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HYPERGRAPH MATCHINGS AND DESIGNS

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Abstract

We survey some aspects of the perfect matching problem in hypergraphs, with particular emphasis on structural characterisation of the existence problem in dense hypergraphs and the existence of designs.

1 Introduction

Matching theory is a rich and rapidly developing subject that touches on many areas of Mathematics and its applications. Its roots are in the work of Steinitz [1894], Egerváry [1931], Hall [1935] and König [1931] on conditions for matchings in bipartite graphs. After the rise of academic interest in efficient algorithms during the mid 20th century, three cornerstones of matching theory were Kuhn's 'Hungarian' algorithm (Kuhn [1955]) for the Assignment Problem, Edmonds' algorithm (Edmonds [1965]) for finding a maximum matching in general (not necessarily bipartite) graphs, and the Gale and Shapley [1962] algorithm for Stable Marriages. For an introduction to matching theory in graphs we refer to Lovász and Plummer [2009], and for algorithmic aspects to parts II and III of Schrijver [2003].

There is also a very large literature on matchings in hypergraphs. This article will be mostly concerned with one general direction in this subject, namely to determine conditions under which the necessary 'geometric' conditions of 'space' and 'divisibility' are sufficient for the existence of a perfect matching. We will explain these terms and discuss some aspects of this question in the next two sections, but first, for the remainder of this introduction, we will provide some brief pointers to the literature in some other directions.

We do not expect a simple general characterisation of the perfect matching problem in hypergraphs, as by contrast with the graph case, it is known to be NP-complete even for 3-graphs (i.e. when all edges have size 3), indeed, this was one of Karp's original 21 NP-complete problems Karp [1972]. Thus for algorithmic questions related to hypergraph

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matching, we do not expect optimal solutions, and may instead consider Approximation Algorithms (see e.g. Williamson and Shmoys [2011], Asadpour, Feige, and Saberi [2008], and Lau, Ravi, and Singh [2011]).

Another natural direction is to seek nice sufficient conditions for perfect matchings. There is a large literature in Extremal Combinatorics on results under minimum degree assumptions, known as 'Dirac-type' theorems, after the classical result of Dirac [1952] that any graph on $n \geq 3$ vertices with minimum degree at least $n/2$ has a Hamiltonian cycle. It is easy to see that $n/2$ is also the minimum degree threshold for a graph on n vertices (with n even) to have a perfect matching, and this exemplifies the considerable similarities between the perfect matching and Hamiltonian problems (but there are also substantial differences). A landmark in the efforts to obtain hypergraph generalisations of Dirac's theorem was the result of Rödl, Ruciński, and Szemerédi [2009] that determined the codegree threshold for perfect matchings in uniform hypergraphs; this paper was significant for its proof method as well as the result, as it introduced the Absorbing Method (see Section 5), which is now a very important tool for proving the existence of spanning structures. There is such a large body of work in this direction that it needs several surveys to describe, and indeed these surveys already exist Rödl and Ruciński [2010], Kühn and Osthus [2009], Kühn and Osthus [2014], Zhao [2016], and Yuster [2007]. The most fundamental open problem in this area is the Erdős Matching Conjecture Erdős [1965], namely that the maximum number of edges in an r-graph[1] on n vertices with no matching of size t is either achieved by a clique of size $tr - 1$ or the set of all edges hitting some fixed set of size $t - 1$ (see Frankl and Tokushige [2016, Section 3] for discussion and a summary of progress).

The duality between matching and covers in hypergraphs is of fundamental important in Combinatorics (see Füredi [1988]) and Combinatorial Optimisation (see Cornuéjols [2001]). A defining problem for this direction of research within Combinatorics is 'Ryser's Conjecture' (published independently by Henderson [1971] and Lovász [1975]) that in any r-partite r-graph the ratio of the covering and matching numbers is at most $r-1$. For $r = 2$ this is König's Theorem. The only other known case is $r = 3$, due to Aharoni [2001], using a hypergraph analogue of Hall's theorem due to Aharoni and Haxell [2000], which has a topological proof. There are now many applications of topology to hypergraph matching, and more generally 'independent transversals' (see the survey Haxell [2016]). In the other direction, the hypergraph matching complex is now a fundamental object of Combinatorial Topology, with applications to Quillen complexes in Group Theory, Vassiliev knot invariants and Computational Geometry (see the survey Wachs [2003]).

From the probabilistic viewpoint, there are (at least) two natural questions:

(i) does a random hypergraph have a perfect matching with high probability (whp)?

[1] An r-graph is a hypergraph in which every edge contains r vertices.

(ii) what does a random matching from a given (hyper)graph look like?

The first question for the usual (binomial) random hypergraph was a longstanding open problem, perhaps first stated by Erdős [1981] (who attributed it to Shamir), finally solved by Johansson, Kahn, and Vu [2008]; roughly speaking, the threshold is 'where it should be', namely around the edge probability at which with high probability every vertex is in at least one edge. Another such result due to Cooper, Frieze, Molloy, and Reed [1996] is that random regular hypergraphs (of fixed degree and edge size) whp have perfect matchings.

The properties of random matchings in lattices have been extensively studied under the umbrella of the 'dimer model' (see Kenyon [2010]) in Statistical Physics. However, rather little is known regarding the typical structure of random matchings in general graphs, let alone hypergraphs. Substantial steps in this direction have been taken by results of Kahn [2000] characterising when the size of a random matching has an approximate normal distribution, and Kahn and Kayll [1997] establishing long-range decay of correlations of edges in random matchings in graphs; the final section of Kahn [2000] contains many open problems, including conjectural extensions to simple hypergraphs.

Prequisite to the understanding of random matchings are the closely related questions of Sampling and Approximate Counting (as established in the Markov Chain Monte Carlo framework of Jerrum and Sinclair, see Jerrum [2003]). An approximate counting result for hypergraph matchings with respect to balanced weight functions was obtained by Barvinok and Samorodnitsky [2011]. Extremal problems also arise naturally in this context, for the number of matchings, and more generally for other models in Statistical Physics, such as the hardcore model for independent sets. Much of the recent progress here appears in the survey Zhao [2017], except for the very recent solution of (almost all cases of) the Upper Matching Conjecture of Friedland, Krop, and Markström [2008] by Davies, Jenssen, Perkins, and Roberts [2017].

2 Space and divisibility

In this section we discuss a result (joint work with Mycroft 2015) that characterises the obstructions to perfect matching in dense hypergraphs (under certain conditions to be discussed below). The obstructions are geometric in nature and are of two types: Space Barriers (metric obstructions) and Divisibility Barriers (arithmetic obstructions).

The simplest illustration of these two phenomena is seen by considering extremal examples for the simple observation mentioned earlier that a graph on n vertices (n even) with minimum degree at least $n/2$ has a perfect matching. One example of a graph with minimum degree $n/2-1$ and no perfect matching is obtained by fixing a set S of $n/2-1$ vertices and taking all edges that intersect S. Then in any matching M, each edge of M

uses at least one vertex of S, so $|M| \le |S| < n/2$; there is no 'space' for a perfect matching. For another example, suppose $n = 2 \bmod 4$ and consider the graph that is the disjoint union of two cliques each of size $n/2$ (which is odd). As edges have size 2, which is even, there is an arithmetic (parity) obstruction to a perfect matching.

There is an analogous parity obstruction to matching in general r-graphs, namely an r-graph G with vertices partitioned as (A, B), so that $|A|$ is odd and $|e \cap A|$ is even for each edge e of G; this is one of the extremal examples for the codegree threshold of perfect matchings (see Rödl, Ruciński, and Szemerédi [2009]).

In general, space barriers are constructions for each $1 \le i \le r$, obtained by fixing a set S of size less than in/r and taking the r-graph of all edges e with $|e \cap S| \ge i$. Then for any matching M we have $|M| \le |S|/i < n/r$, so M is not perfect.

General divisibility barriers are obtained by fixing a lattice (additive subgroup) L in $\mathbb{Z}^d$ for some d, fixing a vertex set partitioned as $(V_1, \dots, V_d)$, with $(|V_1|, \dots, |V_d|) \notin L$, and taking the r-graph of all edges e such that $(|e \cap V_1|, \dots, |e \cap V_d|) \in L$. For example, the parity obstruction corresponds to the lattice $\{(2x, y) : x, y \in \mathbb{Z}\}$.

To state the result of Keevash and Mycroft [2015a] that is most conveniently applicable we introduce the setting of simplicial complexes and degree sequences. We consider a simplicial complex J on $[n] = \{1, \dots, n\}$, write $J_i = \{e \in J : |e| = i\}$ and look for a perfect matching in the r-graph J_r. We define the degree sequence $(\delta_0(J), \dots, \delta_{r-1}(J))$ so that each $\delta_i(J)$ is the least m such that each $e \in J_i$ is contained in at least m edges of J_{i+1}. We define the critical degree sequence $\delta^c = (\delta^c_0, \dots, \delta^c_{r-1})$ by $\delta^c_i = (1-i/r)n$. The space barrier constructions show that for each i there is a complex with $\delta_i(J)$ slightly less than δ^c_i but no perfect matching. An informal statement of Keevash and Mycroft [ibid., Theorem 2.9] is that if J is an r-complex on $[n]$ (where $r \mid n$) with all $\delta_i(J) \ge \delta^c_i - o(n)$ such that J_r has no perfect matching then J is close (in edit distance) to a space barrier or a divisibility barrier.

One application of this result (also given in Keevash and Mycroft [ibid.]) is to determine the exact codegree threshold for packing tetrahedra in 3-graphs; it was surprising that it was possible to obtain such a result given that the simpler-sounding problems of determining the thresholds (edge or codegree) for the existence of just one tetrahedron are open, even asymptotically (the edge threshrold is a famous conjecture of Turán; for more on Turán problems for hypergraphs see the survey Keevash [2011b]). Other applications are a multipartite version of the Hajnal-Szemeredi theorem (see Keevash and Mycroft [2015b]) and determining the 'hardness threshold' for perfect matchings in dense hypergraphs (see Keevash, Knox, and Mycroft [2015] and Han [2017]).

We will describe the hardness threshold in more detail, as it illustrates some important features of space and divisibility, and the distinction between perfect matchings and almost perfect matchings. For graphs there is no significant difference in the thresholds for these problems, whereas for general r-graphs there is a remarkable contrast: the codegree threshold for perfect matchings Rödl, Ruciński, and Szemerédi [2009] is about $n/2$, whereas Han [2015], proving a conjecture from Rödl, Ruciński, and Szemerédi [2009], showed that a minimum codegree of only n/r guarantees a matching of size $n/r - 1$, i.e. one less than perfect. The explanation for this contrast is that the divisibility barrier is no obstacle to almost perfect matching, whereas the space barrier is more robust, and can be continuously 'tuned' to exclude a matching of specified size.

To illustrate this, we consider a 3-graph G_0 on $[n]$ where the edges are all triples that intersect some fixed set S of size $(1/3 - c)n$, for some small $c > 0$. Then the minimum codegree and maximum matching size in G_0 are both equal to $|S|$. Furthermore, if we consider $G = G_0 \cup G_1$ where all edges of G_1 lie within some S' disjoint from S with $|S'| = 3cn$ then G has a perfect matching if and only if G_1 has a perfect matching, which is NP-complete to decide for arbitrary G_1. Thus the robustness of the space barrier provides a reduction showing that the codegree threshold for the existence of an algorithm for the perfect matching is at least the threshold for an approximate perfect matching.

Now consider the decision problem for perfect matchings in 3-graphs on $[n]$ (where $3 \mid n$) with minimum codegree at least δn. For $\delta < 1/3$ the problem is NP-complete, and for $\delta > 1/2$ it is trivial (there is a perfect matching by Rödl, Ruciński, and Szemerédi [ibid.]). For intermediate δ there is a polynomial-time algorithm, and this is in essence a structural stability result: the main ingredient of the algorithm is a result of Keevash, Knox, and Mycroft [2015] that any such 3-graph with no perfect matching is contained in a divisibility barrier. (For general r the structural characterisation is more complicated.)

3 Fractional matchings

The key idea of the Absorbing Method of Rödl, Ruciński, and Szemerédi [2009] mentioned earlier is that the task of finding perfect matchings can often broken into two subproblems: (i) finding almost perfect matchings, (ii) absorbing uncovered vertices into an almost perfect matching until it becomes perfect. We have already seen that the almost perfect matching problem appears naturally as a relaxation of the perfect matching problem in which we eliminate divisibility obstacles but retain space obstacles. This turns out to fit into a more general framework of fractional matchings, in which the relaxed problem is a question of convex geometry, and space barriers correspond to separating hyperplanes.

The fractional (linear programming) relaxation of the perfect matching problem in a hypergraph is to assign non-negative weights to the edges so that for any vertex v, there is a total weight of 1 on all edges incident to v. A perfect matching corresponds to a $\{0, 1\}$-valued solution, so the existence of a fractional perfect matching is necessary for the existence of a perfect matching. We can adopt a similar point of view regarding divisibility conditions. Indeed, we can similarly define the integer relaxation of the perfect matching problem in which we now require the weights to be integers (not necessarily non-negative); then the existence of an integral perfect matching is necessary for the existence of a perfect matching.

The fractional matching problem appears naturally in Combinatorial Optimisation (see Cornuéjols [2001] and Schrijver [2003]) because it brings in polyhedral methods and duality to bear on the matching problem. It has also been studied as a problem in its own right from the perspective of random thresholds (e.g. Devlin and Kahn [2017] and Krivelevich [1996]), and it appears naturally in combinatorial existence problems, as in dense hypergraphs almost perfect matchings and fractional matchings tend to appear at the same threshold. Indeed, for many open problems, such as the Erdős Matching Conjecture Erdős [1965] or the Nash-Williams Triangle Decomposition Conjecture Nash-Williams [1970], any progress on the fractional problem translates directly into progress on the original problem (see Barber, Kühn, Lo, and Osthus [2016]).

This therefore makes the threshold problem for fractional matchings and decompositions a natural problem in its own right. For example, an asymtotic solution of the Nash-Williams Conjecture would follow from the following conjecture: any graph on n vertices with minimum degree at least $3n/4$ has a fractional triangle decomposition, i.e. an assignment of non-negative weights to its triangles so that for any edge e there is total weight 1 on the triangles containing e. An extremal example G for this question can be obtained by taking a balanced complete bipartite graph H and adding a $(n/4-1)$-regular graph inside each part; indeed, this is a space barrier to a fractional triangle decomposition, as any triangle uses at least one edge not in H, but $|H| > 2|G \setminus H|$. The best known upper bound is $0.913n$ by Dross [2016]. More generally, Barber, Kühn, Lo, Montgomery, and Osthus [2017] give the current best known bounds on the thresholds for fractional clique decompositions (in graphs and hypergraphs), but these seem to be far from optimal.

There are (at least) two ways to think about the relationship between almost perfect matchings and fractional matchings. The first goes back to the 'nibble' (semi-random) method of Rödl [1985], introduced to solve the Erdős and Hanani [1963] conjecture on approximate Steiner systems (see the next section), which has since had a great impact on Combinatorics (e.g. Alon, J. H. Kim, and Spencer [1997], Bennett and Bohman [2012], Bohman [2009], Bohman, Frieze, and Lubetzky [2015], Bohman and Keevash [2010, 2013], Pontiveros, Griffiths, and Morris [2013], Frankl and Rödl [1985], Grable [1999], Kahn [1996b,a], J. H. Kim [2001], Kostochka and Rödl [1998], Kuzjurin [1995], Pippenger and Spencer [1989], Spencer [1995], and Vu [2000]). A special case of a theorem of Kahn [1996a] is that if there is a fractional perfect matching on the edges of an r-graph G on $[n]$ such that for any pair of vertices x, y the total weight on edges containing $\{x, y\}$ is $o(1)$ then G has a matching covering all but $o(n)$ vertices. In this viewpoint, it is natural to interpret the weights of a fractional matching as probabilities, and an almost perfect matching as a random rounding; in fact, this random rounding is obtained iteratively, so there are some parallels with the development of iterative rounding algorithms (see Lau, Ravi, and Singh [2011]).

Another way to establish the connection between almost perfect matchings and fractional matchings is via the theory of Regularity, developed by Szemerédi [1978] for graphs and extended to hypergraphs independently by Gowers [2007] and Nagle, Rödl, and Schacht [2006], Rödl and Schacht [2007b,a], and Rödl and Skokan [2004]. (The connection was first established by Haxell and Rödl [2001] for graphs and Rödl, Schacht, Siggers, and Tokushige [2007] for hypergraphs.) To apply Regularity to obtain spanning structures (such as perfect matchings) requires an accompanying result known as a blowup lemma, after the original such result for graphs obtained by Komlós, Sárközy, and Szemerédi [1997]; we proved the hypergraph version in Keevash [2011a]. More recent developments (for graphs) along these lines include the Sparse Blowup Lemmas of Allen, Böttcher, Hàn, Kohayakawa, and Person [2016] and a blowup-up lemma suitable for decompositions (as in the next section) obtained by J. Kim, Kühn, Osthus, and Tyomkyn [2016] (it would be interesting and valuable to obtain hypergraph versions of these results). The technical difficulties of the Hypergraph Regularity method are a considerable barrier to its widespread application, and preclude us giving here a precise statement of Keevash and Mycroft [2015a, Theorem 9.1], which informally speaking characterises the perfect matching problem in dense hypergraphs with certain extendability conditions in terms of space and divisibility.

4 Designs and decompositions

A *Steiner system* with parameters (n, q, r) is a q-graph G on $[n]$ such that any r-set of vertices is contained in exactly one edge. For example, a Steiner Triple System on n points has parameters $(n, 3, 2)$. The question of whether there is a Steiner system with given parameters is one of the oldest problems in combinatorics, dating back to work of Plücker (1835), Kirkman (1846) and Steiner (1853); see R. Wilson [2003] for a historical account.

Note that a Steiner system with parameters (n, q, r) is equivalent to a K_q^r-decomposition of K_n^r (the complete r-graph on $[n]$). It is also equivalent to a perfect matching in the auxiliary $\binom{q}{r}$-graph on $\binom{[n]}{r}$ (the r-subsets of $[n] := \{1, \dots, n\}$) with edge set $\{\binom{Q}{r} : Q \in \binom{[n]}{q}\}$.

More generally, we say that a set S of q-subsets of an n-set X is a *design* with parameters (n, q, r, λ) if every r-subset of X belongs to exactly λ elements of S. (This is often called an 'r-design' in the literature.) There are some obvious necessary 'divisibility conditions' for the existence of such S, namely that $\binom{q-i}{r-i}$ divides $\lambda\binom{n-i}{r-i}$ for every

$0 \leq i \leq r-1$ (fix any i-subset I of X and consider the sets in S that contain I). It is not known who first advanced the 'Existence Conjecture' that the divisibility conditions are also sufficient, apart from a finite number of exceptional n given fixed q, r and λ.

The case $r = 2$ has received particular attention due to its connections to statistics, under the name of 'balanced incomplete block designs'. We refer the reader to Colbourn and Dinitz [2007] for a summary of the large literature and applications of this field. The Existence Conjecture for $r = 2$ was a long-standing open problem, eventually resolved by R. M. Wilson [1972a,b, 1975] in a series of papers that revolutionised Design Theory. The next significant progress on the general conjecture was in the solution of the two relaxations (fractional and integer) discussed in the previous section (both of which are interesting in their own right and useful for the original problem). We have already mentioned Rödl's solution of the Erdős–Hanani Conjecture on approximate Steiner systems. The integer relaxation was solved independently by Graver and Jurkat [1973] and R. M. Wilson [1973], who showed that the divisibility conditions suffice for the existence of integral designs (this is used in R. M. Wilson [ibid.] to show the existence for large λ of integral designs with non-negative coefficients). R. M. Wilson [1999] also characterised the existence of integral H-decompositions for any r-graph H.

The existence of designs with $r \geq 7$ and any 'non-trivial' λ was open before the breakthrough result of Teirlinck [1987] confirming this. An improved bound on λ and a probabilistic method (a local limit theorem for certain random walks in high dimensions) for constructing many other rigid combinatorial structures was recently given by Kuperberg, Lovett, and Peled [2017]. Ferber, Hod, Krivelevich, and Sudakov [2014] gave a construction of 'almost Steiner systems', in which every r-subset is covered by either one or two q-subsets.

In Keevash [2014] we proved the Existence Conjecture in general, via a new method of Randomised Algebraic Constructions. Moreover, in Keevash [2015] we obtained the following estimate for the number $D(n,q,r,\lambda)$ of designs with parameters (n,q,r,λ) satisfying the necessary divisibility conditions: writing $Q = \binom{q}{r}$ and $N = \binom{n-r}{q-r}$, we have

$$D(n,q,r,\lambda) = \lambda!^{-\binom{n}{r}}\left((\lambda/e)^{Q-1}N + o(N)\right)^{\lambda Q^{-1}\binom{n}{r}}.$$

Our counting result is complementary to that in Kuperberg, Lovett, and Peled [2017], as it applies (e.g.) to Steiner Systems, whereas theirs is only applicable to large multiplicities (but also allows the parameters q and r to grow with n, and gives an asymptotic formula when applicable).

The upper bound on the number of designs follows from the entropy method pioneered by Radhakrishnan [1997]; more generally, Luria [2017] has recently established a similar upper bound on the number of perfect matchings in any regular uniform hypergraph with small codegrees. The lower bound essentially matches the number of choices in the

Random Greedy Hypergraph Matching process (see Bennett and Bohman [2012]) in the auxiliary Q-graph defined above, so the key to the proof is showing that this process can be stopped so that whp it is possible to complete the partial matching thus obtained to a perfect matching. In other words, instead of a design, which can be viewed as a K_q^r-decomposition of the r-multigraph λK_n^r, we require a K_q^r-decomposition of some sparse submultigraph, that satisfies the necessary divisibility conditions, and has certain pseudo-randomness properties (guaranteed whp by the random process).

The main result of Keevash [2014] achieved this, and indeed (in the second version of the paper) we obtained a more general result in the same spirit as Keevash and Mycroft [2015a], namely that we can find a clique decomposition of any r-multigraph with a certain 'extendability' property that satisfies the divisibility conditions and has a 'suitably robust' fractional clique decomposition.

Glock, Kühn, Lo, and Osthus [2016, 2017] have recently given a new proof of the existence of designs, as well as some generalisations, including the existence of H-decompositions for any hypergraph H (a question from Keevash [2014]), relaxing the quasirandomness condition from Keevash [ibid.] (version 1) to an extendability condition in the same spirit as Keevash [ibid.] (version 2), and a more effective bound than that in Keevash [ibid.] on the minimum codegree decomposition threshold; the main difference in our approaches lies in the treatment of absorption (see the next section).

5 Absorption

Over the next three sections we will sketch some approaches to what is often the most difficult part of a hypergraph matching or decomposition problem, namely converting an approximate solution into an exact solution. We start by illustrating the Absorbing Method in its original form, namely the determination in Rödl, Ruciński, and Szemerédi [2009] of the codegree threshold for perfect matchings in r-graphs; for simplicity we consider $r = 3$.

We start by solving the almost perfect matching problem. Let G be a 3-graph on $[n]$ with $3 \mid n$ and minimum codegree $\delta(G) = n/3$, i.e. every pair of vertices is in at least $n/3$ edges. We show that G has a matching of size $n/3 - 1$ (i.e. one less than perfect). To see this, consider a maximum size matching M, let $V_0 = V(G) \setminus V(M)$, and suppose $|V_0| > 3$. Then $|V_0| \geq 6$, so we can fix disjoint pairs a_1b_1, a_2b_2, a_3b_3 in V_0. For each i there are at least $n/3$ choices of c such that $a_ib_ic \in E(G)$, and by maximality of M any such c lies in $V(M)$. We define the weight w_e of each $e \in M$ as the number of edges of G of the form a_ib_ic with $c \in e$. Then $\sum_{e \in M} w_e \geq n$, and $|M| < n/3$, so there is $e \in M$ with $w_e \geq 4$. Then there must be distinct c, c' in e and distinct i, i' in $[3]$ such that $a_i b_i c$ and $a_{i'} b_{i'} c'$ are edges. However, deleting e and adding these edges contradicts maximality of M.

Now suppose $\delta(G) = n/2 + cn$, where $c > 0$ and $n > n_0(c)$ is large. Our plan for finding a perfect matching is to first put aside an 'absorber' A, which will be a matching in G with the property that for any triple T in $V(G)$ there is some edge $e \in A$ such that $T \cup e$ can be expressed as the disjoint union of two edges in G (then we say that e absorbs T). Suppose that we can find such A, say with $|A| < n/20$. Deleting the vertices of A leaves a 3-graph G' on $n' = n - |A|$ vertices with $\delta(G') \geq \delta(G) - 3|A| > n'/3$. As shown above, G' has a matching M' with $|M'| = n'/3 - 1$. Let $T = V(G') \setminus V(M')$. By choice of A there is $e \in A$ such that $T \cup e = e_1 \cup e_2$ for some disjoint edges e_1, e_2 in G. Then $M' \cup (A \setminus \{e\}) \cup \{e_1, e_2\}$ is a perfect matching in G.

It remains to find A. The key idea is that for any triple T there are many edges in G that absorb T, and so if A is random then whp many of them will be present. We can bound the number of absorbers for any triple $T = xyz$ by choosing vertices sequentially. Say we want to choose an edge $e = x'y'z'$ so that $x'yz$ and $xy'z'$ are also edges. There are at least $n/2 + cn$ choices for x' so that $x'yz$ is an edge. Then for each of the $n - 4$ choices of $y' \in V(G) \setminus \{x, y, z, x'\}$ there are at least $2cn - 1$ choices for $z' \neq z$ so that $x'y'z'$ and $xy'z'$ are edges. Multiplying the choices we see that T has at least cn^3 absorbers.

Now suppose that we construct A by choosing each edge of G independently with probability $c/(4n^2)$ and deleting any pair that intersect. Let X be the number of deleted edges. There are fewer than n^5 pairs of edges that intersect, so $\mathbb{E}X < c^2 n/16$, so $\mathbb{P}(X < c^2 n/8) \geq 1/2$. Also, the number of chosen absorbers N_T for any triple T is binomial with mean at least $c^2 n/4$, so whp all $N_T > c^2 n/8$. Thus there is a choice of A such that every T has an absorber in A. This completes the proof of the approximate version of Rödl, Ruciński, and Szemerédi [2009], i.e. that minimum codegree $n/2 + cn$ guarantees a perfect matching.

The idea for the exact result is to consider an attempt to construct absorbers as above under the weaker assumption $\delta(G) \geq n/2 - o(n)$. It is not hard to see that absorbers exist unless G is close to one of the extremal examples. The remainder of the proof (which we omit) is then a stability analysis to show that the extremal examples are locally optimal, and so optimal.

In the following two sections we will illustrate two approaches to absorption for designs and hypergraph decompositions, in the special case of triangle decompositions of graphs, which is considerably simler, and so allows us to briefly illustrate some (but not all) ideas needed for the general case. First we will conclude this section by indicating why the basic method described above does not suffice.

Suppose we seek a triangle decomposition of a graph G on $[n]$ with $e(G) = \Omega(n^2)$ in which there is no space or divisibility obstruction: we assume that G is 'tridivisible' (meaning that $3 \mid e(G)$ and all degrees are even) and 'triangle-regular' (meaning that there is a set T of triangles in G such that every edge is in $(1+o(1))tn$ triangles of T, where $t > 0$ and $n > n_0(t)$). This is equivalent to a perfect matching in the auxiliary 3-graph H with $V(H) = E(G)$ and $E(H) = \{\{ab, bc, ca\} : abc \in T\}$. Note that H is 'sparse': we have $e(H) = O(v(H)^{3/2})$. Triangle regularity implies that Pippenger's generalisation (see Pippenger and Spencer [1989]) of the Rödl nibble can be applied to give an almost perfect matching in H, so the outstanding question is whether there is an absorber.

Let us consider a potential random construction of an absorber A in H. It will contain at most $O(n^2)$ triangles, so the probability of any triangle (assuming no heavy bias) will be $O(n^{-1})$. On the other hand, to absorb some fixed (tridivisible) $S \subseteq E(G)$, we need A to contain a set A_S of a edge-disjoint triangles (for some constant a) such that $S \cup A_S$ has a triangle decomposition B_S, so we need $\Omega(n^a)$ such A_S in G. To see that this is impossible, we imagine selecting the triangles of A_S one at a time and keeping track of the number E_S of edges that belong to a unique triangle of $S \cup A_S$. If a triangle uses a vertex that has not been used previously then it increases E_S, and otherwise it decreases E_S by at most 3. We can assume that no triangle is used in both A_S and B_S, so we terminate with $E_S = 0$. Thus there can be at most $3a/4$ steps in which E_S increases, so there are only $O(n^{3a/4})$ such A_S in G.

The two ideas discussed below to overcoming this obstacle can be briefly summarised as follows. In Randomised Algebraic Construction (introduced in Keevash [2014]), instead of choosing independent random triangles for an absorber, they are randomly chosen according to a superimposed algebraic structure that has 'built-in' absorbers. In Iterative Absorption (used for designs and decompositions in Glock, Kühn, Lo, and Osthus [2016, 2017]), instead of a single absorption step, there is a sequence of absorptions, each of which replaces the current subgraph of uncovered edges by an 'easier' subgraph, until we obtain S that is so simple that it can be absorbed by an 'exclusive' absorber put aside at the beginning of the proof for the eventuality that we end up with S. This is a powerful idea with many other applications (see the survey Kühn and Osthus [2014]).

6 Iterative Absorption

Here we will sketch an application of iterative absorption to finding a triangle decomposition of a graph G with no space or divisibility obstruction as in the previous subsection. (We also make certain 'extendability' assumptions that we will describe later when they are needed.) Our sketch will be loosely based on a mixture of the methods used in Barber, Kühn, Lo, and Osthus [2016] and Glock, Kühn, Lo, and Osthus [2016], thus illustrating some ideas of the general case but omitting most of the technicalities.

The plan for the decomposition is to push the graph down a 'vortex', which consists of a nested sequence $V(G) = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\tau$, where $|V_i| = \theta|V_{i-1}|$ for each $i \in [\tau]$ with $n^{-1} \ll \theta \ll t$, and $|V_\tau| = O(1)$ (so τ is logarithmic in $n = v(G)$). Suppose G has a set T of triangles such that every edge is in $(1 \pm c)tn$ triangles of T, where $n^{-1} \ll \theta \ll c, t$. By choosing the V_i randomly we can ensure that each edge of $G[V_i]$ is in $(1 \pm 2c)t|V_i|$ triangles of $T_i = \{f \in T : f \subseteq V_i\}$. At step i with $0 \le i \le t$ we will have covered all edges of G not contained in V_i by edge-disjoint triangles, and also some edges within V_i, in a suitably controlled manner, so that we still have good triangle regularity in $G[V_i]$.

At the end of the process, the uncovered subgraph L will be contained in V_τ, so there are only constantly many possibilities for L. Before starting the process, for each tridivisible subgraph S of the complete graph on V_τ we put aside edge-disjoint 'exclusive absorbers' A_S, i.e. sets of edge-disjoint triangles in G such that $S \cup A_S$ has a triangle decomposition B_S (we omit here the details of this construction). Then L will be equal to one of these S, so replacing A_S by B_S completes the triangle decomposition of G.

Let us now consider the process of pushing G down the vortex; for simplicity of notation we describe the first step of covering all edges not within V_1. The plan is to cover almost all of these edges by a nibble, and then the remainder by a random greedy algorithm (which will also use some edges within V_1). At first sight this idea sounds suspicious, as one would think that the triangle regularity parameter c must increase substantially at each step, and so the process could not be iterated logarithmically many times before the parameters blow up.

However, quite suprisingly, if we make the additional extendability assumption that every edge is in at least $c'n^3$ copies of K_5 (where c' is large compared with c and $t \ge c'$), then we can pass to a different set of triangles which dramatically 'boost' the regularity. The idea (see Glock, Kühn, Lo, and Osthus [2016, Lemma 6.3]) is that a relatively weak triangle regularity assumption implies the existence of a perfect fractional triangle decomposition, which can be interpreted as selection probabilities (in the same spirit as Kahn [1996a]) for a new set of triangles that is much more regular. A similar idea appears in the Rödl-Schacht proof of the hypergraph regularity lemma via regular approximation (see Rödl and Schacht [2007b]). It may also be viewed as a 'guided version' of the self-correction that appears naturally in random greedy algorithms (see Bohman and Keevash [2013] and Pontiveros, Griffiths, and Morris [2013]).

Let us then consider $G^* = G \setminus G[V_1] \setminus H$, where H contains each edge of G crossing between V_1 and $V^* := V(G) \setminus V_1$ independently with some small probability $p \ll c, \theta$. (We reserve H to help with the covering step.) Then whp every edge of G^* is in $(1 \pm c)tn \pm |V_1| \pm 3pn$ triangles of T within G^*. By boosting, we can find a set T^* of triangles in G^* such that every edge of G^* is in $(1 \pm c_0)tn/2$ triangles of T^*, where $c_0 \ll p$. By the nibble, we can choose a set of edge-disjoint triangles in T^* covering all of G^* except for some 'leave' L of maximum degree $c_1 n$, where we introduce new constants $c_0 \ll c_1 \ll c_2 \ll p$.

Now we cover L by two random greedy algorithms, the first to cover all remaining edges in V^* and the second to cover all remaining cross edges. The analysis of these algorithms is not as difficult as that of the nibble, as we have 'plenty of space', in that we only have to cover a sparse graph within a much denser graph, whereas the nibble seeks to cover almost all of a graph. In particular, the behaviour of these algorithms is well-approximated by a 'binomial heuristic' in which we imagine choosing random triangles to cover the uncovered edges without worrying about whether these triangles are edge-disjoint (so we make independent choices for each edge). In the actual algorithm we have to exclude any triangle that uses an edge covered by a previous step of the algorithm, but if we are covering a sparse graph one can show that whp at most half (say) of the choices are forbidden at each step, so any whp estimate in the binomial process will hold in the actual process up to a factor of two. (This idea gives a much simpler proof of the result of Ferber, Hod, Krivelevich, and Sudakov [2014].)

For the first greedy algorithm we consider each remaining edge in V^* in some arbitrary order, and when we consider e we choose a triangle on e whose two other edges are in H, and have not been previously covered. In general we would make this choice uniformly at random, although the triangle case is sufficiently simple that an arbitrary choice suffices; indeed, there are whp at least $p^2\theta n/2$ such triangles in H, of which at most $2c_1 n$ are forbidden due to using a previously covered edge (by the maximum degree of L). Thus the algorithm can be completed with arbitrary choices.

The second greedy algorithm for covering the cross edges is more interesting (the analogous part of the proof for general designs is the most difficult part of Glock, Kühn, Lo, and Osthus [2016]). Let H' denote the subgraph of cross edges that are still uncovered. We consider each $x \in V^*$ sequentially and cover all edges of H' incident to x by the set of triangles obtained by adding x to each edge of a perfect matching M_x in $G[H'(x)]$, i.e. the restriction of G to the H'-neighbourhood of x. We must choose M_x edge-disjoint from $M_{x'}$ for all x' preceding x, so an arbitrary choice will not work; indeed, whp the degree of each vertex y in $G[H'(x)]$ is $(1 \pm c_2)p\theta t n$, but our upper bound on the degree of y in H' may be no better than pn, so previous choices of $M_{x'}$ could isolate y in $G[H'(x)]$.

To circumvent this issue we choose random perfect matchings. A uniformly random choice would work, but it is easier to analyse the process where we fix many edge-disjoint matchings in $G[H'(x)]$ and then choose one uniformly at random to be M_x. We need some additional assumption to guarantee that $G[H'(x)]$ has even one perfect matching (the approximate regularity only guarantees an almost perfect matching).

One way to achieve this is to make the additional mild extendability assumption that every pair of vertices have at least $c'n$ common neighbours in $G[H'(x)]$, i.e. any adjacent pair of edges xy, xy' in G have at least $c'n$ choices of z such that xz, yz and $y'z$ are edges. It is then not hard to see that a random balanced bipartite subgraph of $G[H'(x)]$ whp satisfies Hall's condition for a perfect matching. Moreover, we can repeatedly delete $p^{3/2}\theta c'n$ perfect matchings in $G[H'(x)]$, as this maintains all degrees $(1 \pm 2\sqrt{p})p\theta tn$ and codegrees at least $p\theta c'n/2$.

The punchline is that for any edge e in $G[H'(x)]$ there are whp at most $2p^2n$ earlier choices of x' with e in $G[H'(x')]$, and the random choice of $M_{x'}$ covers e with probability at most $(p^{3/2}\theta c'n)^{-1}$, so e is covered with probability at most $2p^2n(p^{3/2}\theta c'n)^{-1} < p^{1/3}$, say. Thus whp $G[H'(x)]$ still has sufficient degree and codegree properties to find the perfect matchings described above, and the algorithm can be completed. Moreover, any edge of $G[V_1]$ is covered with probability at most $p^{1/3}$, so whp we maintain good triangle regularity in $G[V_1]$ and can proceed down the vortex.

7 Randomised Algebraic Construction

Here we sketch an alternative proof (via our method of Randomised Algebraic Construction from Keevash [2014]) of the same result as in the previous subsection, i.e. finding a triangle decomposition of a graph G with certain extendability properties and no space or divisibility obstruction. Our approach will be quite similar to that in Keevash [2015], except that we will illustrate the 'cascade' approach to absorption which is more useful for general designs.

As discussed above, we circumvent the difficulties in the basic method for absorption by introducing an algebraic structure with built-in absorbers. Let $\pi : V(G) \to \mathbb{F}_{2^a} \setminus \{0\}$ be a uniformly random injection, where $2^{a-2} < n \le 2^{a-1}$. Our absorber (which in this context we call the 'template') is defined as the set T of all triangles in G such that $\pi(x)+\pi(y)+\pi(z) = 0$. Clearly T consists of edge-disjoint triangles. We let $G^* = \bigcup T$ be the underlying graph of the template and suppress π, imagining $V(G)$ as a subset of $\mathbb{F}_{2^a}$.

Standard concentration arguments show that whp $G \setminus G^*$ has the necessary properties to apply the nibble, so we can find a set N of edge-disjoint triangles in $G \setminus G^*$ with leave $L := (G \setminus G^*) \setminus \bigcup N$ of maximum degree $c_1 n$ (we use a similar hierarchy of very small parameters c_i as before). To absorb L, it is convenient to first 'move the problem' into the template: we apply a random greedy algorithm to cover L by a set M^c of edge-disjoint triangles, each of which has one edge in L and the other two edges in G^*. Thus some subgraph S of G^*, which we call the 'spill' has now been covered twice. The binomial heuristic discussed in the previous subsection applies to show that whp this algorithm is successful, and moreover S is suitably bounded. (To be precise, we also ensure that each edge of S belongs to a different triangle of T, and that the union S^* of all such triangles is c_2-bounded.)

The remaining task of the proof is to modify the current set of triangles to eliminate the problem with the spill. The overall plan is to find a 'hole' in the template that exactly matches the spill. This will consist of two sets of edge-disjoint triangles, namely M^o (outer set) and M^i (inner set), such that $M^o \subseteq T$ and $\bigcup M^o$ is the disjoint union of S and $\bigcup M^i$. Then replacing M^o by M^i will fix the problem: formally, our final triangle decomposition of G is $M := N \cup M^c \cup (T \setminus M^o) \cup M^i$.

We break down the task of finding the hole into several steps. The first is a refined form of the integral decomposition theorem of Graver and Jurkat [1973] and R. M. Wilson [1973], i.e. that there is an assignment of integers to triangles so that total weight of triangles on any edge e is 1 if $e \in S$ or 0 otherwise. Our final hole can be viewed as such an assignment, in which a triangle f has weight 1 if $f \in M^o$, -1 if $f \in M^i$, or 0 otherwise. We intend to start from some assignment and repeatedly modify it by random greedy algorithms until it has the properties required for the hole. As discussed above, the success of such random greedy algorithms requires control on the maximum degree, so our refined version of Graver and Jurkat [1973] and R. M. Wilson [1973] is that we can choose the weights w_T on triangles with $\sum_{T:v\in T} |w_T| < c_3 n$ for every vertex v. (The proof is fairly simple, but the analogous statement for general hypergraphs seems to be much harder to prove.) Note that in this step we allow the use of any triangle in K_n (the complete graph on $V(G)$), without considering whether they belong to G^*: 'illegal' triangles will be eliminated later.

Let us now consider how to modify assignments of weights to triangles so as to obtain a hole. Our first step is to ignore the requirement $M^o \subseteq T$, which makes our task much easier, as T is a special set of only $O(n^2)$ triangles. Thus we seek a signed decomposition of S within G^*, i.e. an assignment from $\{-1, 0, 1\}$ to each triangle of G^* so that the total weight on any e is 1 if $e \in S$ or 0 otherwise, and every edge appears in at most one triangle of each sign.

To achieve this, we start from the simple observation that the graph of the octahedron has 8 triangles, which can be split into two groups of 4, each forming a triangle decomposition. For any copy of the octahedron in K_n we can add 1 to the triangles of one decomposition and subtract 1 from the triangles of the other without affecting the total weight of triangles on any edge. We can use this construction to repeatedly eliminate 'cancelling pairs', consisting of two triangles on a common edge with opposite sign. (There is a preprocessing step to ensure that each triangle to be eliminated can be assigned to a unique such pair.) In particular, as edges not in G^* have weight 0, this will eliminate all illegal triangles. The boundedness condition facilitates a random greedy algorithm for choosing edge-disjoint octahedra for these eliminations, which constructs the desired signed decomposition of S.

Now we remember that we wanted the outer triangle decomposition M^o to be contained in the template T. Finally, the algebraic structure will come into play, in absorbing the set

M^+ of positive triangles in the signed decomposition. To see how this can be achieved, consider any positive triangle xyz, recall that vertices are labelled by elements of $\mathbb{F}_{2^a} \setminus \{0\}$, and suppose first for simplicity that xyz is 'octahedral', meaning that G^* contains the 'associated octahedron' of xyz, defined as the complete 3-partite graph O with parts $\{x, y+z\}$, $\{y, z+x\}$, $\{z, x+y\}$. Then xyz is a triangle of O, and we note that O has a triangle decomposition consisting entirely of template triangles, namely $\{x, y, x+y\}$, $\{y+z, y, z\}$, $\{x, z+x, z\}$ and $\{y+z, z+x, x+y\}$. Thus we can 'flip' O (i.e. add and subtract the two triangle decompositions as before) to eliminate xyz while only introducing positive triangles that are in T.

The approach taken in Keevash [2015] was to ensure in the signed decomposition that every positive triangle is octahedral, with edge-disjoint associated octahedra, so that all positive triangles can be absorbed as indicated above without interfering with each other. For general designs, it is more convenient to define a wider class of triangles (in general hypergraph cliques) that can be absorbed by the following two step process, which we call a 'cascade'. Suppose that we want to absorb some positive triangle xyz. We look for some octahedron O with parts $\{x, x'\}$, $\{y, y'\}$, $\{z, z'\}$ such that each of the 4 triangles of the decomposition not using xyz is octahedral. We can flip the associated octahedra of these triangles so as to include them in the template, and now O is decomposed by template triangles, so can play the role of an associated octahedron for xyz: we can flip it to absorb xyz. The advantage of this approach is that whp any non-template xyz has many cascades, so no extra property of the signed decomposition is required to complete the proof. In general, there are still some conditions required for a clique have many cascades, but these are not difficult to ensure in the signed decomposition.

8 Concluding remarks

There are many other questions of Design Theory that can be reformulated as asking whether a certain (sparse) hypergraph has a perfect matching. This suggests the (vague) meta-question of formulating and proving a general theorem on the existence of perfect matchings in sparse 'design-like' hypergraphs (for some 'natural' definition of 'design-like' that is sufficiently general to capture a variety of problems in Design Theory). One test for such a statement is that it should capture all variant forms of the basic existence question, such as general hypergraph decompositions (as in Glock, Kühn, Lo, and Osthus [2016]) or resolvable designs (the general form of Kirkman's original 'schoolgirl problem', solved for graphs by Ray-Chaudhuri and R. M. Wilson [1971] but still open for hypergraphs). But could we be even more ambitious?

To focus the ideas, one well-known longstanding open problem is Ryser's Conjecture Ryser [1967] that every Latin square of odd order has a transversal. (A generalised form of this conjecture by Stein [1975] was recently disproved by Pokrovskiy and Sudakov [2017].) To see the connection with hypergraph matchings, we associate to any Latin square a tripartite 3-graph in which the parts correspond to rows, columns and symbols, and each cell of the square corresponds to an edge consisting of its own row, column and symbol. A perfect matching in this 3-graph is precisely a transversal of the Latin square. However, there is no obvious common structure to the various possible 3-graphs that may arise in this way, which presents a challenge to the absorbing methods described in this article, and so to formulating a meta-theorem that might apply to Ryser's Conjecture. The best known lower bound of $n - O(\log^2 n)$ on a partial transversal (by Hatami and Shor [2008]) has a rather different proof. Another generalisation of Ryser's Conjecture by Aharoni and Berger [2009] concerning rainbow matchings in properly coloured multigraphs has recently motivated the development of various other methods for such problems not discussed in this article (see e.g. Gao, Ramadurai, Wanless, and Wormald [2017], Keevash and Yepremyan [2017], and Pokrovskiy [2016]).

Recalling the theme of random matchings discussed in the introduction, it is unsurprising that it is hard to say much about random designs, but for certain applications one can extract enough from the proof in Keevash [2014], e.g. to show that whp a random Steiner Triple System has a perfect matching (Kwan [2016]) or that one can superimpose a constant number of Steiner Systems to obtain a bounded codegree high-dimensional expander (Lubotzky, Luria, and Rosenthal [2015]). Does the nascent connection between hypergraph matchings and high-dimensional expanders go deeper?

We conclude by recalling two longstanding open problems from the other end of the Design Theory spectrum, concerning q-graphs with q of order $\sqrt{n}$ (the maximum possible), as opposed to the setting $n > n_0(q)$ considered in this article (or even the methods of Kuperberg, Lovett, and Peled [2017] which can allow q to grow as a sufficiently small power of n).

Hadamard's Conjecture. (Hadamard [1893])
There is an $n \times n$ orthogonal matrix H with all entries $\pm n^{-1/2}$ iff n is 1, 2 or divisible by 4?

Projective Plane Prime Power Conjecture. (folklore)
There is a Steiner system with parameters $(k^2 + k + 1, k + 1, 2)$ iff k is a prime power?

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