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BOURGAIN–DELBAEN $\mathscr{L}_\infty$-SPACES, THE SCALAR-PLUS-COMPACT PROPERTY AND RELATED PROBLEMS

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Abstract

We outline a general method of constructing $\mathscr{L}_\infty$-spaces, based on the ideas of Bourgain and Delbaen, showing how the solution to the Scalar-plus-Compact Problem, the embedding theorem of Freeman, Odell and Schlumprecht and other recent developments fit into this framework.

1 Introduction

Bourgain and Delbaen [1980] introduced a new class of separable Banach space that provided counterexamples to a number of open problems. On the one hand, each of these spaces $X_{\alpha,\beta}$ is an "$\mathscr{L}_\infty$-space", which means that its finite-dimensional structure resembles that of a $\mathcal{C}(K)$-space, and the dual $X^*_{\alpha,\beta}$ is isomorphic either to ℓ_1 or $\mathcal{C}[0,1]^*$; on the other hand, each infinite-dimensional subspace of $X_{\alpha,\beta}$ has a further subspace isomorphic to some ℓ_p $(1 \le p < \infty)$, so that the global structure of $X_{\alpha,\beta}$ is very different from that of $\mathcal{C}(K)$-spaces, or complemented subspaces of such spaces.

Much more recently, it has become clear that the spaces $X_{\alpha,\beta}$ are a special case of a more general class, of what we now call Bourgain–Delbaen spaces (or BD-spaces for short). Taking a suitably general definition, it has been shown by Argyros, Gasparis, and Motakis [2016] that every separable $\mathscr{L}_\infty$-space is isomorphic to a BD-space, and BD-constructions have been used both in the solution of the Scalar-plus-Compact Problem by Argyros and Haydon [2011] and in the proof by Freeman, Odell, and Schlumprecht [2011] that every Banach space with separable dual embeds in an isomorphic predual of ℓ_1. The aim of this article is to sketch some general theory of BD constructions, trying to show how recent results fit together, and (hopefully) shedding light on some older theorems by presenting them in a BD framework. We do not have space for detailed proofs, especially

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not of the scalar-plus-compact construction. Readers who are interested in an alternative account of this may wish to consult the Séminaire Bourbaki paper of Grivaux and Roginskaya [2014–2015], or the associated video available on YouTube. We include statements and sketched proofs of certain results that are as yet unpublished, including some from the long-promised paper of Argyros, Freeman, Haydon, Odell, Raikoftsalis, Schlumprecht, and Zisimopoulou [n.d.]. We also include a few easy proofs of known results, where we think that they may aid understanding of what is otherwise a rather abstract narrative.

The theory we sketch here builds on a rich body of literature starting with the paper of Tsirel'son [1974] that introduced the first example of a Banach space with no subspace isomorphic to c_0 or to any ℓ_p. Since Tsirelson norms are essential tools for us, we devote a section to sketching a fairly general theory of such norms, including an account of regular families of subsets of $\mathbb{N}$ and the crucial notion of asymptotic ℓ_1 structure.

We then move on to the mixed-Tsirelson spaces that have provided the raw material for the solution to the Distortion Problem by Odell and Schlumprecht [1994], the remarkable counterexamples of Gowers [1994a,b, 1996] and the whole theory of indecomposable and hereditarily indecomposable spaces as initiated by Gowers and Maurey [1993] and developed further by Argyros and Deliyanni [1997], Argyros and Felouzis [2000], and Argyros and Tolias [2004]. There are already some excellent surveys of this material in the literature and we shall try to avoid excessive duplication with papers such as Argyros and Tolias [2004], Maurey [1994] and Maurey [2003]. In particular, despite their importance, we shall say little about the notions of distortion and hereditary indecomposability.

Next we follow Argyros, Gasparis, and Motakis [2016] by introducing a notion of BD-space that is sufficiently general to embrace all separable $\mathscr{L}_\infty$-spaces, before restricting attention to "standard BD-spaces", a subclass amenable to detailed analysis and admitting norm estimates of Tsirelson type. We look at duality and subspace structure for such spaces and note that the spaces that are constructed in the Embedding Theorem of Freeman, Odell, and Schlumprecht [2011] may be taken to be of this type. We look at some natural notions of sub- and super-objects, and see how two known constructions, one due to Zippin [1977] and one to Cabello Sánchez, Castillo, Kalton, and Yost [2003] emerge naturally from BD methods. Finally, we move on to the scalar-plus-compact construction of Argyros and Haydon [2011] and more recent developments of this kind.

2 Notation

We write $\mathbb{N}$ for the set $\{1, 2, 3, \dots\}$ of natural numbers and ω for $\{0, 1, 2, \dots\}$, which we are usually considering as an ordinal. The cardinality of a set A is denoted $\#A$. When $f : X \to Y$ is a mapping and $A \subseteq X$, we write $f[A]$ for the image of A under f.

In a vector space, we write $\mathrm{sp}\langle x_i : i \in I\rangle$ for the linear span of a set $\{x_i : i \in I\}$; in a normed space $\overline{\mathrm{sp}}\langle x_i : i \in I\rangle$ denotes the corresponding closed linear span. The closed unit ball of a normed space X is denoted ball X. Two Banach spaces X and Y are said to be M-isomorphic if there exists a linear homeomorphism $T : X \to Y$ with $\|T\|\|T^{-1}\| \le M$. Sequences $(x_n)_{n\in\mathbb{N}}$ in X and $(y_n)_{n\in\mathbb{N}}$ in Y are said to be M-equivalent if there is an M-isomorphism $T : \overline{\mathrm{sp}}\langle x_n : n \in \mathbb{N}\rangle \to \overline{\mathrm{sp}}\langle y_n : n \in \mathbb{N}\rangle$ with $T(x_n) = y_n$ for all n. We say that a Banach space X is ℓ_p-saturated if there is a constant M such that every infinite-dimensional subspace of X has a further subspace M-isomorphic to ℓ_p.

A sequence $(M_n)_{n\in\mathbb{N}}$ of closed subspaces of a Banach space X is said to be a Schauder Decomposition if every $x \in X$ admits a unique representation as a norm-convergent sum $x = \sum_{n=1}^\infty x_n$ with $x_n \in M_n$ for all n. This is the case if and only if the linear direct sum $\bigoplus_{n\in\mathbb{N}} M_n$ is dense in X and there is a constant M such that for each N the usual projection $P_N : \bigoplus_{n\in\mathbb{N}} M_n \to \bigoplus_{n\le N} M_n$ extends to an operator on X with norm at most M. If each of the subspaces M_n is finite-dimensional we speak of a finite-dimensional decomposition, or f.d.d. Of course, if each of the subspaces is one-dimensional, with $M_n = \mathrm{sp}\langle x_n\rangle$, we get back to the well-known notion of a Schauder basis.

We use fairly standard notation for function spaces : $\mathbb{R}^\Gamma$ is the space of all scalar valued functions on a set Γ, and for $x \in \mathbb{R}^\Gamma$, the *support* of x is $\mathrm{supp}\, x = \{\gamma \in \Gamma : x(\gamma) \ne 0\}$; $\mathbb{R}^{(\Gamma)}$ is the space of functions of finite support, $\ell_\infty(\Gamma)$ the space of bounded functions, equipped with the supremum norm, $c_0(\Gamma)$ the norm closure of $\mathbb{R}^{(\Gamma)}$ in $\ell_\infty(\Gamma)$, and $\ell_p(\Gamma)$ the space of all functions x for which the norm $\|x\|_p$, defined by $\|x\|_p^p = \sum_{\gamma\in\Gamma} |x(\gamma)|^p$ is finite. For $1 \le p \le \infty$ we write ℓ_p (resp. ℓ_p^n) for $\ell_p(\Gamma)$ with $\Gamma = \mathbb{N}$ (resp. $\Gamma = \{1, 2, \dots, n\}$). When $\Gamma_1 \subset \Gamma_2$ we identify $\mathbb{R}^{\Gamma_1}$ with the subspace of $\mathbb{R}^{\Gamma_2}$ consisting of functions that vanish off Γ_1, and adopt the same convention for other function spaces.

According to the context, an element y of $\mathbb{R}^{(\Gamma)}$ may be regarded either as a vector, or as a functional acting on $\mathbb{R}^\Gamma$ via the duality $\langle y, x\rangle = \sum_\gamma x(\gamma)y(\gamma)$. If we are thinking of y as a functional, we shall generally employ a notation adorned with a star, writing for instance f^* instead of y. In particular, the element of $\mathbb{R}^\Gamma$ that takes the value 1 at a specific $\gamma \in \Gamma$ and is zero elsewhere may be denoted either e_γ if we are thinking of it as a vector, or as e_γ^* if we are thinking of it as the evaluation functional $x \mapsto \langle e_\gamma^*, x\rangle = x(\gamma)$.

We say that finite subsets $E_1, E_2, \dots$ of $\mathbb{N}$ are *successive*, and write $E_1 < E_2 < \cdots$ if $\max E_j < \min E_{j+1}$ for all j. If X is a space with a basis $(d_n)_{n\in\mathbb{N}}$ (resp. a finite-dimensional decomposition $(M_n)_{n\in\mathbb{N}}$) we define the *range* of a vector x, denoted $\mathrm{ran}\, x$, to be the minimal interval $I \subset \mathbb{N}$ such that $x \in \langle d_n : n \in I\rangle$ (resp. $x \in \bigoplus_{n\in I} M_n$). We say that vectors $x_1, x_2, \dots$ are successive, or that (x_n) is a *block sequence*, if $\mathrm{ran}\, x_1 < \mathrm{ran}\, x_2 < \cdots$, and that (x_n) is a *skipped* block sequence if $1 + \max \mathrm{ran}\, x_n < \min \mathrm{ran}\, x_{n+1}$ for all n. The closed linear span of a block sequence is called a *block subspace*. In the context of a normed space, we say that a sequence (x_n) is *normalized* if $\|x_n\| = 1$ for all n.

We say that an infinite-dimensional Banach space X is *indecomposable* if X cannot be expressed as the direct sum of two infinite-dimensional closed subspaces, and *hereditarily indecomposable* if every infinite-dimensional subspace of X is indecomposable. We recall that a bounded linear operator T on a Banach space X is *strictly singular* if there is no infinite dimensional subspace Y of X such that $T\restriction_Y$ is an isomorphism. We say that X has *few operators* if every bounded linear operator on X can be written $T = \lambda I + S$, where S is strictly singular. The Banach space is said to have *very* few operators, or to have the Scalar-plus-Compact Property if, in addition, every strictly singular operator on X is compact.

3 Regular families, Tsirelson norms and asymptotic ℓ_p spaces

Definition 3.1. We say that a collection $\mathcal{M}$ of finite subsets of $\mathbb{N}$ is *regular* if

1. $\mathcal{M}$ is *compact* (for the topology induced by the product topology on $\{0,1\}^{\mathbb{N}}$) and
2. $\mathcal{M}$ is *spreading*, i.e. if $M = \{m_1, m_2, \dots, m_k\} \in \mathcal{M}$ and $n_j \geq m_j$ for all j then $N = \{n_1, n_2, \dots, n_k\}$ is also in $\mathcal{M}$. (Such an N is called a *spread* of M.)

We note that a regular family is also *hereditary* in the sense that $N \subset M \in \mathcal{M}$ implies $N \in \mathcal{M}$, and that any compact family of subsets is contained in a regular family (namely the closure of the set of all its spreads).

Important examples of regular families include $\mathcal{A}_n = \{M \subset \mathbb{N} : \#M \leq n\}$, the Schreier family $\mathcal{S} = \{M \subset \mathbb{N} : \#M \leq \min M\}$ and the higher Schreier families $\mathcal{S}_\alpha$ introduced by Alspach and Argyros [1992] and defined for all countable ordinals α. There is an associative binary operation $*$ defined on the set of regular families by taking $\mathcal{M} * \mathcal{N}$ to be the set of all unions $\{M_1 \cup M_2 \cup \dots M_n$ where M_1, $M_2, \dots$ are successive members of $\mathcal{M}$ and $\{\min M_j : j \leq n\} \in \mathcal{N}$. As with any associative operation we can form powers $\mathcal{M}^{*n} = \mathcal{M} * \mathcal{M} * \cdots * \mathcal{M}$ (with n terms). For finite n the higher Schreier families are given by $\mathcal{S}_n = \mathcal{S}^{*n}$.

It follows from standard results (based on the Hahn–Banach Theorem or Ptak's combinatorial lemma) about weakly null sequences of continuous functions on a compact set that for every regular family and every $\epsilon > 0$ there exist n and a finite sequence $a_1, a_2, \dots, a_n$ such that $a_j \geq 0$ for all j, $\sum_{j=1}^n a_j = 1$ while $\sum_{j\in M} a_j \leq \epsilon$ for all $M \in \mathcal{M}$. We call the vector $a = \sum_n a_n e_n$ a *basic convex combination* that is *ϵ-small* for $\mathcal{M}$. The papers of Alspach and Odell [1988] and Alspach and Argyros [1992] studied this phenomenon in greater detail, introducing ordinal indices that measure the speed of weak convergence of a sequence and the complexity of convex combinations obtained by the method of repeated averages. It was in this context that the special regular families $\mathcal{S}_\alpha$ ($\alpha < \omega_1$) were introduced in Alspach and Argyros [ibid.]. The same ideas yield a general result that is easy to state and sufficient for our present purposes.

Proposition 3.2. *Let $\mathcal{M}$ be a regular family. Then there is a regular family $\mathcal{M}^\#$ such that every maximal member N of $\mathcal{M}^\#$ is the support of a basic convex combination $a_{\mathcal{M},N,\epsilon}$ that is $2^{-\min N+1}$-small for $\mathcal{M}$.*

Let $\mathcal{M}$ be a regular family and let $E_1 < E_2 < \cdots < E_n$ be a sequence of successive subsets of $\mathbb{N}$. We say that the sequence $(E_j)_{j\le n}$ is $\mathcal{M}$-admissible if the set $\{\min E_j : j \le n\}$ is in $\mathcal{M}$ and that sequence $(f_j^*)_{j=1}^a$ in $\mathbb{R}^{(\mathbb{N})}$ is $\mathcal{M}$-admissible if the sequence of supports $\operatorname{supp} f_1^*, \dots, \operatorname{supp} f_n^*$ is.

We are now ready to define Tsirelson norms. There are two ways to to do this: directly by an implicit functional equation for a norm $\|\cdot\|$ on $\mathbb{R}^{(\mathbb{N})}$, or by constructing a norming set $W \subset \mathbb{R}^{(\mathbb{N})}$ and defining $\|x\| = \sup_{f^*\in W} |\langle f^*, x\rangle|$. The first approach was introduced in Figiel and Johnson [1974] and often makes for elegant proofs; the second, closer in spirit to the original construction Tsirel'son [1974], is useful when a more delicate calculation based on an analysis of the functionals f^* is needed. In the context of Tsirilson spaces we shall follow the notation of Figiel and Johnson [1974], writing Ex for the vector given by

$$Ex(n) = \begin{cases} x(n) & \text{when } n \in E \\ 0 & \text{when } n \notin E. \end{cases}$$

Definition 3.3. Let $\mathcal{M}$ be a regular family and let $0 < \theta < 1$ be a real number. We define $W(\mathcal{M}, \theta)$ to be the minimal subset W of $\mathbb{R}^{(\mathbb{N})}$ with $\pm e_n^* \in W$ for all n and such that $\theta \sum_{j=1}^a f_j^* \in W$ whenever $f_1^*, \dots, f_a^*$ is an $\mathcal{M}$-admissible sequence in W. We define the *Tsirelson* norm $\|\cdot\|_{T(\mathcal{M},\theta)}$ on $\mathbb{R}^{(\mathbb{N})}$ by

$$\|x\|_{T(\mathcal{M},\theta)} = \sup_{f^*\in W(\mathcal{M},\theta)} \langle f^*, x\rangle.$$

The space $T(\mathcal{M}, \theta)$ is then defined to be the completion of $\mathbb{R}^{(\mathbb{N})}$ with respect to this norm. An equivalent definition of the norm is to define $\|\cdot\|_{T(\mathcal{M},\theta)}$ directly as the smallest solution to the functional equation

$$\|x\| = \max\Big\{\|x\|_\infty\,,\ \theta \sup \sum_{j=1}^a \|E_j x\|\Big\},$$

where the supremum is taken over all $\mathcal{M}$-admissible sequences $E_1, \dots, E_a$. *The* Tsirelson space T, as defined in Figiel and Johnson [ibid.] and studied extensively in Casazza and Shura [1989], is $T(\mathcal{S}, \frac{1}{2})$.

For general families $\mathcal{M}$, the above definition appears in the preprint ofArgyros and Deliyanni [1992], where results are obtained even in the case where $\mathcal{M}$ is only assumed to be compact (and not necessarily spreading). In the case of a regular family we have the following theorem, in which the second statement is due to Bellenot [1986].

Theorem 3.4. *Let $\mathcal{M}$ be a regular family and let $0 < \theta < 1$. If $\mathcal{M}$ has members of arbitrarily large finite cardinality then $T[\mathcal{M}, \theta]$ is reflexive with no subspace isomorphic to any space ℓ_p. If the members of $\mathcal{M}$ are of bounded cardinality then the unit vector basis of $T[\mathcal{M}, \theta]$ is equivalent to the usual basis of ℓ_p where $\theta = n^{-1/p'}$, $n = \max_{M\in\mathcal{M}} \#M$ and $1/p + 1/p' = 1$.*

We shall give a sketch proof of the first statement in this theorem, not because there is anything new in it (indeed it is very close to that given by Figiel and Johnson), but in order to give a very easy example of the use of special convex combinations, and to introduce the important notion of asymptotic ℓ_p structure.

Definition 3.5. Let X be a Banach space with a finite-dimensional Schauder decomposition $(M_n)_{n\in\mathbb{N}}$, and let $p \in [1, \infty]$. We say that (M_n) is *asymptotic ℓ_p* with constant $C > 1$, if for every n there exists N such that the sequence $(x_1, x_2, \ldots, x_n)$ is C-equivalent to the unit vector basis of ℓ_p^n whenever $(x_j)_{j=1}^n$ is a normalized block sequence with $N \leq \operatorname{ran} x_1 < \operatorname{ran} x_2 < \cdots \operatorname{ran} x_n$. We sometimes risk ambiguity by saying that the space X is asymptotic ℓ_p. There is a related, but weaker, notion close to what was called "asymptotic ℓ_p" by Maurey, Milman, and Tomczak-Jaegermann [1995]. To avoid ambiguity, we shall say that a finite-dimensional decomposition is *skipped-asymptotic ℓ_p* with constant C, if for every n there exists N such that the sequence $(x_1, x_2, \ldots, x_n)$ is C-equivalent to the unit vector basis of ℓ_p^n whenever $(x_j)_{j=1}^n$ is a normalized skipped block sequence with $N \leq \operatorname{ran} x_1$.

A space with a finite-dimensional decomposition that is *skipped-asymptotic ℓ_p* with $p < \infty$ cannot contain c_0 or ℓ_q for $q \neq p$. Theorem 3.4 thus follows from the next proposition.

Proposition 3.6. *Let $\mathcal{M}$ be a regular family with members of arbitrarily large finite cardinality and let $0 < \theta < 1$ be a real number. Then $T(\mathcal{M}, \theta)$ is reflexive and its usual unit-vector basis $(e_n)_{n\in\mathbb{N}}$ is asymptotic ℓ_1.*

Proof. For any n there is a set $M_0 \in \mathcal{M}$ of cardinality n; using the spreading property, there is a natural number N such that every set M with $\#M = n$ and $\min M \geq N$ is in $\mathcal{M}$. Now suppose that $x_1, \ldots, x_n$ is a normalized block sequence with $\min \operatorname{ran} x_1 \geq N$ and set $x = \sum_{j=1}^n a_j x_j$; taking $E_j = \operatorname{ran} x_j$, we see that the sequence $E_1, E_2, \ldots, E_n$

is $\mathcal{M}$-admissible, so that

$$\|\sum_{j=1}^{n} a_j x_j\| = \|x\| \geq \theta \sum_{j=1}^{n} \|E_j x\| = \theta \sum_{j=1}^{n} \|a_j x_j\| = \theta \sum_{j=1}^{n} |a_j|,$$

by the implicit formula for the norm. Thus $T(\mathcal{M}, \beta)$ is asymptotic ℓ_1.

Since $T(\mathcal{M}, \beta)$ has an unconditional basis, to show that it is reflexive it is enough to show that it has no subspace isomorphic to c_0 or ℓ_1, and c_0 is excluded by the asymptotic ℓ_1 property. By a result of James [1964], if a space has a subspace isomorphic to ℓ_1 it has a space nearly isometric to ℓ_1. So it will be enough for us to show that there is no normalized block sequence $(x_n)_{n\in\mathbb{N}}$ in $T(\mathcal{M}, \theta)$ satisfying the lower estimate

$$\|\sum_j a_j x_j\| \geq \theta' \sum_j |a_j|$$

for all scalars a_j, where $\theta' = \frac{1}{2}(1+\theta) < 1$.

To do this we shall employ an easy splitting lemma variants of which underlie many proofs about Tsirelson spaces, as well as the BD-spaces we shall be looking at later.

Lemma 3.7. *Let $\mathcal{M}$ be a regular family and let $E_1, \ldots, E_a$ be an $\mathcal{M}$-admissible sequence of subsets of $\mathbb{N}$. Let I be an interval in $\mathbb{N}$ and let $R_i = [p_i, q_i]$ ($i \in I$) be successive intervals in $\mathbb{N}$. Then we may write $I = I' \cup \bigcup_{k=0}^{a} I_k$ where*

$$\begin{aligned} I' &= \{i \in I : R_i \cap E_k = \varnothing \text{ for all } k\} \\ I_0 &= \{i \in I : R_i \cap E_k \neq \varnothing \text{ for more than one value of } k\} \\ I_k &= \{i \in I : R_i \cap E_l \neq \varnothing \text{ for } l = k \text{ but no other value of } l\}. \end{aligned}$$

The set $\{q_i : i \in I_0\}$ belongs to the family $\mathcal{M}$.

We now consider a normalized block sequence $(x_n)_{n\in\mathbb{N}}$ with $\operatorname{ran} x_j = R_j = [p_j, q_j]$ and form the sum $x = \sum_{i\in I} a_i x_i$ where the coefficients a_i are chosen so that $\sum_{i\in I} a_i e_{q_i}$ is a basic convex combination that is ϵ-small for $\mathcal{M} \cup \mathcal{A}_1$, where $\epsilon = \frac{1}{2}(1-\theta)$. What this means is that $\sum_{i\in J} a_i \leq \epsilon$ whenever $\{q_j : j \in J\} \in \mathcal{M}$.

By the implicit definition of the norm, either $\|x\| = \|x\|_\infty \leq \max_i |a_i| \leq \epsilon$ or there is an $\mathcal{M}$-admissible sequence $E_1, \ldots, E_a$ such that

$$\|x\| = \theta \sum_{k=1}^{a} \|E_k x\|.$$

Applying Lemma 3.7 we obtain

$$\begin{aligned}\|x\| &\leq \sum_{i\in I_0}\|a_i x_i\| + \theta\sum_{k=1}^{a}\sum_{i\in I_k}\|a_i x_i\| \\ &\leq \sum_{i\in I_0} a_i + \theta\sum_{i\in I} a_i \leq \epsilon + \theta = \theta',\end{aligned}$$

because $\{q_j : j \in I_0\} \in \mathcal{M}$. □

The original Tsirelson space has a rich theory and, thanks to special properties of the Schreier family $\mathcal{S}$, we have a good understanding of its subspace structure; the reader is referred to Casazza and Shura [1989] for a comprehensive account of this material. Taken together with the generalized versions $T(\mathcal{S}_\alpha, \theta)$, the Tsirelson space has become much more than an isolated counterexample and now plays a key role in the general theory. For instance, the "subsequential $T(\mathcal{S}_\alpha, \theta)$-estimates" of Odell, Schlumprecht, and Zsák [2007] may be used to study the structure of general separable reflexive spaces.

4 Mixed Tsirelson spaces

The definition we are about to give of a "mixed Tsirelson space" might at first sight seem an idle generalization. But in fact Schlumprecht's space Schlumprecht [1991], which is of this type, opened the way to a new chapter in the theory, making possible Gowers's solutions to a group of previously intractable problems in Gowers [1994a,b, 1996], the theory of hereditarily indecomposable spaces introduced by Gowers and Maurey [1993] and the solution of the Distortion Problem in Hilbert space by Odell and Schlumprecht [1994].

Definition 4.1. Argyros and Deliyanni [1997] Let I be a countable set; for each $i \in I$ let $\mathcal{M}_i$ be a regular family and let $0 < \beta_i < 1$ be a real number. We define the norming set $W[(\mathcal{M}_i, \beta_i)_{i\in I}]$ to be the smallest subset of $\mathbb{R}^{(\mathbb{N})}$ that contains $\pm e_n^*$ for all $n \in \mathbb{N}$ and has the property that the functional f^* given by

$$f^* = \beta_i \sum_{r=1}^{a} f_r^*$$

is in W whenever $f_1^*, \ldots, f_a^*$ is an $\mathcal{M}_i$-admissible sequence with $f_r^* \in W$ for all r. (Such a functional is said to have been created by an *$(\mathcal{M}_i, \beta_i)$-operation* or to be of *weight* β_i.) The space $T[(\mathcal{M}_i, \beta_i)_{i\in I}]$ is defined to be the completion of $\mathbb{R}^{(\mathbb{N})}$ with respect to the norm given by

$$\|x\| = \sup\{\langle f^*, x\rangle : f^* \in W[(\mathcal{M}_i, \beta_i)_{i\in I}]\}.$$

There is of course an alternative definition of the norm using an implicit formula.

It is possible to modify the definition of the norming set $W[(\mathcal{M}_i, \beta_i)_{i\in I}]$ by placing restrictions on the functionals f_r^* that are permitted in an $(\mathcal{M}_i, \beta_i)$-operation. One possibility is to insist that f_r^* must be a functional of a specific weight (determined by i and the preceding functionals $f_1^* \dots, f_{r-1}^*$); this is referred to as a *coding*, and is generally applied for i in some proper subset of I (typically, for i odd when $I = \mathbb{N}$). The idea of coding in this way can be traced back to the paper of Maurey and Rosenthal [1977]. Another possibility is to insist that f_r^* be an average $n_r^{-1}\sum_{k=1}^{n_r} g_{r,k}^*$ where the $g_{r,k}^*$ are successive elements of W; this approach, introduced by Odell and Schlumprecht [1995, 2000], is referred to as *saturation under constraints* and has found more recent applications in Argyros, Beanland, and Motakis [2013], Argyros and Motakis [2014], and Beanland, Freeman, and Motakis [2015].

The first example in the literature of what we now call a mixed Tsirelson space seems to be a space constructed by Tzafriri in Tzafriri [1979] to solve a delicate question about type and cotype. In our notation, this space is $T[(\mathcal{A}_n, \theta/\sqrt{n})_{n\in\mathbb{N}}]$. Schlumprecht's space Schlumprecht [1991] is $T[(\mathcal{A}_n, (\log_2(n+1))^{-1}_{n\in\mathbb{N}}]$. Most subsequent work on hereditary indecomposability and related topics has followed Argyros and Deliyanni [1997] by working not with the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ but with either a highly lacunary subsequence $(\mathcal{A}_{n_k})_{k\in\mathbb{N}}$ or a sequence $(\mathcal{S}_{\alpha_k})_{k\in\mathbb{N}}$ of higher Schreier families. For the applications we are considering here it is convenient to write $\beta_i = m_i^{-1}$ and make some standard assumptions about the natural numbers m_i and the families $\mathcal{M}_i$ that we work with.

Definition 4.2. Let $(m_i)_{i\in\mathbb{N}}$ be a sequence of natural numbers and let $(\mathcal{M}_i)_{i\in\mathbb{N}}$ be a sequence of regular families. We shall say that (m_i) and $(\mathcal{M}_i)$ satisfy the *Standard Assumptions* if

1. each m_i has the form 2^{l_i} with $l_i \in \mathbb{N}$, $l_1 \geq 2$ and $l_{i+1} \geq 2l_i$;
2. for each $i \in \mathbb{N}$, $\mathcal{M}_{i+1} \supseteq \mathcal{M}_i^{*l_{i+1}}$
3. for each $i \in \mathbb{N}$, and each maximal $N \in \mathcal{M}_{i+1}$ there is a basic convex combination supported on N that is m_{i+1}^{-2}-small for $(\mathcal{A}_4 * \mathcal{M}_i)^{*l_{i+1}}$.

The existence, given $\mathcal{M}_i$, of a family $\mathcal{M}_{i+1}$ satisfying (2) and (3) follows from Proposition 3.2. A basic convex combination having the property (3) will be called an $\mathcal{M}_{i+1}$-*special basic convex combination.*

The next proposition is central to the theory of mixed Tsirelson spaces and to many constructions that use coding to achieve hereditary indecomposability and few operators.

Proposition 4.3. *Let (m_i) and $(\mathcal{M}_i)$ satisfy the Standard Assumptions, let $h \geq 2$ be a natural number and let $a = \sum_{n\in N} a_n e_n$ be an $\mathcal{M}_h$-special basic convex combination. For a functional f^* of weight m_i^{-1} in $W[(\mathcal{M}_i, m_i^{-1})_{i\in I}]$ we have*

$$|\langle f^*, a\rangle| \leq \begin{cases} m_i^{-1} & \text{if } i \geq h \\ 2m_h^{-1}m_i^{-1} & \text{if } i < h \end{cases}$$

In particular, $\|a\| = m_h^{-1}$ and the equality $\langle f^*, a\rangle = \|a\|$ can be achieved only for a functional f^* of weight m_h^{-1}.

It is an important property of the Tsirelson space $T(\mathcal{S}, \beta)$ that every normalised block sequence is equivalent to a subsequence of the usual basis; in fact (x_n) is equivalent to (e_{q_n}), where $q_n = \max \operatorname{supp} x_n$ (see Casazza and Shura [1989]). No such result holds for arbitrary normalised block sequences in mixed Tsirelson spaces, and one could even say that it is this "failure" that underlies all their interesting properties. Indeed, provided the Standard Assumptions are satisfied, we can construct block sequences of vectors that do not behave like a subsequence of the usual basis. The following proposition is proved by a finite version of the "ℓ_1-improvement argument" of James [1964] that we used earlier in our discussion of Theorem 3.4.

Proposition 4.4. *Let (m_i) and $(\mathcal{M}_i)$ satisfy the Standard Assumptions, let $h \geq 2$ be a natural number and let $(w_n)_{n\in\mathbb{N}}$ be a block sequence in $T[(\mathcal{M}_i, \beta_i)_{i\in I}]$. Then there exists a maximal member M of $\mathcal{M}_h$, and a normalized finite block subsequence $(x_n)_{n\in I}$ of (w_n) such that $\{\max \operatorname{supp} x_n : n \in I\} = M$ and*

$$\Big\|\sum_{n\in I} \lambda_n x_n\Big\| \geq \tfrac{1}{4} \sum_{n\in I} |\lambda_n|,$$

for all scalars λ_n.

If $(x_n)_{n\in I}$ and M are as in the above proposition, so that $M = \{q_n : n \in I\}$ with $q_n = \max \operatorname{supp} x_n$, we may consider an $\mathcal{M}_h$-special basic convex combination $\sum_{n\in I} a_n e_{q_n}$, noting that

$$\Big\|\sum_{n\in I} a_n e_{q_n}\Big\| \leq m_h^{-1}$$
$$\Big\|\sum_{n\in I} a_n x_n\Big\| \geq \tfrac{1}{4},$$

by Propositions 4.3 and 4.4. Thus there is no constant C such that an inequality

$$\Big\|\sum_{n\in I} a_n x_n\Big\| \leq C \Big\|\sum_{n\in I} a_n e_{q_n}\Big\|$$

holds for an arbitrary normalized block sequence with $q_n = \max \operatorname{supp} x_n$.

However, there are special sequences for which we do have upper estimates of this form. These are the so-called "rapidly increasing sequences", or RIS. The idea is that from an arbitrary block sequence (w_n) we may construct, first of all, successive normalized finite block sequences $(x_n)_{n\in I_h}$ ($h = 2, 3, \dots$) as in Proposition 4.4. We then form the $\mathcal{M}_h$-special convex combinations $y_h = \sum_{n\in I_h} a_n x_n$. A suitable subsequence ("rapidly increasing") of $(y_h)_{h\geq 2}$ is then a RIS and satisfies the desired norm estimates. We shall not go into detail either about rapidly increasing sequences or about the coding that can be introduced into a mixed Tsirelson construction to produce hereditary indecomposability and other exotic behaviours. We refer the reader to original papers such as Gowers and Maurey [1993] and Argyros and Deliyanni [1997], or the survey articles Maurey [1994] and Maurey [2003]. The idea in brief is that, with suitable coding, a construction of this kind results in a space X such that for every bounded linear operator T on X, there is a scalar λ such that $\|Tx_n - \lambda x_n\| \to 0$ for every RIS (x_n). Since every block subspace can be shown to contain a RIS, $T - \lambda I$ is strictly singular. The space X constructed in this way thus has few operators. In a later section we shall see what coding means in the context of a BD construction, and attempt to show why we are able to get from strict singularity to compactness of an operator $T - \lambda I$.

5 $\mathcal{L}_\infty$ spaces and Bourgain–Delbaen constructions

The $\mathcal{L}_p$-spaces were introduced some 50 years ago by Lindenstrauss and Pełczyński [1968] and were studied further in Lindenstrauss and Rosenthal [1969] (see also Nielsen and Wojtaszczyk [1973]). They provide an early, and striking, example of how conditions placed on finite-dimensional subspaces can have strong consequences for the isomorphic structure of an infinite-dimensional space, and remain one of the key areas of interest in Banach spaces.

Definition 5.1. Let X be a Banach space, let $1 \leq p \leq \infty$ and let $M > 1$ be a real number. We say that X is a $\mathcal{L}_{p,M}$-space if for every finite-dimensional subspace E of X there is a finite-dimensional subspace $F \supseteq E$ which is M-isomorphic to $\ell_p^{\dim F}$. If X is $\mathcal{L}_{p,M}$ for some M we say it is a $\mathcal{L}_p$-space.

Despite their definition in terms of finite-dimensional structure, the $\mathcal{L}_p$-spaces have many good infinite-dimensional properties, of which we now recall a few. For $1 \leq p \leq \infty$ every separable $\mathcal{L}_p$-space has a Schauder basis, and when $p < \infty$ every such space contains an isomorphic copy of ℓ_p. When $1 < p < \infty$ a separable Banach space is $\mathcal{L}_p$ if and only if it is isomorphic to a non-hilbertian complemented subspace of the Lebesgue space

$L_p(0,1)$. A useful characterization of $\mathcal{L}_\infty$-spaces in terms of extensions of compact operators was given in Lindenstrauss and Rosenthal [1969] (see also Theorem 4.2 of Zippin [2003]).

Proposition 5.2. *A Banach space X is an $\mathcal{L}_{\infty,M}$-space X if and only if, for every Banach space Z, every closed subspace $Y \subset Z$ and every compact operator $K : Y \to X$, there is a compact extension $L : Z \to X$ with $\|L\| \le M\|K\|$.*

In other respects, the structure of $\mathcal{L}_\infty$-spaces is more complicated than what we have for the case $p < \infty$. One of the main achievements of the original paper of Bourgain and Delbaen [1980] was to exhibit $\mathcal{L}_\infty$-spaces without subspaces isomorphic to c_0. We are now ready to look at the Bourgain–Delbaen construction, and its generalizations, in more detail.

It is immediate from the definition that a separable Banach space X is a $\mathcal{L}_\infty$-space if and only if there is a constant M and an increasing sequence of finite-dimensional subspaces $E_1 \subseteq E_2 \subseteq \cdots$ with $\overline{\bigcup_n E_n} = X$ and such that, for each n, E_n is M-isomorphic to $\ell_\infty(\Gamma_n)$ for some finite set Γ_n. In this set-up, the inclusion $E_n \hookrightarrow E_{n+1}$ corresponds to an isomorphic embedding $i_{n+1,n} : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma_{n+1})$, and the structure of a separable $\mathcal{L}_\infty$-space is determined by this sequence of embeddings.

There is a particular class of isomorphic embeddings that is convenient to work with: we say that $i : \ell_\infty(\Gamma_1) \to \ell_\infty(\Gamma_2)$ is an *extension operator* if Γ_1 is a subset of Γ_2 and, for all $u \in \ell_\infty(\Gamma_1)$, $(iu)\restriction_{\Gamma_1} = u$. A quick way to describe Bourgain–Delbaen spaces is to say that they are $\mathcal{L}_\infty$-spaces constructed from a sequence of extension operators. Before giving a formal definition, we briefly forget about norms and boundedness, and consider the (very easy) linear algebra of such a sequence of extension operators.

Let $\Gamma_1 \subset \Gamma_2 \subset \cdots$ be an increasing sequence of finite sets, with $\Gamma = \bigcup_n \Gamma_n$; for each n, let r_n be the restriction mapping $\mathbb{R}^\Gamma \to \mathbb{R}^{\Gamma_n}$. We shall say that a sequence $(i_n)_{n\in\mathbb{N}}$ of linear mappings $i_n : \mathbb{R}^{\Gamma_n} \to \mathbb{R}^\Gamma$ is a *compatible sequence of extension mappings* if the following are true:

1. each i_n is an extension mapping, that is to say $r_n i_n r_n = r_n$;
2. the compatibility condition $i_n r_n i_m = i_m$ holds whenever $m < n$.

Given such a compatible sequence, each of the "one-step" mappings $i_{n+1,n} = r_{n+1} i_n$ is an extension mapping $\mathbb{R}^{\Gamma_n} \to \mathbb{R}^{\Gamma_{n+1}}$, and conversely, if we are given a sequence of one-step extension mappings $i_{n+1,n}$ as in the earlier discussion, there is a unique compatible family of extensions $i_n : \mathbb{R}^{\Gamma_n} \to \mathbb{R}^\Gamma$ satisfying $i_{n+1,n} = r_{n+1} i_n$.

We now introduce some notation that will be used consistently in the rest of this section. We consider a countably infinite set Γ, expressed as the union of an increasing sequence of finite subsets Γ_n, and write Δ_n for the difference set $\Gamma_n \setminus \Gamma_{n-1}$ when $n > 1$; for the case

$n = 1$ we set $\Delta_1 = \Gamma_1$. We say that an element γ is of *rank* n if $\gamma \in \Delta_n$. We suppose Γ to be equipped with a compatible sequence of linear extension mappings i_n. The images $E_n = i_n[\mathbb{R}^{\Gamma_n}]$ form an increasing sequence of subspaces of $\mathbb{R}^\Gamma$ and, for each n, we may define a projection $P_n = i_n r_n$ from $\mathbb{R}^\Gamma$ onto E_n. The dual projection P_n^* takes $\mathbb{R}^{(\Gamma)}$ onto $\mathbb{R}^{\Gamma_n}$, regarded as a subspace of $\mathbb{R}^{(\Gamma)}$. When $\gamma \in \Delta_{n+1}$ for some n we define $c_\gamma^* = P_n^* e_\gamma^*$, a functional supported on Γ_n, while if $\gamma \in \Gamma_1$ we set $c_\gamma^* = 0$. In either case we define $d_\gamma^* = e_\gamma^* - c_\gamma^*$. For any n and any $\gamma \in \Delta_n$ we define $d_\gamma = i_n e_\gamma \in \mathbb{R}^\Gamma$.

Proposition 5.3. *Let Γ be a countably infinite set, equipped with a compatible sequence of linear extension mappings as in the discussion above. The family $(d_\gamma^*)_{\gamma\in\Gamma}$ is an algebraic basis of $\mathbb{R}^{(\Gamma)}$ and d_γ $(\gamma \in \Gamma)$ are the unique elements of $\mathbb{R}^\Gamma$ such that $(d_\gamma, d_\gamma^*)_{\gamma\in\Gamma}$ is a biorthogonal system. The extension mappings i_n, and the projections P_n and P_n^* are given by*

$$i_n u = \sum_{\gamma\in\Gamma_n} \langle d_\gamma^*, u\rangle d_\gamma \qquad (u \in \mathbb{R}^{\Gamma_n})$$

$$P_n x = \sum_{\gamma\in\Gamma_n} \langle d_\gamma^*, x\rangle d_\gamma \qquad (x \in \mathbb{R}^{\Gamma})$$

$$P_n^* f^* = \sum_{\gamma\in\Gamma_n} \langle f^*, d_\gamma\rangle d_\gamma^* \qquad (f^* \in \mathbb{R}^{(\Gamma)}).$$

If we think of Δ_m as being "to the left" of Δ_n when $m < n$, then $(d_\gamma^*)_{\gamma\in\Gamma}$ is a left-triangular basis of $\mathbb{R}^{(\Gamma)}$, while the family $(d_\gamma)_{\gamma\in\Gamma}$ is right-triangular.

Definition. Let Γ be a countably infinite set, expressed as the union of an increasing sequence of finite subsets Γ_n and equipped with a compatible family of linear extension mappings $i_n : \mathbb{R}^{\Gamma_n} \to \mathbb{R}^\Gamma$. We shall say that Γ is a *Bourgain–Delbaen set*, or more briefly a *BD-set*, if the mappings i_n take values in $\ell_\infty(\Gamma)$ and are uniformly bounded as operators from $\ell_\infty(\Gamma_n)$ to $\ell_\infty(\Gamma)$. We define $X(\Gamma)$ to be the closure in $\ell_\infty(\Gamma)$ of the union $\bigcup_{n\in\mathbb{N}} i_n[\ell_\infty(\Gamma_n)]$ and call $X(\Gamma)$ a *Bourgain–Delbaen space*.

When Γ is a BD-set then for any n and any $u \in \ell_\infty(\Gamma_n)$ we have $\|u\| \le \|i_n u\| \le M\|u\|$, where $M = \sup_n \|i_n\|$. Thus each of the increasing sequence of subspaces $E_n = i_n[\ell_\infty(\Gamma_n)]$ is M-isomorphic to $\ell_\infty(\Gamma_n)$, so that $X(\Gamma)$ is an $\mathscr{L}_{\infty,M}$-space. The subspaces of $X(\Gamma)$ defined by $M_n = i_n[\ell_\infty(\Delta_n)] = \mathrm{sp}\langle d_\gamma : \gamma \in \Delta_n\rangle$ form a Schauder decomposition of $X(\Gamma)$, the associated projection onto $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ being P_n. In fact, suitably ordered, the vectors d_γ form a Schauder basis of $X(\Gamma)$ but we do not often need to use this finer structure. Similarly, the functionals d_γ^* form a ("left-triangular") Schauder basis of $\ell_1(\Gamma)$, but it is usually convenient to work with a coarser structure, the Schauder decomposition formed by the subspaces $M_n^* = \mathrm{sp}\langle d_\gamma^* : \gamma \in \Delta_n\rangle$; the corresponding projection onto the finite direct sum $M_1^* \oplus M_2^* \oplus \cdots \oplus M_n^* = \mathrm{sp}\langle d_\gamma^* : \gamma \in \Gamma_n\rangle = \ell_1(\Gamma_n)$

is P_n^*. It is convenient in this context to write $\Gamma_0 = \varnothing$ and P_0 (resp. P_0^*) for the zero operator on $\ell_\infty(\Gamma)$ (resp. $\ell_1(\Gamma)$). When $E = [m,n]$ is an interval in $\mathbb{N}$ we shall write $P_E = P_n - P_{m-1}$ and $P_E^* = P_n^* - P_{m-1}^*$. When we talk about "block-sequences" in $X(\Gamma)$ it will always be with respect to the above Schauder decomposition: thus the *range* $\operatorname{ran} x$ of of an vector x in $X(\Gamma)$ is the smallest interval E such that $P_E x = x$ and (x_n) is a block-sequence if $\operatorname{ran} x_n < \operatorname{ran} x_{n+1}$ for all n. We adopt similar notation for the range of a functional $f^* \in \ell_1$, but we give the word "support" its usual meaning $\operatorname{supp} f^* = \{\gamma \in \Gamma : f^*(\gamma) \neq 0\}$.

Since the extension mappings that are used to build BD-spaces form a rather special subclass of the class of all isomorphic embeddings of finite-dimensional ℓ_∞-spaces, it might be natural to guess that BD-spaces form a rather special sort of $\mathscr{L}_\infty$-space. So the following recent result of Argyros, Gasparis, and Motakis [2016] is perhaps surprising.

Theorem 5.4. *Every infinite-dimensional separable $\mathscr{L}_\infty$-space is isomorphic to a BD-space.*

We have already noted that a BD-structure on a set Γ is determined by specifying either the extension operators i_n or the functionals c_γ^*. When we use this method to carry out interesting constructions, it is usually most convenient to work with the c_γ^* and we need a criterion for norm-boundedness of the mappings i_n expressed in terms of these functionals.

Proposition 5.5. *Let $\Gamma = \bigcup_n \Gamma_n$ be a set equipped with a compatible sequence of extension mappings $(i_n)_{n\in\mathbb{N}}$ and let $M \geq 1$. Using our standard notation, the following are equivalent:*

(1) *Γ is a BD-set with constant M; this is to say $\|i_n\| \leq M$ for all n;*
(2) *For the norm of operators on $\ell_1(\Gamma)$, $\|P_n^*\| \leq M$ for all n;*
(3) *For every n, every $m < n$ and every $\gamma \in \Delta_n$, $\|P_m^* c_\gamma^*\|_1 \leq M$.*

As well as introducing the idea of building $\mathscr{L}_\infty$-spaces by successive extensions, the original paper of Bourgain and Delbaen gave a neat way to construct functionals c_γ^* satisfying condition (3) above. Modifying slightly the definitions that have appeared in earlier papers, and eliminating some special cases, we give a definition that captures the crucial idea.

Definition 5.6. Let Γ be a countably infinite set, equipped as usual with a compatible sequence of linear extension mappings (i_n). Let $\beta < \frac{1}{2}$ be a positive constant and let n be a natural number. We shall say that an element c^* of $\ell_1(\Gamma_n)$ is a *BD-functional*, with weight (at most) β, if c^* has one of the forms

$$c^* = \begin{cases} \beta P_{[s,n]}^* b^* & \text{or} \\ \alpha e_\xi^* + \beta P_{[s,n]}^* b^* \end{cases} \tag{5-1}$$

with $0 \le \alpha \le 1$, $s \le n+1$, $b^* \in \operatorname{ball} \ell_1(\Gamma_n \setminus \Gamma_{s-1})$ and (in the second case) $\xi \in \Delta_m$ for some $m < s$. We note that in the case where $s = n+1$, we have $b^* = 0$, so that c^* is either 0 or αe^*_ξ. A trivial generalization of the proof given by Bourgain and Delbaen [1980] gives us the following theorem.

Theorem 5.7. *Let Γ be a countably infinite set, equipped with a compatible sequence of linear extension mappings (i_n). Suppose that there is a constant $\beta < \frac{1}{2}$ such that, for each n and each $\gamma \in \Delta_{n+1}$, the functional c^*_γ is a BD-functional with weight at most β. Then Γ is a BD-set with $\sup_n \|i_n\| \le M = (1-2\beta)^{-1}$.*

Bourgain and Delbaen considered pairs of scalars α, β with $\beta < \frac{1}{2}$, $\alpha \le 1$ and $\alpha + \beta > 1$, constructing for each such pair an $\mathscr{L}_\infty$-space $X_{\alpha,\beta}$ with the Radon–Nikodým Property; in particular, $X_{\alpha,\beta}$ has no subspace isomorphic to c_0. When $\alpha = 1$ the space $X_{1,\beta}$ has the Schur Property and in particular is ℓ_1-saturated. For the case $\alpha < 1$, Haydon [2000] established ℓ_p-saturation, where $\alpha^q + \beta^q = 1$ and $1/p + 1/q = 1$. For a space $X(\Gamma)$ to have these properties (for a given pair α, β) it is sufficient that

1. for every $\gamma \in \Gamma$, c^*_γ is a BD functional with the given values of α and β;
2. for every $m < n < p$, every $\xi \in \Delta_m$, every $\eta \in \Delta_n$ and every choice of sign $\pm$ there exists $\gamma \in \Delta_p$ with $c^*_\gamma = \alpha e^*_\xi \pm \beta P^*_{(m,\infty)} e^*_\eta$,

It is easy to see that, for a general BD-space $X(\Gamma)$, the evaluation functionals e^*_γ form a system equivalent to the usual basis of $\ell_1(\Gamma)$. So $X(\Gamma)^*$ has a subspace $\overline{\mathrm{sp}}\langle e^*_\gamma : \gamma \in \Gamma\rangle$ naturally isomorphic to $\ell_1(\Gamma)$. It was Alspach [2000] who first observed that for the spaces $X_{\alpha,\beta}$ of Bourgain and Delbaen, when $\alpha < 1$, this subspace makes up the whole of the dual space. It seems hard to arrive at straightforward conditions on the functionals c^*_γ for this to be true in a general BD construction.

There remain open problems about the original BD spaces $X_{\alpha,\beta}$, for instance whether $X_{\alpha,\beta}$ and $X_{\alpha',\beta'}$ are non-isomorphic when α, β and α', β' are distinct pairs with $\alpha^q + \beta^q = 1 = \alpha'^q + \beta'^q$. But subsequent developments have concentrated on constructions involving BD-functionals with $\alpha = 1$. In the next section we sketch a fairly general framework within which a lot of constructions are possible, and where precise analysis can be carried out, including an exact description of the dual space.

6 Standard BD-spaces

We shall say that a BD-set Γ is a *standard* BD-set if there is a constant $\beta \in (0, \frac{1}{2})$, called the *weight* of Γ, such that, for each n and each $\gamma \in \Delta_{n+1}$, we have

$$c^*_\gamma = \begin{cases} \beta P^*_{[s,\infty)} b^* & \text{or} \\ e^*_\xi + \beta P^*_{[s,\infty)} b^* \end{cases}$$

with $s \le n+1$, $b^* \in \operatorname{ball} \ell_1(\Gamma_n \setminus \Gamma_{s-1})$ and (in the second case) $\xi \in \Gamma_{s-1}$. We call s the "cut", b^* the "top" and ξ (when it exists) the "base" of γ.

The key tool in the study of the structure of standard BD-spaces is what we call the *evaluation analysis* which expresses the evaluation e^*_γ as a sum of terms that are adapted to the finite-dimensional decomposition $(M^*_n)_{n\in\mathbb{N}}$ of $\ell_1(\Gamma)$. If base γ is undefined this is easy to write down:

$$e^*_\gamma = c^*_\gamma + d^*_\gamma = \qquad \beta P^*_{[s,\infty)} b^* + d^*_\gamma.$$

If $\xi = \operatorname{base}\gamma$ is defined then we have

$$e^*_\gamma = c^*_\gamma + d^*_\gamma = e^*_\xi + \beta P^*_{[s,\infty)} b^* + d^*_\gamma,$$

and we may continue by expressing e^*_ξ as $c^*_\xi + d^*_\xi$ and so on. What we end up with is the following

Proposition 6.1. *Let Γ be a standard BD-space of weight β and let γ be an element of Γ. Then there exist a natural number a and elements $\xi_1, \xi_2, \dots, \xi_a$ of Γ such that* base ξ_1 *is undefined, $\xi_a = \gamma$ and $\xi_k = \operatorname{base}\xi_{k+1}$ when $1 \le k < a$. We have the evaluation analysis*

$$e^*_\gamma = \sum_{k=1}^{a} (\beta P^*_{[s_k,\infty)} b^*_k + d^*_{\xi_k}),$$

where $s_k = \operatorname{cut}\xi_k$ and $b^*_k = \operatorname{top}\xi_k$.

A good way to investigate duality of standard BD spaces is to introduce a tree-order $\preccurlyeq$ on Γ. We can define this recursively by saying that $\xi \preccurlyeq \gamma$ if and only if either $\xi = \gamma$ or $\xi \preccurlyeq \operatorname{base}\gamma$. What this amounts to is that the elements ξ with $\xi \preccurlyeq \gamma$ are exactly the $\xi_1, \xi_2, \dots, \xi_a$ that occur in the evaluation analysis of Proposition 6.1. The elements γ with no base are minimal in this tree. In accordance with standard terminology, we shall say that a standard BD-set is *well-founded* if it has no infinite branch, that is to say, if there is no infinite sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n = \operatorname{base}\xi_{n+1}$ for all n.

Proposition 6.2. *Let Γ be a standard BD-set. Then $X(\Gamma)^* = \overline{\operatorname{sp}}\langle e^*_\gamma : \gamma \in \Gamma\rangle$ if and only if Γ is well-founded.*

One of the implications in the above proposition is easy to see: if $\beta = \{\xi_1 \prec \xi_2 \prec \xi_3 \prec \cdots\}$ is an infinite branch of the tree Γ then we can define a functional f^* by $\langle f^*, x\rangle = \lim_{n\to\infty} x(\xi_n)$. This functional, which it is natural to denote by e^*_β, is not in $\ell_1(\Gamma)$ since $\langle e^*_\beta, d_{\xi_n}\rangle = 1$ for all n, while $\langle g^*, d_{\xi_n}\rangle \to 0$ as $n \to \infty$ for any $g^* \in \ell_1(\Gamma)$. One way of proving the converse implication uses the fact that the set of extreme points of the unit ball of $X(\Gamma)^*$ is contained in the weak*-closure of the evaluations e^*_γ and applies Choquet's integral representation theorem.

Although in current applications we have need of only one very special sort of BD-set with infinite branches it may be interesting to note that there is a general duality result here too. We first note that the set B of infinite branches of Γ has a natural topology as a Polish space and that for any bounded Radon measure μ on B we may define a functional $R^*\mu \in X(\Gamma)^*$ by $\langle R^*\mu, x\rangle = \int_{\mathrm{B}} \langle e^*_\beta, x\rangle \mathrm{d}\mu(\beta)$. Subject to modest additional hypotheses on weights (of which the following proposition gives one example) we have a nice extension of Proposition 6.2, again provable by a Choquet argument.

Proposition 6.3. *Let Γ be a standard BD set of weight $\beta < \frac{1}{4}$. The dual space $X(\Gamma)^*$ is naturally isomorphic to $\ell_1(\Gamma) \oplus \mathcal{M}^{\mathrm{b}}(\mathrm{B})$.*

It is of course an elementary fact that every separable Banach space is isomorphic to a quotient of ℓ_1. The striking and unexpected result proved by Freeman, Odell, and Schlumprecht [2011] is that, when Y^* is a separable *dual* space, the quotient operator can be chosen to be the dual of an isomorphic embedding of Y into a space X with X^* isomorphic to ℓ_1. Unsurprisingly, they use a BD-construction, though one that does not quite fit with our definition of a standard BD-space. Nonetheless, their idea carries over to this framework yielding the following.

Theorem 6.4. *Let Y be a Banach space with separable dual and let $\beta < \frac{1}{2}$ be a positive real number. Then Y embeds isomorphically into a standard BD-space $X(\Gamma)$ of weight β with $X(\Gamma)^*$ naturally isomorphic to $\ell_1(\Gamma)$.*

7 Tsirelson-type estimates for standard BD spaces

Let Γ be a standard BD-set and let $\gamma \in \Gamma$ be an element with evaluation analysis

$$e^*_\gamma = \sum_{k=1}^{a} (\beta P^*_{[s_k,\infty)} b^*_k + d^*_{\xi_k})$$

as in Proposition 6.1. We shall say that a is the *age* of γ and that the set $\{s_1, s_2, \ldots, s_a\}$ is the *history* of γ. If all branches of the tree structure on Γ are finite then the collection of all histories $\{\operatorname{hist}\gamma : \gamma \in \Gamma\}$ is a compact family of finite subsets of $\mathbb{N}$ and so is contained in some regular family $\mathcal{M}$.

It becomes clear that there may be a connection with Tsirelson norms if we consider γ with the above evaluation analysis and a sequence $(x_i)_{i=1}^a$ in $X(\Gamma)$ such that $\operatorname{ran} x_i \subseteq$

$[s_i, \operatorname{rank}\xi_i)$. We then have $\langle d^*_{\xi_k}, x_i\rangle = 0$ for all i and k, and

$$\langle \beta P^*_{[s_k,\infty)} b^*_k, x_i\rangle = \begin{cases} \langle b^*_k, x_k\rangle & \text{if } i = k \\ 0 & \text{otherwise.}\end{cases}$$

This leads to

$$\langle e^*_\gamma, \sum_{i=1}^a x_i\rangle = \beta \sum_{i=1}^a \langle b^*_i, x_i\rangle \le \beta \sum_{i=1}^a \|x_i\|$$

while if the b^*_i can be chosen i such a way that $\langle b^*_i, x_i\rangle \ge \delta\|x_i\|$, we obtain

$$\langle e^*_\gamma, \sum_{i=1}^a x_i\rangle \ge \beta\delta \sum_{i=1}^a \|x_i\|,$$

a formula highly suggestive of Tsirelson norms.

Of course dealing with the general case where the sequence (x_i) does not fit so nicely with the evaluation analysis of γ requires some extra effort, but by making use of an appropriate version of Lemma 3.7 we obtain an upper Tsirelson estimate that is valid for all block sequences.

Theorem 7.1 (First Basic Inequality)**.** *Let $\mathcal{M}$ be a regular family and let Γ be a standard BD-set of weight β such that $\operatorname{hist}\gamma \in \mathcal{M}$ for all $\gamma \in \Gamma$. Then for any normalized block sequence $(x_j)_{j=1}^n$ in $X(\Gamma)$ we have the upper Tsirelson estimate*

$$\|\sum_{j=1}^n a_j x_j\| \le \beta^{-1} \|\sum_{j=1}^n a_j e_{q_j}\|_{T(\mathcal{A}_3 * \mathcal{M}, \beta)},$$

where $q_j = \max\operatorname{ran} x_j$.

The above inequality gives an alternative way of proving Proposition 6.2, and also provides insight into the subspace structure of the “Bourgain–Tsirelson” space introduced in Haydon [2006].

Proposition 7.2. *Let Γ be a standard BD-set of weight $\beta < \frac{1}{2}$ with the following properties:*

1. *for every $\gamma \in \Gamma$ the history $\operatorname{hist}\gamma$ is in the Schreier family $\mathcal{S}$;*
2. *if $s < n$ are natural numbers, ξ, η are elements of Γ with $\operatorname{rank}\xi < s \le \operatorname{rank}\eta < n$ and $\{s\} \cup \operatorname{hist}\xi \in \mathcal{S}$ then for each choice of sign $\pm$ there exists an element γ with $\operatorname{rank}\gamma = n$, $\operatorname{base}\gamma = \xi$ and $\operatorname{top}\gamma = \pm e^*_\eta$.*

Then $X(\Gamma)$ is skipped-asymptotic ℓ_1 and every infinite-dimensional subspace of $X(\Gamma)$ contains a sequence equivalent to some subsequence of the unit-vector basis of the Tsirelson space $T(\mathcal{S},\beta)$.

A sketch of the proof proceeds as follows. Consider a normalized block sequence $(x_n)_{n\in\mathbb{N}}$ in $X(\Gamma)$, setting $\operatorname{ran} x_n = [p_n, q_n]$. Assumption (1), together with the First Basic Inequality, yields an upper estimate of the form

$$\|\sum_{j=1}^{n} a_j x_j\| \le C\|\sum_{j=1}^{n} a_j e_{q_j}\|_{T(\mathcal{A}_3 * \mathcal{S},\beta)}.$$

Provided (x_n) is a skipped-block sequence, Assumption (2) allows us to construct an element $\gamma\in\Gamma$ whose analysis does fit nicely with (x_n), leading to a lower estimate

$$\|\sum_{j=1}^{n} a_j x_j\| \ge \beta(1-2\beta)\sum_{j=1}^{n}|a_n|$$

for a skipped block sequence with $\min\operatorname{ran} x_1 \ge n$. This gives the skipped asymptotic ℓ_1 property. For "sufficiently skipped" block sequences we can get a better lower estimate

$$\|\sum_{j=1}^{n} a_j x_j\| \ge (1-\epsilon)\|\sum_{j=1}^{n} a_j e_{p_j}\|_{T(\mathcal{S},\beta)}.$$

To finish, we need two standard results from Casazza and Shura [1989] about the standard Tsirelson space: first that the $T(\mathcal{A}_3 * \mathcal{S},\beta)$-norm is equivalent to the $T(\mathcal{S},\beta)$-norm and secondly that the sequences $(e_{p_n})_{n\in\mathbb{N}}$ and $(e_{q_n})_{n\in\mathbb{N}}$ are equivalent in $T(\mathcal{S},\beta)$ whenever $p_1\le q_1<p_2\le q_2<\cdots$.

It may be helpful at this point to describe explicitly a BD set satisfying the conditions of Proposition 7.2. The recursive definition that follows is the simple prototype on which more complicated BD constructions, including the one in Argyros and Haydon [2011] are modeled.

Definition 7.3. The set Γ^{BT} is defined as $\bigcup_{n\in\mathbb{N}}\Delta_n$, where the sets Δ_n and the function hist are given recursively by setting $\Delta_1=\{1\}$, $\Gamma_n=\bigcup_{m\le n}\Delta_m$ and

$$\begin{aligned}\Delta_{n+1} = \{&(n+1,0,s,\pm,\eta) : 1\le s\le n,\ \eta\in\Gamma_n\setminus\Gamma_{s-1}\}\\ &\cup\{(n+1,\xi,s,\pm,\eta) : 2\le s\le n,\ \xi\in\Gamma_{s-1},\ \{s\}\cup\operatorname{hist}\xi\in\mathcal{S},\eta\in\Gamma_n\setminus\Gamma_{s-1}\},\\ \operatorname{hist}(n+1,0,s,\pm,\eta)&=\{s\}\qquad \operatorname{hist}(n+1,\xi,s,\pm,\eta)=\{s\}\cup\operatorname{hist}\xi.\end{aligned}$$

We then introduce a standard BD structure (of weight $\beta < \frac{1}{2}$) on Γ^{BT} by setting

$$\begin{aligned} &\operatorname{base}(n+1, 0, s, \pm, \eta) = \text{undefined} && \operatorname{top}(n+1, *, s, \pm, \eta) = \pm e_\eta^* \\ &\operatorname{base}(n+1, \xi, s, \pm, \eta) = \xi && \operatorname{cut}(n+1, *, s, \pm, \eta) = s\,. \end{aligned}$$

8 Two recent examples

We have already mentioned the original spaces $X_{\alpha,\beta}$ of Bourgain and Delbaen. In the case where $\alpha = 1$, these are standard BD-spaces in our terminology, and the branches of the tree structure of the corresponding BD-set are all infinite. When $\alpha < 1$, the ℓ_p-saturated space $X_{\alpha,\beta}$ is not a standard BD-space, and it is not immediately obvious how to construct a standard BD-space that is ℓ_p-saturated for $1 < p < \infty$. Such a construction has, however, been carried out by Gasparis, Papadiamantis, and Zisimopoulou [2010]. Rephrasing their result, we have the following.

Theorem 8.1. *For every real number p with $1 < p < \infty$ there is a well-founded standard BD-set Γ such that $X(\Gamma)$ is ℓ_p-saturated.*

It is worth noting that in this example there is an upper bound N on the ages of members of Γ (that is to say on the lengths of branches in the tree-structure of Γ). Another example with this property has been constructed by Argyros, Gasparis, and Motakis [2016]; this space lies at the opposite end of the spectrum from the space $X(\Gamma^{\mathrm{BT}})$, which is skipped-asymptotic ℓ_1.

Theorem 8.2. *There is a well-founded standard BD-set Γ such that the standard basis of $X(\Gamma)$ is asymptotic-ℓ_∞ but $X(\Gamma)$ does not contain c_0.*

The example above is of interest because it is relevant to problems about uniform homeomorphisms. It is not known whether a Banach space X that is uniformly homeomorphic to c_0 must be linearly homeomorphic (i.e. isomorphic) to c_0, but it is shown in Godefroy, Kalton, and Lancien [2001] that any such space X must be an isomorphic predual of ℓ_1 and have "summable Szlenk index"; Godefroy, Kalton and Lancien ask whether these properties already imply that X is isomorphic to c_0. The example of Argyros, Gasparis and Motakis shows that the answer is negative; it is not clear whether this space is uniformly homeomorphic to c_0. Very recently, this example has also found an application in descriptive set theory, in the proof by Kurka [2017] that the isomorphism class of c_0 is not Borel.

9 Self-determining subsets and BD augmentations

We have noted that the structure of a BD-set may be thought of in two (equivalent) ways, either in terms of the extension mappings $i_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma)$ and the right-triangular basis $(d_\gamma)_{\gamma\in\Gamma}$ of $X(\Gamma)$ or in terms of the left-triangular basis functionals $d^*_\gamma = e^*_\gamma - c^*_\gamma$ in $\ell_1(\Gamma)$. This gives two different ways in which certain subsets of Γ may be naturally equipped with an induced BD structure. Introducing terminology that we shall use only temporarily, we shall say that a subset Γ' of Γ is *left-closed* if $\operatorname{supp} d^*_\gamma \subseteq \Gamma'$ whenever $\gamma \in \Gamma'$, and that a subset Γ'' is *right-closed* if $\operatorname{supp} d_\gamma \subseteq \Gamma''$ whenever $\gamma \in \Gamma''$.

When Γ' is left-closed, the functionals d^*_γ $(\gamma \in \Gamma')$ form a left triangular basis of $\ell_1(\Gamma')$ and so yield a BD-structure on Γ'; we then write $X(\Gamma')$ for the corresponding BD-space. On the other hand, when $\Gamma'' \subset \Gamma$ is right-closed, we have $i_n[\ell_\infty(\Gamma''_n)] \subset \ell_\infty(\Gamma'')$ for all n, so that Γ'' has its own BD-structure, defined by the extension operators $i''_n = i_n \restriction_{\ell_\infty(\Gamma''_n)}$. The connection between our two notions of closedness was established by Argyros and Motakis [2014].

Theorem 9.1. *Let Γ be a BD-set, let Γ' be a subset of Γ and let $\Gamma'' = \Gamma \setminus \Gamma'$. Then Γ' is left-closed if and only if Γ'' is right-closed. When this is the case, the restriction mapping $R' : x \mapsto x\restriction_{\Gamma'}$ is a quotient operator from $X(\Gamma)$ onto $X(\Gamma')$ and* $\ker R' = X(\Gamma'')$.

We note that in the set-up of Theorem 9.1, the BD-space $X(\Gamma'')$ is a subspace of $X(\Gamma)$, but that $X(\Gamma')$ typically is not. Indeed this happens only when Γ' is both left- and right-closed, and in this case $X(\Gamma)$ is just the direct sum of the disjointly supported spaces $X(\Gamma')$ and $X(\Gamma'')$. In general, $X(\Gamma)$ is a twisted sum of $X(\Gamma')$ and $X(\Gamma'')$, and the interesting cases are where this twisted sum is non-trivial.

In the terminology of Argyros and Motakis, left-closed subsets are called *self-determining* and we shall use that term in the rest of this paper. Once we have constructed a suitable "large" BD-set Γ then defining suitable self-determined subsets can be a useful and economical way of generating further examples with different properties. Thus a number of examples use self-determined subsets of a certain BD-set constructed by Argyros and Haydon [2011] and denoted $\Gamma^{\max}$ in that work: these include the scalar-plus-compact space of Argyros and Haydon [ibid.], the spaces constructed by Tarbard [2012, 2013], the main example of Argyros and Motakis [2016] and the recent construction due to Manoussakis, Pelczar-Barwacz, and Świętek [2017]. We look at the space of Argyros and Haydon [2011] in greater detail in a later section.

In order to prove refinements of Theorem 6.4, Freeman, Odell, and Schlumprecht [2011] introduced the tool of "augmenting" a BD-set, by adding extra elements to change chosen aspects of the structure of the associated BD-space. Let $\Gamma' = \bigcup_{n\in\mathbb{N}} \Delta'_n$ be a general BD-set; a BD-set $\Gamma = \Gamma' \cup \Gamma''$ that contains Γ' as a self-determining subset will be called an *augmentation* of Γ'. It is very easy to build augmentations as we have great freedom in choosing the functionals c^*_γ for the new elements $\gamma \in \Gamma''$. In particular, if Γ' is a standard BD-set of weight β, we can recursively build a standard augmentation, also of weight β, by adding new elements $\gamma \in \Delta''_{n+1}$ with complete freedom of choice of $\xi = \operatorname{base}\gamma \in \Gamma_n = \Gamma'_n \cup \Gamma''_n$, $s = \operatorname{cut}\gamma \le n+1$ and $\operatorname{top}\gamma \in \operatorname{ball}\ell_1(\Gamma_n \setminus \Gamma_{s-1})$. Augmentations will be used repeatedly in the rest of this article.

10 BD-sets with zero weight

A special case arises if we consider a BD-set such that, for each $\gamma \in \Gamma$, we have either $c^*_\gamma = 0$ or $c^*_\gamma = e^*_\xi$ for some ξ with $\operatorname{rank}\xi < \operatorname{rank}\gamma$. We may think of such a set as a standard BD-set with weight $\Gamma = \beta = 0$. It is not a surprise to find that the spaces $X(\Gamma)$ that arise in this way are actually objects with which we are already familiar.

Proposition 10.1. *Let Γ be a BD-set of zero weight. If Γ is well-founded, there is a locally compact topology on the countable set Γ such that $X(\Gamma) = \mathscr{C}_0(\Gamma)$. If Γ is ill-founded and B is the set of infinite branches of Γ then there is a locally compact topology on $\Gamma \cup \mathrm{B}$ such that $X(\Gamma)$ is naturally identifiable with $\mathscr{C}_0(\Gamma \cup \mathrm{B})$.*

The topologies in the above proposition are natural ones defined by the tree structure introduced in Section 6. In the well-founded case, the topology is the coarsest such that all the sets $U_\gamma = \{\delta \in \Gamma : \gamma \preccurlyeq \delta\}$ are open and closed; in the general case we take the sets V_γ $(\gamma \in \Gamma)$ as basic clopen sets, where $V_\gamma = U_\gamma \cup \{\beta \in \mathrm{B} : \gamma \in \beta\}$. Two special cases are worth mentioning: if Γ is of weight zero and, moreover, no element of Γ has a base, then $c^*_\gamma = 0$ for all γ and $X(\Gamma)$ is just $c_0(\Gamma)$; if Γ is of weight 0 and $\operatorname{hist}\gamma \in \mathscr{S}$ for all $\gamma \in \Gamma$ then $X(\Gamma)$ is isometric to $\mathscr{C}_0(\alpha)$ for some ordinal $\alpha \le \omega^\omega$.

The technique of BD augmentations gives a neat way to construct certain examples of twisted sums, due originally to Cabello Sánchez, Castillo, Kalton, and Yost [2003]. The following is an example.

Theorem 10.2. *For every $\epsilon > 0$ there is a Banach space X and a subspace Y of X, isometric to $\mathscr{C}_0(\omega^\omega)$, such that X/Y is $(1+\epsilon)$-isomorphic to c_0 while the quotient mapping $X \to X/Y$ is strictly singular.*

To prove this, we start with the set $\Gamma' = \mathbb{N}$, equipped with the trivial BD-structure where $\operatorname{rank} n = n$ and $c^*_n = 0$ for all n; as we noted above, $X(\Gamma') = c_0$. We construct an augmentation of Γ' by adjoining new elements $\gamma \in \Gamma''$ as in the the construction of Γ^{BT} above, but subject to the condition that $\operatorname{supp}\operatorname{top}\gamma \subseteq \Gamma'$ for all γ (that is to say, $\eta \in \Gamma'$ in the notation of Definition 7.3). By Theorem 9.1 the restriction mapping $R' : x \mapsto x\restriction_{\Gamma'}$ is a quotient mapping from $X(\Gamma)$ onto $X(\Gamma') = c_0$ and the kernel of R' is the space $X(\Gamma'')$. But Γ'', considered as a BD-space in its own right, is of weight zero, because of the condition we placed on the tops. Because of the role of the Schreier family in the construction of Γ'', we see that $X(\Gamma'')$ is isometric to $\mathcal{C}_0(\omega^\omega)$. To prove the strict singularity of the quotient mapping R', we show that asymptotic ℓ_1 estimates of the kind used in Proposition 7.2 are valid for skipped block sequences (x_n) in $X(\Gamma)$ provided that $\inf_n \|R'x_n\| > 0$.

There is a natural way to associate with an arbitrary standard BD-set Γ, a BD-set $\mathring{\Gamma}$ of weight zero. We take $\mathring{\Gamma}$ to have the same underlying set as Γ, and retain the same definition of base γ, while redefining top γ to be 0 for all γ. While $X(\Gamma)$ and $X(\mathring{\Gamma})$ typically have very different Banach space structures, these two subspaces of $\ell_\infty(\Gamma)$ are quite close to each other in the Hausdorff metric.

Proposition 10.3. *Let Γ be a standard BD-space of weight β and let $\mathring{\Gamma}$ be the associated set of weight zero. For every $x \in X(\Gamma)$ (resp. $X(\mathring{\Gamma})$) and every $\epsilon > 0$, there exists $y \in X(\mathring{\Gamma})$ (resp. $X(\Gamma)$) with $\|x - y\| \le (2\beta M + \epsilon)\|x\|$, where $M = (1-2\beta)^{-1}$.*

Combining Proposition 10.3 with the embedding given by Theorem 6.4, we retrieve a result of Zippin [1977].

Theorem 10.4. *Zippin [ibid.] Let Y be a Banach space with separable dual and let ϵ be a positive real number. Then there is a countable locally compact space Γ and a subspace Z of $\ell_\infty(\Gamma)$ which is $(1+\epsilon)$-isomorphic to Y, such that for all $z \in Z$ there exists $h \in \mathcal{C}_0(\Gamma)$ with $\|z - h\| \le \epsilon\|z\|$.*

11 BD-sets with mixed weights

In order to introduce finer structure into our BD-sets and exploit the theory of mixed Tsirelson spaces, we make the following definition.

Definition 11.1. Let Γ be a countably infinite set and let $(m_i)_{i\in\mathbb{N}}$ be a sequence of natural numbers satisfying Definition 4.2(1). A weighted BD structure on Γ consists of the following mappings:

1. rank $: \Gamma \to \mathbb{N}$ such that each inverse image $\Delta_n = \operatorname{rank}^{-1}\{n\}$ is finite;
2. cut $: \Gamma \to \mathbb{N}$ satisfying $\operatorname{cut}\gamma \le \operatorname{rank}\gamma$;
3. top $: \Gamma \to \operatorname{ball}\ell_1(\Gamma)$ such that $\operatorname{supp}\operatorname{top}\gamma \subseteq \{\eta \in \Gamma : \operatorname{cut}\gamma \le \operatorname{rank}\eta < \operatorname{rank}\gamma\}$;
4. base $: \Gamma \to \Gamma \cup \{\text{undefined}\}$ such that $\operatorname{rank}\operatorname{base}\gamma < \operatorname{cut}\gamma$ whenever it is defined;
5. weight $: \Gamma \to \{m_i^{-1} : i \in \mathbb{N}\}$ such that $\operatorname{weight}\gamma = \operatorname{weight}\operatorname{base}\gamma$ when this is defined.

Given such a structure we set $\Gamma_n = \bigcup_{m \le n} \Delta_m = \{\gamma \in \Gamma : \operatorname{rank} \gamma \le n\}$ and define BD-functionals

$$c^*_\gamma = \qquad m_i^{-1}(I - P^*_{s-1})b^*,$$

where $m_i^{-1} = \operatorname{weight} \gamma$, $s = \operatorname{cut} \gamma$, $b^* = \operatorname{top} \gamma$ and $\operatorname{base} \gamma$ is undefined, or

$$c^*_\gamma = e^*_\xi + m_i^{-1}(I - P^*_{s-1})b^*,$$

with $\xi = \operatorname{base} \gamma$ when this is defined. We call Γ a *weighted BD-set*.

Provided Γ is well-founded, there are regular families $\mathcal{M}_i$ such that $\operatorname{hist} \gamma \in \mathcal{M}_i$ whenever γ is of weight m_i^{-1}. Our first task is to seek a class of sequences in $X(\Gamma)$ for which we can establish upper mixed-Tsirelson estimates. These will be the BD versions of the rapidly increasing sequences mentioned earlier in the context of mixed Tsirelson spaces. Although we define them in terms of evaluation estimates, rather than a particular method of construction, we shall shall continue to use the term RIS.

Definition 11.2. Let Γ be a well-founded BD-set with weights taking values in the sequence (m_i^{-1}). We shall say that a block sequence $(x_k)_{k \in \mathbb{N}}$ in $X(\Gamma)$ is a RIS if there exist a constant C and an increasing sequence (i_n) of natural numbers such that the following hold:

1. $\|x_k\| \le C$ for all n;
2. $|x_k(\gamma)| \le C m_h^{-1}$ whenever $\operatorname{weight} \gamma = m_h^{-1}$ and $h < i_k$;
3. $i_{k+1} > \max\{i : \exists \gamma \text{ with } \operatorname{rank} \gamma \le \max \operatorname{ran} x_k \text{ and } \operatorname{weight} \gamma = m_i^{-1}\}$.

We note that from a block sequence satisfying (1) and (2) we can always extract a subsequence satisfying (3) as well.

The next result is central to many subsequent developments. The proof is slightly intricate but is based on ideas that can be traced back to the splitting lemma, presented earlier as Lemma 3.7.

Theorem 11.3 (Second Basic Inequality)**.** *Let $(m_i)_{i \in \mathbb{N}}$ satisfy Definition 4.2(1), let $(\mathcal{M}_i)_{i \in \mathbb{N}}$ be sequence of regular families and let Γ be a weighted BD-set such that* $\operatorname{hist} \gamma \in \mathcal{M}_i$ *whenever* $\operatorname{weight} \gamma = m_i^{-1}$. *Let $(x_k)_{k \in \mathbb{N}}$ be a C-RIS in $X(\Gamma)$ with* $\max \operatorname{ran} x_k = q_k$. *If $I \subset \mathbb{N}$ is a finite interval, λ_k ($k \in I$) are scalars and $\gamma \in \Gamma$ then there exist $k_0 \in I$ and $g^* \in W[(\mathcal{A}_4 * \mathcal{M}_i, m_i^{-1})_{i \in \mathbb{N}}]$ such that either $g^* = 0$ or* $\operatorname{weight} g^* = \operatorname{weight} \gamma$ *and* $\operatorname{supp} g^* \subseteq \{q_k : k_0 < k \in I\}$, *and such that*

$$\Big|\sum_{k \in I} \lambda_k x_k\Big| \le 2C|\lambda_{k_0}| + 2C\langle g^*, \sum_{k_0 < k \in I} \lambda_k e_{q_k}\rangle.$$

In particular, we have the upper estimate

$$\|\sum_k \lambda_k x_k\| \leq 4C \|\sum_k \lambda_k e_{q_k}\|_{T[(\mathcal{M}_i * \mathcal{A}_4, m_i^{-1})_{i\in\mathbb{N}}]}.$$

The second crucial property of weighted BD-spaces, the one that will enable us to pass from "few operators" to "very few operators" in the next section, is the existence of two types of RIS that do not have analogues in the usual mixed Tsirelson framework. These are defined in terms of "local support". If $x \in X(\Gamma)$ has finite range $\operatorname{ran} x_k = [p, q]$ then we can write $x = i_q u$, where $u \in \ell_\infty(\Gamma_q)$ and $\operatorname{supp} u \subseteq \Gamma_q \setminus \Gamma_{p-1}$; we call $\operatorname{supp} u$ the *local support* of x. We say that a bounded block sequence (x_k) in $X(\Gamma)$ has *bounded local weight* if $\inf_k \min\{\text{weight}\,\gamma : \gamma \in \operatorname{loc\,supp} x_k\} > 0$, and that (x_k) has rapidly decreasing local weight if $\max\{\text{weight}\,\gamma : \gamma \in \operatorname{loc\,supp} x_k\}$ tends to 0 sufficiently fast. The interest of these definitions is in the following two propositions, the first of which is proved using the Evaluation Analysis 6.1.

Proposition 11.4. *Let Γ be a BD-set with weights in the sequence $(m_i^{-1})_{i\in\mathbb{N}}$ and let (x_k) be a bounded block sequence in $X(\Gamma)$. If (x_k) has either bounded local weight, or rapidly decreasing local weight, (x_k) is a RIS.*

Proposition 11.5. *Let Γ be a well-founded BD set, let Y be a Banach space and let $T : X(\Gamma) \to Y$ be a bounded linear operator. If $\|T(x_n)\| \to 0$ for every RIS in $X(\Gamma)$ then $\|T(x_n)\| \to 0$ for every bounded block sequence in $X(\Gamma)$; hence T is a compact operator.*

The proof of this second proposition is worth including, since it is easy and exploits the local ℓ_∞-structure of $X(\Gamma)$. We consider a bounded block sequence (x_k) with $\operatorname{ran} x_k = [p_k, q_k]$ and assume if possible that $\|Tx_k\| > \delta > 0$ for all k. We can write $x_k = i_{q_k} u_k$ where $\operatorname{supp} u_k \subseteq \Gamma_{q_k} \setminus \Gamma_{p_k-1}$ and $\|u_k\| \leq \|x_k\|$. For any $h \in \mathbb{N}$ we define $v_k^h \in \ell_\infty(\Gamma_{q_k})$ by setting

$$v_k^h(\gamma) = \begin{cases} u_k(\gamma) & \text{if weight}\,\gamma \geq m_h^{-1} \\ 0 & \text{otherwise} \end{cases}$$

For any h, the sequence (y_k^h) given by $y_k^h = i_{q_k} v_k^h$ is bounded ($\|y_k^h\| \leq \|i_{q_k}\|\|v_k^h\| \leq M\|x_k\|$) and has bounded local weight; hence it is a RIS and $\|Ty_k^h\| \to 0$ by hypothesis. This means that if we write $z_k^h = x_k^h - y_k^h = i_{q_k}(u_k - v_k^h)$, we must have $\|Tz_k^h\| > \frac{1}{2}\delta$ for all large enough k. It is now easy to construct sequences $h(r)$ and $k(r)$ tending to ∞ with r such that $\|Tz_{k(r)}^{h(r)}\| > \delta$ for all r. This contradicts our hypothesis, since a suitable subsequence of $(z_{k(r)}^{h(r)})$ has rapidly decreasing local weight and so is a RIS.

12 The scalar-plus-compact property and invariant subspaces

In Argyros and Haydon [2011], where $\mathcal{M}_i$ is taken to be $\mathcal{A}_{n_i}$, for an appropriately fast-growing sequence n_i, a large BD-set, denoted $\Gamma^{\max}$ was introduced, providing a framework in which other constructions can be made by taking self-determining subsets. The same can be done in general, using a recursive construction like that of Γ^{BT} given earlier.

Definition 12.1. We define $\Gamma^{\max}[(\mathcal{M}_i, m_i^{-1})_{i\in\mathbb{N}}]$ to be the union $\bigcup_{n\in\mathbb{N}} \Delta_n$, where the sets Δ_n and the functions hist and weight are given recursively by setting $\Delta_1 = \{1\}$, $\Gamma_n = \bigcup_{m\le n} \Delta_m$ and

$$\begin{aligned}\Delta_{n+1} = \{&(n+1, i, 0, s, b^*) : i \le n+1,\ 1 \le s \le n,\ b^* \in B(s,n)\}\\ &\cup \{(n+1, i, \xi, s, b^*) : 2 \le s \le n,\ \xi \in \Gamma_{s-1},\ \ \text{weight}\, \xi = m_i^{-1},\ \{s\} \cup \text{hist}\, \xi \in \mathcal{M}_i,\\ &b^* \in B(s,n)\}, \text{hist}\,(n+1, i, 0, s, b^*) = \{s\}, \quad \text{hist}\,(n+1, i, \xi, s, b^*) = \{s\} \cup \text{hist}\, \xi,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{weight}\,(n+1, i, *, s, b^*) = m_i^{-1}.\end{aligned}$$

The differences with Definition 7.3 are first that we introduce i to allow for the mixed weights, and secondly that we allow the top b^* of an element to be a general element of ball $\ell_1(\Gamma_n \setminus \Gamma_{s-1})$, rather than restricting it to be an evaluation functional $\pm e_\eta^*$. For the sets Δ_{n+1} to be finite, we do need to place some restrictions on b^*, requiring it to lie in some finite ϵ-net $B(s,n)$ (which, of course, ought really to have been included in the recursive definition).

As we have said, the main role of $\Gamma^{\max}$ is to provide a framework for other constructions, but the $X(\Gamma^{\max})$, a mixed-Tsirelson version of the space $X(\Gamma^{\mathrm{BT}})$ is of some interest in its own right. It is natural to ask about its subspace structure, and whether there is an analogue of Proposition 7.2. It has been shown by Świętek that such an analogue does hold, at least for the version of $\Gamma^{\max}$ given in Argyros and Haydon [ibid.]; the reader is referred to Świętek [2017] for details.

In order to construct a space with the scalar-plus-compact property, we define a self determining subset Γ' of $\Gamma^{\max}$ by following the same recursive construction, while restricting the choice of the tops b^* of odd-weight elements. First we fix a *coding function* σ, an increasing injection from the set of odd-weight elements of $\Gamma^{\max}$ into $\mathbb{N}$. Then we insist that for an element $(n+1, 2j-1, 0, s, b^*)$ to be in Δ'_{n+1}, b^* must have the form e_η^* for some $\eta \in \Gamma'_n$ with weight of the form m_{4i-2}^{-1}, while for an element $(n+1, 2j-1, \xi, s, b^*)$ b^* must equal e_η^* for some $\eta \in \Gamma'$ of weight $m_{4\sigma(\xi)}$. We call the resulting BD-set $\Gamma^{\mathrm{K}}[[(\mathcal{M}_i, m_i^{-1})_{i\in\mathbb{N}}]$.

Theorem 12.2. *Provided the sequences (m_i) and $(\mathcal{M}_i)$ satisfy the Standard Assumptions, the space $X(\Gamma^{\mathrm{K}})$ has the scalar-plus-compact property.*

While the full strength of the Standard Assumptions was not needed in the previous section, we really need it here, in order to form special convex combinations and apply mixed-Tsirelson estimates such as Proposition 4.3. The proof, while complicated, is closely modeled on earlier proofs that certain spaces have few operators, eventually showing that for any bounded linear operator T on $X(\Gamma^K)$ there is a scalar λ such that $\|Tx_k - \lambda x_k\| \to 0$ for every RIS. The difference here is that we can now deduce compactness of $T - \lambda I$ by Proposition 11.5.

By the theorem of Aronszajn and Smith [1954], every bounded linear operator on $X(\Gamma^K)$ has a proper, non-trivial invariant subspace; $X(\Gamma^K)$ was the first infinite-dimensional space known to have this Invariant Subspace Property (though, of course, it is a famously open problem whether the more familiar space ℓ_2 does). A class of spaces with the Invariant Subspace Property but not the Scalar-plus-Compact Property was constructed by Tarbard [2012].

Theorem 12.3. *Tarbard [ibid.] For each natural number N there is a weighted BD-set Γ^N and a strictly singular operator S on $X(\Gamma^N)$ such that*

1. *S^N is non-compact*
2. *$S^{N+1} = 0$*
3. *every bounded linear operator T on $X(\Gamma^N)$ can be written uniquely as $T = \lambda I + \sum_{k=1}^{N} \lambda_k S^k + K$ with K compact.*

Tarbard's spaces have the Invariant Subspace Property, by Lomonosov's theorem, because any operator can be written $\lambda I + U$ with U^{N+1} compact.

No example is known of an infinite-dimensional reflexive space with the Scalar-plus-Compact Property (indeed, all known examples are $\mathcal{L}_\infty$-spaces). We must mention, however, the paper Argyros and Motakis [2014] which uses the method of saturation with constraints to construct an infinite-dimensional reflexive space all of whose subspaces have the Invariant Subspace Property.

The space $X(\Gamma^{\mathrm{K}})$ of Argyros and Haydon [2011] is hereditarily indecomposable and saturated with reflexive subspaces. Recent work has shown that the scalar-plus-compact property can hold in spaces with very different subspace structure: the space constructed by Manoussakis, Pelczar-Barwacz, and Świętek [2017] has the scalar-plus-compact property but is saturated with unconditional basic sequences; Argyros and Motakis [2016] combine the BD construction with the method of saturation with constraints to construct a space that has the scalar-plus-compact property, but no infinite-dimensional reflexive subspaces.

13 Calkin algebras

The Calkin algebra of a Banach space is the quotient $\mathcal{L}(X)/\mathcal{K}(X)$ where $\mathcal{L}(X)$ is the algebra of all bounded linear operators on X and $\mathcal{K}(X)$ the ideal of compact operators. Obviously the Calkin algebra of $X(\Gamma^K)$ is one-dimensional and that of Tarbard's space $X(\Gamma^N)$ is $N+1$-dimensional. It is natural to pose a question about the structure of Calkin algebras as Banach algebras:

Which unital Banach algebras arise as Calkin algebras of Banach spaces?

Tarbard [2013] gave a further example.

Theorem 13.1. *There is a well-founded BD-set Γ^∞ such that the Calkin algebra of $X(\Gamma^\infty)$ is isometrically isomorphic to the convolution algebra $\ell_1(\omega)$.*

Direct sums of versions of the example of Argyros and Haydon [2011] are used in Kania and Laustsen [2017] to show that every finite-dimensional semisimple algebra can be realised as a Calkin algebra. A major advance has come from Motakis, Puglisi, and Zisimopoulou [2016] who build on the interesting theory of BD-direct sums developed by Zisimopoulou [2014].

Theorem 13.2. *For every countable compact space K there is a $\mathcal{L}_\infty$ Banach space X with Calkin algebra isomorphic to $\mathcal{C}(K)$.*

Most recently, Motakis, Puglisi, and Tolias [2017] give a broad class of algebras of diagonal operators that may be realized as Calkin algebras, including examples that are hereditarily indecomposable and quasi-reflexive as Banach spaces. We are however still a long way from an answer to our question and we are still waiting for a first example of a unital Banach algebra that is *not* isomorphic to a Calkin algebra.

14 Indecomposable extensions of Banach spaces with separable duals

In this section and the next we shall be investigating the question of when a Banach space Y can be embedded as a subspace of a separable indecomposable space, or, more ambitiously, a separable space with the scalar-plus-compact property. The analogous question of when Y can be expressed as a quotient of a separable, hereditarily indecomposable space has been completely solved: it was shown by Argyros and Tolias [2004] that this is the case if and only if Y has no subspace isomorphic to ℓ_1. We conjecture that there is a similar best-possible for irreducible extensions.

Conjecture 14.1. For a Banach space Y with separable dual the following are equivalent:

1. Y embeds isomorphically into a separable space X with the scalar-plus-compact property;
2. Y embeds isomorphically into a separable indecomposable space X ;
3. Y has no subspace isomorphic to c_0.

It is clear that (1) $\implies$ (2) $\implies$ (3) because a c_0 subspace of a separable space is automatically complemented by Sobczyk's theorem. We note that the non-separable space $\mathcal{C}(K)$ constructed by Koszmider [2004] is indecomposable but contains $\mathcal{C}[0,1]$ (and hence copies of all separable spaces). Thus, to have a sensible conjecture we certainly need the word "separable" in (2). In passing we remark that we know of no non-separable Banach space with the scalar-plus-compact property, nor whether there is an upper bound on the size of such a space. While a hereditarily indecomposable space necessarily embeds in ℓ_∞ (Argyros and Tolias [2004]), it has recently been shown (subject to GCH) by Koszmider, Shelah, and Świętek [2018] that there exist arbitrarily large indecomposable Banach spaces. At the risk of drifting seriously off-topic, we wonder whether every Banach space not containing ℓ_∞ embeds in some indecomposable space.

While we cannot prove Conjecture 14.1, some partial results in this direction do exist. First, it was shown in Argyros, Freeman, Haydon, Odell, Raikoftsalis, Schlumprecht, and Zisimopoulou [2012] that every separable superreflexive Banach space Y embeds into a BD-space with the scalar-plus-compact property. The same authors can now do a little better, replacing the best-possible hypothesis that c_0 does not embed in Y with the same condition applied to the double dual Y^{**}.

Theorem 14.2. *Let Y be a Banach space such that Y^* is separable and c_0 does not embed into Y^{**}. Then Y embeds into a BD-space with the scalar-plus-compact property.*

The plan of the proof is to start by applying Theorem 6.4 to embed Y into $X(\Gamma')$ where Γ' is a well-founded standard BD-set of weight $m_1^{-1} = \frac{1}{4}$. Since Γ' is well-founded, there is a regular family $\mathcal{M}_1$ that contains all the histories $\operatorname{hist}\gamma$. We introduce $m_2, m_3, \ldots$ and $\mathcal{M}_2, \mathcal{M}_r, \ldots$ so that the Standard Assumptions are satisfied. The aim is then to build an augmentation $\Gamma = \Gamma' \cup \Gamma''$, such that $X(\Gamma)$ has the scalar-plus-compact property and contains the subspace Y. Even this last part is not as straightforward as it sounds, since, as we have noticed, $X(\Gamma')$ need not be a subspace of $X(\Gamma)$. For its subspace Y to be contained in $X(\Gamma)$ we shall need a rather special sort of augmentation.

Definition 14.3. Let Γ' be a standard BD-set, let Y be a subspace of $X(\Gamma')$ and let $\Gamma = \Gamma' \cup \Gamma''$ be a standard augmentation of Γ. Identify Y with the subspace

$$\{x \in \ell_\infty(\Gamma) : x\restriction_{\Gamma'} \in Y \text{ and } x\restriction_{\Gamma''} = 0\}.$$

of $\ell_\infty(\Gamma)$. We shall say that the augmentation *respects the subspace* Y if, for every $\gamma \in \Gamma''$, we have

$$P^*_{[s,\infty)} b^* \in Y^\perp \quad \text{and} \quad \xi \in \Gamma'' \text{ (if it exists)},$$

where, as usual $s = \operatorname{cut} \gamma$, $b^* = \operatorname{top} \gamma$ and $\xi = \operatorname{base} \gamma$.

Proposition 14.4. *If Γ is a standard augmentation of a standard BD-set Γ' that respects the subspace Y of $X(\Gamma)$ then Y is a subspace of $X(\Gamma)$.*

So our plan to prove Theorem 14.2 will be to construct an exotic augmentation of Γ' that respects the subspace Y. We shall need plenty of choice in selecting the tops of newly added $\gamma \in \Gamma''$ in order to achieve the scalar-plus-compact property, while ensuring that the condition $P^*_{[s,\infty)} b^* \in Y^\perp$ is satisfied. Theorem 6.4 does not necessarily have quite the property we want, so we may need to adjust it slightly, using a standard M-basis argument.

Proposition 14.5. *Let Γ' be a well-founded standard BD-set and let Y be a closed subspace of $X(\Gamma')$. Then there is small perturbation Z of Y such that $\mathbb{Q}\text{-sp}\langle d^*_\gamma : \gamma \in \Gamma' \rangle \cap Z^\perp$ is norm-dense in $\ell_1(\Gamma') \cap Z^\perp$.*

We assume that Y already has the property of Z in the above proposition and construct an augmentation that respects Y but is otherwise modeled on Theorem 12.2. We arrange that for every bounded linear operator $T : X(\Gamma) \to X(\Gamma)$ there is a scalar λ such that $Q(T - \lambda I)$ is compact, where $Q : X(\Gamma) \to X(\Gamma)/Y$ is the quotient operator. It is only now that we need the hypothesis on the original space Y in order to apply the following result, which we call the "Quotient-Compact Property".

Proposition 14.6. *Let Z be a Banach space, let Y be a subspace of Z such that c_0 does not embed in Y^{**}, let X be a Banach space with X^* isomorphic to ℓ_1 and let $S : X \to Z$ be a bounded linear operator. If QS is compact, where $Q : Z \to Z/Y$ is the quotient operator, then S is compact.*

Applying this proposition with $X = Z = X(\Gamma)$ and $S = T - \lambda I$ allows us to finish the proof of Theorem 14.2.

15 A space with the scalar-plus-compact property that contains ℓ_1

In this section, based on joint work of the authors with Th. Raikoftsalis, we sketch the construction of a Banach space that contains ℓ_1 and has the scalar-plus-compact property.

We start with a simple case of an ill-founded BD-set of zero weight, taking Γ^{d} to be the dyadic tree

$$\Gamma^{\mathrm{d}} = 2^{<\omega} = \bigcup_{n \in \omega} 2^n = \{(), (0), (1), (00), (01), \ldots\}.$$

We use fairly standard notation for this tree. When $\gamma \in 2^n$ we write $n = \operatorname{length}\gamma$ (which is also the *domain* of γ if we are thinking of γ as a function) and for $\xi, \gamma \in 2^{<\omega}$ we write $\xi \prec \gamma$ if $\operatorname{length}\xi < \operatorname{length}\gamma$ and $\xi(i) = \gamma(i)$ for $0 \le i < \operatorname{length}\xi$, that is to say ξ is the restriction of γ to the domain of ξ. If $\gamma \in 2^{n+1}$ we write γ^- for the restriction of γ to n ; so γ^- is the unique member of 2^n with $\gamma^- \prec \gamma$.

To endow Γ^{d} with a BD-structure, we define $\operatorname{rank}\gamma = 1 + \operatorname{length}\gamma$ and

$$\operatorname{base}\gamma = \gamma^-, \qquad \operatorname{top}\gamma = 0,$$

when $\operatorname{length}\gamma > 0$. Of course, the only reason for the term "+1" in the definition of rank is our earlier decision that the rank function on a BD-set should take values in $\mathbb{N}$, rather than ω. As in Section 10 we see that $X(\Gamma^{\mathrm{d}})$ may be identified naturally with the space $\mathcal{C}(2^{\le\omega})$, where $2^{\le\omega}$ is the compact metric space of all finite and infinite sequences in $\{0,1\}$. The space $\mathcal{C}(2^{\le\omega})$ has a subspace isomorphic to ℓ_1 and is a quotient of $X(\Gamma)$ whenever Γ is an augmentation of Γ^{d}. By the lifting property, $X(\Gamma)$ contains ℓ_1 for all such Γ.

Theorem 15.1. *There is an augmentation of Γ^{d} that has the scalar-plus-compact property.*

The idea is very simple: we take sequences (m_i) and $\mathcal{M}_i)$ satisfying the Standard Assumptions, and with $\mathcal{M}_1 = \mathcal{S}$, and then augment the BD-set $\Gamma' = \Gamma^{\mathrm{d}}$ by adding elements $\gamma \in \Gamma''$ as in the construction of Theorem 12.2. We do not have good upper norm estimates for sequences of vectors in $X(\Gamma)$, since Γ is not well-founded. But on the subspace $X(\Gamma'')$ the machinery of Theorem 12.2 can still be applied, leading to the following.

Proposition 15.2. *Let $\Gamma' = \Gamma^d$ and let $\Gamma = \Gamma' \cup \Gamma''$ be an augmentation as descibed above. For every bounded linear operator $U : X(\Gamma'') \to X(\Gamma)$ there is a scalar λ such that $U - \lambda J$ is compact, where $J : X(\Gamma'') \to X(\Gamma)$ is the inclusion operator.*

Once we have this proposition, we consider a bounded linear operator $T : X(\Gamma) \to X(\Gamma)$ and set $U = T\restriction_{X(\Gamma'')}$. There exists a scalar λ such that $U - \lambda J$ is compact; since $X(\Gamma)$ is a $\mathscr{L}_\infty$-space, there is a compact operator $S_1 : X(\Gamma) \to X(\Gamma)$ extending $U - \lambda J$ by Proposition 5.2. The operator $T - \lambda I - S_1$ vanishes on the subspace $X(\Gamma'')$, which is the kernel of the restriction quotient mapping $R' : X(\Gamma) \to X(\Gamma')$. So we can write $T = \lambda I + S_1 + S_2 R'$, where $S_2 : X(\Gamma') \to X(\Gamma)$ is some operator. Now $X(\Gamma')$ is a $\mathcal{C}(K)$-space, and $X(\Gamma)$ is skipped-asymptotic ℓ_1 (because of our choice $\mathcal{M}_1 = \mathcal{S}$). We finish the proof of Theorem 15.1 by using a general result about operators between such spaces.

Proposition 15.3. *If X is skipped-asymptotic ℓ_1 then every bounded linear operator from $\mathcal{C}(K)$ to X is compact.*

A more elaborate construction involving a Y-respecting augmentation of Γ^{d} leads to the analogue of Theorem 14.2 for spaces with nonseparable dual.

Theorem 15.4. *Let Y be a separable Banach space such that Y^{**} does not have a subspace isomorphic to c_0. Then Y may be embedded isomorphically into an indecomposable BD-$\mathcal{L}_\infty$ space $X(\Gamma)$. If every bounded linear operator from $\mathcal{C}[0,1]$ into Y is compact, then $X(\Gamma)$ may be chosen to have the scalar-plus-compact property.*

We have already noted in Proposition 10.9 of Argyros and Haydon [2011] that some extra condition is needed in Theorem 15.4 in order to get the scalar-plus-compact property, rather than just indecomposability. We can make this a little more precise with the following easy proposition, which seems to be a good note on which to end.

Proposition 15.5. *Let Y be a separable Banach space with non-separable dual. If Y embeds isomorphically in a separable $\mathcal{L}_\infty$-space with the scalar-plus-compact property then every bounded linear operator from $\mathcal{C}[0,1]$ to Y is compact.*

Proof. Assume that Y is a subspace of a $\mathcal{L}_\infty$-space X with the scalar-plus-compact property. Since Y^* is non-separable, so is X^*, which implies that X has a subspace isomorphic to ℓ_1 by a theorem of Lewis and Stegall [1973]; by a theorem of Pelczynski there is a quotient operator $Q : X \to \mathcal{C}[0,1]$. If $S : \mathcal{C}[0,1] \to Y$ is non-compact then so is $T = SQ : X \to Y \subset X$. On the other hand, Y cannot contain c_0 and so S is strictly singular. So T is strictly singular and non-compact contradicting the assumption about X. □

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