec{x}_{\hat{v}})\|_{\mathrm{BMO}}\}.$$

On the bi-disk, this becomes

$$\|b\|_{\mathrm{bmo}} = \max\{\sup_{x_1} \|b(x_1, \cdot)\|_{\mathrm{BMO}}, \sup_{x_2} \|b(\cdot, x_2)\|_{\mathrm{BMO}}\},$$

the space discussed in Ferguson and Sadosky [ibid.]. All other cases are an obvious generalisation, at the cost of notational inconvenience.

2.3 Little product BMO. In this section we define a BMO space which is in between little BMO and product BMO. As mentioned in the introduction, we aim at characterising BMO spaces consisting for example of those functions $b(x_1, x_2, x_3)$ such that $b(x_1, \cdot, \cdot)$ and $b(\cdot, \cdot, x_3)$ are uniformly in product BMO in the remaining two variables.

Definition 2.2. *Let $b : \mathbb{R}^{\vec{d}} \to \mathbb{C}$ with $\vec{d} = (d_1, \cdots, d_t)$. Take a partition $\mathfrak{l} = \{I_s : 1 \leq s \leq l\}$ of $\{1, 2, ..., t\}$ so that $\dot{\cup}_{1 \leq s \leq l} I_s = \{1, 2, ..., t\}$. We say that $b \in BMO_{\mathfrak{l}}(\mathbb{R}^{\vec{d}})$ if for any choices $\mathbf{v} = (v_s)$, $v_s \in I_s$, b is uniformly in product BMO in the variables indexed by v_s. We call a BMO space of this type a 'little product BMO'. If for any $\vec{x} = (x_1, ..., x_t) \in \mathbb{R}^{\vec{d}}$, we define $\vec{x}_{\hat{\mathbf{v}}}$ by removing those variables indexed by v_s, the little product BMO norm becomes*

$$\|b\|_{BMO_{\mathfrak{l}}} = \max_{\mathbf{v}} \{\sup_{\vec{x}_{\hat{\mathbf{v}}}} \|b(\vec{x}_{\hat{\mathbf{v}}})\|_{BMO}\}$$

where the BMO norm is product BMO in the variables indexed by v_s.

When $\vec{d}$ and $\vec{s}$ have dimension one, the definition recovers that of little BMO. When $\vec{d}$ and $\vec{s}$ have dimension $t > 1$ and $\vec{s} = \vec{1}$, then we recover the t-parameter product BMO space in $\mathbb{R}^{\vec{d}}$. The following simple example captures the essence of the intermediary spaces: $\mathrm{BMO}_{(1,1)\cdot(2,1)}$ is a class of functions defined on $(\mathbb{R}^1 \times \mathbb{R}^1) \times (\mathbb{R}^1)$ and is uniformly in two-parameter product BMO in variables 1 and 3 as well as 2 and 3.

3 Upper Bounds

In this section we describe upper norm estimates for commutators in terms of BMO norms of their symbol.

3.1 Hilbert transform. The easiest such estimate is

$$\|[H, b]\|_{2\to 2} \le C\|b\|_{BMO}.$$

There are very simple proofs of this fact, using the projection structure of the Hilbert transform. Let us revisit a different proof using the seminal idea of Haar shift, a strategy started by the author in Petermichl [2000] to address a question by Pisier on the dimensional growth of Hankel operators with matrix symbol. We will see that this proof restricted to the scalar case enjoys the generalisations we are seeking. For historic reasons we detail the object in its original form.

We will be using a variety of dyadic grids in $\mathbb{R}$. The standard dyadic grid, starting at 0 with intervals of length $1 \cdot 2^n$, will be called $\mathfrak{D}^{0,1}$.

$$\mathfrak{D}^{0,1} = \{2^{-k}([0,1) + m) : k, m \in \mathbb{Z}\}.$$

Then $h_J^{0,1}$ is the Haar function for $J \in \mathfrak{D}^{0,1}$, namely

$$h_J^{0,1} = 1/\sqrt{|J|}\,(\chi_{J_-} - \chi_{J_+})$$

where J_- is the left half of J and J_+ the right half of J. We obtain a variation of $\mathfrak{D}^{0,1}$ by first shifting the starting point 0 to $\alpha \in \mathbb{R}$ and secondly choosing intervals of length $r \cdot 2^n$ for positive r. The resulting grid is called $\mathfrak{D}^{\alpha,r}$, and the corresponding Haar functions $h^{\alpha,r}$ are chosen so that they are still normalized in L^2. We often omit the indices α, r in our notations for the Haar functions. For $f \in L^2(\mathbb{R})$ we have

$$f(x) = \sum_{I \in \mathfrak{D}^{\alpha,r}} (f, h_I) h_I(x) \quad \forall \alpha \in \mathbb{R},\ r > 0.$$

We define for such α, r a dyadic shift operator $S^{\alpha,r}$ by

$$(S^{\alpha,r} f)(x) = \sum_{I \in \mathfrak{D}^{\alpha,r}} (f, h_I)(h_{I_-}(x) - h_{I_+}(x)).$$

Its L^2 operator norm is $\sqrt{2}$ and its representing kernel is

(2) $$K^{\alpha,r}(t,x) = \sum_{I \in \mathfrak{D}^{\alpha,r}} h_I(t)(h_{I_-}(x) - h_{I_+}(x)).$$

Through elementary methods one can show (see Petermichl [ibid.])

Lemma 3.1. *For $x \neq t$ let*

$$K(t,x) = \lim_{L\to\infty} \frac{1}{2\log L} \int_{1/L}^{L} \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} K^{\alpha,r}(t,x)\, d\alpha \frac{dr}{r}.$$

The limits exist pointwise and the convergence is bounded for $|x-t| \geq \delta$ for every $\delta > 0$ and $K(t,x) = c_0/(t-x)$ for some $c_0 > 0$.

Notice that $1/(t-x)$ is, up to a constant, the kernel of the Hibert transform. This fundamental lemma allows one to estimate commutators with Haar shifts instead of the Hilbert transform. The latter is the correct tool to capture the cancellation of the commutator.

We show that for all $\alpha \in \mathbb{R}$ and for all $r > 0$

(3) $$\|S^{\alpha,r}b - bS^{\alpha,r}\|_{L^2\to L^2} \leq C\|b\|_{BMO}.$$

In the following α, r will be omitted because all estimates do not depend on the dyadic grid. Consider formally

(4) $$b(x) = \sum_{I\in\mathfrak{D}} (b, h_I)h_I(x)$$

and

(5) $$f(x) = \sum_{I\in\mathfrak{D}} (f, h_I)h_I(x).$$

By multiplying the sums (4) and (5) formally one gets $bf = A_b(f) + \Pi_b(f) + R_b(f)$, where

$$A_b(f) = \sum_{I\in\mathfrak{D}} (b, h_I)(f, h_I)h_I^2$$

$$\Pi_b(f) = \sum_{I\in\mathfrak{D}} (b,h_I)\langle f\rangle_I \, h_I$$
$$R_b(f) = \sum_{I\in\mathfrak{D}} \langle b\rangle_I (f,h_I) h_I$$

The expressions can be made meaningful in a standard way. Hence

$$Sb - bS = SA_b - A_bS + S\Pi_b - \Pi_bS + SR_b - R_bS$$

and we can estimate the terms separately.

The term Π_b is a paraproduct with symbol b and $\|\Pi_b\|_{L^2\to L^2} \leq C\|b\|_{BMO}$. Also $A_B^* = \Pi_{B^*}$, so this term is bounded. We estimate the last term as commutator, noting that

$$SR_bf - R_bSf = 1/2\sum_I (\langle b\rangle_{I_+} - \langle b\rangle_{I_-})(f,h_I)(h_{I_-} - h_{I_+}).$$

Observe that $|(\langle b\rangle_{I_+} - \langle b\rangle_{I_-})| \lesssim \|b\|_{BMO}$. We have therefore shown that $\|[H,b]\|_{2\to 2} \lesssim \|b\|_{BMO}$.

3.2 Calderón-Zygmund operators. The idea of Haar shift and representation theorems for singular operators has found deep generalisations. To obtain a proof of the estimate

$$\|[T,b]\|_{2\to 2} \leq C\|b\|_{BMO}$$

with T a Calderón-Zygmund operator that will generalise to the iterated case, we use a famous theorem by Hytönen Hytönen [2012]. The original argument in Coifman, Rochberg, and Weiss [1976] does not generalise to the multi-parameter case.

Recall that a Calderón-Zygmund operator T acts on test functions and has a kernel representation for $x \notin \operatorname{supp} f$

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy.$$

Here the kernel K satisfies the standard estimates such as for example

$$|K(x,y)| \leq \frac{c_0}{|x-y|^d}$$

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \frac{c_1}{|x-y|^d}\left(\frac{|x-x'|}{|x-y|}\right)^\delta$$

for all x, x' with $|x-y| > 2[x-x'|$ for some $0 < \delta \leq 1$. We say that T is bounded if in addition it acts boundedly in L^2.

To obtain a representation formula for T, consider instead of simple translates and dilates of the dyadic grid as in the Hilbert transform case, the randomised grid due to Nazarov, Treil, and Volberg [2003] with parameter $\omega \in (\{0,1\}^d)^{\mathbb{Z}}$ and

$$I \dot{+} \omega = I + \sum_{j:2^{-j}<\ell(I)} 2^{-j}\omega_j$$

where I belongs to the standard dyadic grid and $\ell(I)$ is the side length. The space is endowed with the natural probability measure.

A dyadic shift with parameters $i, j \in \mathbb{N}$ is an operator $Sf = \sum_{K\in\mathfrak{D}} A_K f$ where

$$A_K f = \sum_{I,J\in\mathfrak{D}, I,J\subset K, \ell(I)=2^{-i}\ell(K), \ell(J)=2^{-j}\ell(K)} a_{IJK}(f, h_I)h_J$$

with coefficients $|a_{IJK}| \leq \frac{(|I||J|)^{1/2}}{|K|}$. It is called cancellative if all Haar functions in the representation are cancellative, otherwise non-cancellative.

Let T be a bounded Calderón-Zygmund operator. Then it was proved by Hytönen that it has an expansion for test functions f, g

$$(g, Tf) = c_T \mathbb{E}_\omega \sum_{i,j=0}^{\infty} \tau(i,j)(g, S_\omega^{i,j} f)$$

where $S_\omega^{i,j}$ is a dyadic shift of parameters i, j on the dyadic system $\mathfrak{D}^\omega$. Except possibly $S_\omega^{0,0}$ all are cancellative. τ has exponential decay with respect to the complexity parameters i, j with some dependence on the characteristics of the operator T.

This representation, along with careful consideration allows one to obtain the upper estimate

$$\|[T,b]\|_{2\to 2} \leq C\|b\|_{BMO}$$

through the use of paraproducts. The specificity of this proof is its applicability to the more difficult multi-parameter situation. One obtains the theorem below

Theorem 3.2. *(*Dalenc-Ou*) Let us consider $\mathbb{R}^{\vec{d}}$ with $\vec{d} = (d_1, \ldots, d_t)$. Let $b \in BMO$ and let T_s denote a Calderón-Zygmund operator acting on function defined on $\mathbb{R}^{d_s}$. Then we have the estimate*

$$\|[T_1, \ldots [T_l, b] \ldots]\|_{L^2(\mathbb{R}^{\vec{d}})\hookleftarrow} \lesssim \|b\|_{BMO},$$

where on the right hand side the product BMO norm stands.

This estimate was first proved under a restriction on the kernel by Coifman, Rochberg, and Weiss [1976] in the one-parameter case and by Lacey, Petermichl, Pipher, and Wick [2009] and the author in the multi-parameter case. Through the use of Haar shift this last proof was simplified considerably and restrictions on the kernel were removed by Dalenc and Ou [2014]. It is now known that this estimate also holds in L^p for $1 < p < \infty$. These recent proofs make use of multi-parameter paraproducts and estimates at their endpoint, considered by Journé Journé [1985] and Muscalu, Pipher, Tao, and Thiele [2004] and Muscalu, Pipher, Tao, and Thiele [2006].

For $j = 1, 2, 3$ let $\{\varphi_{j,R} \mid R \in \mathfrak{D}_{\vec{d}}\}$ be three families of functions adapted to the dyadic rectangles in $\mathfrak{D}_{\vec{d}}$ (we consider here products of cancellative or non-cancellative Haar functions, the actual theorems hold in greater generality). We say $\varphi_{j,R}$ has zero in a coordinate if the corresponding Haar function in that coordinate is cancellative. Then define

$$\mathrm{B}(f_1, f_2) := \sum_{R \in \mathfrak{D}_{\vec{d}}} \frac{(f_1, \varphi_{1,R})}{|R|^{1/2}} (f_2, \varphi_{2,R}) \varphi_{3,R}.$$

Theorem 3.3. *Assume that the family $\{\varphi_{1,R}\}$ has zeros in all coordinates. For every other coordinate s, assume that there is a choice of $j = 2, 3$ for which the the family $\{\varphi_{j,R}\}$ has zeros in the sth coordinate. Then the operator* B *enjoys the property*

$$\mathrm{B} : \mathrm{BMO} \times L^p \longrightarrow L^p, \qquad 1 < p < \infty.$$

3.3 Journé operators. To pass to the little BMO case, we observe that the generality of the upper estimate holds for Calderón-Zygmund operators of the multi-parameter type (or Journé operators).

The first generation of multi-parameter singular integrals that are not of tensor product type goes back to Fefferman [1981] and was generalised by Journé in Journé [1985] to the non-convolution type in the framework of his $T(1)$ theorem in this setting. We restrict ourselves for clarity to the bi-parameter case.

The class of bi-parameter singular integral operators treated in this section is that of any Journé type operator (not necessarily a tensor product and not necessarily of convolution type) satisfying a certain weak boundedness property, which we define as follows:

Definition 3.4. *A continuous linear mapping $T : C_0^\infty(\mathbb{R}^n) \otimes C_0^\infty(\mathbb{R}^m) \to [C_0^\infty(\mathbb{R}^n) \otimes C_0^\infty(\mathbb{R}^m)]'$ is called a bi-parameter Calderón-Zygmund operator if the following conditions are satisfied:*

1. T is a Journé type bi-parameter δ-singular integral operator, i.e. there exists a pair (K_1, K_2) of δCZ-δ-standard kernels so that, for all $f_1, g_1 \in C_0^\infty(\mathbb{R}^n)$ and $f_2, g_2 \in$

$C_0^\infty(\mathbb{R}^m)$,

$$\langle T(f_1 \otimes f_2), g_1 \otimes g_2 \rangle = \int f_1(y_1) \langle K_1(x_1, y_1) f_2, g_2 \rangle g_1(x_1)\, dx_1 dy_1$$

when $spt f_1 \cap spt g_1 = \varnothing$;

$$\langle T(f_1 \otimes f_2), g_1 \otimes g_2 \rangle = \int f_2(y_2) \langle K_2(x_2, y_2) f_1, g_1 \rangle g_2(x_2)\, dx_2 dy_2$$

when $spt f_2 \cap spt g_2 = \varnothing$.

2. T satisfies the weak boundedness property $|\langle T(\chi_I \otimes \chi_J), \chi_I \otimes \chi_J \rangle| \lesssim |I||J|$, for any cubes $I \subset \mathbb{R}^n$, $J \in \mathbb{R}^m$.

T is called paraproduct free if $T(1 \otimes \cdot) = T(\cdot \otimes 1) = T^*(1 \otimes \cdot) = T^*(\cdot \otimes 1) = 0$.

Recall that δCZ-δ-standard kernel is a vector valued standard kernel taking values in the Banach space consisting of all Calderón-Zygmund operators. It is easy to see that an operator defined as above satisfies all the characterizing conditions in Martikainen's paper Martikainen [2012], hence is L^2 bounded and can be represented as an average of bi-parameter dyadic shift operators together with dyadic paraproducts. This is the generalisation of Hytönen's theorem to the bi-parameter case. See also higher order Journé operators treated by Ou in Ou [2014]. To be precise, for test functions f, g, one has the following representation:

$$\langle Tf, g \rangle = C \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \sum_{i_1, j_1 = 0}^{\infty} \sum_{i_2, j_2 = 0}^{\infty} 2^{-\max(i_1, j_1)} 2^{-\max(i_2, j_2)} \langle S^{i_1 j_1 i_2 j_2} f, g \rangle. \tag{6}$$

where expectation is with respect to a certain parameter of the dyadic grids. The dyadic shifts $S^{i_1 j_1 i_2 j_2}$ are defined as

$$S^{i_1 j_1 i_2 j_2} f := \sum_{K_1 \in \mathfrak{D}_1} \sum_{\substack{I_1, J_1 \subset K_1, I_1, J_1 \in \mathfrak{D}_1 \\ \ell(I_1) = 2^{-i_1} \ell(K_1) \\ \ell(J_1) = 2^{-j_1} \ell(K_1)}} \sum_{K_2 \in \mathfrak{D}_2} \sum_{\substack{I_2, J_2 \subset K_2, I_2, J_2 \in \mathfrak{D}_2 \\ \ell(I_2) = 2^{-i_2} \ell(K_2) \\ \ell(J_2) = 2^{-j_2} \ell(K_2)}} a_{I_1 J_1 K_1 I_2 J_2 K_2} \langle f, h_{I_1} \otimes h_{I_2} \rangle h_{J_1} \otimes h_{J_2}$$

$$=: \sum_{K_1} \sum_{I_1, J_1 \subset K_1}^{(i_1, j_1)} \sum_{K_2} \sum_{I_2, J_2 \subset K_2}^{(i_2, j_2)} a_{I_1 J_1 K_1 I_2 J_2 K_2} \langle f, h_{I_1} \otimes h_{I_2} \rangle h_{J_1} \otimes h_{J_2}.$$

The coefficients above satisfy $a_{I_1 J_1 K_1 I_2 J_2 K_2} \leq \frac{\sqrt{|I_1||J_1||I_2||J_2|}}{|K_1||K_2|}$, which also guarantees that $\|S^{i_1 j_1 i_2 j_2}\|_{L^2 \to L^2} \leq 1$. Moreover, if T is paraproduct free, all the Haar functions appearing above are cancellative. The theorem below was proved partially in the author's work with Ou and Strouse Yumeng, Petermichl, and Strouse [2016] for the paraproduct-free case and in full generality as part of the author's work with Holmes and Wick in Holmes, Petermichl, and Wick [2018]. We obtained the estimate below.

Theorem 3.5. *Let us consider* $\mathbb{R}^{\vec{d}}$, $\vec{d} = (d_1, \dots, d_t)$ *with a partition* $\mathcal{I} = (I_s)_{1\le s\le l}$ *of* $\{1, \dots, t\}$ *as discussed before. Let* $b \in BMO_{\mathcal{I}}(\mathbb{R}^{\vec{d}})$ *and let* T_s *denote a multi-parameter Journé operator acting on function defined on* $\bigotimes_{k\in I_s} \mathbb{R}^{d_k}$. *Then we have the estimate*

$$\|[T_1, \dots [T_l, b] \dots]\|_{L^2(\mathbb{R}^{\vec{d}})\hookleftarrow} \lesssim \|b\|_{BMO_{\mathcal{I}}(\mathbb{R}^{\vec{d}})}.$$

The same estimate holds in L^p for $1 < p < \infty$.

This last estimate is more general than all previously mentioned commutator estimates.

4 Lower Bounds

In this section we bring a list of notable theorems under one roof. The theorem by Nehari in its formulation through a Hilbert transform commutator:

Theorem 4.1. *(*Nehari) There holds

$$\|b\|_{BMO} \lesssim \|[H, b]\|_{2\to 2} \lesssim \|b\|_{BMO}$$

as well as Ferguson and Sadosky's theorem on the commutator with the double Hilbert using the little BMO space:

Theorem 4.2. *(*Ferguson-Sadosky) There holds

$$\|b\|_{bmo} \lesssim \|[H_1 H_2, b]\|_{2\to 2} \lesssim \|b\|_{bmo}$$

as well as the iterated Hilbert commutators by Ferguson, Lacey, Terwilleger using product BMO:

Theorem 4.3. *(*Ferguson, Lacey, Terwilleger)There holds

$$\|b\|_{BMO} \lesssim \|[H_1, \dots [H_t, b] \dots]\|_{2\to 2} \lesssim \|b\|_{BMO}.$$

. In the real variable situation it includes the characterisation of Coifman, Rochberg and Weiss:

Theorem 4.4. *(*Coifman, Rochberg, Weiss) There holds

$$\|b\|_{BMO} \lesssim \sup_j \|[R_j, b]\|_{2\to 2} \lesssim \|b\|_{BMO}$$

It also includes the characterisation of Lacey, Pipher, Wick and the author using product BMO:

Theorem 4.5. (Lacey, Pipher, Petermichl, Wick) There holds

$$\|b\|_{BMO} \lesssim \sup_{\vec{j}} \|[R_{1,j_1}, \dots [R_{t,j_t}, b] \dots]\|_{2\to 2} \lesssim \|b\|_{BMO}.$$

To be precise, we prove a characterisation theorem of the space $\mathrm{BMO}_{\mathfrak{l}}(\mathbb{R}^{\vec{d}})$. We model the exposition after the formulation of the result by Ferguson and Sadosky.

Theorem 4.6. (Ferguson-Sadosky) *For $b \in L^1(\mathbb{T}^2)$ the following are equivalent with linear relations of their norms:*

(1) $b \in bmo$

(2) The commutators $[H_1, b]$ and $[H_2, b]$ are bounded on $L^2(\mathbb{T}^2)$

(3) The commutator $[H_2 H_1, b]$ is bounded on $L^2(\mathbb{T}^2)$.

Corollary 4.7. (Ferguson-Sadosky) *There is the equivalence of norms*

$$\|b\|_{bmo} \lesssim \|[H_1 H_2, b]\|_{2\to 2} \lesssim \|b\|_{bmo}.$$

The punch line in their beautiful argument is the use of Toeplitz forms. Indeed, typical terms of simple commutators, say with H_1 in this setting are of the form $P_{1,-} b P_{1,+}$ while the double commutator has typical terms of the form $P_{2,+} P_{1,-} b P_{1,+} P_{2,+}$. The norms of these are equal, when regarded as a Toeplitz operator with Hankel symbol. Further, the $L^\infty(BMO)$ characterisation arises naturally, admitting Nehari's theorem as a base.

This theorem in the iterated real variable setting and in its most general form reads as follows. See Yumeng, Petermichl, and Strouse [2016].

Theorem 4.8. *The following are equivalent with linear dependence in the respective norms.*

(1) $b \in BMO_{\mathfrak{l}}(\mathbb{R}^{\vec{d}})$

(2) All commutators of the form $[R_{k_1, j_{k_1}}, \dots, [R_{k_l, j_{k_l}}, b] \dots]$ are bounded in $L^2(\mathbb{R}^{\vec{d}})$ where $k_s \in I_s$ and $R_{k_s, j_{k_s}}$ is the one-parameter Riesz transform in direction j_{k_s}.

(3) All commutators of the form $[\vec{R}_{1,\vec{j}^{(1)}}, \dots, [\vec{R}_{l,\vec{j}^{(l)}}, b] \dots]$ are bounded in $L^2(\mathbb{R}^{\vec{d}})$ where $\vec{j}^{(s)} = (j_k)_{k \in I_s}$, $1 \le j_k \le d_k$ and the operators $\vec{R}_{s,\vec{j}^{(s)}}$ are a tensor product of Riesz transforms $\vec{R}_{s,\vec{j}^{(s)}} = \bigotimes_{k \in I_s} R_{k,j_k}$.

Corollary 4.9. *Let $\vec{j} = (j_1, \dots, j_t)$ with $1 \le j_k \le d_k$ and let for each $1 \le s \le l$, $\vec{j}^{(s)} = (j_k)_{k \in I_s}$ be associated a tensor product of Riesz transforms $\vec{R}_{s,\vec{j}^{(s)}} = \bigotimes_{k \in I_s} R_{k,j_k}$; here R_{k,j_k} are j_k^{th} Riesz transforms acting on functions defined on the k^{th} variable. We have the two-sided estimate*

$$\|b\|_{BMO_{\mathfrak{L}}(\mathbb{R}^{\vec{d}})} \lesssim \sup_{\vec{j}} \|[\vec{R}_{1,\vec{j}^{(1)}}, \dots, [\vec{R}_{t,\vec{j}^{(t)}}, b] \dots]\|_{L^2(\mathbb{R}^{\vec{d}}) \hookleftarrow} \lesssim \|b\|_{BMO_{\mathfrak{L}}(\mathbb{R}^{\vec{d}})}.$$

Such two-sided estimates also hold in L^p for $1 < p < \infty$.

We make some remarks about the strategy of the proof.

In the Hilbert transform case, Toeplitz operators with operator symbol arise naturally.

While Riesz transforms in $\mathbb{R}^d$ are a good generalisation of the Hilbert transform, there is absence of analytic structure and tools relying on analytic projection or orthogonal spaces are not readily available. We overcome this difficulty through a first intermediate passage via tensor products of Calderón-Zygmund operators whose Fourier multiplier symbols are adapted to cones. This idea is inspired by Lacey, Petermichl, Pipher, and Wick [2009].

A class of operators of this type classifies little product BMO through two-sided commutator estimates, but it does not allow the passage to a classification through iterated commutators with tensor products of Riesz transforms. In a second step, we find it necessary to consider upper and lower commutator estimates using a well-chosen family of Journé operators that are not of tensor product type. These operators are constructed to resemble the multiple Hilbert transform. A two-sided estimate of iterated commutators involving operators of this family facilitates a passage to iterated commutators with tensor products of Riesz transforms. There is an increase in difficulty when the arising tensor products involve more than two Riesz transforms and when the dimension is greater than two.

The actual passage to the Riesz transforms requires for us to prove a stability estimate in commutator norms for the multi-parameter singular integrals in terms of the mixed BMO class (see the section on upper bounds). In this context, we prove a qualitative upper estimate for iterated commutators using Journé operators.

To give a flavour of the argument, let us focus on $\mathbb{R}^2$ for simplicity. Riesz transforms and cone operators are homogeneous and their Fourier symbols are determined through their values on $\mathbb{S}^1$. Riesz transforms have the symbols of the coordinates, while cone operators have value 1 on an interval on the sphere covering less than half of the sphere and 0 else. The cone multipliers have to be mollified to result in Calderón-Zygmund operators, a fact we will omit. Through polynomial approximation, the symbols of (mollified) cone operators can be expressed via Riesz transforms symbols. One uses the simple fact

$$[AB, b] = A[B, b] + [A, b]B$$

to pass from lower cone transform estimates to lower Riesz transform estimates. This was one of the essential points in Lacey, Petermichl, Pipher, and Wick [ibid.]. The Toeplitz forms that arise in the tensor product case create an additional difficulty. Most polynomial representations, such as obtained when using tensor products of cone operators, are no longer enough. Other cone operators have to be considered that we try to describe.

Cone functions based on the two oblique strips containing $\vec{\xi}$ are averaged as illustrated below. The cone multiplier is 1 where the two oblique strips containing $\vec{\xi}$ intersect, it is 1/2 in sections with just one of the two strips and 0 else.

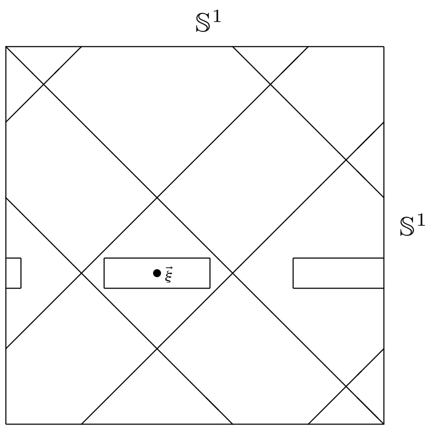

Figure 1: $\mathbb{R}^2 \times \mathbb{R}^2$

The rectangle around $\vec{\xi}$ with sides parallel to the axes illustrates the support of the tensor product of cone operators with direction $\vec{\xi}$. The longer side is the aperture that arises from the Hankel part Lacey, Petermichl, Pipher, and Wick [ibid.]. The short sides can be chosen freely as they arise from the Toeplitz part and is chosen small so that the rectangle fits into the oblique square. The other small rectangle corresponds to the Fourier support of the test function f.

This picture generalises to multiple copies of higher order spheres through the use of zonal harmonics and their identities. An averaging technique on products of spheres comes into play.

Using this intermediate tool, one can obtain lower commutator estimates with tensor products of Riesz transforms in accordance to the model of Ferguson and Sadosky in the Hilbert transform case.

5 Weak Factorization

It is well known that two-sided commutator estimates have an equivalent formulation in terms of *weak factorization* of Hardy space; indeed, this equivalence was important to the part of the proof of the two sided estimates of the iterated commutator. Let us recall the theorem of Lacey, Pipher, Wick and the author Lacey, Petermichl, Pipher, and Wick [2009].

Theorem 5.1. *We have the two-sided estimate*

$$\|b\|_{BMO} \lesssim \sup_{\vec{j}} \|[R_{1,j_1}, \dots [R_{t,j_t}, b] \dots]\|_{2\to 2} \lesssim \|b\|_{BMO}.$$

For $\vec{j}$ the vector above with $1 \le j_s \le d_s$, and $s = 1, \dots, t$, let $\Pi_{\vec{j}}$ be the bilinear operator defined by the following equation

$$\langle C_{\vec{j}}(b, f), g\rangle := \langle b, \Pi_{\vec{j}}(f, g)\rangle.$$

One can express $\Pi_{\vec{j}}$ as a linear combination of products of iterates of Riesz transforms, R_{s,j_s}, applied to the f and g. It follows immediately by duality from the two sided estimate for iterated Riesz commutators Lacey, Petermichl, Pipher, and Wick [ibid.] that for sequences $f_k^{\vec{j}}, g_k^{\vec{j}} \in L^2(\mathbb{R}^{\vec{d}})$ with $\sum_{\vec{j}} \sum_{k=1}^{\infty} \|f_k^{\vec{j}}\|_2 \|g_k^{\vec{j}}\|_2 < \infty$ we have

$$\sum_{\vec{j}} \sum_{k=1}^{\infty} \Pi_{\vec{j}}(f_k^{\vec{j}}, g_k^{\vec{j}}) \in H^1(\mathbb{R}^{\vec{d}}).$$

With this observation, we define

$$L^2(\mathbb{R}^{\vec{d}}) \widehat{\odot} L^2(\mathbb{R}^{\vec{d}}) := \left\{ f \in L^1(\mathbb{R}^{\vec{d}}) : f = \sum_{\vec{j}} \sum_{k=1}^{\infty} \Pi_{\vec{j}}(f_k^{\vec{j}}, g_k^{\vec{j}}) \right\}. \tag{7}$$

This is the projective tensor product given by

$$\|f\|_{L^2(\mathbb{R}^{\vec{d}}) \widehat{\odot} L^2(\mathbb{R}^{\vec{d}})} := \inf \left\{ \sum_{\vec{j}} \sum_{k} \|f_k^{\vec{j}}\|_2 \|g_k^{\vec{j}}\|_2 \right\}$$

where the infimum is taken over all decompositions of f as in (7). We have the following corollary.

We have $H^1(\mathbb{R}^{\vec{d}}) = L^2(\mathbb{R}^{\vec{d}}) \widehat{\odot} L^2(\mathbb{R}^{\vec{d}})$. Namely, for any $f \in H^1(\mathbb{R}^{\vec{d}})$ there exist sequences $f_k^{\vec{j}} \in L^2(\mathbb{R}^{\vec{d}})$ and $g_k^{\vec{j}} \in L^2(\mathbb{R}^{\vec{d}})$ such that

$$f = \sum_{\vec{j}} \sum_{k=1}^{\infty} \Pi_{\vec{j}}(f_k^{\vec{j}}, g_k^{\vec{j}})$$

with

$$\|f\|_{H^1} \simeq \sum_{\vec{j}} \sum_{k} \|f_k^{\vec{j}}\|_2 \|g_k^{\vec{j}}\|_2.$$

Similar results hold when replacing the exponent 2 by $1 < p < \infty$.

6 Div-Curl Lemma

Suppose $E, B \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ are vector fields. It is immediate that their dot product $E \cdot B \in L^1(\mathbb{R}^n)$ with

$$\|E \cdot B\|_{L^1(\mathbb{R}^n)} \leq \|E\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \|B\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}.$$

If in addition they satisfy

$$\text{div} E(x) = 0 \text{ and } \text{curl} B(x) = 0,$$

then we have more cancellation: $E \cdot B \in H^1(\mathbb{R}^n)$ with

$$\|E \cdot B\|_{H^1(\mathbb{R}^n)} \lesssim \|E\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \|B\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}.$$

Indeed, this fact is included in the paper by Coifman, Lions, Meyer, and Semmes [1993]. We sketch their elegant proof, using an upper Riesz commutator estimate. There exists a function ϕ such that $B_j = R_j \phi$ with $\|B\|_{L^2(\mathbb{R}^n,\mathbb{R}^n)} \sim \|\phi\|_{L^2(\mathbb{R}^n)}$. We then have point wise

$$E \cdot B = \sum_j E_j B_j = \sum_j E_j R_j \phi + \phi R_j E_j - \phi R_j E_j = \sum_j E_j R_j \phi + \phi R_j E_j.$$

The last equality is due to E being divergence free and $\sum_j R_j E_j = 0$. Now test this equality over $b \in BMO$ and obtain

$$(E \cdot B, b) = \sum_j (b, E_j R_j \phi + \phi R_j E_j) = \sum_j ([b, R_j](E_j), \phi).$$

Thanks to the BMO condition we know that $[b, R_j]$ is bounded. Thus

$$|(E \cdot B, b)| \lesssim \|b\|_{BMO} \|E\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \|\phi\|_{L^2(\mathbb{R}^n)} \sim \|b\|_{BMO} \|E\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \|B\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}.$$

The duality between H^1 and BMO then yields

$$\|E \cdot B\|_{H^1(\mathbb{R}^n)} \lesssim \|E\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \|B\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}.$$

There are several possible generalisations of this result to the multi-parameter case. See Lacey, Pipher, Wick and the author in Lacey, Petermichl, Pipher, and Wick [2012]. We state one possible generalisation, that uses the upper estimate of the iterated Riesz commutator in terms of product BMO.

Theorem 6.1. *Suppose* $E \in L^2(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ *and* $B \in L^2(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ *have*

$$div_x E(x,y) = 0 \quad curl_x B(x,y) = 0 \ \ \forall y \in \mathbb{R}^n$$

and

$$div_y E(x,y) = 0 \quad curl_y B(x,y) = 0 \ \ \forall x \in \mathbb{R}^n.$$

Then

$$\int_{\mathbb{R}^n} \|E(x,\cdot) \cdot B(x,\cdot)\|_{H^1} dx \lesssim \|E\|_{L^2} \|B\|_{L^2}$$

and

$$\int_{\mathbb{R}^n} \|E(\cdot,y) \cdot B(\cdot,y)\|_{H^1} dy \lesssim \|E\|_{L^2} \|B\|_{L^2}.$$

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