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ON THE CLASSIFICATION OF FUSION CATEGORIES

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Abstract

We report, from an algebraic point of view, on some methods and results on the classification problem of fusion categories over an algebraically closed field of characteristic zero.

> For I could not count or name the multitude who came to Troy, though I had ten tongues and a tireless voice, and lungs of bronze as well...
>
> Homer, *Iliad, Book II (The catalogue of ships)*

1 Introduction

Fusion categories arise from many areas of mathematics and mathematical physics encoding symmetries of structures of different nature and in this sense they can be regarded as a generalization of (finite) groups. This makes the problem of classifying fusion categories both an exciting and at the same time a colossal task. Some classes of examples of fusion categories with distinct features come from such structures like finite groups themselves, quantum groups at roots of 1, subfactors, vertex algebras... A unifying systematic approach to the theory of fusion categories was initiated in the paper Etingof, Nikshych, and Ostrik [2005]. The classification is still in an early age and perhaps awaiting for its monsters to wake up. Some progress has been made however towards the classification of fusion categories in certain classes. What we want to present here is an overview of some constructions, results and open questions related to the classification problem that we think are interesting. The world of fusion categories is a vast one and there are also important constructions, results and questions that we are not going to discuss here, mainly due to space constraints. Our approach concerns the algebraic aspect of fusion categories

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and it is for the most part motivated by different notions of extensions. The perspectives we present are the fruit of the efforts of many and the list of references at the end of the paper is not exhaustive.

We shall work over an algebraically closed base field k. Except in Sections 2.1, 2.2 and Section 4, we assume that k is of characteristic zero. We refer the reader to the book Etingof, Gelaki, Nikshych, and Ostrik [2015] and references therein for most notions on tensor and fusion categories appearing throughout.

2 Fusion categories

We start by recalling some basic definitions and notation regarding monoidal and tensor categories and the relevant functors between them. A detailed study can be found in the books Bakalov and Kirillov [2001], Etingof, Gelaki, Nikshych, and Ostrik [2015], Kassel [1995], Majid [1995], V. G. Turaev [1994].

2.1 Basic notions. A *monoidal category* is a collection $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor, $\mathbf{1}$ is an object of $\mathcal{C}$, called the *unit object*,

$$a : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \to \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes), \quad l : \mathbf{1} \otimes \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}, \quad r : \mathrm{id}_{\mathcal{C}} \otimes \mathbf{1} \to \mathrm{id}_{\mathcal{C}},$$

are natural isomorphisms called, respectively, the associativity and left and right unit constraints, subject to the so-called *pentagon* and *triangle* axioms. For the sake of brevity, we shall simply speak of 'the monoidal category $\mathcal{C}$'.

Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories. A *monoidal functor* $\mathcal{C} \to \mathcal{D}$ is a triple (F, F^2, F^0)[1], where $F : \mathcal{C} \to \mathcal{D}$ is a functor, $F^0 : \mathbf{1} \to F(\mathbf{1})$ is an isomorphism compatible with the unit constraints, and $F^2 : \otimes \circ (F \times F) \to F \circ \otimes$ is a natural isomorphism such that, for all objects X, Y, Z of $\mathcal{C}$,

$$(F^2_{X,Y\otimes Z})(\mathrm{id}_{F(X)} \otimes F^2_{Y,Z}) a_{F(X),F(Y),F(Z)} = F(a_{X,Y,Z}) F^2_{X\otimes Y,Z}(F^2_{X,Y} \otimes \mathrm{id}_{F(Z)}).$$

An *equivalence of monoidal categories* is a monoidal functor (F, F^2, F^0) such that F is an equivalence of categories.

A monoidal category $\mathcal{C}$ is called *strict* if the associativity and unit constraints of $\mathcal{C}$ are identities. A famous result of Mac Lane states that every monoidal category is monoidally equivalent to a strict monoidal category. This allows us in (most of) what follows to suppress the associativity and unit isomorphisms.

[1] The explicit mention of F^2 and F^0 will be often omitted in what follows.

A *braiding* in a monoidal category $\mathcal{C}$ is a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$, $X, Y \in \mathcal{C}$, subject to the so-called hexagon axioms. A *braided monoidal category* is a monoidal category endowed with a braiding Joyal and Street [1993]. Braided monoidal categories such that $c_{Y,X} c_{X,Y} = \mathrm{id}_{X \otimes Y}$, for all objects $X, Y \in \mathcal{C}$, are called *symmetric*.

Let $\mathcal{C}$ be a monoidal category. Then the *Drinfeld center* $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is the braided monoidal category whose objects are pairs (Z, σ_Z), where Z is an object of $\mathcal{C}$ and $\sigma_Z : Z \otimes - \to - \otimes Z$ is a natural isomorphism satisfying appropriate compatibility conditions. The tensor product of $\mathcal{Z}(\mathcal{C})$ is induced from that of $\mathcal{C}$.

Let $\mathcal{C}$ be a monoidal category. A *left dual* of an object X of $\mathcal{C}$ is a triple $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$, where X^* is an object of $\mathcal{C}$ and $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$, $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$, are morphisms in $\mathcal{C}$ called, respectively, evaluation and coevaluation morphisms such that the following compositions are identities:

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X, \quad X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*,$$

A *right dual* of X is defined as a triple $({}^*X, \mathrm{ev}'_X, \mathrm{coev}'_X)$, where $\mathrm{ev}'_X : X \otimes {}^*X \to \mathbf{1}$, $\mathrm{coev}'_X : \mathbf{1} \to {}^*X \otimes X$ are morphisms satisfying similar axioms. The monoidal category $\mathcal{C}$ is called *rigid* if every object X has left and right duals X^*, ${}^*X$.

A *tensor category* over the field k is a k-linear abelian category with finite dimensional Hom spaces and objects of finite length, endowed with a rigid monoidal category structure, such that the monoidal product is k-linear in each variable and the unit object is simple. In a tensor category the monoidal product is exact in each variable. A *tensor functor* between tensor categories is a k-linear exact monoidal functor. Every tensor functor preserves duals and it is automatically faithful. A tensor functor $F : \mathcal{C} \to \mathfrak{D}$ between tensor categories $\mathcal{C}$ and $\mathfrak{D}$ is *dominant* if every object of $\mathfrak{D}$ is a subobject of $F(X)$, for some $X \in \mathcal{C}$.

A tensor category over k is called *finite* if it is equivalent as a k-linear category to the category of finite dimensional left modules over a finite dimensional k-algebra. A *fusion category over* k is a semisimple finite tensor category.

Example 2.1. Examples of tensor categories over k are given by the categories of finite dimensional left (resp. right) modules and finite dimensional left (resp. right) comodules over a *Hopf algebra* over k with bijective antipode. The tensor product in these examples is $\otimes_k$ and the associativity and unit constraints are the canonical vector space isomorphisms. These categories will be denoted, respectively, by H-mod, mod-H, H-comod, comod-H. Finite tensor categories $\mathcal{C}$ equivalent to H-mod, for some finite-dimensional Hopf algebra H, are exactly those that admit a *fiber functor*, that is, a tensor functor $\mathcal{C} \to \mathrm{Vect}_k$, where Vect_k is the tensor category of finite dimensional k-vector spaces.

More generally, if H is a *quasi-Hopf algebra* over k, then the category H-mod of finite dimensional H-modules is a tensor category over k; here the tensor product is $\otimes_k$ but the associativity constraint is induced by the associator $\Phi \in H^{\otimes 3}$ V. G. Drinfeld

[1989b]. Let H_1, H_2 be finite dimensional quasi-Hopf algebras. The tensor categories H_1- mod and H_2- mod are equivalent if and only if H_1 and H_2 are *gauge equivalent*, that is, $H_2 \cong (H_1)_F$ as quasi-Hopf algebras, where $(H_1)_F$ is certain quasi-Hopf algebra such that $(H_1)_F = H_1$ as an algebra with comultiplication $\Delta_F(h) = F\Delta(h)F^{-1}$, $h \in H_1$, and associator $\Phi_F = (1 \otimes F)(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})(F^{-1})(F^{-1} \otimes 1)$.

When $\mathcal{C}$ is the representation category of a finite dimensional (quasi-)Hopf algebra H, then the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is equivalent to the category $D(H)$-mod, where $D(H)$ is the quantum double of H V. G. Drinfeld [1989a], Hausser and Nill [1999], Majid [1998].

The previous example admits several generalizations. In the next example we give an outline of a construction from Hopf monads Bruguières and Virelizier [2007], Bruguières, Lack, and Virelizier [2011].

Example 2.2. A *monad* on a category $\mathcal{C}$ is an algebra in the monoidal category of endofunctors of $\mathcal{C}$. We refer the reader to Mac Lane [1998] for a study of this notion and its relation with adjunctions of functors. A *bimonad* on $\mathcal{C}$ (introduced previously by Moerdijk under the name 'Hopf monad') is a monad T endowed with a structure of a (lax) comonoidal endofunctor, that is, a natural transformation $T_2 : T \circ \otimes \to \otimes \circ (T \times T)$ and a morphism $T_0 : T(\mathbf{1}) \to \mathbf{1}$ satisfying certain compatibility conditions. A *Hopf monad* on a rigid monoidal category $\mathcal{C}$ is a bimonad equipped with a left and a right antipode Bruguières and Virelizier [2007]. If T is a Hopf monad on a rigid monoidal category $\mathcal{C}$, then the category $\mathcal{C}^T$ of T-modules in $\mathcal{C}$ is a rigid monoidal category and the forgetful functor $U : \mathcal{C}^T \to \mathcal{C}$ is a strict monoidal functor. Furthermore, suppose that $\mathcal{C}$ is a tensor category over k, and let T be a k-linear right exact Hopf monad on $\mathcal{C}$. Then $\mathcal{C}^T$ is a tensor category over k, and the forgetful functor $U : \mathcal{C}^T \to \mathcal{C}$ is a tensor functor Bruguières and Natale [2011]. Further, $\mathcal{C}^T$ is a fusion category if and only if $\mathcal{C}$ is a fusion category and T is a semisimple Hopf monad in the sense of Bruguières and Virelizier [2007].

Let $F : \mathcal{C} \to \mathfrak{D}$ be a tensor functor between tensor categories. Suppose F admits a left adjoint (which is always the case if $\mathcal{C}$ and $\mathfrak{D}$ are finite tensor categories). Since F is faithful exact, as a consequence of results of Beck on monadicity of adjunctions (see Mac Lane [1998]), there exists a Hopf monad T on $\mathfrak{D}$ such that $\mathcal{C} \cong \mathfrak{D}^T$ as tensor categories.

Let G be a finite group. The next two basic examples are special cases of those in Example 2.1.

Example 2.3. The category $\operatorname{Rep} G$ of finite dimensional k-linear representations of G is a finite tensor category over k with the usual tensor product of representations and whose unit object is the trivial representation. Thus $\operatorname{Rep} G = kG$- mod, where kG is the group (Hopf) algebra of G. By Maschke theorem, $\operatorname{Rep} G$ is a fusion category if and only if the order of G is coprime to the characteristic of k.

Two finite groups G_1 and G_2 are called *isocategorical* if the categories $\operatorname{Rep} G_1$ and $\operatorname{Rep} G_2$ are equivalent as tensor categories. This notion was introduced in Etingof and Gelaki [2001], where necessary and sufficient conditions for two finite groups to be isocategorical were given. In particular, isocategorical groups need not be isomorphic when their (common) order is divisible by 4.

Example 2.4. Let $\omega : G \times G \times G \to k$ be a 3-cocycle on G. The category Vect_G^ω of finite dimensional G-graded k-vector spaces is a fusion category with tensor product $\otimes_k$, unit object $\mathbf{1} = k$ (graded in degree $1 \in G$), and associativity constraint induced by ω. Indeed, $\mathrm{Vect}_G^\omega = H$-mod, where H is the quasi-Hopf algebra k^G of k-valued functions on G with the usual comultiplication and associator $\omega \in k^{G\times G\times G} \cong (k^G)^{\otimes 3}$. The category Vect_G^ω admits a fiber functor if and only if the class of ω is trivial in $H^3(G, k^\times)$. Equivalence classes of fusion categories of the form Vect_G^ω are in bijection with the orbit space $H^3(G, k^\times)/\operatorname{Out} G$ with respect to the natural action of the group $\operatorname{Out} G$ of outer automorphisms of G in the third cohomology group $H^3(G, k^\times)$.

Suppose that $\mathcal{C}$ is a tensor category. An object X of $\mathcal{C}$ is called *invertible* if the evaluation ev_X and the coevaluation coev_X are isomorphisms. The set G of isomorphism classes of invertible objects of $\mathcal{C}$ is a group with multiplication induced by the tensor product of $\mathcal{C}$. The tensor category $\mathcal{C}$ is called *pointed* if every simple object of $\mathcal{C}$ is invertible. Every pointed fusion category is equivalent to a category Vect_G^ω, for some 3-cocycle ω, where G is the group of invertible objects of $\mathcal{C}$.

2.2 Quantum groupoids.

A *weak Hopf algebra* (or *quantum groupoid*) over k is an associative algebra H endowed with a coassociative coalgebra structure (H, Δ, ϵ) such that Δ is multiplicative, that is, $\Delta(ab) = \Delta(a)\Delta(b)$, for all $a, b \in H$, and

$$(\Delta \otimes \mathrm{id})\Delta(1) = (\Delta(1) \otimes 1)\,(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))\,(\Delta(1) \otimes 1)\,.$$
$$\epsilon(abc) = \epsilon(ab_{(1)})\epsilon(b_{(2)}c) = \epsilon(ab_{(2)})\epsilon(b_{(1)}c), \qquad \forall a, b, c \in H,$$

where $b_{(1)} \otimes b_{(2)} = \Delta(b)$. The existence of an *antipode* is also required: this is a linear map $S : H \to H$ satisfying appropriate conditions. See Böhm, Nill, and Szlachányi [1999], Bòhm and Szlachónyi [1996], Nikshych and Vainerman [2002].

A quantum groupoid H gives rise to the tensor category H-mod of its finite dimensional representations. Here, the tensor product $\otimes$ is defined as

$$V \otimes W = \Delta(1)\; V \otimes_k W,$$

for objects $V, W \in H$-mod, and by restriction of the tensor product on morphisms. The unit object is the so-called *base* subalgebra of H, and the left and right duals of an object

$V \in H$-mod are defined using the antipode. Examples of quantum groupoids from certain so-called *double groupoids* were constructed in Andruskiewitsch and Natale [2006]. Semisimple finite dimensional quantum groupoids give rise to fusion categories. The key fact about them is that every fusion category is the representation category of a quantum groupoid, in view of results of Hayashi and Ostrik:

Theorem 2.5. *Hayashi [2000], Ostrik [2003a]. Let $\mathcal{C}$ be a (multi-)fusion[2] category over k. Then there exists a finite semisimple quantum groupoid H over k such that $\mathcal{C}$ is equivalent to H-mod. Moreover, it is always possible to choose H such that its base is a commutative algebra.*

Theorem 2.5 can be generalized to finite tensor categories; in this case the result states that every finite tensor category is equivalent to H-mod for some finite dimensional left *Hopf algebroid* (a more complicated structure than a quantum groupoid that will not be discussed here). This is proved in Bruguières, Lack, and Virelizier [2011] using Hopf monads.

For the rest of this section we assume that k is the field of complex numbers.

2.3 Fusion categories from quantum groups at roots of 1. Let $\mathfrak{g}$ be a simple complex Lie algebra. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$ and let $q \in \mathbb{C}$ such that q^2 is a primitive root of unity of order $\ell \geq h^\vee$. We sketch here a celebrated construction, due to Andersen and Paradowski Andersen and Paradowski [1995], of a fusion category, called *Verlinde category*, associated to the pair $(\mathfrak{g}, q)$.

Let $U_q(\mathfrak{g})$ denote the Lusztig's quantized enveloping algebra specialized at q Lusztig [1993]. A $U_q(\mathfrak{g})$-module T is called a *tilting module* if both T and its dual T^* have composition series whose factors are Weyl modules. The category $\mathcal{T}$ of tilting modules, although not abelian, is a k-linear *ribbon* category (c.f. SubSection 6.1), which allows to define the trace of an endomorphism $f : T \to T$. A morphism $f : T_1 \to T_2$ in $\mathcal{T}$ is called *negligible* if $\mathrm{Tr}(fg) = 0$ for every morphism $g : T_2 \to T_1$. The *Verlinde fusion category* associated to the pair $(\mathfrak{g}, q)$ is defined as the category whose objects are tilting modules and the morphism spaces are defined by modding out negligible morphisms. We refer the reader to Andersen and Paradowski [1995], Bakalov and Kirillov [2001], Sawin [2006], V. G. Turaev [1994] and references therein for a detailed exposition about this construction.

Other celebrated construction of fusion (in fact modular) categories arising from the simple complex Lie algebra $\mathfrak{g}$ are the categories $\mathcal{C}(\mathfrak{g}, k)$ of integrable highest weight modules of level $k \in \mathbb{Z}_+$ over the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$. See Bakalov and Kirillov

[2] The definition of a multi-fusion category is like that of a fusion category, but dropping the assumption of simplicity of the unit object.

[2001, Chapter 7] for a proof of this fact as well as for the relevant references and a historical overview. Alternatively, the category $\mathcal{C}(\mathfrak{g}, k)$ can be described as the category of finite length modules over the simple vertex algebra $V(\mathfrak{g}, k)$ associated with the vacuum $\widehat{\mathfrak{g}}$-module of level k Huang and Lepowsky [1999]. The relation between these categories and Verlinde categories is given by a theorem of Finkelberg [1996] that asserts that $\mathcal{C}(\mathfrak{g}, k)$ is equivalent as a modular category to the Verlinde category associated to the pair $(\mathfrak{g}, q)$, where $q = \exp(\pi i / m(k + h^\vee))$, such that $m := \langle\langle \alpha, \alpha \rangle\rangle / 2$ for a long root α of $\mathfrak{g}$, $\langle\langle\ ,\ \rangle\rangle$ being an invariant bilinear form on $\mathfrak{g}$ normalized so that $\langle\langle \beta, \beta \rangle\rangle = 2$ for short roots β. See Huang [2008].

2.4 Fusion categories from subfactors. A *subfactor* is an inclusion $A \subseteq B$ of von Neumann algebras with trivial centers. A subfactor $A \subseteq B$ is called of *finite depth* if the tensor powers of the A-bimodule B contain a finite number of isomorphism classes of simple bimodules. A subfactor $A \subseteq B$ of finite index Jones [1983] and finite depth gives rise to a (unitary) fusion category $\mathcal{C}$, called its *principal even part*: this is the full subcategory of the category of A-bimodules generated by the tensor powers of the A-bimodule B. The full subcategory of the category of B-bimodules generated by the tensor powers of the B-bimodule $B \otimes_A B$ is also a fusion category, called the *dual even part* of $A \subseteq B$, which is categorically Morita equivalent to the principal even part (c.f. SubSection 3.3).

Certain examples of fusion categories associated to subfactors do not arise from quantum groups or finite groups by means of any known contruction. Such exotic examples appear related to the Haagerup subfactor, the Asaeda-Haagerup subfactor Asaeda and Haagerup [1999], Haagerup [1994] and the extended Haagerup subfactor Bigelow, Peters, Morrison, and Snyder [2012] and have been intesively studied in the literature; see Grossman, Izumi, and Snyder [2015], Izumi [2001], Jones, Morrison, and Snyder [2014], Peters [2010], and references therein.

The Haagerup and the extended Haagerup subfactors give rise to examples of fusion categories that cannot be defined over a cyclotomic field Morrison and Snyder [2012]. Nevertheless, a result of Etingof, Nikshych, and Ostrik [2005] known as *Ocneanu rigidity* implies that every fusion category can always be defined over an algebraic number field.

3 Some invariants of a fusion category

3.1 Grothendieck ring. Let $\mathcal{C}$ be a fusion category over k and let $\mathrm{Irr}(\mathcal{C})$ denote the set of isomorphism classes of simple objects of $\mathcal{C}$. The Grothendieck group $\mathrm{Gr}(\mathcal{C})$ is the free abelian group with basis $\mathrm{Irr}(\mathcal{C})$. The cardinality of $\mathrm{Irr}(\mathcal{C})$ is called the *rank* of $\mathcal{C}$. The tensor product of $\mathcal{C}$ endows $\mathrm{Gr}(\mathcal{C})$ with a ring structure with unit element $[\mathbf{1}]$ such that, for all objects X and Y, $[X][Y] = [X \otimes Y]$, where $[X]$ denotes the isomorphism class of the object X.[3] For all $X, Y \in \operatorname{Irr}(\mathcal{C})$ we have decompositions

$$XY = \sum_{Z \in \operatorname{Irr}(\mathcal{C})} N_{X,Y}^{Z}\, Z,$$

called the *fusion rules* of $\mathcal{C}$, where the *fusion coefficients* $N_{X,Y}^{Z}$ of $\mathcal{C}$ are the non-negative integers given by $N_{X,Y}^{Z} = \dim_k \operatorname{Hom}_{\mathcal{C}}(Z, X \otimes Y)$, for all $X, Y, Z \in \operatorname{Irr}(\mathcal{C})$.

The duality of $\mathcal{C}$ induces an anti-involution of the Grothendieck ring $\operatorname{Gr}(\mathcal{C})$ that makes it into a *fusion ring* Etingof, Gelaki, Nikshych, and Ostrik [2015, Definition 3.1.3]. In particular, the fusion coefficients satisfy the relations

$$N_{X,Y}^{\mathbf{1}} = \delta_{Y,X^*}, \quad N_{X,Y}^{Z^*} = N_{Z,X}^{Y^*} = N_{Y,Z}^{X^*}, \quad \text{for all } X, Y, Z \in \operatorname{Irr}(\mathcal{C}).$$

Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are *Grothendieck equivalent* (or *have the same fusion rules*) if there is a bijection between $\operatorname{Irr}(\mathcal{C})$ and $\operatorname{Irr}(\mathcal{D})$ that induces a unit preserving ring isomorphism $\operatorname{Gr}(\mathcal{C}) \cong \operatorname{Gr}(\mathcal{D})$. Non-equivalent fusion categories may share the same fusion rules: examples are the fusion categories of finite dimensional representations of the non-isomorphic non-abelian groups of order 8 Tambara and Yamagami [1998].

A *fusion subcategory* of a fusion category $\mathcal{C}$ is a full abelian subcategory closed under subquotients and tensor products. A fusion subcategory is automatically closed under duality, whence it contains the unit object of $\mathcal{C}$, and thus it is a fusion category Etingof, Gelaki, Nikshych, and Ostrik [2015, Corollary 4.11.4]. Fusion subcategories of $\mathcal{C}$ are in bijective correspondence with unital subrings of $\operatorname{Gr}(\mathcal{C})$ spanned by subsets of $\operatorname{Irr}(\mathcal{C})$. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck equivalent fusion categories, then the lattices of fusion subcategories of $\mathcal{C}$ and $\mathcal{D}$ are isomorphic.

The following theorem is a consequence of a result known as *Ocneanu rigidity*, which states that fusion categories do not admit nontrivial deformations.

Theorem 3.1. *Etingof, Nikshych, and Ostrik [2005]. Up to equivalence, there is a finite number of fusion categories with a given Grothendieck ring.*

The following question remains open, although it has been established in Bruillard, Ng, E. C. Rowell, and Wang [2016b] for modular categories (see SubSection 6.1 below).

Question 3.2. Ostrik [2003b]. Are there finitely many equivalence classes of fusion categories with a given finite rank?

The answer to this question is known to be affirmative for fusion categories whose Frobenius-Perron dimension (as defined in SubSection 3.2 below) is an integer Etingof, Nikshych, and Ostrik [2005]. A related result of Etingof says that there is a finite number,

[3] By abuse of notation, we shall also write X to indicate the class of X in $\operatorname{Gr}(\mathcal{C})$.

up to Hopf algebra isomorphism, of semisimple Hopf algebras with a given finite number of irreducible representations Ostrik [2003b, Appendix]. The answer is also affirmative in small rank under certain restrictions: this was proved by Ostrik for rank 2 Ostrik [ibid.] and for rank 3 *pivotal* fusion categories Ostrik [2015].

3.2 Frobenius-Perron dimension. Let $\mathcal{C}$ be a fusion category. The *Frobenius-Perron dimension* of a simple object $X \in \mathcal{C}$ is the Frobenius-Perron eigenvalue of the matrix of left multiplication by the class of X in the basis $\operatorname{Irr}(\mathcal{C})$ of the Grothendieck ring of $\mathcal{C}$ consisting of isomorphism classes of simple objects. The *Frobenius-Perron dimension of* $\mathcal{C}$ is the number $\operatorname{FPdim}\mathcal{C} = \sum_{X\in\operatorname{Irr}(\mathcal{C})}(\operatorname{FPdim} X)^2$. The Frobenius-Perron dimension extends to a ring homomorphism $\operatorname{FPdim} : \operatorname{Gr}(\mathcal{C}) \to \mathbb{R}$ which is characterized by the fact that $\operatorname{FPdim} X > 0$, for all $X \in \operatorname{Irr}(\mathcal{C})$.

Let $\mathcal{C}$ and $\mathfrak{D}$ be fusion categories. Suppose that $F : \mathcal{C} \to \mathfrak{D}$ is a dominant tensor functor. Then the number $\operatorname{FPdim}\mathcal{C}/\operatorname{FPdim}\mathfrak{D}$ is an algebraic integer. This is also true if $\mathfrak{D}$ is a fusion subcategory of $\mathcal{C}$. See Etingof, Nikshych, and Ostrik [2005]. In particular, if $\mathcal{C}$ is integral, then $\operatorname{FPdim}\mathfrak{D}$ divides $\operatorname{FPdim}\mathcal{C}$.

Let $\mathcal{C}$ be any fusion category. For every object X of $\mathcal{C}$, $\operatorname{FPdim} X$ is a cyclotomic integer ≥ 1. Furthermore, it is known that if $\operatorname{FPdim}(X) < 2$, for some $X \in \operatorname{Irr}(\mathcal{C})$, then $\operatorname{FPdim}(X) = 2\cos(\pi/n)$, for some integer $n \geq 3$. See Etingof, Nikshych, and Ostrik [ibid.]. We have the following result on small dimensions of objects in a fusion category:

Theorem 3.3. *Calegari, Morrison, and Snyder [2011]. Let X be an object in a fusion category such that* $\operatorname{FPdim} X$ *belongs to the interval* $(2, 76/33]$. *Then* $\operatorname{FPdim} X$ *is equal to one of the following:*

$$\frac{\sqrt{7}+\sqrt{3}}{2}, \quad \sqrt{5}, \quad 1+2\cos(\frac{2\pi}{7}), \quad \frac{1+\sqrt{5}}{\sqrt{2}}, \quad \frac{1+\sqrt{13}}{2}.$$

Moreover, each of these numbers occurs as the Frobenius-Perron dimension of an object of a fusion category.

A fusion category $\mathcal{C}$ is called *integral* if $\operatorname{FPdim} X \in \mathbb{Z}$, for all simple object $X \in \mathcal{C}$. A fusion category over k is integral if and only if it is equivalent to the category of finite dimensional representations of a finite dimensional semisimple quasi-Hopf algebra over k. If this is the case, then for every H-module V we have $\operatorname{FPdim} V = \dim_k V$. Every fusion category of odd integer Frobenius-Perron dimension is integral Gelaki and Nikshych [2008].

Let $\mathcal{C}$ be an integral fusion category. One can attach some graphs to the set $\operatorname{cd}(\mathcal{C})$ of Frobenius-Perron dimensions of simple objects in the category $\mathcal{C}$: the *prime graph* $\Delta(\mathcal{C})$, whose vertices are the prime divisors of elements of $\operatorname{cd}(\mathcal{C})$ such that two vertices p and q

are joined by an edge if and only if the product pq divides some element of $\mathrm{cd}(\mathcal{C})$, and the *common divisor* graph $\Gamma(\mathcal{C})$, whose vertices are the elements of $\mathrm{cd}(\mathcal{C})-\{1\}$ such that two vertices are joined by an edge if and only if they are not coprime. These graphs extend the corresponding graphs associated to the irreducible character degrees and the conjugacy class sizes of a finite group. Some generalizations of known results on the number of connected components of these graphs for finite groups hold as well in the context of fusion categories Natale and Pacheco Rodríguez [2016].

Remark 3.4. The *categorical* or *global dimension* of a fusion category $\mathcal{C}$ is defined as $\dim \mathcal{C} = \sum_{X \in \mathrm{Irr}(\mathcal{C})} |X|^2$ where, for each simple object X of $\mathcal{C}$, $|X|^2 = \mathrm{Tr}_L(a_X)\mathrm{Tr}_L((a_X^{-1})^*)$, $a_X : X \to X^{**}$ being a fixed isomorphism (which necessarily exists in a fusion category), and $\mathrm{Tr}_L(a_X) = \mathrm{ev}_{X^*}(a_X \otimes \mathrm{id}_{X^*})\mathrm{coev}_X$ Müger [2003a]. A *spherical* structure on $\mathcal{C}$ is an isomorphism of tensor functors $\tau : \mathrm{id}_{\mathcal{C}} \to (\ )^{**}$ such that $\mathrm{Tr}_L(f) = \mathrm{Tr}_R(f)$, for any endomorphism $f : X \to X$ in $\mathcal{C}$, where Tr_L and Tr_R are certain left and right traces induced by τ. A fusion category $\mathcal{C}$ is called *pseudo-unitary* if $\dim(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C})$; this is always the case if $\mathrm{FPdim}\,\mathcal{C}$ is an integer. Every pseudo-unitary fusion category $\mathcal{C}$ has a unique *spherical* structure whose categorical dimensions coincide with the Frobenius-Perron dimensions Etingof, Nikshych, and Ostrik [2005].

In the next examples we discuss some classes of non-pointed fusion categories with distinguished fusion rules.

Example 3.5. *(Near group fusion categories.)* A *near-group* fusion category is a fusion category $\mathcal{C}$ with exactly one non-invertible simple object X up to isomorphism. The fusion rules of $\mathcal{C}$ are determined by the multiplication in the group G of invertible objects of $\mathcal{C}$ and an additional relation

$$X^2 = \sum_{g \in G} g + nX, \tag{3-1}$$

where n is a non-negative integer Siehler [2003]. We say in this case that $\mathcal{C}$ has fusion rules *of type* (G, n). Near-group fusion categories of type $(1, 1)$ are called *Yang-Lee categories*: they fall into two equivalence classes and were classified by Moore and Seiberg [1989a]. Not all pairs (G, n) arise from the fusion rules of a fusion category: Ostrik proved in Ostrik [2003b] that if $n \geq 2$ then there exist no near-group fusion category with fusion rules of type $(1, n)$. Examples and results on near-group fusion categories, including restrictions on the possible values of n and its relation with the structure of G, have been obtained by Evans and Gannon and Izumi. See Evans and Gannon [2014] and references therein.

Example 3.6. *(Tambara-Yamagami fusion categories.)* A fusion category $\mathcal{C}$ is called a *Tambara-Yamagami* fusion category if $\mathcal{C}$ has near-group fusion rules of type $(G, 0)$ for some (necessarily abelian) group G. We have $\operatorname{FPdim} X = \sqrt{|G|}$ and $\operatorname{FPdim} \mathcal{C} = 2|G|$. The classification of these fusion categories was given by Tambara and Yamagami in Tambara and Yamagami [1998]: for each finite abelian group G, they are parameterized, up to equivalence, by isomorphism classes of non-degenerate symmetric bilinear forms $\chi : G \times G \to k$ and elements $\tau \in k$ such that $\tau^2 = |G|^{-1}$.

A Tambara-Yamagami fusion category is integral if and only if the order of G is a square. Tambara-Yamagami fusion categories such that G is of order 2 are called *Ising fusion categories*. These are the only non-pointed fusion categories of Frobenius-Perron dimension 4.

3.3 Categorical Morita equivalence. Let $\mathcal{C}$ be a fusion category over k. A (*right*) $\mathcal{C}$-*module category* is a finite semisimple k-linear abelian category $\mathfrak{M}$ equipped with a k-bilinear functor $\bar{\otimes} : \mathfrak{M} \times \mathcal{C} \to \mathfrak{M}$ and natural isomorphisms $\mu : \bar{\otimes} \circ (\mathrm{id}_{\mathfrak{M}} \times \otimes) \to \bar{\otimes} \circ (\bar{\otimes} \times \mathrm{id}_{\mathcal{C}})$, $r : -\bar{\otimes}\mathbf{1} \to \mathrm{id}_{\mathfrak{M}}$, satisfying certain coherence conditions similar to the pentagon and triangle axioms of a monoidal category.

Let A be an algebra in $\mathcal{C}$. Then the category ${}_A\mathcal{C}$ of left A-modules in $\mathcal{C}$ is a right $\mathcal{C}$-module category with action $\bar{\otimes} :_A \mathcal{C} \times \mathcal{C} \to_A \mathcal{C}$, given by $M \bar{\otimes} X = M \otimes X$ endowed with the natural left A-module structure. The associativity constraint of ${}_A\mathcal{C}$ is induced from that of $\mathcal{C}$.

Let $(\mathfrak{M}, \bar{\otimes})$ and $(\mathfrak{M}', \bar{\otimes}')$ be right $\mathcal{C}$-module categories. A $\mathcal{C}$-module functor $\mathfrak{M} \to \mathfrak{M}'$ is a pair (F, ζ), where $F : \mathfrak{M} \to \mathfrak{M}'$ is a functor and $\zeta_{M,X} : F(M \bar{\otimes} X) \to F(M) \bar{\otimes}' X$ is a natural isomorphism such that, for all $M \in \mathfrak{M}$, $X, Y \in \mathcal{C}$,

$$(\zeta_{M,X} \otimes \mathrm{id}_Y)\ \zeta_{M \bar{\otimes} X, Y}\ F(\mu_{M,X,Y}) = {\mu'}_{F(M),X,Y}\ \zeta_{M, X\otimes Y}, \quad {r'}_{F(M)}\ \zeta_{M,\mathbf{1}} = F(r_M).$$

An *equivalence* of $\mathcal{C}$-module categories $\mathfrak{M} \to \mathfrak{M}'$ is a $\mathcal{C}$-*module functor* $(F, \zeta) : \mathfrak{M} \to \mathfrak{M}'$ such that F is an equivalence of categories. A $\mathcal{C}$-module category is called *indecomposable* if it is not equivalent to a direct sum of two nontrivial $\mathcal{C}$-submodule categories.

Let $\mathfrak{M}, \mathfrak{M}'$ be indecomposable $\mathcal{C}$-module categories. The category $\mathrm{End}_{\mathcal{C}}(\mathfrak{M})$ of $\mathcal{C}$-module endofunctors of $\mathfrak{M}$ is a fusion category with tensor product induced by composition of functors. In particular, this gives a tool for building new examples of fusion categories from 'basic' ones. The category $\mathrm{Fun}_{\mathcal{C}}(\mathfrak{M}, \mathfrak{M}')$ of $\mathcal{C}$-module functors $\mathfrak{M} \to \mathfrak{M}'$ is an indecomposable module category over $\mathrm{End}_{\mathcal{C}}(\mathfrak{M})$ in a natural way. If A and B are indecomposable algebras in $\mathcal{C}$ such that $\mathfrak{M} \cong {}_A\mathcal{C}$ and $\mathfrak{M}' \cong {}_B\mathcal{C}$, then $\mathrm{End}_{\mathcal{C}}(\mathfrak{M})^{op}$ is equivalent to the fusion category ${}_A\mathcal{C}_A$ of (A, A)-bimodules in $\mathcal{C}$ and there is an equivalence of ${}_A\mathcal{C}_A$-module categories ${}_B\mathcal{C}_A \cong \mathrm{Fun}_{\mathcal{C}}(\mathfrak{M}, \mathfrak{M}')$, where ${}_B\mathcal{C}_A$ is the category of (B, A)-bimodules in $\mathcal{C}$.

Two fusion categories $\mathcal{C}$ and $\mathfrak{D}$ are called *categorically Morita equivalent* if $\mathfrak{D} \cong \mathrm{End}_{\mathcal{C}}(\mathfrak{M})^{op}$ for some indecomposable module category $\mathfrak{M}$. If $\mathcal{C}$ and $\mathfrak{D}$ are categorically

Morita equivalent fusion categories, then $\operatorname{FPdim} \mathcal{C} = \operatorname{FPdim} \mathcal{D}$. In addition the class of integral fusion categories is closed under categorical Morita equivalence. See Müger [2003a], Etingof, Nikshych, and Ostrik [2005]. An important characterization of the notion of categorical Morita equivalence is given by the following theorem.

Theorem 3.7. *Etingof, Nikshych, and Ostrik [2011]. Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are categorically Morita equivalent if and only if $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{D})$ are equivalent as braided fusion categories.*

4 Extensions of tensor categories

4.1 Exact sequences of tensor categories. Let $\mathcal{C}, \mathcal{D}$ be tensor categories and let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. Let $\mathfrak{Ker}_F \subset \mathcal{C}$ denote the full subcategory of $\mathcal{C}$ whose objects are those X such that $F(X)$ is a *trivial object* of $\mathcal{D}$, that is, such that $F(X)$ is isomorphic to a direct sum of copies of the unit object $\mathbf{1}$ of $\mathcal{D}$.

A tensor functor $F : \mathcal{C} \to \mathcal{D}$ is called *normal* if for every object X of $\mathcal{C}$ there exists a subobject X_0 such that $F(X_0)$ is the largest trivial subobject of $F(X)$ in $\mathcal{D}$. The notion of normal tensor functor was introduced in Bruguières and Natale [2011] as a generalization of the notion of normal Hopf subalgebra. When $\mathcal{C}$ is a fusion category, the functor F is *normal* if and only if for every simple object $X \in \mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{D}}(\mathbf{1}, F(X)) \neq 0$, we have that $X \in \mathfrak{Ker}_F$.

Definition 4.1. Bruguières and Natale [ibid.]. An *exact sequence of tensor categories* is a sequence of tensor functors

(4-1) $$\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

such that the functor F is dominant and normal, and i is a full embedding whose essential image is $\mathfrak{Ker}_F$. In this case we say that $\mathcal{C}$ is an extension of $\mathcal{C}''$ by $\mathcal{C}'$.

Remark 4.2. An exact sequence of tensor categories (4-1) defines a fiber functor $\mathcal{C}' \to \operatorname{Vect}_k$, since the composition $F \circ f$ maps $\mathcal{C}'$ to the trivial subcategory of $\mathcal{C}''$. By Tannaka reconstruction, (4-1) gives rise to a finite dimensional semisimple Hopf algebra H (the induced Hopf algebra of (4-1)), such that $\mathcal{C}' \cong \text{comod-}H$.

Exact sequences of tensor categories (4-1) are classified in terms of algebraic data under suitable conditions. We say that a k-linear right exact Hopf monad T on a tensor category $\mathcal{C}$ is *normal* if $T(\mathbf{1})$ is a trivial object. If T is such a Hopf monad, and if T is faithful, then it gives rise to an exact sequence of tensor categories comod-$H \to \mathcal{C}^T \to \mathcal{C}$, where, roughly, H is the Hopf algebra such that $T|_{\langle \mathbf{1} \rangle} \cong H \otimes -$ (the induced Hopf algebra of T).

Theorem 4.3. *Bruguières and Natale [ibid.]. Let $\mathcal{C}'$, $\mathcal{C}''$ be tensor categories and assume that $\mathcal{C}'$ is finite. Then the following data are equivalent:*

(i) An exact sequence (4-1)*;*

(ii) A normal faithful k-linear right exact Hopf monad T on $\mathcal{C}''$, with induced Hopf algebra H, endowed with a tensor equivalence $K : \mathcal{C}' \cong$ comod-H.

Let $\mathcal{C}$ and $\mathfrak{D}$ be finite tensor categories. In this case exact sequences (4-1) such that F has an exact right adjoint are also classified by commutative algebras (A, σ) in the center $\mathcal{Z}(\mathcal{C})$ Bruguières and Natale [ibid., Section 6] which are *self-trivializing*, that is, such that $A \otimes A \cong A^n$ as right A-modules in $\mathcal{C}$, for some $n \geq 1$.

An exact sequence of finite tensor categories (4-1) is called *central* if, denoting by (A, σ) the corresponding commutative algebra in $\mathcal{Z}(\mathcal{C})$, the tensor functor $i : \mathcal{C}' \to \mathcal{C}$ lifts to a tensor functor $\tilde{i} : \mathcal{C}' \to \mathcal{Z}(\mathcal{C})$ such that $\tilde{i}(A) = (A, \sigma)$ Bruguières and Natale [2014].

4.2 Hopf algebra extensions.

Let H, H', H'' be Hopf algebras over the field k. A sequence of Hopf algebra maps

$$k \longrightarrow H' \xrightarrow{i} H \xrightarrow{f} H'' \longrightarrow k, \tag{4-2}$$

is called a (strictly) *exact sequence of Hopf algebras* if i is injective, π is surjective, $i(H') = H^{co\pi} = \{h \in H : (\mathrm{id} \otimes \pi)\Delta(h) = h \otimes 1\}$, and H is right faithfully flat over $i(H')$ (the last condition is automatic in the finite dimensional case). Letting $\iota = i_*$ and $F = f_*$ to be the functors induced by restriction along i and f, respectively, the exact sequence (4-2) induces an exact sequence of tensor categories

$$\text{comod-}H' \xrightarrow{\iota} \text{comod-}H \xrightarrow{F} \text{comod-}H''.$$

Suppose H is finite dimensional. If (4-2) is an exact sequence, then $i(H')$ is a *normal* Hopf subalgebra of H, that is, a Hopf subalgebra stable under the adjoint actions of H. Conversely, every normal Hopf subalgebra H' of H gives rise to an exact sequence (4-2), where $i : H' \longrightarrow H$ is the inclusion and $f : H \longrightarrow H/H(H')^+ =: H''$ is the canonical projection. In this case H can be recovered as a bicrossed product $H' \# H''$ with respect to suitable cohomological data Andruskiewitsch and Devoto [1995]. A Hopf algebra is called *simple* if it contains no proper normal Hopf subalgebra. The following result implies that the simplicity of a Hopf algebra is not a categorical notion.

Theorem 4.4. *Galindo and Natale [2007]. There exists a simple Hopf algebra H such that H-mod $\cong$ Rep G, where G is a solvable group.*

In fact in the examples of Galindo and Natale [2007], one may take H of dimension p^2q^2, where p and q are distinct prime numbers, thus showing that the analogue of Burnside's p^aq^b-Theorem does not extend to the context of semisimple Hopf algebras with the natural definition of normal Hopf subalgebra. In the (characteristic zero) semisimple case, the only simple examples with dimension ≤ 60 arise in dimensions 36 and 60 and they are twistings of group algebras Natale [2007], Natale [2010].

Composition series of a finite dimensional Hopf algebra H were introduced by Andruskiewitsch and Müller as a sequence of finite dimensional simple Hopf algebras $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ defined recursively as follows: If H is simple, then $n = 1$ and $\mathfrak{H}_1 = H$. If, on the other hand, $k \subsetneq A \subsetneq H$ is a normal Hopf subalgebra, and $\mathfrak{A}_1, \dots, \mathfrak{A}_m$, $\mathfrak{B}_1, \dots, \mathfrak{B}_l$, are composition series of A and $B = H/HA^+$, respectively, then we let $n = m + l$ and $\mathfrak{H}_i = \mathfrak{A}_i$, if $1 \leq i \leq m$, $\mathfrak{H}_i = \mathfrak{B}_{i-m}$, if $m < i \leq m + l$.

The following analogue of the Jordan-Hölder theorem holds for finite dimensional Hopf algebras; see Andruskiewitsch [2002, Question 2.1].

Theorem 4.5. *Natale [2015]. Let $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ and $\mathfrak{H}'_1, \dots, \mathfrak{H}'_m$ be two composition series of H. Then there exists a bijection $f : \{1, \dots, n\} \to \{1, \dots, m\}$ such that $\mathfrak{H}_i \cong \mathfrak{H}'_{f(i)}$ as Hopf algebras.*

4.3 Exact sequences with respect to a module category. The notion of exact sequence of tensor categories was generalized in Etingof and Gelaki [2017] to that of exact sequence of finite tensor categories with respect to a module category.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be finite k-linear abelian categories. Their *Deligne tensor product* is a finite tensor category denoted $\mathfrak{C} \boxtimes \mathfrak{D}$ endowed with a functor $\boxtimes : \mathfrak{C} \times \mathfrak{D} \to \mathfrak{C} \boxtimes \mathfrak{D}$ exact in both variables such that for any k-bilinear right exact functor $F : \mathfrak{C} \times \mathfrak{D} \to \mathfrak{A}$, where $\mathfrak{A}$ is a k-linear abelian category, there exists a unique right exact functor $\tilde{F} : \mathfrak{C} \boxtimes \mathfrak{D} \to \mathfrak{A}$ such that $\tilde{F} \circ \boxtimes = F$. Such a category exists and it is unique up to equivalences. In fact, if $\mathfrak{C} \cong A$-mod and $\mathfrak{D} \cong B$-mod, for some finite dimensional k-algebras A and B, then $\mathfrak{C} \boxtimes \mathfrak{D} \cong (A \otimes B)$-mod. See Deligne [1990]. The tensor product of two finite tensor categories $\mathfrak{C}$ and $\mathfrak{D}$ is again a finite tensor category and if $\mathfrak{C}$ and $\mathfrak{D}$ are fusion categories, then so is $\mathfrak{C} \boxtimes \mathfrak{D}$.

Let $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{C}$ be finite tensor categories and let $\mathfrak{M}$ be an exact indecomposable left $\mathfrak{A}$-module category[4]; in particular, $\mathfrak{M}$ is finite. Let $\operatorname{End}(\mathfrak{M})$ denote the category of k-linear right exact endofunctors of $\mathfrak{M}$, which is a monoidal category with tensor product given by composition of functors. Let also $i : \mathfrak{A} \to \mathfrak{B}$ denote the inclusion functor.

[4]Exactness of $\mathfrak{M}$ means that $P \otimes M$ is projective for any projective object $P \in \mathfrak{A}$ and any object $M \in \mathfrak{M}$.

Definition 4.6. Etingof and Gelaki [2017]. An exact sequence of tensor categories *with respect to* $\mathfrak{M}$ is a sequence of tensor functors

$$\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{F} \mathcal{C} \boxtimes \operatorname{End}(\mathfrak{M}),$$

such that F is dominant, $\mathcal{A}$ coincides with the subcategory of $\mathcal{B}$ mapped to $\operatorname{End}(\mathfrak{M})$ under F and, for every object X of $\mathcal{B}$, there exists a subobject X_0 of X such that $F(X_0)$ is the largest subobject of $F(X)$ contained in $\operatorname{End}(\mathfrak{M})$. In this case $\mathcal{B}$ is said an extension of $\mathcal{C}$ by $\mathcal{A}$ with respect to $\mathfrak{M}$.

It was shown in Etingof and Gelaki [ibid.] that the Deligne tensor product of two tensor categories gives rise to an exact sequence in the sense of the previous definition. The notion of exact sequence with respect to a module category is self-dual in an appropriate sense. In addition, if $\mathcal{A}$ and $\mathcal{C}$ are fusion categories and $\mathfrak{M}$ is an indecomposable exact (thus semisimple) module category over $\mathcal{A}$, then any extension of $\mathcal{C}$ by $\mathcal{A}$ with respect to $\mathfrak{M}$ is also a fusion category.

The answers to the following natural questions are at the moment not known:

Question 4.7. Does an analogue of the Jordan-Hölder theorem hold for finite tensor (fusion) categories? Is it possible to classify *simple* tensor (fusion) categories?

5 Fusion categories from finite groups

A fusion category $\mathcal{C}$ is called *group-theoretical* if it is categorically Morita equivalent to a pointed fusion category. Let $\mathcal{C}$ be a pointed fusion category, so that there exist a finite group G and a 3-cocycle $\omega : G\times G\times G \to k^\times$ such that $\mathcal{C} \cong \operatorname{Vect}_G^\omega$, c.f. Example 2.4. Every indecomposable module category over $\operatorname{Vect}_G^\omega$ arises from a pair (F, α), where F is a subgroup of G and $\alpha : F \times F \to k^\times$ is a 2-cochain on F such that $d\alpha = \omega|_{F\times F\times F}$. Thus, the restriction $\omega|_F$ represents the trivial cohomology class in $H^3(F, k^\times)$. The (left) module category associated to such pair (F, α) is the category $\mathfrak{M}_0(F, \alpha) = (\operatorname{Vect}_G^\omega)_{k_\alpha F}$ of (right) $k_\alpha F$-modules in $\operatorname{Vect}_G^\omega$, where $k_\alpha F$ is the twisted group algebra of F. The group-theoretical category $(\operatorname{Vect}_G^\omega)^*_{\mathfrak{M}_0(F,\alpha)}$ is denoted $\mathcal{C}(G, \omega, F, \alpha)$. Every group-theoretical fusion category is integral. Necessary and sufficient conditions for a group-theoretical category to be equivalent to the representation category of a (semisimple) Hopf algebra were given in Ostrik [2003c].

The class of group-theoretical fusion categories is quite well-understood. It is almost tautologically closed under categorical Morita equivalence, as well as under Deligne tensor products and Drinfeld centers. Moreover, a fusion category $\mathcal{C}$ is equivalent to a group-theoretical fusion category $\mathcal{C}(G, \omega, F, \alpha)$ if and only if its Drinfeld center $\mathcal{Z}(\mathcal{C})$ is equivalent as a braided fusion category to the category of finite-dimensional representations of the *twisted quantum double* $D^\omega G$ Dijkgraaf, Pasquier, and Roche [1991], Majid [1998].

The next theorem illustrates that certain restrictions on the dimension of an integral fusion category force it to be group-theoretical.

Theorem 5.1. *Let p, q, r be distinct prime numbers. Then:*

(i) *V. Drinfeld, Gelaki, Nikshych, and Ostrik [2007]. Every integral fusion category of Frobenius-Perron dimension p^n, $n \geq 0$, is group-theoretical.*

(ii) *Etingof, Gelaki, and Ostrik [2004], Etingof, Nikshych, and Ostrik [2011]. Every integral fusion category of Frobenius-Perron dimension pq or pqr is group-theoretical.*

Example 5.2. *(Abelian extensions of Hopf algebras.)* Exact sequences of finite dimensional Hopf algebras (4-2) such that $H' \cong k^\Gamma$ and $H'' \cong kF$, for some finite groups F and Γ, are called *abelian extensions*. An abelian extension of kF by k^Γ arises from mutual actions by permutations $\Gamma \overset{\triangleleft}{\leftarrow} \Gamma \times F \overset{\triangleright}{\rightarrow} F$ that make (F, Γ) into a *matched pair* of finite groups. This amounts to the existence of a group G together with an exact factorization $G = F\Gamma$: the actions $\triangleleft, \triangleright$ are this case determined by the relations $gx = (g \triangleright x)(g \triangleleft x)$, for all $x \in F$, $g \in \Gamma$.

Let (F, Γ) be a matched pair of finite groups. Let also $\sigma : F \times F \to (k^*)^\Gamma$ and $\tau : \Gamma \times \Gamma \to (k^*)^F$ be normalized 2-cocycles. Under suitable conditions, one can associate a Hopf algebra $H = k^\Gamma {}^\tau\#_\sigma kF$ (with crossed product algebra structure and crossed coproduct coalgebra structure) that fits into an exact sequence of Hopf algebras $k \to k^\Gamma \to H \to kF \to k$. Moreover, every Hopf algebra H fitting into an exact sequence of this form is isomorphic to $k^\Gamma {}^\tau\#_\sigma kF$ for appropriate data $\triangleleft, \triangleright, \sigma, \tau$. Equivalence classes of such extensions associated to a fixed pair $(\triangleleft, \triangleright)$ form an abelian group whose unit element is the class of the *split* extension $k^\Gamma \# kF$.

Abelian extensions are among the first non-commutative and non-cocommutative examples of Hopf algebras in the literature; they were studied by G.I. Kac in the 60's Kac [1962]. We refer to Masuoka [2002] for results on the cohomology underlying an abelian exact sequence and generalizations.

Every abelian extension is group-theoretical. Indeed, if $H \cong k^\Gamma {}^\tau\#_\sigma kF$, then there is equivalence of tensor categories mod-$H \cong \mathcal{C}(G, \omega, F, 1)$, where $\omega : G \times G \times G \to k^\times$ is a 3-cocycle coming from the class of H in an exact sequence due to G. I. Kac. However, group-theoretical Hopf algebras are not closed under taking extensions. Examples of semisimple Hopf algebras H which are not group-theoretical were constructed by Nikshych, answering a question of Etingof, Nikshych, and Ostrik [2005]:

Theorem 5.3. *Nikshych [2008]. There exist semisimple Hopf algebras which are not group-theoretical.*

The examples of Nikshych [ibid.] have dimension $4p^2$, where p is an odd prime number. These Hopf algebras H fit into a central exact sequence $k \to k^{\mathbb{Z}_2} \to H \to A \to k$, where A is certain group-theoretical Hopf algebra of dimension $2p^2$. The 36 dimensional example arising from Nikshych [ibid.] gives the smallest semisimple Hopf algebra which is not group-theoretical.

5.1 Group extensions and equivariantization. Let G be a finite group. A G-grading on a fusion category $\mathcal{C}$ is a decomposition $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$, such that $\mathcal{C}_g \otimes \mathcal{C}_h \subseteq \mathcal{C}_{gh}$, for all $g, h \in G$. The fusion category $\mathcal{C}$ is called a *G-extension* of a fusion category $\mathfrak{D}$ if there is a faithful grading $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ with neutral component $\mathcal{C}_1 \cong \mathfrak{D}$. Group extensions of a fusion category were classified in Etingof, Nikshych, and Ostrik [2010] in terms of various data related to some low degree cohomology groups.

If $\mathcal{C}$ is any fusion category, there exist a finite group $U(\mathcal{C})$, called the *universal grading group* of $\mathcal{C}$, and a canonical faithful grading $\mathcal{C} = \oplus_{g\in U(\mathcal{C})}\mathcal{C}_g$, with neutral component $\mathcal{C}_{ad}$, where $\mathcal{C}_{ad}$ is the adjoint subcategory of $\mathcal{C}$, that is, the fusion subcategory generated by $X \otimes X^*$, $X \in \operatorname{Irr}(\mathcal{C})$. In addition, if $\operatorname{FPdim}\mathcal{C} \in \mathbb{Z}$, then $\mathcal{C}$ is faithfully graded by an elementary abelian 2-group E. Moreover, there is a set of distinct square-free integers n_g, $g \in E$, such that $n_1 = 1$ and $\operatorname{FPdim} X \in \mathbb{Z}\sqrt{n_g}$, for every simple object X of $\mathcal{C}_g$. The neutral component of this grading is the unique maximal integral fusion subcategory of $\mathcal{C}$. See Gelaki and Nikshych [2008].

A fusion category $\mathcal{C}$ is (cyclically) *nilpotent* if there exists a sequence of fusion categories $\operatorname{Vect}_k = \mathcal{C}_0 \subseteq \mathcal{C}_1 \cdots \subseteq \mathcal{C}_n = \mathcal{C}$, and finite (cyclic) groups $G_1, \dots, G_n$, such that for all $i = 1, \dots, n$, $\mathcal{C}_i$ is a G_i-extension of $\mathcal{C}_{i-1}$.

Another kind of 'group extension' of a fusion category is provided by the equivariantization under a finite group action. An *action* of a finite group G on a fusion category $\mathcal{C}$ by tensor autoequivalences is a monoidal functor $\rho : \underline{G} \to \operatorname{Aut}_{\otimes} \mathcal{C}$, where $\underline{G}$ is the strict monoidal category with objects $g \in G$, identities as its only morphisms, and group multiplication as tensor product, and $\operatorname{Aut}_{\otimes} \mathcal{C}$ is the monoidal category of tensor autoequivalences of $\mathcal{C}$ where morphisms are isomorphisms of tensor functors. The *equivariantization* of $\mathcal{C}$ with respect to the action ρ, denoted $\mathcal{C}^G$, is a fusion category whose objects are pairs (X, μ), such that X is an object of $\mathcal{C}$ and $\mu = (\mu^g)_{g\in G}$, is a collection of isomorphisms $\mu^g : \rho^g X \to X$, $g \in G$, satisfying

$$\mu^g \rho^g(\mu^h) = \mu^{gh} \rho^{g,h}_{2_X}, \quad \mu_1 \rho_{0X} = \operatorname{id}_X,$$

for all $g, h \in G$, where ρ^2 and ρ^0 denote the monoidal structure of ρ.

Remark 5.4. If $\rho : \underline{G} \to \operatorname{Aut}_{\otimes} \mathcal{C}$ is an action of a group G by tensor autoequivalences, then the forgetful functor $F : \mathcal{C}^G \to \mathcal{C}$, $F(X, \mu) = X$, is a normal dominant tensor functor and it gives rise to a central exact sequence of fusion categories $\operatorname{Rep} G \to \mathcal{C}^G \to \mathcal{C}$

Bruguières and Natale [2014]. Thus equivariantization provides examples of exact sequences of tensor categories. The notion of equivariantization explained above can be extended to define equivariantization of tensor categories under finite group scheme actions. Some characterizations of an exact sequence of tensor categories arising from equivariantization where given in Bruguières and Natale [ibid.]. In particular, an exact sequence of finite tensor categories arises from an equivariantization if and only if the sequence is central (see SubSection 4.1), which is an alternative formulation of a previous characterization given in Etingof, Nikshych, and Ostrik [2011] in the context of fusion categories.

A generalization of the equivariantization construction was given in Natale [2016b]: the input for this construction is a *crossed action* of a matched pair of finite groups (G, Γ) in a tensor category $\mathcal{C}$, which means a k-linear action of G on $\mathcal{C}$ (not necessarily by tensor autoequivalences) and a Γ-grading on $\mathcal{C}$ satisfying certain compatibility conditions. The resulting tensor category $\mathcal{C}^{(G,\Gamma)}$ also fits into an exact sequence $\operatorname{Rep} G \to \mathcal{C}^{(G,\Gamma)} \to \mathcal{C}$. The representation categories of abelian extensions (c.f. Example 5.2) are also contained in this construction.

Associated to an action of a group on a fusion category, there is another fusion category, called a *crossed product*, and denoted $\mathcal{C} \rtimes G$ Tambara [2001]. As a k-linear abelian category $\mathcal{C} \rtimes G = \mathcal{C} \boxtimes \operatorname{Vect}_G$ with tensor product defined by

$$(X \boxtimes g) \otimes (Y \boxtimes h) = (X \otimes \rho^g(Y)) \boxtimes gh,$$

for all $X, Y \in \mathcal{C}$, $g, h \in G$, unit object $\mathbf{1} \boxtimes k$ and associativity and unit constraints induced from those of $\mathcal{C}$. The category $\mathcal{C} \rtimes G$ is a G-extension of $\mathcal{C}$ with homogeneous components $(\mathcal{C} \rtimes G)_g = \mathcal{C} \boxtimes g$, $g \in G$. The relation with the equivariantized fusion category is given by an equivalence of tensor categories $(\mathcal{C}^G)^*_{\mathcal{C}} \cong \mathcal{C} \rtimes G$ Nikshych [2008]. In particular, every equivariantization is categorically Morita equivalent to a (graded) group extension.

Remark 5.5. The representation category of the non-group-theoretical examples of Nikshych are constructed in Nikshych [ibid.] as an equivariantization of a Tambara-Yamagami category of dimension $2p^2$ under the action of the group $\mathbb{Z}_2$. Thus, *a fortiori*, the class of group-theoretical fusion categories is not closed under group equivariantizations and neither under group extensions. Necessary and sufficient conditions for a group extension of a fusion category to be group-theoretical were given in Nikshych [ibid.].

Example 5.6. (*Equivariantization of pointed fusion categories.*) Let $\mathcal{C} = \operatorname{Vect}_\Gamma^\omega$ be a pointed fusion category, where Γ is a finite group and $\omega : \Gamma \times \Gamma \times \Gamma \to k^\times$ is a normalized 3-cocycle. Let also G be a finite group. An action $\rho : \underline{G} \to \operatorname{Aut}_\otimes \mathcal{C}$ corresponds to an action by group automorphisms of G on Γ, $x \mapsto {}^g x$, $x \in \Gamma$, $g \in G$ and maps $\tau : G \times \Gamma \times \Gamma \to k^\times$,

$\sigma : G \times G \times \Gamma \to k^\times$, obeying, for all $x, y, z \in \Gamma$, $g, h, l \in G$, the following conditions:

$$\sigma(h, l; x)\,\sigma(g, hl; x) = \sigma(gh, l; x)\,\sigma(g, h; {}^l x)$$
$$\frac{\omega(x, y, z)}{\omega({}^g x, {}^g y, {}^g z)} = \frac{\tau(g; xy, z)\tau(g; x, y)}{\tau(g; y, z)\,\tau(g; x, yz)}$$
$$\frac{\tau(gh; x, y)}{\tau(g; {}^h x, {}^h y)\,\tau(h; x, y)} = \frac{\sigma(g, h; x)\,\sigma(g, h; y)}{\sigma(g, h; xy)}$$

In this example, the category $\mathcal{C} \rtimes G$ is pointed: indeed, $\mathcal{C} \rtimes G \cong \mathrm{Vect}^{\tilde{\omega}}_{\Gamma \rtimes G}$, where $\Gamma \rtimes G$ is the semidirect product associated to the given action by group automorphisms of G on Γ and $\tilde{\omega}$ is a certain 3-cocycle on $\Gamma \rtimes G$ Tambara [2001]. In particular, any equivariantization of a pointed fusion category is group-theoretical.

5.2 Weakly group-theoretical fusion categories. A fusion category $\mathcal{C}$ is called *weakly group-theoretical* (respectively, *solvable*) if it is categorically Morita equivalent to a nilpotent (respectively, cyclically nilpotent) fusion category Etingof, Nikshych, and Ostrik [2011]. Weakly group-theoretical fusion categories can be described by means of group-theoretical data. The notion of solvability extends that of finite groups: the fusion categories $\mathrm{Rep}\, G$ and Vect^{ω}_G are solvable if and only if G is solvable. However, not every nilpotent fusion category is solvable: for instance, Vect_G is always nilpotent.

The class of weakly group-theoretical fusion categories is stable under the operations of taking extensions, equivariantizations, Morita equivalent categories, tensor products, Drinfeld center, fusion subcategories and components of quotient categories. Also, the class of solvable fusion categories is stable under taking extensions and equivariantizations by solvable groups, Morita equivalent categories, tensor products, Drinfeld center, fusion subcategories and components of quotient categories. See Etingof, Nikshych, and Ostrik [ibid.].

Every weakly group-theoretical fusion category has integer Frobenius-Perron dimension. The following is the most important open question regarding the classification of this class of fusion categories:

Question 5.7. Etingof, Nikshych, and Ostrik [ibid.]. Is every fusion category with integer Frobenius-Perron dimension weakly group-theoretical?

A related open question is the following:

Question 5.8. Is the class of weakly group-theoretical fusion categories closed under extensions?

We summarize in the next theorem some results related to Question 5.7. Part (ii) is a generalization of a well-known theorem of Bunrside for finite groups.

Theorem 5.9. *Let $\mathcal{C}$ be a fusion category. Then the following hold:*

(i) *Etingof, Nikshych, and Ostrik [2005]. If* $\operatorname{FPdim} \mathcal{C} = p^n$, *$p$ prime, $n \geq 0$, then $\mathcal{C}$ is nilpotent.*

(ii) *Etingof, Nikshych, and Ostrik [2011]. If* $\operatorname{FPdim} \mathcal{C} = p^n q^m$, *$p, q$ primes, $n, m \geq 0$, then $\mathcal{C}$ is solvable.*

(iii) *Natale [2014]. If $\mathbb{C}$ is braided non-degenerate and* $\operatorname{FPdim} \mathcal{C} = dq^n$, *$p$ prime, $n \geq 0$ and d a square-free integer, then $\mathcal{C}$ is solvable.*

It was shown in Etingof, Nikshych, and Ostrik [2011] that a fusion category $\mathcal{C}$ is weakly group-theoretical if and only if there exists a series of fusion categories

$$\text{Vect} = \mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_n = \mathcal{C}, \tag{5-1}$$

such that for all $1 \leq i \leq n$, the Drinfeld center $\mathcal{Z}(\mathcal{C}_i)$ contains a Tannakian subcategory $\mathcal{E}_i$ and the de-equivariantization of the Müger centralizer $\mathcal{E}_i'$ in $\mathcal{Z}(\mathcal{C}_i)$ by $\mathcal{E}_i$ is equivalent to $\mathcal{Z}(\mathcal{C}_{i-1})$ as braided fusion categories. Since the categories $\mathcal{E}_i$ are Tannakian, then, for all $i = 1, \dots, n$, there exist finite groups $G_1, \dots, G_n$, such that $\mathcal{E}_i \cong \operatorname{Rep} G_i$ as symmetric fusion categories (c.f. Section 6 below). A *composition series* of $\mathcal{C}$ is a series (5-1) whose *factors* $G_1, \dots, G_n$, are simple groups.

The following theorem is an analogue of the Jordan-Hölder theorem. Its proof relies on the structure of a crossed braided fusion category (c.f. SubSection 6.3).

Theorem 5.10. *Natale [2016a]. Let $\mathcal{C}$ be a weakly group-theoretical fusion category. Then two composition series of $\mathcal{C}$ have, up to isomorphisms, the same factors counted with multiplicities. Thus they are invariants of $\mathcal{C}$ under categorical Morita equivalence.*

The solvability of a finite group G is known to be determined by its character table or, equivalently, by the fusion rules of the category $\operatorname{Rep} G$. The answer to the analogous question for fusion categories is not known.

Question 5.11. Escañuela González and Natale [2017]. Is the solvability of a fusion category determined by its fusion rules?

6 Braided fusion categories

Let $\mathcal{C}$ be a braided fusion category and let $\mathfrak{D}$ be a fusion subcategory of $\mathcal{C}$. The *Müger centralizer* of $\mathfrak{D}$ in $\mathcal{C}$, denoted $\mathfrak{D}'$, is the full fusion subcategory generated by all objects $X \in \mathcal{C}$ such that $c_{Y,X} c_{X,Y} = \mathrm{id}_{X \otimes Y}$, for all objects $Y \in \mathfrak{D}$. The category $\mathcal{C}'$ is called the *Müger center* (or *symmetric center*) of $\mathcal{C}$. If $\mathcal{C}$ is any braided fusion category, its Müger center $\mathcal{C}'$ is a symmetric fusion subcategory of $\mathcal{C}$. On the opposite extreme, $\mathcal{C}$ is called *non-degenerate* if $\mathcal{C}' \cong \text{Vect}$. See V. Drinfeld, Gelaki, Nikshych, and Ostrik [2010], Müger [2003b].

For a fusion category $\mathcal{C}$, the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is a non-degenerate braided fusion category of Frobenius-Perron dimension $\operatorname{FPdim} \mathcal{Z}(\mathcal{C}) = (\operatorname{FPdim} \mathcal{C})^2$. Drinfeld centers of fusion categories are characterized as those non-degenerate braided fusion categories $\mathcal{Z}$ containing a Lagrangian algebra, that is, a separable commutative algebra A such that $\operatorname{Hom}_{\mathcal{C}}(\mathbf{1}, A) \cong k$ and $\operatorname{FPdim}(A)^2 = \operatorname{FPdim}(\mathcal{Z})$ Davydov, Müger, Nikshych, and Ostrik [2013].

Example 6.1. (*Group-theoretical braided fusion categories.*) Let G be a finite group and let ω be a normalized 3-cocycle on G. Let also $D^\omega G$-mod be the twisted quantum double Dijkgraaf, Pasquier, and Roche [1991]. Then there is an equivalence of braided fusion categories $D^\omega G\text{-mod} \cong \mathcal{Z}(\operatorname{Vect}_G^\omega)$. Suppose that $\mathcal{C}$ is a braided group-theoretical fusion category. Then there are equivalences of braided fusion categories $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\operatorname{Vect}_G^\omega) \cong D^\omega G$-mod, for some finite group G and 3-cocycle ω. Thus, every braided group-theoretical fusion category can be embedded in $D^\omega G$-mod. A description of the fusion subcategories of $D^\omega G$-mod was given in Naidu, Nikshych, and Witherspoon [2009] in terms of subgroups of G and certain so-called *ω-bicharacters* on them.

Let G be a finite group. The fusion category $\operatorname{Rep} G$ is a symmetric fusion category with respect to the canonical braiding given by the flip of vector spaces. A braided fusion category $\mathcal{E}$ is called *Tannakian* if $\mathcal{E} \cong \operatorname{Rep} G$ for some finite group G as symmetric fusion categories. More generally, if $u \in G$ is a central element such that $u^2 = 1$, then the category $\operatorname{Rep}(G, u)$ of representations of G on finite dimensional super-vector spaces where u acts as the parity operator is a symmetric fusion category. The category $\operatorname{Rep}(\mathbb{Z}_2, u)$, where $1 \neq u \in \mathbb{Z}_2$ is denoted sVect.

The following result of Deligne is a crucial ingredient in the approach to the classification of braided fusion categories in the literature. Related results in a C^*-context are due to Doplicher and Roberts [1989].

Theorem 6.2. *Deligne [1990], Deligne [2002]. Let $\mathcal{C}$ be a symmetric fusion category. Then $\mathcal{C}$ is equivalent as a braided fusion category to the category* $\operatorname{Rep}(G, u)$ *for some finite group G and central element $u \in G$ such that $u^2 = 1$, which are, up to isomorphism, uniquely determined by $\mathcal{C}$.*

Let $\mathcal{C} \cong \operatorname{Rep}(G, u)$ be a symmetric fusion category. Then $\mathcal{C}$ is a $\mathbb{Z}_2$-extension of the Tannakian subcategory $\mathcal{E} = \operatorname{Rep}(G/(u))$. Thus if $\operatorname{FPdim} \mathcal{C} > 2$, then $\mathcal{C}$ contains a Tannakian subcategory, and a non-Tannakian symmetric fusion category of dimension 2 is equivalent to the category sVect.

Let $\mathcal{C}$ be a fusion category and suppose that $\mathcal{E} \cong \operatorname{Rep} G$ is a Tannakian subcategory of the center $\mathcal{Z}(\mathcal{C})$. The *de-equivariantization* of $\mathcal{C}$ with respect to G (or with respect to $\mathcal{E}$), denoted $\mathcal{C}_G$, is the category $\mathcal{C}_A$ of right A-modules in $\mathcal{C}$, where $A \in \mathcal{E}$ is the (commutative, separable) algebra corresponding to $k^G \in \operatorname{Rep} G$ under an equivalence of braided fusion categories $\operatorname{Rep} G \to \mathcal{E}$. This is a fusion category with tensor product $\otimes_A$ and unit object A. De-equivariantization and equivariantization are inverse processes. Indeed, the natural action by translations of G on A induces an action of G on $\mathcal{C}_G$ by tensor auto-equivalences such that $\mathcal{C} \cong (\mathcal{C}_G)^G$. See Bruguières [2000], Müger [2000], V. Drinfeld, Gelaki, Nikshych, and Ostrik [2010, Section 4].

6.1 Modular categories. A *premodular* category (Bruguières [2000]) is a braided fusion category endowed with a natural isomorphism $\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$, called a *ribbon structure*, satisfying

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) c_{Y,X} c_{X,Y}, \quad \theta_X^* = \theta_{X^*}, \tag{6-1}$$

for all objects X, Y of $\mathcal{C}$. Equivalently, $\mathcal{C}$ is a braided fusion category endowed with a spherical structure. The ribbon structure of $\mathcal{C}$ allows to consider the *quantum trace* of an endomorphism $f : X \to X$ of an object X of $\mathcal{C}$ and in particular the *quantum dimension* of X defined as $\dim X = \mathrm{Tr}(\mathrm{id}_X)$.

Suppose $\mathcal{C}$ is a premodular category. Let X, Y be simple objects of $\mathcal{C}$ and let $S_{X,Y} \in k$ denote the quantum trace of the squared braiding $c_{Y,X} c_{X,Y} : X \otimes Y \to Y \otimes X$. The S*-matrix* of $\mathcal{C}$ is defined in the form $S = (S_{XY})_{X,Y \in \mathrm{Irr}(\mathcal{C})}$.

A premodular category $\mathcal{C}$ is called *modular* if its S-matrix is invertible. Equivalently, a premodular category $\mathcal{C}$ is modular if and only if it is non-degenerate. Every non-degenerate fusion category with integral Frobenius-Perron dimension is a modular category with its canonical spherical structure (c.f. Remark 3.4).

Let $\mathcal{C}$ be a modular category. The following relation, known as *Verlinde formula*, gives the fusion coefficients of $\mathcal{C}$ in terms of its S-matrix:

$$N_{XY}^Z = \frac{1}{\dim \mathcal{C}} \sum_{T \in \mathrm{Irr}(\mathcal{C})} \frac{S_{XT}\, S_{YT}\, S_{Z^*T}}{d_T},$$

for all $X, Y, Z \in \mathrm{Irr}(\mathcal{C})$, where d_T is the categorical dimension of the object T and $\dim \mathcal{C} = \sum_{T \in \mathrm{Irr}(\mathcal{C})} d_T^2$ is the categorical dimension of $\mathcal{C}$; c.f. Bakalov and Kirillov [2001].

Let $\mathcal{C}$ be a premodular category. A *modularization* of $\mathcal{C}$ is a dominant tensor functor $F : \mathcal{C} \to \mathcal{C}_0$ compatible with the braiding and the ribbon structures, where $\mathcal{C}_0$ is a modular category Bruguières [2000]. If such a modularization exists, then $\mathcal{C}$ is called *modularizable*. In Bruguières [ibid.], Müger [2000], Bruguières and Müger showed that a premodular category is modularizable if and only if its symmetric center $\mathcal{C}'$ is a Tannakian category. In this case, $\mathcal{C}' \cong \operatorname{Rep} G$ for some finite group G and $\mathcal{C}_0$ is the de-equivariantization $\mathcal{C}_G$. In this context, the group G acts on $\mathcal{C}_0$ by braided auto-equivalences and there is an equivalence of braided fusion categories $\mathcal{C} \cong \mathcal{C}_0^G$.

The following theorem was conjectured by Z. Wang. It implies the feasibility of the classification of modular fusion categories of a given rank:

Theorem 6.3. *Bruillard, Ng, E. C. Rowell, and Wang [2016b]. For every natural number r there is, up to equivalence, a finite number of modular categories of rank r.*

As pointed out by Etingof, the number of modular categories of rank r is, however, not bounded by any polynomial in r. Up to monoidal equivalence, the classification of modular categories has been achieved up to rank 5 Bruillard, Ng, E. C. Rowell, and Wang [2016a], E. Rowell, Stong, and Wang [2009].

6.2 Witt group of non-degenerate braided fusion categories. The (abelian) group $\mathcal{W}$ of Witt classes of non-degenerate braided fusion categories was introduced in Davydov, Müger, Nikshych, and Ostrik [2013]. Two non-degenerate braided fusion categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are called *Witt equivalent* if there exist fusion categories $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{D}_1) \cong \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{D}_2)$ as braided tensor categories.

The Witt group $\mathcal{W}$ consists of equivalence classes of non-degenerate braided fusion categories under this equivalence relation with multiplication induced by Deligne's tensor product $\boxtimes$. The unit element is the class of the category Vect of finite-dimensional vector spaces over k and the inverse of the class of a non-degenerate braided fusion category $\mathcal{C}$ is the class of the *reverse* braided fusion category $\mathcal{C}^{rev}$ (this is the fusion category $\mathcal{C}$ with the *reversed* braiding $c^{rev}_{X,Y} = c^{-1}_{Y,X}$). The explicit determination of the relations in $\mathcal{W}$ is a relevant problem in connection with the classification of fusion categories.

Let $\mathcal{W}_{pt}$ and $\mathcal{W}_{Ising}$ denote the subgroups of Witt classes of pointed non-degenerate fusion categories and Ising braided categories, respectively. The subgroups $\mathcal{W}_{pt}$ and $\mathcal{W}_{Ising}$ were described in V. Drinfeld, Gelaki, Nikshych, and Ostrik [2010]. Let $\widetilde{\mathcal{W}}$ be the subgroup of $\mathcal{W}$ generated by the Witt classes of the modular categories $\mathcal{C}(\mathfrak{g}, l)$ of integrable highest weight modules of level $l \in \mathbb{Z}_+$ over the affinization of a simple finite-dimensional Lie algebra $\mathfrak{g}$ (see SubSection 2.3). It was shown in Davydov, Müger, Nikshych, and Ostrik [2013] that $\mathcal{W}_{pt}, \mathcal{W}_{Ising} \subseteq \widetilde{\mathcal{W}}$. The following open question was raised in Davydov, Müger, Nikshych, and Ostrik [ibid.] as a mathematical formulation of a conjecture stated by Moore and Seiberg [1989b]:

Question 6.4. Davydov, Müger, Nikshych, and Ostrik [2013]. Does $\widetilde{\mathcal{W}}$ coincide with be the subgroup $\mathcal{W}_{un}$ of Witt classes of pseudo-unitary non-degenerate braided fusion categories?

In relation with Question 5.7, we have:

Theorem 6.5. *Natale [2014]. The Witt class of a weakly group-theoretical non-degenerate braided fusion category belongs to the subgroup generated by $\mathcal{W}_{Ising}$ and $\mathcal{W}_{pt}$, whence also to $\widetilde{\mathcal{W}}$.*

6.3 Tannakian categories and braided G-crossed fusion categories. Let G be a finite group. A *G-crossed braided fusion category* is a fusion category $\mathfrak{D}$ endowed with a G-grading $\mathfrak{D} = \oplus_{g\in G}\mathfrak{C}_g$ and an action of G by tensor autoequivalences $\rho : \underline{G} \to \mathrm{Aut}_{\otimes}\ \mathfrak{D}$, such that $\rho^g(\mathfrak{D}_h) \subseteq \mathfrak{D}_{ghg^{-1}}$, for all $g, h \in G$, and a *G-braiding* $c : X \otimes Y \to \rho^g(Y) \otimes X$, $g \in G$, $X \in \mathfrak{D}_g$, $Y \in \mathfrak{D}$, subject to certain compatibility conditions. This notion was introduced by Turaev; see V. Turaev [2010].

The equivariantization $\mathfrak{D}^G$ of a G-crossed braided fusion category is a braided fusion category. The canonical embedding $\mathrm{Rep}\, G \to \mathfrak{D}^G$ of fusion categories is fact an embedding of braided fusion categories, with respect to the canonical braiding in $\mathrm{Rep}\, G$. The G-braiding on $\mathfrak{D}$ restricts to a braiding in the neutral component $\mathfrak{D}_1$ of the G-grading. Furthermore, the group G acts by restriction on $\mathfrak{D}_1$ and this action is by braided tensor autoequivalences. This makes the equivariantization $(\mathfrak{D}_1)^G$ into a braided fusion subcategory of $\mathfrak{D}^G$. This fusion subcategory coincides with the centralizer $\mathcal{E}'$ of the Tannakian subcategory $\mathcal{E} \cong \mathrm{Rep}\, G$ in $\mathfrak{D}^G$.

Conversely, if $\mathcal{C}$ is a braided fusion category containing a Tannakian subcategory $\mathcal{E} \cong \mathrm{Rep}\, G$, then the de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ with respect to $\mathcal{E}$ is a braided G-crossed fusion category. Thus equivariantization defines a bijective correspondence between equivalence classes of braided fusion categories containing $\mathrm{Rep}\, G$ as a Tannakian subcategory and G-crossed braided fusion categories Jr [2001], Müger [2004].

Let $\mathfrak{D}$ be a G-crossed braided fusion category. The braided fusion category $\mathfrak{D}^G$ is non-degenerate if and only if the neutral component $\mathfrak{D}_1$ is non-degenerate and the G-grading of $\mathfrak{D}$ is faithful. If this is the case, then there is an equivalence of braided fusion categories $\mathcal{Z}(\mathfrak{D}) \cong \mathfrak{D}^G \boxtimes \mathfrak{D}_1^{rev}$ Davydov, Müger, Nikshych, and Ostrik [2013].

An important invariant of a braided fusion category is its *core*, introduced in V. Drinfeld, Gelaki, Nikshych, and Ostrik [2010]. As a braided fusion category, the core of a braided fusion category $\mathcal{C}$ is the neutral homogeneous component $\mathcal{C}_G^0$ of the de-equivariantization of $\mathcal{C}$ by a maximal Tannakian subcategory $\mathcal{E} \cong \mathrm{Rep}\, G$. The core of $\mathcal{C}$ is independent of $\mathcal{E}$. Furthermore, the core of a braided fusion category is *weakly anisotropic*, that is, it contains no Tannakian subcategories stable under all braided auto-equivalences. In addition, the core of $\mathcal{C}$ is non-degenerate if $\mathcal{C}$ is non-degenerate. A complete classification of pointed weakly anisotropic braided fusion categories has been proposed in V. Drinfeld, Gelaki, Nikshych, and Ostrik [ibid.].

A braided fusion category is weakly group-theoretical if and only if it can be obtained from a crossed braided fusion category with pointed neutral component, thus from a pointed category by means of suitable group extensions and equivariantizations:

Theorem 6.6. *Natale [2017]. Let $\mathcal{C}$ be a weakly group-theoretical braided fusion category. Then the core of $\mathcal{C}$ is equivalent to a Deligne tensor product $\mathfrak{B} \boxtimes \mathfrak{D}$, where $\mathfrak{D}$ is a* *pointed weakly anisotropic braided fusion category and* $\mathcal{B} \cong \mathrm{Vect}$ *or* $\mathcal{B}$ *is an Ising category. In particular, if* $\mathcal{C}$ *is integral, then its core is a pointed weakly anisotropic braided fusion category.*

The following theorem summarizes some known results related to Question 5.7 whose proofs rely on the existence of Tannakian subcategories and its relation with G-crossed braided fusion categories outlined above.

Theorem 6.7. *Let* $\mathcal{C}$ *be a braided fusion category. Then the following hold:*

(i) *Bruguières and Natale [2011]. If* $\mathrm{FPdim}\,\mathcal{C}$ *is an odd square-free integer, then* $\mathcal{C}$ *is equivalent to* $\mathrm{Rep}\,G$ *as a fusion category for some finite group* G.

(ii) *Natale [2014]. If* $\mathcal{C}$ *is non-degenerate and* $\mathrm{FPdim}\,\mathcal{C}$ *is a natural number less than* 1800*, or an odd natural number less than* 33075*, then* $\mathcal{C}$ *is weakly group-theoretical.*

(iii) *Natale and Pacheco Rodríguez [2016]. If* $\mathrm{FPdim}\,\mathcal{C} \in \mathbb{Z}$ *and the Frobenius-Perron dimensions of any simple object of* $\mathcal{C}$ *is a* p_i*-power, for some* $1 \leq i \leq r$*, where let* $p_1, \dots, p_r$ *be prime numbers, then* $\mathcal{C}$ *is weakly group-theoretical. Moreover, it is solvable is either* $r \leq 2$*, or* $p_i > 7$*, for all* $i = 1, \dots, r$.

(iv) *Natale [2017]. If* $\mathcal{C}$ *is integral and non-degenerate such that* $\mathrm{FPdim}\,X \leq 2$*, for every simple object* X*, then* $\mathcal{C}$ *is group-theoretical.*

(v) *Dong and Natale [2017]. If* $\mathcal{C}$ *is a non-degenerate and* $\mathrm{FPdim}\,\mathcal{C} = dq^n$*, where* $n \geq 0$*,* d *is a square-free natural number and* q *is an odd prime number, then* $\mathcal{C}$ *is group-theoretical.*

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