Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 2 (381–394)

THE SUBCONVEXITY PROBLEM FOR L-FUNCTIONS

Ritabrata Munshi

Abstract

Estimating the size of automorphic L-functions on the critical line is a central problem in analytic number theory. An easy consequence of the standard analytic properties of the L-function is the convexity bound, whereas the generalised Riemann Hypothesis predicts a much sharper bound. Breaking the convexity barrier is a hard problem. The moment method has been used to surpass convexity in the case of L-functions of degree one and two. In this talk I will discuss a different method, which has been quite successful to settle certain longstanding open problems in the case of degree three.

At the 1994 International Congress at Zürich, J. B. Friedlander [1995] briefly described the essence of the amplified moment method which he was developing in a series of joint works with Duke and Iwaniec, with the aim of obtaining non-trivial bounds for L-functions. Since then the amplification technique has proved to be very effective in a number of scenarios involving $GL(2)$ L-functions (see J. Friedlander and Iwaniec [1992], Duke, J. B. Friedlander, and Iwaniec [1993, 1994, 1995, 2001, 2002], Kowalski, Michel, and VanderKam [2002], Michel [2004], Harcos and Michel [2006], and Blomer and Harcos [2008]). But there are major hurdles in extending the method far beyond. In the last decade the automorphic period approach has been developed in great detail and generality (over number fields), by Michel, Venkatesh and others (see Bernstein and Reznikov [2010], Michel and Venkatesh [2010], Wu [2014]). This puts the moment method in a proper perspective and gives a satisfactory explanation to the 'mysterious identities between families of L-functions' that already occurs in the study of the moments of the Rankin-Selberg L-functions Harcos and Michel [2006], Michel [2004]. This has been the topic of Michel's address at the 2006 International Congress at Madrid Michel and Venkatesh [2006]. Here I will briefly describe a new approach to tackle subconvexity, which has not only settled some of the longstanding open problems in the field, but has also matched in strength the existing benchmarks. As there are several excellent accounts

MSC2010: primary 11F66; secondary 11M41.
Keywords: L-functions, subconvexity, circle method.

on the subconvexity problem for general automorphic L-functions and on its importance in number theory, equidistribution and beyond (see J. B. Friedlander [1995], Iwaniec and Sarnak [2000], Michel [2007], Michel and Venkatesh [2006], Sarnak [1998]), I will discuss some specific cases which will easily bring out the new features of the method in contrast to the amplified moment method of Friedlander-Iwaniec.

First let us recall the subconvexity problem and the amplification technique with the aid of an example. The Ramanujan Δ-function

$$\Delta(z) = \eta(z)^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$$

is a modular form of weight 12 for the full modular group $SL(2, \mathbb{Z})$, and is the prototype for all modular forms. Let χ be a primitive Dirichlet character with modulus M, then the twist $\Delta \otimes \chi$ is a modular form of weight 12 for the congruence group $\Gamma_0(M^2)$ with nebentypus χ^2. It was conjectured by Ramanujan, and later proved by Deligne, that $\tau(n) \ll n^{\frac{11}{2}+\varepsilon}$. Accordingly we define the normalized Fourier coefficients $\lambda_\Delta(n) = \tau(n)/n^{11/2}$, so that the associated Hecke L-function

$$L(s, \Delta \otimes \chi) = \sum_{n=1}^{\infty} \frac{\lambda_\Delta(n)\chi(n)}{n^s}$$

is absolutely convergent for $\mathrm{Re}(s) > 1$ and satisfies the Riemann type functional equation $s \to 1 - s$ with center at $s = 1/2$. The multiplicativity of the τ function, again conjectured by Ramanujan and proved shortly thereafter by Mordell (and by Hecke in general), leads to a degree two Euler product representation of this L-function. The basic analytic properties of this type of L-functions - analytic continuation, functional equation - were established by Hecke. This is an example of an automorphic L-function of degree two. In general, understanding the behaviour of automorphic L-functions inside the critical strip $0 \le \mathrm{Re}(s) \le 1$, and in particular on the central line $\mathrm{Re}(s) = 1/2$, is the main problem in this field.

There are few results in complex analysis which dictate the behaviour of holomorphic functions inside a domain, once their behaviour is known on the boundary. One such result is the convexity principle of Phragmén-Lindelöf, which when applied to the above L-function, yields the bound

$$L\left(\tfrac{1}{2} + it, \Delta \otimes \chi\right) \ll C^{1/4+\varepsilon}$$

for any $t \in \mathbb{R}$ and any $\varepsilon > 0$, where $C = [M(1+|t|)]^2$. (The notation '$\ll$' here means that there exists a constant $c(\varepsilon)$ depending only on ε such that the absolute value of the left-hand side is smaller than $c(\varepsilon)$ times the right-hand side.) The same bound can be obtained through the approximate functional equation which gives a Dirichlet series approximation to the L-value. It is now understood that the length of this approximation can not be smaller than the square-root of the size of the analytic conductor J. B. Friedlander [1995], Iwaniec and Sarnak [2000], which for the particular example we are considering is given by C. A direct consequence of the approximate functional equation is the bound

$$L(\tfrac{1}{2}+it, \Delta \otimes \chi) \ll C^{\varepsilon} \sup_{N \ll C^{1/2+\varepsilon}} \frac{|S(N)|}{N^{1/2}} + C^{-2018},$$

where $S(N)$ are Dirichlet polynomials of the form

$$S(N) = \sum_{n=1}^{\infty} \lambda_{\Delta}(n)\chi(n)n^{it}\, W\left(\frac{n}{N}\right)$$

with W a smooth bump function. A trivial estimation of this sum recovers the above convexity bound $O(C^{1/4+\varepsilon})$. This is far from what one expects to be the truth. Indeed the Generalized Riemann Hypothesis (GRH) implies the Generalized Lindelöf Hypothesis which predicts a bound with exponent 0 in place of $1/4$. Any bound with exponent $1/4 - \delta$ for some $\delta > 0$ is called a subconvex bound. Such bounds have several striking applications Michel [2007], Sarnak [1995].

In the example we are considering there are two distinct parameters of interest, namely t, which is allowed to take any real value, and M which can take any positive integral value. In other words, there are two types of subfamilies of L-values of interest, viz.

$$\{L(\tfrac{1}{2}+it, \Delta \otimes \chi) : t \in \mathbb{R}\}$$

where the character χ is kept fixed, and

$$\{L(\tfrac{1}{2}+it, \Delta \otimes \chi) : \chi \bmod M \ \text{ primitive}, M \in \mathbb{N}\}$$

where t is held fixed and the character varies with the modulus tending to infinity. A subconvex bound for the former type of subfamilies is called t-aspect subconvexity and that for the latter type is called M-aspect (or twist aspect). Often, especially in arithmetic applications, the parameter M is of interest, and one is required to the break the convexity barrier in the M-aspect only. One such application is the uniform distribution of rational points on sphere (Linnik's problem) Michel [2007].

The t-aspect subconvexity for $L(s,\Delta)$ (χ principal character) was first established by Good using the spectral theory of Maass forms. Good's approach is via the moment method and is based on getting a strong error term in the asymptotic expansion for the second moment

$$\int_T^{2T} |L(1/2+it,\Delta)|^2 \mathrm{d}t.$$

In Good [1982] he establishes that the above integral is asymptotically

$$TP(\log T) + O((T\log T)^{2/3})$$

where P is a linear polynomial. This strong error term is achieved by studying the more concentrated second moment

$$\int_{T-T/U}^{T+T/U} |L(1/2+it,\Delta)|^2 \mathrm{d}t.$$

where one strives to get U as big as possible. Expanding the absolute value square using the approximate functional equation, and then executing the t-integral one is left with a shifted convolution sum problem which may be tackled by studying the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\tau(n)\overline{\tau(n+h)}}{(n+h/2)^s}.$$

The automorphic forms enter the picture as the above series can be realised as the Petersson inner product of the form Δ and the h-th Poincare series. The analytic continuation of this L-series beyond the region of absolute convergence was first obtained by Selberg [1965]. But it was Good who first effectively used the spectral interpretation of the above Dirichlet series to obtain estimates for moments of L-function. The estimates obtained by Good give the optimal choice $U = (T\log T)^{1/3}$, and the above asymptotic expansion follows. This in turn yields

$$L(\tfrac{1}{2}+it,\Delta) \ll (1+|t|)^{1/3+\varepsilon},$$

which is the $GL(2)$ analogue of the famous estimate of Hardy-Littlewood-Weyl for the Riemann zeta function $\zeta(1/2+it) \ll (1+|t|)^{1/6+\varepsilon}$. The most straightforward way to prove the Weyl bound for the zeta function is through exponential sums, but a proof based on the moment method, similar in spirit as above, was obtained by Iwaniec [1980].

The twist aspect subconvexity for $L(s,\Delta\otimes\chi)$ is a harder problem. One reason being that there is no simple way to recover a subconvex bound for an individual L-value from an asymptotic for the second moment

$$\sum_{\psi \bmod M} |L(1/2+it, \Delta \otimes \psi)|^2.$$

There is no natural way (while retaining some sort of spectral completeness of the family) to shorten the outer sum so as to obtain a second moment concentrated around χ, the particular character we are interested in. (However this is again possible for the weight/spectral aspect, e.g. Lau, Liu, and Ye [2006], Sarnak [2001].) Of course, one way will be to estimate a higher moment, say the fourth, but this turns out to be a much harder problem. Though one would not need an asymptotic expansion in this case, but even getting an appropriately strong bound is difficult. The amplification technique originated to bypass this problem. The simple idea being to consider the weighted moment

$$\sum_{\psi \bmod M} w(\psi)|L(1/2+it, \Delta \otimes \psi)|^2,$$

where the weights are necessarily non-negative, and $w(\chi)$ is in some sense larger than the average weight. One way to assign such weights (amplifier) is to consider sums of the form

$$w(\psi) = \left|\sum_{\ell \sim L} a(\ell)\psi(\ell)\right|^2,$$

where the coefficients $a(\ell)$ are allowed to depend on χ, Δ and t. In the particular example we are looking at, the choice is rather simple $a(\ell) = \bar{\chi}(\ell)$. Indeed $|\chi(\ell)| = 1$, hence bounded away from 0, if $(\ell, M) = 1$. (But such simple effective lower bounds are not available for Fourier coefficients of modular or Maass forms, and the construction of the amplifier, in the more general setup, has to go through deeper arithmetic structure like Hecke relations.) Using this amplifier Duke, J. B. Friedlander, and Iwaniec [1993] obtained the subconvex bound

$$L(1/2, \Delta \otimes \chi) \ll M^{\frac{1}{2}-\frac{1}{22}+\varepsilon}.$$

This bound has been improved and extended to cover twists of any $GL(2)$ automorphic form. The strongest bound is due to Blomer and Harcos [2008], where they get the exponent $1/2 - 1/8$ pushing to the limit the amplification method and utilising ideas of Bykovskii. This exponent corresponds to the classic result of Burgess $L(1/2, \chi) \ll M^{3/16+\varepsilon}$ for the Dirichlet L-function Burgess [1963]. In the same paper Blomer and Harcos get a hybrid subconvex bound

$$L(\tfrac{1}{2}+it, \Delta \otimes \chi) \ll [M(1+|t|)]^{\frac{1}{2}-\frac{1}{40}+\varepsilon}.$$

A strong hybrid subconvex bound for Dirichlet L-function was obtained by Heath-Brown [1980] extending the work of Burgess [1963].

In general, the basic philosophy of the moment method and the amplification technique can be described as follows. Suppose one seeks to get a bound for $L(\pi_0) = L(1/2, \pi_0)$ where π_0 is an automorphic form. The approximate functional equation reduces the problem to getting cancellation in a sum of the form

$$S_{\pi_0} = \sum_{n \sim C^{1/2}} \lambda_{\pi_0}(n)$$

where λ_{π_0} are the Whittaker-Fourier coefficients and C is the conductor of π_0. As a first step, we need to find a 'spectrally complete family' $\mathfrak{F}$ containing π_0, where all the objects in $\mathfrak{F}$ have 'comparable conductors'. Let

$$\mathfrak{M} = \sum_{\pi \in \mathfrak{F}} |S_\pi|^2,$$

then dropping all terms in the sum except the particular object we are interested in, we conclude the bound $S_{\pi_0} \ll \mathfrak{M}^{1/2}$. Since $\mathfrak{M}$ cannot be smaller than $|\mathfrak{F}|$, the diagonal contribution, for subconvexity we at least need that $|\mathfrak{F}| \ll C^{1/2-\delta}$. So the family cannot be too big. Indeed, larger the family less likely it is, even with amplification, to obtain a non-trivial bound for a particular L-value $L(\pi_0)$, as the individual contribution gets washed out in the large average. Now let us consider the problem of estimating the moment $\mathfrak{M}$. The usual way for the moment method is to open the absolute value and then execute the sum over forms

$$\sum_{\pi \in \mathfrak{F}} \lambda_\pi(n)\overline{\lambda_\pi(m)},$$

using some sort of quasi-orthogonality (e.g. Petersson trace formula). So to get a good estimate we need the family to be big enough - e.g. the off-diagonal contribution in the Petersson trace formula is easily seen to be smaller for larger families, say when the level is bigger. This dichotomy on the size of the family puts a severe restriction on the choice of $\mathfrak{F}$. It is probably 'an accident' that such families exist for L-functions of degree one and two. The amplification technique kicks in when one stumbles upon a family, where $|\mathfrak{F}| = C^{1/2}$. Then one has to manufacture a suitable amplifier A_π, and has to estimate the amplified moment

$$\mathfrak{M}^\sharp = \sum_{\pi \in \mathfrak{F}} |A_\pi|^2 |S_\pi|^2,$$

instead of $\mathfrak{M}$.

I shall now explain a different approach to subconvexity in the context of the above example. We are seeking cancellation in the sum $S(N) = \sum_{n\sim N} \lambda_\Delta(n)\chi(n)n^{it}$, where $N = C^{1/2}$. The basic idea is to separate the oscillation of the Fourier coefficient $\lambda_\Delta(n)$ from that of $\chi(n)n^{it}$. We do this bluntly by introducing a new variable, and rewriting the sum as

$$S(N) = \sum\sum_{\substack{m,n\sim N\\ m=n}} \lambda_\Delta(n)\chi(m)m^{it}.$$

Next we detect the equation $m = n$ using the circle method to arrive at

$$S(N) = \frac{1}{Q^2}\sum_{q\sim Q}\ \sum_{\substack{a \bmod q\\ (a,q)=1}} \Big[\sum_{n\sim N} \lambda_\Delta(n)e_q(an)\Big]\Big[\sum_{m\sim N} \chi(m)m^{it}e_q(-am)\Big]$$

with $Q = N^{1/2} = C^{1/4}$, where we are using the standard shorthand notation $e_q(z) = e^{2\pi i z/q}$. Summation formulas of Poisson and Voronoi, reduces the expression to a bilinear form with Kloosterman fractions

$$\sum\sum_{\substack{m,q\sim Q\\ (m,q)=1}} \chi(q\bar{m})(qm^{-1})^{it}e_q(-M\bar{m})$$

where $\bar{m}$ is the multiplicative inverse of m modulo q. The trivial estimation of this sum yields the convexity bound. A deep result of Duke, Friedlander and Iwaniec gives some cancellation in such sums. But in Munshi [2014] I proceeded in a different direction which paved the way for further developments in the method. Suppose by some means we had reached a similar expression with a factorizable modulus $q = q_1q_2$ with $q_i \sim Q_i$ and $Q_1Q_2 = Q$. Then one could apply the Cauchy inequality to dominate the above expression by

$$Q^{1/2}\sum_{q_2\sim Q_2}\left[\sum_{\substack{m\sim Q\\ (m,q_2)=1}}\Big|\sum_{\substack{q_1\sim Q_1\\ (m,q_1)=1}} \chi(q_1)q_1^{it}e_{q_1q_2}(-M\bar{m})\Big|^2\right]^{1/2},$$

and now it would be possible to get more cancellation by opening the absolute value square and applying the Poisson summation formula on the sum over m. The new input that one would need is the Weil bound for Kloosterman sums. But how does one get this desired structure on the modulus? In Munshi [ibid.] I used Jutila's version of the circle method to achieve this goal, and was able to prove

$$L(\tfrac{1}{2}+it, \Delta\otimes\chi) \ll [M(1+|t|)]^{\frac{1}{2}-\frac{1}{18}+\varepsilon},$$

which was an improvement over the above mentioned result of Blomer-Harcos. There is another interesting way to achieve the same structure - 'congruence-equation trick'. Pick a set of primes $\mathbb{Q}_1$ from the interval $[Q_1, 2Q_1]$. Then for any $q_1 \in \mathbb{Q}_1$ we split the integral equation $m = n$ into a congruence $m \equiv n \bmod q_1$ and a smaller integral equation $(m-n)/q_1 = 0$. The last equation can be detected by a circle method with modulus $Q_2 = \sqrt{N/Q_1}$. Detecting the congruence using additive characters modulo q_1, we arrive at an expression similar to one we got above with $q = q_1 q_2$. Of course the price we pay to get this factorization is the increase in the size of the modulus $q \sim \sqrt{Q_1 N}$. But the structural advantage compensates this loss adequately, and even provides the desired extra saving. (For applications of this trick to Diophantine problems, see Browning and Munshi [2013] and Munshi [2015a].)

Unfortunately, this simple approach does not work for the twisted $GL(3)$ L-functions $L(s, \pi \otimes \chi)$ where π is a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$. Previously Li [2011] and Blomer [2012] had studied the t-aspect and the M-aspect subconvexity problems for these L-functions in the special case where π is a symmetric square lift of a $SL(2, \mathbb{Z})$ Hecke-Maass cusp form. (A p-adic version of Li's result was established in Munshi [2013b] using the ideas of Munshi [2013a].) Their approach is an extension of Conrey and Iwaniec [2000] method where non-negativity of certain L-values plays a crucial role. As such their results could not be extended to cover generic $SL(3, \mathbb{Z})$ forms. In Munshi [2015d] I partially succeeded in the M-aspect. Suppose the character χ factorises as $\chi_1 \chi_2$ where χ_i is primitive modulo M_i with $(M_1, M_2) = 1$, $M_1 < M_2$. We are seeking cancellation in the sum

$$S_3(N) = \sum_{n \sim N} \lambda_\pi(n, 1) \chi(n),$$

where $\lambda_\pi(n, r)$ are the normalised Whittaker-Fourier coefficients of the form π, and $N = M^{3/2}$ - square-root of the size of the conductor. As before we separate the oscillation of the Fourier coefficients from that of the character by introducing a new variable and an equation,

$$S_3(N) = \sum\sum_{\substack{m,n \sim N \\ m=n}} \lambda_\pi(n, 1) \chi(m).$$

Now we use the congruence-equation trick to split the integral equation $m = n$ as $m \equiv n \bmod M_1$ and the integral equation $(m-n)/M_1 = 0$. Here this trick acts as a level lowering mechanism as the modulus M_1 was intrinsic to the problem. The remaining integral equation is detected using the circle method with modulus of smaller size $Q =$

$\sqrt{N/M_1}$. The resulting expression now looks like

$$S_3(N) = \frac{1}{Q^2 M_1} \sum_{q\sim Q} \sum_{\substack{a \bmod qM_1 \\ (a,q)=1}} \Big[\sum_{n\sim N} \lambda_\pi(n,1) e_{qM_1}(an)\Big] \times \Big[\sum_{m\sim N} \chi(m) e_{qM_1}(-am)\Big],$$

and one is again able to win, as long as $\sqrt{M_2} < M_1 < M_2$, by applying summation formulas - Poisson summation and $GL(3)$ Voronoi summation - followed by an application of Cauchy to escape from the trap of involution, and another application of Poisson summation. Here one needs to use Deligne's bound for complete exponential sums.

Though it was clear that this approach would not extend to general characters, the t-aspect subconvexity for $L(s,\pi)$ for π a $SL(3,\mathbb{Z})$ form, became tractable. Indeed the t-aspect is related to twists by highly factorizable characters. In Munshi [2015b] I established the following subconvex bound.

Theorem 1. *Let π be a Hecke-Maass cusp form for $SL(3,\mathbb{Z})$. Then we have*

$$L(\tfrac{1}{2}+it,\pi) \ll (1+|t|)^{\frac{3}{4}-\frac{1}{16}+\varepsilon}.$$

Curiously the exponent matches with Li's bound in Li [2011] for symmetric square lifts, though the two approaches are totally different. The above result is proved using the technique outlined above, but now one have to use the archimedean analogue of the congruence-equation trick. Imagine $M_1 = p^r$ with p a fixed prime and $r \to \infty$, then the congruence condition $m \equiv n \bmod M_1$ corresponds to a condition on the p-adic size of $m-n$. This translates as $|m-n| \ll N/M_1$ in the archimedean situation of t-aspect. In Munshi [2015b] we choose a suitable parameter V and factorise the integral equation $m-n=0$ (of size N) into a distance condition $|m-n| \ll N/V$ and the smaller integral equation $m-n=0$ (of size N/V). The size restriction $|m-n| \ll N/V$ is then detected using an integral involving $(m/n)^{iv}$.

For the twist aspect one needs to introduce higher order harmonics. The usual circle methods and the DFI delta method are based on $GL(1)$ harmonics, or the harmonics of the abelian circle group S^1. These are the trigonometric functions $e(z)$. The delta method gives a Fourier resolution of the delta symbol $\delta : \mathbb{Z} \to \{0,1\}$, $\delta(0)=1$ and $\delta(n)=0$ for $n \neq 0$. A very rough version of this formula looks like

$$\delta(n) \approx \frac{1}{C^2} \sum_{c\sim C} \sum_{\substack{a \bmod c \\ (a,c)=1}} e_c(an).$$

Usually to detect the event $n = 0$ with n varying in the range $[-N, N]$ one takes $C = \sqrt{N}$ so as to minimise the total conductor - the arithmetic modulus c and the amplitude of the oscillation in the weight function (which we have ignored above) - in the circle method formula. In several cases, it turns out that all one needs is a circle method formula with C slightly smaller than $\sqrt{N}$. For example, the subconvexity of $L(s, \Delta \otimes \chi)$ follows quite easily if one has such an elusive circle method.

In the above applications we always needed a souped up version of the usual circle method formula, which we achieved by putting extra structural conditions on the fractions a/c that parametrise the outer sum. In all cases we ended up relating the L-value under focus to an average of products of L-values, e.g.

$$L(1/2 + it, \Delta \otimes \chi) \leftrightsquigarrow \sum_{f \in \mathfrak{F}} L(s_1, \Delta \otimes \upsilon_f) L(s_2 + it, \chi \otimes \upsilon_f)$$

where $\mathfrak{F}$ is a collection of (Farey) fractions with suitable factorization of denominator, and for $f = (a, c)$, υ_f is the additive character $n \to e_c(an)$. Hence at the end we are still computing moments of certain products of L-values (but not sizes of L-values). Since clearly the $GL(1)$ harmonics will not suffice for higher degree L-functions, one looks for expansions of delta involving higher order harmonics. The trace formulas of non-abelian groups are natural sources for such expansions. For example, the Petersson trace formula for modular forms gives

$$\begin{aligned}\delta(m-n) = &\sum_{f \in H_k(q,\psi)} w_f^{-1} \lambda_f(n) \overline{\lambda_f(m)} \\ &- 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{S_\psi(m,n;cq)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{cq}\right)\end{aligned}$$

for $m, n > 0$. Here $H_k(q, \psi)$ is an orthogonal Hecke basis for the space of cusp forms of weight k level q and nebentypus ψ, $S_\psi(m, n; c)$ are Kloosterman sums and $J_{k-1}(x)$ is the J-Bessel function. Since the left hand side does not depend on ψ, q or k, one may take suitable averages to jazz up the formula a bit and make it more suitable for application. In Munshi [2015c] and Munshi [2016], where I was considering the arithmetic twist aspect, it was more natural to take averages over q and ψ. One would imagine that when the problem in focus is in the t-aspect or spectral aspect, one would need to take average over k. Taking averages over nebentypus and level, executing the sum over ψ in the second part, and applying the reciprocity relations, we get

$$\delta(m-n) \approx \frac{1}{Q^2} \sum_{q\sim Q} \sum_{\substack{\psi \bmod q \\ \psi(-1)=(-1)^k}} \sum_{f\in H_k(q,\psi)} w_f^{-1}\lambda_f(n)\overline{\lambda_f(m)}$$
$$-\frac{2\pi i^{-k}}{Q} \sum_{q\sim Q} \Big[\frac{1}{N}\sum_{c\sim C} \sum_{\substack{a \bmod c \\ (a,c)=1}} e_c((a+1)\bar{q}m + (\bar{a}+1)\bar{q}n)\Big],$$

with $C = N/Q$. The first part of the formula involves $GL(2)$ harmonics, or Fourier coefficients of modular forms, the second part is like the usual circle method formula, and in fact matches in length to that if we take $Q = \sqrt{N}$. There are however two advantages - first one can take larger Q and thereby make c smaller, and secondly one can take advantage of the extra averaging over q. The cost one pays is of course the introduction of the more complicated $GL(2)$ harmonics in the formula. This version of delta method was introduced in Munshi [2015c] to tackle the M-aspect subconvexity for $L(s,\pi\otimes\chi)$ with π a $SL(3,\mathbb{Z})$ form. In the follow up paper Munshi [2016] a much simplified version of the approach was given. The second version of the proof is more in line with Munshi [2015d] and Munshi [2015b].

Theorem 2. *Let π be a Hecke-Maass cusp form for $SL(3,\mathbb{Z})$ and χ is a primitive character modulo M. Then we have*

$$L(\tfrac{1}{2},\pi\otimes\chi) \ll M^{\frac{3}{4}-\theta},$$

for some explicitly computable $\theta > 0$.

The reader perhaps has already realised that the above method is robust enough and that these results should generalise to Hecke-Maass cusp forms for any congruence subgroups of $SL(3,\mathbb{Z})$. Also in the last stated theorem we can have a subconvex bound at any point on the critical line with polynomial dependence on t. The work of Blomer [2012] for quadratic twists of symmetric square lifts, in contrast, is only for the central point. There are other trace formulas which can be utilised in the same fashion. For example the Kuznetsov trace formula can be used to give an expansion of the delta involving Fourier coefficients of Maass forms and Einsenstein series. I believe that this would be the key in settling the weight/spectral aspect subconvexity for the symmetric square L-functions or the t-aspect subconvexity for $GL(4)$ L-functions. My recent preprint Munshi [2017b] addresses the level aspect subconvexity for the symmetric square L-function. Let me also mention that a preprint of Blomer and Buttcane [2015] gives a partial result towards settling the spectral aspect subconvexity for $GL(3)$ L-functions. They do not use the above method.

Few recent works have used the above method to get strong bounds for lower degree L-functions. First Aggarwal [2017] and Singh [2017] have independently revisited Good's problem using the method of Munshi [2015b]. Here it is likely that this method is strong enough to yield the Weyl exponent. I have shown that the $GL(2)$ delta method can be used to prove the Burgess exponent both for twists of $GL(2)$ forms and for classical Dirichlet L-functions Munshi [2017a]. New ideas are still required to break the Burgess barrier. However in a joint work with Singh Munshi and Singh [2017], we have shown that the Weyl bound holds in the M-aspect for twists of $GL(2)$ L-functions when the modulus is a suitable prime power, for example if $M = p^3$ for a prime p. This work again uses the congruence-equation trick. In this context, let me mention that Milićević [2016] and Blomer and Milićević [2015] have developed a p-adic version of the Van der Corput method, which yields a sub-Weyl bound for Dirichlet L-functions when the modulus is a suitably high p power, and gives a sub-Burgess bound (which asymptotically decreases to Weyl bound) for twists of $GL(2)$ L-functions by similar characters.

Finally I would like to mention that Holowinsky and Nelson [n.d.] have come up with a much 'abridged version' of the method. This have simplified and shortened the proofs substantially. Their work is pivoted on a crucial observation that there is a 'central identity' which makes the circle method approach to subconvexity work. They have shown that in many cases this central identity can be derived using simpler summation formulas, completely avoiding the circle method. However one drawback that still remains in their work is that it is not clear how to predict this central identity without taking recourse to the circle method approach. This is now an active topic of research. First Holowinsky-Nelson have given a simpler proof for the twists of $GL(3)$ L-functions. Y. Lin [n.d.] have used this approach to give a hybrid bound for $GL(3)$ L-functions, and Aggarwal, Holowinsky, Lin, and Sun [n.d.] have given the non-circle method version of Munshi [2017a]. Apart from the problem of breaking the longstanding Burgess barrier, the following two problems should be the main focus of the circle method approach.

Problem 1: Weight/spectral aspect subconvexity for symmetric square L-functions.

Problem 2: t-aspect subconvexity for $GL(4)$ L-functions.

I hope one of these would be solved before the next International Congress.

References

K. Aggarwal, R. Holowinsky, Y. Lin, and Q. Sun (n.d.). "Burgess bound for twisted L-functions via a trivial delta method" (cit. on p. 392).

Keshav Aggarwal (July 2017). "t-aspect subconvexity for $GL(2)$ L-functions". arXiv: 1707.07027 (cit. on p. 392).

Joseph Bernstein and Andre Reznikov (2010). "Subconvexity bounds for triple L-functions and representation theory". *Ann. of Math. (2)* 172.3, pp. 1679–1718. MR: 2726097 (cit. on p. 381).

Valentin Blomer (2012). "Subconvexity for twisted L-functions on $\mathrm{GL}(3)$". *Amer. J. Math.* 134.5, pp. 1385–1421. MR: 2975240 (cit. on pp. 388, 391).

Valentin Blomer and Jack Buttcane (Apr. 2015). "On the subconvexity problem for L-functions on GL(3)". arXiv: 1504.02667 (cit. on p. 391).

Valentin Blomer and Gergely Harcos (2008). "Hybrid bounds for twisted L-functions". *J. Reine Angew. Math.* 621, pp. 53–79. MR: 2431250 (cit. on pp. 381, 385).

Valentin Blomer and Djordje Milićević (2015). "p-adic analytic twists and strong subconvexity". *Ann. Sci. Éc. Norm. Supér. (4)* 48.3, pp. 561–605. MR: 3377053 (cit. on p. 392).

T. D. Browning and R. Munshi (2013). "Rational points on singular intersections of quadrics". *Compos. Math.* 149.9, pp. 1457–1494. MR: 3109729 (cit. on p. 388).

D. A. Burgess (1963). "On character sums and L-series. II". *Proc. London Math. Soc. (3)* 13, pp. 524–536. MR: 0148626 (cit. on pp. 385, 386).

V. A. Bykovskiĭ (1996). "A trace formula for the scalar product of Hecke series and its applications". *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 226.Anal. Teor. Chisel i Teor. Funktsiĭ. 13, pp. 14–36, 235–236. MR: 1433344.

J. B. Conrey and H. Iwaniec (2000). "The cubic moment of central values of automorphic L-functions". *Ann. of Math. (2)* 151.3, pp. 1175–1216. MR: 1779567 (cit. on p. 388).

W. Duke, J. B. Friedlander, and H. Iwaniec (1993). "Bounds for automorphic L-functions". *Invent. Math.* 112.1, pp. 1–8. MR: 1207474 (cit. on pp. 381, 385).

– (1994). "Bounds for automorphic L-functions. II". *Invent. Math.* 115.2, pp. 219–239. MR: 1258904 (cit. on p. 381).

– (1995). "Class group L-functions". *Duke Math. J.* 79.1, pp. 1–56. MR: 1340293 (cit. on p. 381).

– (2001). "Bounds for automorphic L-functions. III". *Invent. Math.* 143.2, pp. 221–248. MR: 1835388 (cit. on p. 381).

– (2002). "The subconvexity problem for Artin L-functions". *Invent. Math.* 149.3, pp. 489–577. MR: 1923476 (cit. on p. 381).

J. Friedlander and H. Iwaniec (1992). "A mean-value theorem for character sums". *Michigan Math. J.* 39.1, pp. 153–159. MR: 1137896 (cit. on p. 381).

John B. Friedlander (1995). “Bounds for L-functions”. In: *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*. Birkhäuser, Basel, pp. 363–373. MR: 1403937 (cit. on pp. 381–383).

Anton Good (1982). “The square mean of Dirichlet series associated with cusp forms”. *Mathematika* 29.2, 278–295 (1983). MR: 696884 (cit. on p. 384).

Gergely Harcos and Philippe Michel (2006). “The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points. II”. *Invent. Math.* 163.3, pp. 581–655. MR: 2207235 (cit. on p. 381).

D. R. Heath-Brown (1980). “Hybrid bounds for Dirichlet L-functions. II”. *Quart. J. Math. Oxford Ser. (2)* 31.122, pp. 157–167. MR: 576334 (cit. on p. 386).

R. Holowinsky and P. Nelson (n.d.). “Subconvex bounds on GL3 via degeneration to frequency zero” (cit. on p. 392).

H. Iwaniec and P. Sarnak (2000). “Perspectives on the analytic theory of L-functions”. *Geom. Funct. Anal.* Special Volume, Part II. GAFA 2000 (Tel Aviv, 1999), pp. 705–741. MR: 1826269 (cit. on pp. 382, 383).

Henryk Iwaniec (1980). “Fourier coefficients of cusp forms and the Riemann zeta-function”. In: *Seminar on Number Theory, 1979–1980 (French)*. Univ. Bordeaux I, Talence, Exp. No. 18, 36. MR: 604215 (cit. on p. 384).

E. Kowalski, P. Michel, and J. VanderKam (2002). “Rankin-Selberg L-functions in the level aspect”. *Duke Math. J.* 114.1, pp. 123–191. MR: 1915038 (cit. on p. 381).

Yuk-Kam Lau, Jianya Liu, and Yangbo Ye (2006). “A new bound $k^{2/3+\epsilon}$ for Rankin-Selberg L-functions for Hecke congruence subgroups”. *IMRP Int. Math. Res. Pap.* Art. ID 35090, 78. MR: 2235495 (cit. on p. 385).

Xiaoqing Li (2011). “Bounds for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ L-functions and $\mathrm{GL}(3)$ L-functions”. *Ann. of Math. (2)* 173.1, pp. 301–336. MR: 2753605 (cit. on pp. 388, 389).

Y. Lin (n.d.). “Bounds for twists of $GL(3)$ L-functions” (cit. on p. 392).

P. Michel (2004). “The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points”. *Ann. of Math. (2)* 160.1, pp. 185–236. MR: 2119720 (cit. on p. 381).

Philippe Michel (2007). “Analytic number theory and families of automorphic L-functions”. In: *Automorphic forms and applications*. Vol. 12. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, pp. 181–295. MR: 2331346 (cit. on pp. 382, 383).

Philippe Michel and Akshay Venkatesh (2006). “Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik”. In: *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, pp. 421–457. MR: 2275604 (cit. on pp. 381, 382).

– (2010). “The subconvexity problem for GL_2”. *Publ. Math. Inst. Hautes Études Sci.* 111, pp. 171–271. MR: 2653249 (cit. on p. 381).

Djordje Milićević (2016). "Sub-Weyl subconvexity for Dirichlet L-functions to prime power moduli". *Compos. Math.* 152.4, pp. 825–875. MR: 3484115 (cit. on p. 392).
Ritabrata Munshi (2013a). "Bounds for twisted symmetric square L-functions". *J. Reine Angew. Math.* 682, pp. 65–88. MR: 3181499 (cit. on p. 388).
– (2013b). "Bounds for twisted symmetric square L-functions—III". *Adv. Math.* 235, pp. 74–91. MR: 3010051 (cit. on p. 388).
– (2014). "The circle method and bounds for L-functions—I". *Math. Ann.* 358.1-2, pp. 389–401. MR: 3158002 (cit. on p. 387).
– (2015a). "Pairs of quadrics in 11 variables". *Compos. Math.* 151.7, pp. 1189–1214. MR: 3371491 (cit. on p. 388).
– (2015b). "The circle method and bounds for L-functions—III: t-aspect subconvexity for $GL(3)$ L-functions". *J. Amer. Math. Soc.* 28.4, pp. 913–938. MR: 3369905 (cit. on pp. 389, 391, 392).
– (2015c). "The circle method and bounds for L-functions—IV: Subconvexity for twists of $\mathrm{GL}(3)$ L-functions". *Ann. of Math. (2)* 182.2, pp. 617–672. MR: 3418527 (cit. on pp. 390, 391).
– (2015d). "The circle method and bounds for L-functions, II: Subconvexity for twists of $\mathrm{GL}(3)$ L-functions". *Amer. J. Math.* 137.3, pp. 791–812. MR: 3357122 (cit. on pp. 388, 391).
– (Apr. 2016). "Twists of $GL(3)$ L-functions". arXiv: 1604.08000 (cit. on pp. 390, 391).
– (Oct. 2017a). "A note on Burgess bound". arXiv: 1710.02354 (cit. on p. 392).
– (Sept. 2017b). "Subconvexity for symmetric square L-functions". arXiv: 1709.05615 (cit. on p. 391).
Ritabrata Munshi and Saurabh Kumar Singh (June 2017). "Weyl bound for p-power twist of $GL(2)$ L-functions". arXiv: 1706.03985 (cit. on p. 392).
Peter Sarnak (1995). "Arithmetic quantum chaos". In: *The Schur lectures (1992) (Tel Aviv)*. Vol. 8. Israel Math. Conf. Proc. Bar-Ilan Univ., Ramat Gan, pp. 183–236. MR: 1321639 (cit. on p. 383).
– (1998). "L-functions". In: *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*. Extra Vol. I, pp. 453–465. MR: 1648042 (cit. on p. 382).
– (2001). "Estimates for Rankin-Selberg L-functions and quantum unique ergodicity". *J. Funct. Anal.* 184.2, pp. 419–453. MR: 1851004 (cit. on p. 385).
Atle Selberg (1965). "On the estimation of Fourier coefficients of modular forms". In: *Proc. Sympos. Pure Math., Vol. VIII*. Amer. Math. Soc., Providence, R.I., pp. 1–15. MR: 0182610 (cit. on p. 384).
Saurabh Kumar Singh (June 2017). "t-aspect subconvexity bound for $GL(2)$ L-functions". arXiv: 1706.04977 (cit. on p. 392).
Akshay Venkatesh (2010). "Sparse equidistribution problems, period bounds and subconvexity". *Ann. of Math. (2)* 172.2, pp. 989–1094. MR: 2680486.

Han Wu (2014). "Burgess-like subconvex bounds for $\mathrm{GL}_2 \times \mathrm{GL}_1$". *Geom. Funct. Anal.* 24.3, pp. 968–1036. MR: 3213837 (cit. on p. 381).

Received 2017-11-27.

RITABRATA MUNSHI
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
1 DR. HOMI BHABHA ROAD
COLABA, MUMBAI 400005
INDIA
and
INDIAN STATISTICAL INSTITUTE
208 B.T. ROAD
KOLKATA 700108
INDIA
rmunshi@math.tifr.res.in