PROC. INT. CONG. OF MATH. – 2018
Rio de Janeiro, Vol. 2 (959–986)

GLOBAL SURFACES OF SECTION FOR REEB FLOWS IN DIMENSION THREE AND BEYOND

PEDRO A. S. SALOMÃO AND UMBERTO L. HRYNIEWICZ

Dedicated to the memory of Professor Kris Wysocki

Abstract

We survey some recent developments in the quest for global surfaces of section for Reeb flows in dimension three using methods from Symplectic Topology. We focus on applications to geometry, including existence of closed geodesics and sharp systolic inequalities. Applications to topology and celestial mechanics are also presented.

1 Introduction

The idea of a global surface of section goes back to Poincaré and the planar circular restricted three-body problem.

Definition 1.1. *Let ϕ^t be a smooth flow on a smooth closed 3-manifold M. An embedded surface $\Sigma \hookrightarrow M$ is a global surface of section for ϕ^t if:*

(i) Each component of $\partial\Sigma$ is a periodic orbit of ϕ^t.

(ii) ϕ^t is transverse to $\Sigma \setminus \partial\Sigma$.

(iii) For every $p \in M \setminus \partial\Sigma$ there exist $t_+ > 0$ and $t_- < 0$ such that $\phi^{t_+}(p)$ and $\phi^{t_-}(p)$ belong to $\Sigma \setminus \partial\Sigma$.

Every $p \in \Sigma \setminus \partial\Sigma$ has a first return time $\tau(p) = \inf\{t > 0 \mid \phi^t(p) \in \Sigma\}$ and the dynamics of the flow are encoded in the first return map

$$\psi : \Sigma \setminus \partial\Sigma \to \Sigma \setminus \partial\Sigma, \qquad \psi(p) = \phi^{\tau(p)}(p). \tag{1}$$

In Poincaré [1912] Poincaré described annulus-like global surfaces of section for the planar circular restricted three-body problem (PCR3BP) for certain values of the Jacobi

MSC2010: primary 37JXX; secondary 53DXX.

constant and mass ratio. Poincaré's global sections motivated his celebrated *last geometric theorem*. The associated first return map preserves an area form, extends up to boundary, and satisfies a twist condition in the range of parameters considered. The exciting discovery made by Poincaré was that the twist condition implies the existence of infinitely many periodic points, i.e., infinitely many periodic orbits for the PCR3BP. In one stroke Poincaré gave a strong push towards a qualitative point of view for studying differential equations, and stated a fixed point theorem intimately connected to the Arnold conjectures and the foundations of Floer Theory.

The recent success of Floer theory and other methods from Symplectic Geometry prompted Hofer to coin the term *Symplectic Dynamics* Bramham and Hofer [2012]. In this note we are concerned with the success of these methods to study Reeb flows in dimension three, with an eye towards applications to geometry.

Our first goal is to discuss existence results for global sections. This will be done in Section 2. After stating Birkhoff's theorem, we focus on Hofer's theory of pseudo-holomorphic curves Hofer [1993]. We survey some published and also some unpublished results, without giving proofs.

Section 3 is devoted to some applications to systolic geometry that were obtained in collaboration with Abbondandolo and Bramham. We will explain how global surfaces of section open the door for symplectic methods in the study of sharp systolic inequalities. We focus on Riemannian two-spheres and on a special case of a conjecture of Viterbo. In Section 4 we present the planar circular restricted three-body problem in more detail. A conjecture due to Birkhoff on the existence of disk-like global surfaces of section for retrograde orbits is discussed.

We intend to convince the reader that there are many positive results for global sections in large classes of flows. However, there are situations where it might be hard to decide whether they exist or not. In sections 5 and 6 we discuss results designed to handle some of these situations. In Section 5 we present deep results of Hofer, Wysocki, and Zehnder [2003] concerning the existence of transverse foliations, and its use in the study of Hamiltonian dynamics near critical levels. In Section 6 we present a Poincaré-Birkhoff theorem for tight Reeb flows on S^3 proved in Hryniewicz, Momin, and Salomão [2015]. It concerns Reeb flows with a pair of closed orbits exactly as those in the boundary of Poincaré's annulus, i.e. forming a Hopf link.

The appendix A discusses a new proof of the existence of infinitely many closed geodesics on any Riemannian two-sphere, which is alternative to the classical arguments of Bangert [1993] and Franks [1992]. It relies on the work of Hingston [1993].

2 Existence results for global surfaces of section

Poincaré constructed his annulus map for a specific family of systems close to integrable[1]. One of the first statements for a large family of systems which can be quite far from integrable is due to Birkhoff.

Theorem 2.1 (Birkhoff [1966]). *Let γ be a simple closed geodesic of a positively curved Riemannian two-sphere. Consider the set A_γ of unit vectors along γ pointing towards one of the hemispheres determined by γ. Then A_γ is a global surface of section for the geodesic flow.*

In other words, every geodesic ray not contained in γ visits both hemispheres infinitely often. We call the embedded annulus A_γ the *Birkhoff annulus*. The family of geodesic flows on positively curved two-spheres is large, making the above statement quite useful. The proof heavily relies on Riemannian geometry and sheds little light on the general existence problem.

A very general theory to attack the existence problem of global surfaces of section exists, and nowadays goes by the name of Schwartzman-Fried-Sullivan theory, see Ghys [2009] or the original works Fried [1982], Schwartzman [1957], and Sullivan [1976]. It produces beautiful theorems with strong conclusions for general flows in dimension three, or even in higher dimensions. The drawback is that these conclusions often require hypotheses which are hard to check, limiting the range of applications. This should not be a surprise because the set of all flows on a 3-manifold is just too wild.

Hofer's pseudo-holomoprhic curve theory deals with the more restrictive class of Reeb flows. However, the results obtained require more reasonable hypotheses which one can often check, as we intend to demonstrate in the next paragraphs. Sometimes results apply automatically for classes of Reeb flows that are large enough to provide applications in topology and geometry. Consider $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) and its standard symplectic form $\omega_0 = \sum_{j=1}^{2} dx_j \wedge dy_j$. Here are two examples of such *unconditional* theorems.

Theorem 2.2 (Hofer, Wysocki, and Zehnder [1998]). *The Hamiltonian flow on a smooth, compact and strictly convex energy level in $(\mathbb{R}^4, \omega_0)$ admits a disk-like global surface of section.*

We see Theorem 2.2 as one of the pinnacles of Symplectic Dynamics, it is the guiding application of this theory to the study of global surfaces of section. All results to be discussed in this section are proved using the methods from Hofer, Wysocki, and Zehnder [ibid.].

[1] Angular momentum is preserved in the rotating Kepler problem.

Theorem 2.3 (Hryniewicz [2012, 2014]). *A periodic orbit of the Hamiltonian flow on a smooth, compact and strictly convex energy level in $(\mathbb{R}^4, \omega_0)$ bounds a disk-like global surface of section if, and only if, it is unknotted and has self-linking number* -1.

To explain the connection between the above statements and Reeb flows, and to describe further results of this theory, we need first to review basic notions. A contact form λ on a 3-manifold M is a 1-form such that $\lambda \wedge d\lambda$ defines a volume form. Its Reeb vector field R_λ is implicitly defined by

$$d\lambda(R_\lambda, \cdot) = 0, \qquad \lambda(R_\lambda) = 1. \tag{2}$$

The distribution $\xi = \ker \lambda$ is a contact structure, the pair (M, ξ) is a contact manifold. More precisely, these are the co-orientable contact manifolds since λ orients TM/ξ. We only work here with co-orientable contact structures. By a Reeb flow on (M, ξ) we mean one associated to a contact form λ on M such that $\xi = \ker \lambda$. Contact manifolds are the main objects of study in contact topology. Our interest here is shifted towards dynamics.

A knot is called transverse if at every point its tangent space is transverse to the contact structure. A transverse knot with a Seifert surface has a self-linking number, which is invariant under transverse isotopies. It is defined as follows: choose a non-vanishing section of the contact structure along the Seifert surface, then use this section to push the knot off from itself, and finally count intersections with the Seifert surface. The vector bundle $(\xi, d\lambda)$ is symplectic and has a first Chern class $c_1(\xi) \in H^2(M; \mathbb{Z})$. If $c_1(\xi)$ vanishes on $H_2(M; \mathbb{Z})$ then the self-linking number does not depend on the Seifert surface. The book Geiges [2008] by Geiges is a nice reference for these concepts.

Finally, we describe the Conley-Zehnder index in low-dimensions following Hofer, Wysocki, and Zehnder [2003]. Let γ be a periodic trajectory of the flow ϕ^t of the Reeb vector field R_λ, and let $T > 0$ be a period of γ. Since $(\phi^t)^*\lambda = \lambda$, we get a path of $d\lambda$-symplectic linear maps $d\phi^t : \xi_{\gamma(0)} \to \xi_{\gamma(t)}$. The orbit γ is called degenerate in period T if 1 is an eigenvalue of $d\phi^T : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$, otherwise it is called non-degenerate in period T. The contact form λ is called non-degenerate when every periodic trajectory is non-degenerate in every period. When T is the primitive period we may simply call γ degenerate or non-degenerate accordingly.

Since T is a period, we get a well-defined map $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ still denoted by γ without fear of ambiguity. Choose a symplectic trivialization Φ of $\gamma^*\xi$. Then the linearized flow $d\phi^t : \xi_{\gamma(0)} \to \xi_{\gamma(t)}$ gets represented as a path of symplectic matrices $M : \mathbb{R} \to Sp(2)$ satisfying $M(0) = I$, $M(t+T) = M(t)M(T)$ $\forall t$. For every non-zero $u \in \mathbb{R}^2$ we write $M(t)u = (r(t)\cos\theta(t), r(t)\sin\theta(t))$ in polar coordinates, for some continuous lift of argument $\theta : \mathbb{R} \to \mathbb{R}$, and define the rotation function $\Delta_M : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ by $\Delta_M(u) = \frac{\theta(T)-\theta(0)}{2\pi}$. The image of Δ_M is a compact interval of length strictly less than $1/2$. The rotation interval J_M is defined as the image of Δ_M.

Consider the following function $\tilde{\mu}(J)$ defined on closed intervals J of length less than $1/2$. If $\partial J \cap \mathbb{Z} = \varnothing$ then set $\tilde{\mu}(J) = 2k$ when $k \in J$, or $\tilde{\mu}(J) = 2k+1$ when $J \subset (k, k+1)$. If $\partial J \cap \mathbb{Z} \neq \varnothing$ then set $\tilde{\mu}(J) = \lim_{\epsilon \to 0^+} \tilde{\mu}(J - \epsilon)$. The Conley-Zehnder index can be finally defined as $\mathrm{CZ}^{\Phi}(\gamma, T) = \tilde{\mu}(J_M)$. We omit the period when it is taken to be the primitive period. If $c_1(\xi)$ vanishes on spheres and $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ is contractible then we write $\mathrm{CZ}^{\mathrm{disk}}$ for the index computed with a trivialization that extends to a capping disk.

The Conley-Zehnder index is an extremely important tool. It is related to Fredholm indices of solutions of many of the elliptic equations from Symplectic Topology, in particular to dimensions of moduli spaces of holomorphic curves.

Definition 2.4 (Hofer, Wysocki and Zehnder). *A contact form λ on a 3-manifold M is dynamically convex if $c_1(\ker \lambda)$ vanishes on spheres and contractible periodic Reeb orbits $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ satisfy $\mathrm{CZ}^{\mathrm{disk}}(\gamma, T) \geq 3$.*

The terminology is justified as follows. The standard contact structure ξ_0 on the unit sphere $S^3 \subset \mathbb{R}^4$ is defined as the kernel of $\lambda_0 = \frac{1}{2} \sum_{j=1}^{2} x_j\, dy_j - y_j\, dx_j$ restricted to S^3. More generally, λ_0 restricts to a contact form on any smooth, compact hypersurface S in $(\mathbb{R}^4, \omega_0)$ that is (strictly) star-shaped with respect to the origin. The associated Reeb flow reparametrizes the Hamiltonian flow on S for any Hamiltonian realizing S as a regular energy level. Moreover, it is smoothly conjugated to a Reeb flow on (S^3, ξ_0). Conversely, every Reeb flow on (S^3, ξ_0) is smoothly conjugated to the Reeb flow of λ_0 restricted to some S. When S is strictly convex we get dynamical convexity in view of

Theorem 2.5 (Hofer, Wysocki, and Zehnder [1998]). *The Hamiltonian flow on a smooth, compact and strictly convex energy level in $(\mathbb{R}^4, \omega_0)$ is smoothly conjugated to a dynamically convex Reeb flow on (S^3, ξ_0).*

A Reeb flow will be called dynamically convex when it is induced by a dynamically convex contact form. The next result and Theorem 2.5 together imply Theorem 2.3.

Theorem 2.6 (Hryniewicz [2012, 2014]). *Let γ be a periodic orbit of a dynamically convex Reeb flow on (S^3, ξ_0). Then γ bounds a disk-like global surface of section if, and only if, it is unknotted and has self-linking number -1. Moreover, such an orbit binds an open book decomposition whose pages are disk-like global surfaces of section.*

These statements are powered by a non-trivial input.

Theorem 2.7 (Hofer, Wysocki, and Zehnder [1996b]). *Any Reeb flow on (S^3, ξ_0) has an unknotted periodic orbit with self-linking number -1.*

Putting together theorems 2.6 and 2.7 we obtain a more general version of Theorem 2.2.

Theorem 2.8 (Hofer, Wysocki, and Zehnder [1998]). *Any dynamically convex Reeb flow on (S^3, ξ_0) admits a disk-like global surface of section.*

Global sections open the door for tools in two-dimensional dynamics. Here is a strong application in this direction taken from Hofer, Wysocki, and Zehnder [ibid.]. The return map of the disk obtained from Theorem 2.8 preserves an area form with finite total area. Brouwer's translation theorem provides a periodic orbit simply linked to the boundary of the disk. If the fixed point corresponding to this orbit is removed then we end up with a return map on the open annulus. Results of John Franks [1992] complete the proof of the following statement.

Corollary 2.9 (Hofer, Wysocki, and Zehnder [1998]). *Dynamically convex Reeb flows on (S^3, ξ_0) admit either two or infinitely many periodic orbits.*

To push Theorem 2.6 beyond dynamical convexity one needs to introduce linking assumptions with certain periodic orbits. This is aligned to Schwartzman-Fried-Sullivan theory where one makes linking assumptions with invariant measures.

Theorem 2.10 (Hryniewicz, Licata, and Salomão [2015] and Hryniewicz and Salomão [2011]). *A periodic orbit γ of a Reeb flow on (S^3, ξ_0) binds an open book decomposition whose pages are disk-like global surfaces of section if it matches the following conditions:*

(a) *γ is unknotted, has self-linking number -1 and satisfies $\mathrm{CZ}^{\mathrm{disk}}(\gamma) \geq 3$.*

(b) *γ is linked to all periodic orbits $\gamma' : \mathbb{R}/T\mathbb{Z} \to S^3 \setminus \gamma$ such that either $\mathrm{CZ}^{\mathrm{disk}}(\gamma', T) = 2$, or $\mathrm{CZ}^{\mathrm{disk}}(\gamma', T) = 1$ and γ' is degenerate in period T.*

Conversely, if γ is non-degenerate (in its primitive period) then these assumptions are necessary for γ to bound a disk-like global surface of section.

After all these results on the 3-sphere we would like to discuss more general Reeb flows. Can we recover and generalize Birkhoff's Theorem 2.1? To make a statement in this direction we need to recall a few concepts.

The notion of fibered link has a contact topological analogue. If λ is a contact form and L is a transverse link then the right notion of fibered is that L binds an open book decomposition satisfying

(i) $d\lambda$ is an area form on each page, and

(ii) the boundary orientation induced on L by the pages oriented by $d\lambda$ coincides with the orientation induced on L by λ.

Such an open book is said to support the contact structure $\xi = \ker\lambda$. We may call them *Giroux open books* because of their fundamental role in the classification of contact structures due to Giroux [2002]. An open book decomposition is said to be *planar* if pages have no genus. A contact structure orients the underlying 3-manifold by $\lambda \wedge d\lambda$, where λ is any defining contact form. A global surface of section will be called *positive* if the orientation induced on it by the flow and the ambient orientation turns out to orient its boundary along the flow.

Theorem 2.11. *Let (M,ξ) be a closed, connected contact 3-manifold. Let the link $L \subset M$ bind a planar Giroux open book decomposition Θ of M. Denote by $f \in H_2(M,L;\mathbb{Z})$ the class of a page of Θ, and by $\gamma_1,\dots,\gamma_n$ the components of L. Let the contact form λ define ξ and realize L as periodic Reeb orbits, and consider the following assertions:*

(i) *L bounds a positive genus zero global surface of section for the λ-Reeb flow representing the class f.*

(ii) *L binds a planar Giroux open book whose pages are global surfaces of section for the λ-Reeb flow and represent the class f.*

(iii) *The following hold:*

(a) $\mathrm{CZ}^{\Theta}(\gamma_k) > 0$ *for all k.*

(b) *Every periodic λ-Reeb orbit in $M \setminus L$ has non-zero intersection number with f.*

Then (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Moreover, (i) $\Rightarrow$ (iii) provided a certain C^∞-generic condition holds.

In (iii-a) $\mathrm{CZ}^{\Theta}(\gamma_k)$ is the Conley-Zehnder index of γ_k in its primitive period computed with a trivialization aligned to the normal of a page of Θ. The genericity needed for (i) $\Rightarrow$ (iii) is implied by non-degeneracy of the contact form. Theorem 2.11 is fruit of joint work with Kris Wysocki and will be proved in Hryniewicz, Salomão, and Wysocki [n.d.]. It heavily relies on Siefring's intersection theory Siefring [2011].

As a first test note that Birkhoff's Theorem 2.1 follows as a consequence. Indeed, the unit sphere bundle of S^2 has a contact form induced by pulling back the tautological 1-from on T^*S^2 via Legendre transform. Reeb flow is geodesic flow. A simple closed geodesic lifts to two closed Reeb orbits, which form a link that binds a supporting open book. Pages are annuli that are isotopic to the Birkhoff annulus. Positivity of the curvature and the Gauss-Bonnet theorem imply that (iii-a) and (iii-b) hold. Birkhoff's theorem follows.

Theorem 2.11 has applications to Celestial Mechanics. The following statement is the abstract result needed for these applications. The standard primitive λ_0 of ω_0 is symmetric by the antipodal map. Identifying antipodal points we obtain $\mathbb{R}P^3 = S^3/\{\pm 1\}$. The restriction of λ_0 to S^3 descends to a contact form on $\mathbb{R}P^3$ defining its standard contact structure, still denoted ξ_0. The Hopf link

$$\widetilde{l}_0 = \{(x_1, y_1, x_2, y_2) \in S^3 \mid x_1 = y_1 = 0 \text{ or } x_2 = y_2 = 0\}$$

is antipodal symmetric and descends to a transverse link l_0 on $\mathbb{R}P^3$. Any transverse link in $(\mathbb{R}P^3, \xi_0)$ transversely isotopic to l_0 will be called a Hopf link. Any transverse knot in $(\mathbb{R}P^3, \xi_0)$ transversely isotopic to a component of l_0 will be called a Hopf fiber.

Theorem 2.12 (Hryniewicz and Salomão [2016], Hryniewicz, Salomão, and Wysocki [n.d.])**.** *Consider an arbitrary dynamically convex Reeb flow on* $(\mathbb{R}P^3, \xi_0)$*. Any periodic orbit which is a Hopf fiber binds an open book decomposition whose pages are rational disk-like global surfaces of section. Any pair of periodic orbits forming a Hopf link binds an open book decomposition whose pages are annulus-like global surfaces of section.*

These techniques have applications to existence of elliptic periodic orbits. A periodic orbit is elliptic if all Floquet multipliers lie in the unit circle.

Theorem 2.13 (Hryniewicz and Salomão [2016])**.** *Any Reeb flow on* $(\mathbb{R}P^3, \xi_0)$ *which is sufficiently* C^∞*-close to a dynamically convex Reeb flow admits an elliptic periodic orbit. This orbit binds a rational open book decomposition whose pages are disk-like global surfaces of section. Its double cover has Conley-Zehnder index equal to* 3*.*

When combined with a result of Harris and Paternain [2008] relating pinched flag curvatures to dynamical convexity, Theorem 2.13 refines the main result from Rademacher [2007].

Corollary 2.14. *Consider a Finsler metric on the two-sphere with reversibility* r*. If all flag curvatures lie in* $(r^2/(r+1)^2, 1]$ *then there exists an elliptic closed geodesic. Moreover, its velocity vector defines a periodic orbit of the geodesic flow that bounds a rational disk-like global surface of section. A fixed point of the return map gives a second closed geodesic.*

We end this section with a topological application. We look for characterizations of contact 3-manifolds in terms of Reeb dynamics, motivated by early fundamental results of Hofer, Wysocki, and Zehnder [1995a, 1999a].

Identify $\mathbb{R}^4 \simeq \mathbb{C}^2$ by $(x_1, y_1, x_2, y_2) \simeq (z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$ and fix relatively prime integers $p \geq q \geq 1$. The action of $\mathbb{Z}/p\mathbb{Z}$ generated by the map $(z_1, z_2) \mapsto (e^{i2\pi/p} z_1, e^{i2\pi q/p} z_2)$ is free on S^3, and the lens space $L(p, q)$ is defined as its orbit space. The 1-form $\lambda_0 = \frac{1}{2}\sum_{j=1}^2 x_j dy_j - y_j dx_j$ is invariant and descends to a contact form on $L(p,q)$. The induced contact structure is called standard, we still denote it by ξ_0 with no fear of ambiguity.

A knot K on a closed 3-manifold M is p-unknotted if there is an immersion $u : \mathbb{D} \to M$ such that $u|_{\mathbb{D}\setminus\partial\mathbb{D}}$ defines a proper embedding $\mathbb{D} \setminus \partial\mathbb{D} \to M \setminus K$, and $u|_{\partial\mathbb{D}}$ defines a p-covering map $\partial\mathbb{D} \to K$. The map u is called a p-disk for K. The Hopf fiber $S^1 \times 0 \subset S^3$ is $\mathbb{Z}/p\mathbb{Z}$ invariant and descends to the simplest example of a p-unknotted knot in $L(p,q)$. The case $p = 2$ has the following geometric meaning: if we identify $L(2,1)$ with the unit tangent bundle of the round two-sphere then the velocity vector of a great circle is 2-unknotted.

In the presence of a contact structure a transverse p-unknotted knot has a rational self linking number. In the examples given above the knots are transverse and their rational self-linking numbers are equal to $-1/p$. These notions play a role in the following dynamical characterization of standard lens spaces.

Theorem 2.15 (Hofer, Wysocki, and Zehnder [1995a, 1999a] and Hryniewicz, Licata, and Salomão [2015])**.** *Let (M,ξ) be a closed connected contact 3-manifold, and let $p \geq 1$ be an integer. Then (M,ξ) is contactomorphic to some $(L(p,q),\xi_0)$ if, and only if, it carries a dynamically convex Reeb flow with a p-unknotted self-linking number $-1/p$ periodic orbit.*

This is a special case of more general statements where linking assumptions with certain periodic orbits are used. The existence of a p-unknotted self-linking number $-1/p$ periodic orbit implies that $(M,\xi) = (L(p,q),\xi_0)\#(M',\xi')$ for some contact 3-manifold (M',ξ'). Dynamical convexity forces $(M',\xi') = (S^3,\xi_0)$.

Using that $(L(2,1),\xi_0)$ is contactomorphic to the unit sphere bundle of any Finsler metric on S^2 we get a geometric application. Consider the set $\mathcal{I}$ of immersions $S^1 \to S^2$ with no positive self-tangencies. Two immersions are declared equivalent if they are homotopic through immersions in $\mathcal{I}$. This defines an equivalence relation $\sim$ and an element of $\mathcal{I}/\sim$ will be called a *weak flat knot type*. This notion is related to Arnold's J^+-theory of plane curves. Note that a closed geodesic on a Finsler two-sphere has a well-defined weak flat knot type. Let k_8 be the weak flat knot type of a curve with precisely one self-intersection which is transverse. Clearly there are curves representing k_8 with an arbitrarily large number of self-intersections.

Theorem 2.16 (Hryniewicz and Salomão [2013])**.** *If a Finsler two-sphere with reversibility r has flag curvatures in $(r^2/(r+1)^2, 1]$ then no closed geodesic represents k_8.*

This statement follows from Theorem 2.15. In fact, the pinching of the curvature forces dynamical convexity (Harris and Paternain [2008]), and the velocity vector of a closed geodesic of type k_8 is unknotted with self-linking number -1 in the unit sphere bundle. Since $\mathbb{R}P^3$ is not the 3-sphere we conclude that such a closed geodesic does not exist.

3 Global surfaces of section applied to systolic geometry

Our first goal in this section is to explain how Birkhoff's annulus-like global surfaces of section (Theorem 2.1) allow for the possibility that symplectic and Riemannian methods be combined to get sharp systolic inequalities on the two-sphere. Our second goal is to describe how disk-like global surfaces of section can be used to prove a special case of Viterbo's conjecture Viterbo [2000]. The results described here were obtained in collaboration with Alberto Abbondandolo and Barney Bramham Abbondandolo, Bramham, Hryniewicz, and Salomão [2017a, 2018, 2017b,c].

The 1-systole $\mathrm{sys}_1(X,g)$ of a closed non-simply connected Riemannian manifold (X,g) is defined as the length of the shortest non-contractible loop. Systolic geometry has its origins in the following results.

Theorem 3.1 (Löwner)**.** *The inequality* $(\mathrm{sys}_1)^2/\mathrm{Area} \leq 2/\sqrt{3}$ *holds for every Riemannian metric on the two-torus. Equality is achieved precisely for the flat torus defined by an hexagonal lattice.*

Theorem 3.2 (Pu)**.** *The inequality* $(\mathrm{sys}_1)^2/\mathrm{Area} \leq \pi/2$ *holds for every Riemannian metric on* $\mathbb{R}P^2$*. Equality is achieved precisely for the round geometry.*

Systolic geometry is a huge and active field, it developed quite a lot since the results of Löwner and Pu. We emphasize Gromov's celebrated paper Gromov [1983].

To include simply connected manifolds one considers the length $\ell_{\min}(X,g)$ of the shortest non-constant closed geodesic of a closed Riemannian manifold (X,g). The systolic ratio is defined by

$$\rho_{\mathrm{sys}}(X,g) = \frac{(\ell_{\min}(X,g))^n}{\mathrm{Vol}(X,g)} \qquad (n = \dim X) \tag{3}$$

The systolic ratio of two-spheres is far from being well understood. An important statement is due to Croke.

Theorem 3.3 (Croke [1988])**.** *The function* $g \mapsto \rho_{\mathrm{sys}}(S^2,g)$ *is bounded among all Riemannian metrics on* S^2*.*

In view of Pu's inequality it is tempting to hope that a round two-sphere (S^2,g_0) maximizes the systolic ratio. Its value is $\rho_{\mathrm{sys}}(S^2,g_0) = \pi$. However, the Calabi-Croke sphere shows that the supremum of $\rho_{\mathrm{sys}}(S^2,g)$ is at least $2\sqrt{3} > \pi$. This is a singular metric constructed by glueing two equilateral triangles along their sides to form a "flat" two-sphere. It can be approximated by smooth positively curved metrics with systolic ratio close to $2\sqrt{3}$.

Question 1. *What is the value of* $\sup_{(S^2,g)} \rho_{\text{sys}}(S^2, g)$*? Are there restrictions on the kinds of geometry that approximate this supremum?*

It has been conjectured that the answer to Question 1 is $2\sqrt{3}$. In Balacheff [2010] Balacheff shows that the Calabi-Croke sphere can be seen as some kind of local maximum if non-smooth metrics with a certain type of singular behavior are included.

A *Zoll metric* is one such that all geodesic rays are closed and have the same length. It is interesting that all Zoll metrics on S^2 have conjugated geodesic flows, and have systolic ratio equal to π.

It becomes a natural problem that of understanding the geometry of the function ρ_{sys} near (S^2, g_0). This problem was considered by Babenko and studied by Balacheff. In Balacheff [2006] Balacheff shows that (S^2, g_0) can be seen as a critical point of ρ_{sys} and conjectured that it is a local maximum. We will refer to this conjecture as the Babenko-Balacheff conjecture.

Contact geometry is a natural set-up to study systolic inequalities. This point of view was advertised and used by Álvarez Paiva and Balacheff [2014]. Let α be a contact form on a closed manifold M of dimension $2n-1$ oriented by $\alpha \wedge (d\alpha)^{n-1}$. We denote by $T_{\min}(M, \alpha)$ the minimal period among closed orbits of the Reeb flow. Existence of closed orbits is taken for granted. The contact volume of (M, α) is defined as

$$\text{Vol}(M, \alpha) := \int_M \alpha \wedge (d\alpha)^{n-1}$$

and the systolic ratio of (M, α) as

$$\rho_{\text{sys}}(M, \alpha) := \frac{T_{\min}(M, \alpha)^n}{\text{Vol}(M, \alpha)}$$

Note that $\rho_{\text{sys}}(M, \alpha)$ is invariant under re-scalings of α.

To see the connection to systolic geometry, consider a Riemannian n-manifold (X, g). The pull-back of the tautological form on T^*X by Legendre transform restricts to a contact form α_g on the unit sphere bundle T^1X. Since the Reeb flow of α_g is the geodesic flow of g, we get $T_{\min}(T^1X, \alpha_g) = \ell_{\min}(X, g)$. It turns out that $\text{Vol}(X, g)$ and $\text{Vol}(T^1X, \alpha_g)$ are proportional by a constant depending only on n. Hence $\rho_{\text{sys}}(T^1X, \alpha_g) = C_n\, \rho_{\text{sys}}(X, g)$ for every Riemannian metric g on X, where C_n depends only on n.

A Zoll contact form is one such that all Reeb trajectories are periodic and have the same period. These are usually called regular in the literature, but we prefer the term Zoll in view of the above connection to the Riemannian case.

A convex body in $\mathbb{R}^{2n}$ is a compact convex set with non-empty interior. In Viterbo [2000] Viterbo conjectured that

$$\frac{c(K)^n}{n!\mathrm{Vol}(K)} \le 1 \tag{4}$$

holds for every convex body $K \subset \mathbb{R}^{2n}$ and every symplectic capacity c, where $\mathrm{Vol}(K)$ denotes euclidean volume. We end by discussing a special case of the conjecture. Let K be a convex body in $(\mathbb{R}^{2n}, \omega_0)$ with smooth and strictly convex boundary, with the origin in its interior. Denote by $\iota : \partial K \to \mathbb{R}^{2n}$ the inclusion map, and by λ_0 the standard Liouville form $\lambda_0 = \frac{1}{2}\sum_{j=1}^n x_j dy_j - y_j dx_j$. Then $\iota^*\lambda_0$ is a contact form on ∂K. In Hofer and Zehnder [1994] it is claimed that the Hofer-Zehnder capacity of K is equal to $T_{\min}(\partial K, \iota^*\lambda_0)$. In this case (4) is restated as

$$\rho_{\mathrm{sys}}(\partial K, \iota^*\lambda_0) \le 1 \tag{5}$$

which is supposed to be an equality if, and only if, $\iota^*\lambda_0$ is Zoll.

Having described our problems, we move on to state some results. Recall that for $\delta \in (0, 1]$, a positively curved closed Riemannian manifold is said to be δ-pinched if the minimal and maximal values $K_{\min}, K_{\max}$ of the sectional curvatures satisfy $K_{\min}/K_{\max} \ge \delta$. On a positively curved two-sphere we write $\ell_{\max}$ for the length of the longest closed geodesic without self-intersections. Note that $\ell_{\max}$ is finite.

Theorem 3.4 (Abbondandolo, Bramham, Hryniewicz, and Salomão [2017a])**.** *If (S^2, g) is δ-pinched for some $\delta > (4+\sqrt{7})/8 = 0.8307...$ then $\ell_{\min}(S^2, g)^2 \le \pi\mathrm{Area}(S^2, g) \le \ell_{\max}(S^2, g)^2$. Moreover, any of these inequalities is an equality if, and only if, the metric is Zoll.*

This first inequality confirms the Babenko-Balacheff conjecture on an explicit and somewhat large C^2-neighborhood of the round geometry. It seems that the upper bound involving $\ell_{\max}$ was not known before.

We discuss some related problems before explaining the role of global surfaces of section in the proof of Theorem 3.4. The pinching constant δ seems to be a helpful parameter. For instance, one could consider the non-increasing bounded (Theorem 3.3) function

$$\rho : (0, 1] \to \mathbb{R} \qquad \rho(\delta) = \sup\{\rho_{\mathrm{sys}}(S^2, g) \mid (S^2, g) \text{ is } \delta\text{-pinched}\}$$

to study the positively curved case.

Question 2. *Is it true that $\rho(1/4) = \pi$? What does the graph of $\rho(\delta)$ look like?*

The Calabi-Croke sphere shows that $\lim_{\delta\to 0^+} \rho(\delta) \ge 2\sqrt{3}$. Theorem 3.4 implies that $\rho(\delta) = \pi$ for all $\delta > (4+\sqrt{7})/8$. One must try to understand among which metrics does the round metric maximize systolic ratio. Assuming positive curvature it might be reasonable to expect that $\inf\{\delta \mid \rho(\delta) = \pi\} \leq 1/4$.

If curvature assumptions are dropped then the situation might be much harder. What about symmetry assumptions? Here is a result in this direction that answers a question by Álvarez-Paiva and Balacheff.

Theorem 3.5. *Inequality $\rho_{\text{sys}} \leq \pi$ holds for every sphere of revolution, with equality precise when the metric is Zoll.*

Global surfaces of section show up in the proofs of theorems 3.4 and 3.5 to connect systolic inequalities to a quantitative fixed point theorem for symplectic maps of the annulus. We outline the proof to make this point precise.

Let (S^2, g) be δ-pinched. If $\delta > 1/4$ then $\ell_{\min}$ is only realized by simple closed geodesics. Let γ be a closed geodesic of length $\ell_{\min}$. By Theorem 2.1 the Birkhoff annulus A_γ is a global surface of section. Let λ be the 1-form on A_γ given by restricting the contact form α_g. Then $d\lambda$ is an area form on the interior of A_γ, and vanishes on ∂A_γ. The total $d\lambda$-area of A_γ is $2\ell_{\min}$.

The first return map ψ and the first return time τ are defined on the interior of A_γ, but it turns out that they extend smoothly to A_γ. Moreover, ψ preserves boundary components. Santaló's formula reads

$$2\pi \operatorname{Area}(S^2, g) = \int_{T^1 S^2} \alpha_g \wedge d\alpha_g = \int_{A_\gamma} \tau \, d\lambda \tag{6}$$

Since ψ preserves the 2-form $d\lambda$, it follows that $\psi^*\lambda - \lambda$ is closed.

We now need to consider lifts of ψ to the universal covering of A_γ. If ψ admits a lift in the kernel of the FLUX homomorphism then $\psi^*\lambda - \lambda$ is exact. The unique primitive σ of $\psi^*\lambda - \lambda$ satisfying

$$\sigma(p) = \int_p^{\psi(p)} \lambda \quad \forall p \in \partial A_\gamma$$

is called the *action of ψ*. Here the integral is taken along the boundary according to the lift with zero FLUX. The Calabi invariant is defined as

$$\operatorname{CAL}(\psi) = \frac{1}{\int_{A_\gamma} d\lambda} \int_{A_\gamma} \sigma \, d\lambda = \frac{1}{2\ell_{\min}} \int_{A_\gamma} \sigma \, d\lambda$$

Of course, we need to worry about whether ψ admits a lift of zero FLUX, but this follows from reversibility of the geodesic flow.

It is a very general fact that τ is also a primitive of $\psi^*\lambda - \lambda$. Toponogov's theorem proves that if $\delta > 1/4$ then

$$\tau = \sigma + \ell_{\min} \tag{7}$$

Combining (7) with (6) we finally get

$$2\pi \,\mathrm{Area}(S^2, g) = 2(\ell_{\min})^2 + 2\ell_{\min}\mathrm{CAL}(\psi) \tag{8}$$

Equations (7) and (8) should be seen as some kind of dictionary between geometry and dynamics: action corresponds to length, Calabi invariant corresponds to area.

We are now in position to make the link to the quantitative fixed point theorem and conclude the argument. Roughly speaking, the theorem states:

If ψ admits a generating function (of a specific kind), $\mathrm{CAL}(\psi) \le 0$ *and $\psi \neq id$, then there exists a fixed point p_0 satisfying $\sigma(p_0) < 0$.*

Arguing indirectly, suppose that either $\pi\,\mathrm{Area} < (\ell_{\min})^2$, or $\pi\,\mathrm{Area} = (\ell_{\min})^2$ and g is not Zoll. It follows from (8) and a little more work that either $\mathrm{CAL}(\psi) < 0$, or $\mathrm{CAL}(\psi) = 0$ and ψ is not the identity. Toponogov's theorem comes into play again to show that ψ admits the required generating function provided $\delta > (4 + \sqrt{7})/8$. The fixed point theorem applies to give a fixed point of negative action. By (7) this fixed point corresponds to a closed geodesic of length strictly smaller than $\ell_{\min}$. This contradiction finishes the proof.

The above argument reveals how global surfaces of section can serve as bridge between systolic geometry and symplectic dynamics. The same strategy proves a special case of Viterbo's conjecture in dimension 4.

Theorem 3.6 (Abbondandolo, Bramham, Hryniewicz, and Salomão [2018]). *There exists a C^3-neighborhood $\mathcal{U}$ of the space of Zoll contact forms on S^3 such that $\alpha \in \mathcal{U} \Rightarrow \rho_{\mathrm{sys}}(S^3, \alpha) \le 1$ with equality if, only if, α is Zoll.*

The proof again strongly relies on global surfaces of sections. Namely, if a contact form is C^3-close to the standard contact form λ_0 then its Reeb flow admits a disk-like global surface of section whose first return map extends up to the boundary and is C^1-close to the identity. We have a dictionary between maps and flows just as in the proof of Theorem 3.4: contact volume corresponds to Calabi invariant, return time corresponds to action. The quantitative fixed point theorem applies to give the desired conclusion.

One could see the constants in sharp systolic inequalities for Riemannian surfaces as invariants. Similarly, one could hope to construct contact invariants from sharp systolic inequalities for contact forms. The following statement shows that this is not possible in dimension three: systolic inequalities are not purely contact topological phenomena. For example, inequalities such as (5) must depend on the convexity assumption.

Theorem 3.7 (Abbondandolo, Bramham, Hryniewicz, and Salomão [2018, 2017b]). *For every co-orientable contact 3-manifold (M, ξ) and every $c > 0$ there exists a contact form α on M satisfying $\xi = \ker\alpha$ and $\rho_{\mathrm{sys}}(M, \alpha) > c$.*

Hofer, Wysocki, and Zehnder [1999a, 1998] introduced the notion of dynamically convex contact forms, see Section 2 for a detailed discussion. It plays a crucial role in the construction of global surfaces of section (theorems 2.6, 2.8). Dynamical convexity is automatically satisfied on the boundary of a smooth convex body with strictly convex boundary. It becomes relevant to decide whether (5) holds for dynamically convex contact forms on S^3.

Theorem 3.8 (Abbondandolo, Bramham, Hryniewicz, and Salomão [2017c]). *Given any $\epsilon > 0$ there is a dynamically convex contact form α on S^3 such that $\rho_{\mathrm{sys}}(S^3, \alpha) > 2 - \epsilon$.*

A narrow connection between high systolic ratios and negativity of Conley-Zehnder indices is quantified in Abbondandolo, Bramham, Hryniewicz, and Salomão [ibid.].

Observe that Theorem 3.8 implies that either Viterbo's conjecture is not true, or there exists a dynamically convex contact form on S^3 whose Reeb flow is not conjugated to the Reeb flow on a strictly convex hypersurface of $(\mathbb{R}^4, \omega_0)$. Unfortunately we can not decide which alternative holds. It also proves that there are smooth compact star-shaped domains U in $(\mathbb{R}^4, \omega_0)$ with the following property: the value $c(U)$ of any capacity realized as the action of some closed characteristic on ∂U is strictly larger than the Gromov width of U.

Global surfaces of section continue to play essential role in the proofs of Theorem 3.7 and Theorem 3.8. Both start by constructing global sections for certain Reeb flows with well-controlled return maps. Then the Reeb flows are modified by carefully changing the return maps in order to make the systolic ratio increase.

4 The planar circular restricted three-body problem

The three-body problem is that of understanding the motion of three massive particles which attract each other according to Newton's law of gravitation. Some simplifying assumptions turn this problem into a two-degree-of-freedom Hamiltonian system:

- The three particles move in a fixed plane.
- The mass of the third body (satellite) is neglected and so the first two particles (primaries) move according to the two-body problem.
- The primaries move on circular trajectories about their center of mass.

In inertial coordinates where the center of mass of the primaries rests at the origin one gets $z_1 = r_1 e^{i\omega t}$ and $z_2 = -r_2 e^{i\omega t}$ for some ω, where $r_1, r_2 > 0$ satisfy $m_1 r_1 - m_2 r_2 = 0$ and $(r_1 + r_2)^3 \omega^2 = m_1 + m_2$. It is harmless to assume that $\omega = r_1 + r_2 = m_1 + m_2 = 1$ which makes the mass ratio $\mu := m_1 = r_2 \in (0, 1)$ the unique parameter of the system.

In rotating (non-inertial) coordinates the position $q(t) \in \mathbb{C}$ of the satellite relative to the second primary is given by $z_3(t) = (q(t) - \mu)e^{it}$, from where it follows that

$$\ddot{q} + 2i\dot{q} - (q - \mu) = -\mu \frac{q-1}{|q-1|^3} - (1-\mu)\frac{q}{|q|^3}. \tag{9}$$

As is well known, if we set $p = \dot{q} + i(q - \mu)$ and consider

$$H_\mu(q,p) = \frac{1}{2}|p|^2 + \langle q - \mu, ip\rangle - \frac{\mu}{|q-1|} - \frac{1-\mu}{|q|}, \tag{10}$$

then (9) becomes Hamilton's equations

$$\dot{q} = \nabla_p H_\mu, \qquad \dot{p} = -\nabla_q H_\mu. \tag{11}$$

The function H_μ has five critical points. A sublevel set below its lowest critical value defines three *Hill regions* in the configuration plane, two of which are bounded while the third is a neighborhood of ∞. Each bounded Hill region is topologically a punctured disk and contains a primary, namely, one of them is a punctured neighborhood of the origin and the other is a punctured neighborhood of 1. The boundaries of the Hill regions are called *ovals of zero velocity*, since there we have $(q - \mu) = -ip \Leftrightarrow \dot{q} = 0$. From now on we restrict to *subcritical cases*, i.e. energy levels $H_\mu = -c$ where $-c$ is below the lowest critical value of H_μ. We focus on the bounded Hill region near the origin.

Following Poincaré, mathematicians first tried to understand the limiting behavior as $\mu \to 0^+$ or as $\mu \to 1^-$. The limit as $\mu \to 0^+$ is in some ways better behaved than the limit $\mu \to 1^-$, but sometimes it is just the other way around. In the limit $\mu = 0$ the system describes the so-called rotating Kepler problem, where all mass is concentrated at the origin. The boundary of the bounded Hill region about the origin converges to a circle of definite radius. As $\mu \to 1^-$ the bounded Hill region about the origin collapses, and we face a somewhat more singular situation.

Definition 4.1. *A retrograde orbit is a periodic orbit $t \mapsto (q(t), p(t))$ such that $q(t)$ is in the Hill region about the origin, and describes a curve without self-intersections with winding number -1 around the origin. Analogously, a direct orbit is a periodic orbit $t \mapsto (q(t), p(t))$ such that $q(t)$ is in the Hill region about the origin, and describes a curve without self-intersections with winding number $+1$ around the origin.*

The difficulty in finding direct orbits led Birkhoff to consider the following strategy in Birkhoff [1914, section 19]. Firstly one should try to find a disk-like global surface of section bounded by a (doubly covered) retrograde orbit. For this to make sense collision orbits need to be regularized. Secondly, due to preservation of an area form with finite total area, one can apply Brouwer's translation theorem to the first return map and find a fixed point that should correspond to a direct orbit. Two main difficulties are: (1) for an arbitrary mass ratio it is hard to find global surfaces of section, and (2) it might be hard to check that the fixed point corresponds to a direct orbit. The following is extracted from Birkhoff [ibid., section 19]:

"This state of affairs seems to me to make it probable that the restricted problem of three bodies admit of reduction to the transformation of a discoid into itself as long as there is a closed oval of zero velocity about J, and that in consequence there always exists at least one direct periodic orbit of simple type."

More recently this has been called a conjecture, which perhaps should be read as following: For any value of μ and any subcritical energy value, there must be a way of finding a disk-like global surface of section in order to understand the movement of the satellite inside the Hill region about the origin. To implement the strategy of Birkhoff this disk should be spanned by the retrograde orbit, in particular fixed points could be good candidates for direct orbits. Again, all this only makes sense if collision orbits are regularized.

Note that the smallest critical value of H_μ converges to $-\frac{3}{2}$ both when $\mu \to 0^+$ or $\mu \to 1^-$. Here is a good point to state and discuss our result concerning Birkhoff's conjecture.

Theorem 4.2. *For every $c > \frac{3}{2}$ there exists $\epsilon > 0$ such that the following holds.*

(a) If $1 - \mu < \epsilon$ and collisions are regularized via Levi-Civita regularization, then the double cover of every retrograde orbit inside the Hill region about the origin bounds a disk-like global surface of section. Moreover, if we quotient by antipodal symmetry then this disk descends to a rational disk-like global surface of section.

(b) If $\mu < \epsilon$ and collisions are regularized via Moser regularization, then every retrograde orbit inside the Hill region about the origin bounds a rational disk-like global surface of section.

Results of Albers, Fish, Frauenfelder, Hofer and van Koert from Albers, Fish, Frauenfelder, Hofer, and van Koert [2012] imply that if $1 - \mu$ is small enough then Levi-Civita regularization lifts the Hamiltonian flow on the corresponding component of $H_\mu^{-1}(-c)$ to the characteristic flow on a strictly convex hypersurface $\widetilde{\Sigma}_{\mu,c}$, up to time reparametrization. Moreover, $\widetilde{\Sigma}_{\mu,c}$ is antipodal symmetric and each state is represented twice as a pair of antipodal points. Results from Hofer, Wysocki, and Zehnder [1998] apply and give disk-like global surfaces of section in $\widetilde{\Sigma}_{\mu,c}$. Statement (a) above says that there is such a global section in $\widetilde{\Sigma}_{\mu,c}$ spanned by the lift of every doubly covered retrograde orbit, and that it descends to a global section in the quotient $\Sigma_{\mu,c} = \widetilde{\Sigma}_{\mu,c}/\{\pm 1\}$. If $\mu = 0$ then Moser regularization applies to the rotating Kepler problem to compactify the Hamiltonian flow on $H_\mu^{-1}(-c)$ to the characteristic flow on a fiberwise starshaped hypersurface $\Sigma_{\mu,c}$ inside TS^2, up to time reparametrization. A proof of this statement can be found in the paper Albers, Frauenfelder, van Koert, and Paternain [2012] where the contact-type property of energy levels of the PCR3BP is studied. Again we have $\Sigma_{\mu,c} \simeq \mathbb{R}P^3$. Statement (b) above says that every retrograde orbit bounds a rational disk-like global surface of section in $\Sigma_{\mu,c}$. A proof in this case would rely on the dynamical convexity obtained in Albers, Fish, Frauenfelder, and van Koert [2013] for $\mu = 0$. Hence, for μ close to 0 or 1 we can always apply Theorem 2.13 and obtain a pair of periodic orbits which are 2-unknotted and have self-linking number $-1/2$. These orbits are transversely isotopic to (a quotient of) a Hopf link. Theorem 2.12 can also be applied and an annulus-like global surface of section is obtained.

We end with a sketch of proof of (a) in Theorem 4.2. Fix $c > 3/2$. The component $\dot{\Sigma}_{\mu,c} \subset H_\mu^{-1}(-c)$ which projects to the Hill region surrounding $0 \in \mathbb{C}$ contains collision orbits. These orbits are regularized with the aid of Levi-Civita coordinates $(v,u) \in \mathbb{C}\times\mathbb{C}$ given by $q = 2v^2$ and $p = -\frac{u}{\bar{v}}$, which are symplectic up to a constant factor. The regularized Hamiltonian is

$$\begin{aligned} K_{\mu,c}(v,u) &:= |v|^2(H_\mu(p,q)+c) \\ &= \frac{1}{2}|u|^2 + 2|v|^2\langle u, iv\rangle - \mu\Im(uv) - \frac{1-\mu}{2} - \mu\frac{|v|^2}{|2v^2-1|} + c|v|^2, \end{aligned} \tag{12}$$

and there is a two-to-one correspondence between a centrally symmetric sphere-like component $\widetilde{\Sigma}_{\mu,c} \subset K_{\mu,c}^{-1}(0)$ and $\dot{\Sigma}_{\mu,c}$, up to collisions.

Now we consider the re-scaled coordinates $v = \hat{v}\sqrt{1-\mu}$ and $u = \hat{u}\sqrt{1-\mu}$, with Hamiltonian

$$\begin{aligned} \hat{K}_{\mu,c}(\hat{v},\hat{u}) &:= \frac{1}{1-\mu}K_{\mu,c}(v,u) \\ &= \frac{1}{2}|\hat{u}|^2 + 2(1-\mu)|\hat{v}|^2\langle \hat{u}, i\hat{v}\rangle - \mu\Im(\hat{u}\hat{v}) - \frac{1}{2} - \mu\frac{|\hat{v}|^2}{|2(1-\mu)\hat{v}^2-1|} + c|\hat{v}|^2. \end{aligned} \tag{13}$$

The component $\widetilde{\Sigma}_{\mu,c} \subset K_{\mu,c}^{-1}(0)$ gets re-scaled and we denote it by $\hat{\Sigma}_{\mu,c} \subset \hat{K}_{\mu,c}^{-1}(0)$.

Taking $\mu \to 1^-$ we see from (13) that $\hat{\Sigma}_{\mu,c}$ converges in the C^∞ topology to a hypersurface satisfying

$$\frac{1}{2}|\hat{u}|^2 - \Im(\hat{u}\hat{v}) + (c-1)|\hat{v}|^2 = \frac{1}{2}. \tag{14}$$

In order to have a better picture of the hypersurface in (14), we denote, for simplicity, $\hat{v} = \hat{v}_1 + i\,\hat{v}_2$ and $\hat{u} = \hat{u}_1 + i\,\hat{u}_2$. Then (14) is equivalent to

$$(\hat{u}_1 - \hat{v}_2)^2 + (\hat{u}_2 - \hat{v}_1)^2 + 2\left(c - \frac{3}{2}\right)(\hat{v}_1^2 + \hat{v}_2^2) = 1. \tag{15}$$

Taking new coordinates $(w = w_1 + i w_2, z = z_1 + i z_2) \in \mathbb{C} \times \mathbb{C}$ with $w_1 = \hat{u}_1 - \hat{v}_2$, $w_2 = \hat{u}_2 - \hat{v}_1$ and $z = \hat{v}\sqrt{2c-3}$, which are symplectic up to a constant factor, we see that (15) is equivalent to $w_1^2 + w_2^2 + z_1^2 + z_2^2 = 1$.

We conclude that the regularized Hamiltonian flow on $\hat{\Sigma}_{\mu,c}$ converges smoothly to the standard Reeb flow on (S^3, ξ_0) as $\mu \to 1^-$ up to reparametrizations. Its orbits are the Hopf fibers. Since the projection of the retrograde orbit winds once around $0 \in \mathbb{C}$ in q-coordinates, it is doubly covered by a simple closed orbit $P_{\mu,c} \subset \widetilde{\Sigma}_{\mu,c}$, which in z-coordinates winds once around $0 \in \mathbb{C}$. Hence, $P_{\mu,c}$ converges smoothly to a Hopf fiber in (w,z) and, in particular, it is unknotted and has self-linking number -1. The dynamical convexity of the Hamiltonian flow on $\hat{\Sigma}_{\mu,c}$ and Theorem 2.6 imply that it is the boundary of a disk-like global surface of section. In view of Theorem 2.12, we may assume that this global section descends to a rational disk-like global section on $\Sigma_{\mu,c} = \widetilde{\Sigma}_{\mu,c}/\{\pm 1\}$.

5 Transverse foliations

We discuss the idea of transverse foliations adapted to a 3-dimensional flow based on the concepts introduced by Hofer, Wysocki and Zehnder in Hofer, Wysocki, and Zehnder [2003]. This generalizes the notion of open books and global sections.

Definition 5.1. *Let ϕ^t be a smooth flow on an oriented closed 3-manifold M. A transverse foliation for ϕ^t is formed by:*

(i) A finite set $\mathcal{P}$ of primitive periodic orbits of ϕ^t, called binding orbits.

(ii) A smooth foliation of $M \setminus \cup_{P \in \mathcal{P}} P$ by properly embedded surfaces. Every leaf $\dot{\Sigma}$ is transverse to ϕ^t, has an orientation induced by ϕ^t and M, and there exists a compact embedded surface $\Sigma \hookrightarrow M$ so that $\dot{\Sigma} = \Sigma \setminus \partial\Sigma$ and $\partial\Sigma$ is a union of components of $\cup_{P \in \mathcal{P}} P$. An end z of $\dot{\Sigma}$ is called a puncture. To each puncture z there is an associated component $P_z \in \mathcal{P}$ of $\partial\Sigma$ called the asymptotic limit of $\dot{\Sigma}$ at z. A puncture z of $\dot{\Sigma}$ is called positive if the orientation on P_z induced by $\dot{\Sigma}$ coincides with the orientation induced by ϕ^t. Otherwise z is called negative.

The following theorem is a seminal result on the existence of transverse foliations for Reeb flows on the tight 3-sphere. It is based on pseudo-holomorphic curve theory in symplectic cobordisms.

Theorem 5.2 (Hofer, Wysocki, and Zehnder [2003]). *Let ϕ^t be a nondegenerate Reeb flow on (S^3, ξ_0). Then ϕ^t admits a transverse foliation. The binding orbits have self-linking number -1 and their Conley–Zehnder indices are 1, 2 or 3. Every leaf $\dot{\Sigma}$ is a punctured sphere and has precisely one positive puncture. One of the following conditions holds:*

- *The asymptotic limit of $\dot{\Sigma}$ at its positive puncture has Conley-Zehnder index 3 and the asymptotic limit of $\dot{\Sigma}$ at any negative puncture has Conley-Zehnder index 1 or 2. There exists at most one negative puncture whose asymptotic limit has Conley-Zehnder index 2.*
- *The asymptotic limit of $\dot{\Sigma}$ at its positive puncture has Conley-Zehnder index 2 and the asymptotic limit of $\dot{\Sigma}$ at any negative puncture has Conley-Zehnder index 1.*

The open books with disk-like pages constructed in Hofer, Wysocki, and Zehnder [1995a, 1999a, 1998], Hryniewicz [2012, 2014], Hryniewicz, Licata, and Salomão [2015], and Hryniewicz and Salomão [2011] for Reeb flows on (S^3, ξ_0) are particular cases of transverse foliations with a single binding orbit. The main obstruction for the existence of such an open book with a prescribed binding orbit P is the presence of closed orbits with Conley-Zehnder index 2 which are unlinked to P. One particular transverse foliation of interest which deals with such situations is the so called 3-2-3 foliation.

Definition 5.3. *A 3-2-3 foliation for a Reeb flow ϕ^t on (S^3, ξ_0) is a transverse foliation for ϕ^t with precisely three binding orbits P_3, P_2 and P_3'. They are unknotted, mutually unlinked and their respective Conley-Zehnder indices are 3, 2 and 3. The leaves are punctured spheres and consist of*

- *A pair of planes U_1 and U_2, both asymptotic to P_2 at their positive punctures.*
- *A cylinder V asymptotic to P_3 at its positive puncture and to P_2 at its negative puncture; a cylinder V' asymptotic to P_3' at its positive puncture and to P_2 at its negative puncture.*
- *A one parameter family of planes asymptotic to P_3 at their positive punctures; a one parameter family of planes asymptotic to P_3' at their positive punctures.*

The 3-2-3 foliations are the natural objects to consider if one studies Hamiltonian dynamics near certain critical energy levels.

Take a Hamiltonian H on $\mathbb{R}^4$ which has a critical point $p \in H^{-1}(0)$ with Morse index 1 and of saddle-center type. Its center manifold is foliated by the so called Lyapunoff orbits $P_{2,E} \subset H^{-1}(E)$, $E > 0$ small. Each one of them is unknotted, hyperbolic inside its energy level and has Conley-Zehnder index 2.

Assume that for every $E < 0$ the energy level $H^{-1}(E)$ contains two sphere-like components S_E and S_E' which develop a common singularity at p as $E \to 0^-$. This means that S_E converges in the Hausdorff topology to $S_0 \subset H^{-1}(0)$ as $E \to 0^-$, where S_0 is homeomorphic to the 3-sphere and contains p as its unique singularity. The analog holds for S'_E. Therefore, $S_0 \cap S'_0 = \{p\}$ and, for $E > 0$ small, $H^{-1}(E)$ contains a sphere-like component W_E close to $S_0 \cup S'_0$. We observe that W_E contains the Lyapunoff orbit $P_{2,E}$ and is in correspondence with the connected sum of S_E and S'_E.

Definition 5.4. *We say that S_0 is strictly convex if S_0 bounds a convex domain in $\mathbb{R}^4$ and all the sectional curvatures of $S_0 \setminus \{p\}$ are positive. We say that S'_0 is strictly convex if analogous conditions hold.*

The following theorem is inspired by results in Hofer, Wysocki, and Zehnder [2003].

Theorem 5.5 (de Paulo and Salomão [2018, n.d.]). *If H is real analytic and both S_0 and S'_0 are strictly convex then, for every $E > 0$ small, the Hamiltonian flow on the sphere-like component $W_E \subset H^{-1}(E)$ admits a 3-2-3 foliation. The Lyapunoff orbit $P_{2,E}$ is one of the binding orbits and there exist infinitely many periodic orbits and infinitely many homoclinics to $P_{2,E}$ in W_E.*

One difficulty in proving Theorem 5.5 is that there are no non-degeneracy assumptions of any kind. A criterium for checking strict convexity of the subsets S_0 and S'_0 is found in Salomão [2003].

The notion of 3-2-3 foliation is naturally extended to Reeb flows on connected sums $\mathbb{R}P^3 \# \mathbb{R}P^3$. In this case the binding orbits P_3 and P'_3 are non-contractible and the families of planes are asymptotic to their respective double covers. The existence of 3-2-3 foliations for Reeb flows on $\mathbb{R}P^3 \# \mathbb{R}P^3$ is still an object of study and it is conjectured that they exist for some Hamiltonians in celestial mechanics such as the Euler's problem of two centers in the plane and the planar circular restricted three body problem for energies slightly above the first Lagrange value.

A more general theory of transverse foliations for Reeb flows still needs to be developed. If one wishes to use holomorphic curves then one step is implemented by Fish and Siefring [2013], who showed persistence under connected sums. Transverse foliations on mapping tori of disk-maps were constructed by Bramham [2015a,b] to study questions about rigidity of pseudo-rotations.

6 A Poincaré-Birkhoff theorem for tight Reeb flows on S^3

Poincaré's last geometric theorem is nowadays known as the Poincaré-Birkhoff theorem. In its simplest form it is a statement about fixed points of area-preserving annulus homeomorphisms $f : \mathbb{R}/\mathbb{Z} \times [0,1] \to \mathbb{R}/\mathbb{Z} \times [0,1]$ preserving orientation and boundary components. The map f can be lifted to the universal covering $\mathbb{R} \times [0,1]$. Let us denote projection onto the first coordinate by $p : \mathbb{R} \times [0,1] \to \mathbb{R}$. Then f is said to satisfy a *twist condition* on the boundary if it admits a lift to the universal covering $F : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ such that the rotation numbers

$$\lim_{n\to\infty} \frac{p \circ F^n(x,0)}{n} \qquad \lim_{n\to\infty} \frac{p \circ F^n(x,1)}{n}$$

differ. We call the open interval I bounded by these numbers the twist interval.

Theorem 6.1 (Poincaré-Birkhoff Birkhoff [1913] and Poincaré [1912]). *If $I \cap \mathbb{Z} \neq \varnothing$ then f has at least two fixed points.*

Poincaré [1912] found annulus-like global surfaces of section for the PCR3BP for energies below the lowest critical value of the Hamiltonian, and when the mass is almost all concentrated in the primary around which the satellite moves. The boundary orbits form a Hopf link in the three-sphere. For generic values of the parameters, the Poincaré-Birkhoff theorem applies to the associated return map and proves the existence of infinitely many periodic orbits.

One also finds such pair of orbits for the Hamiltonian flow on a smooth, compact and strictly convex energy level inside $(\mathbb{R}^4, \omega_0)$. In fact, the fundamental result of Hofer, Wysocki, and Zehnder [1998] provides an unknotted periodic orbit P_0 that bounds a disk-like global surface of section. Brouwer's translation theorem yields a second periodic orbit P_1 simply linked to P_0, but much more can be said. The orbit P_0 is the binding of an open book decomposition whose pages are disk-like global surfaces of section. It turns out that the following statement follows: the flow is smoothly conjugated to a Reeb flow on (S^3, ξ_0) in such a way that $P_0 \cup P_1$ corresponds to a link transversely isotopic to the standard Hopf link

$$\widetilde{l}_0 = \{(x_1, y_1, x_2, y_2) \in S^3 \mid x_1 = y_1 = 0 \text{ or } x_2 = y_2 = 0\}$$

If the fixed point corresponding to P_1 is removed from the disk-like global section spanned by P_0, then we obtain a diffeomorphism of the open annulus that preserves a standard area-form and can be continuously extended to the boundary. It is interesting to study the twist condition for this map. We need to consider the transverse rotation numbers θ_0 and θ_1 of P_0 and P_1 with respect to Seifert surfaces (disks). In terms of Conley-Zehnder indices, these can be read as follows:

$$1 + \theta_0 = \lim_{n\to\infty} \frac{\mathrm{CZ}(P_0^n)}{2n} \qquad 1 + \theta_1 = \lim_{n\to\infty} \frac{\mathrm{CZ}(P_1^n)}{2n} \tag{16}$$

Here $\mathrm{CZ}(P_i^n)$ denotes the Conley-Zehnder index of the n-iterated orbit P_i computed with respect to a global trivialization of ξ_0. The open book singles out a lift of the map to the strip such that the rotation numbers on the boundary are precisely $1/\theta_0$ and θ_1. The Poincaré-Birkhoff theorem proves the following non-trivial statement: *If $\theta_1 \neq 1/\theta_0$ then there are infinitely many periodic orbits in the complement of $P_0 \cup P_1$. These orbits are distinguished by their homotopy classes in the complement of $P_0 \cup P_1$.*

One motivation for the main result of this section is to study the possibility of extending the above discussion to situations where neither P_0 nor P_1 bound global sections. Before the statement we need some notation. The term *Hopf link* will be referred to any transverse link in (S^3, ξ_0) that is transversely isotopic to the standard Hopf link $\widetilde{l}_0$. Given non-zero vectors $u, v \in \mathbb{R}^2$ in the complement of the third quadrant, we write $u > v$ (or $v < u$) if the argument of u is larger than that of v in the counter-clockwise sense.

Theorem 6.2 (Hryniewicz, Momin, and Salomão [2015]). *Consider a Reeb flow on (S^3, ξ_0) that admits a pair of periodic orbits P_0, P_1 forming a Hopf link. Denote by θ_0, θ_1 their transverse rotation numbers computed with respect to Seifert surfaces. If (p, q) is a pair of relatively prime integers satisfying*

$$(\theta_0, 1) < (p, q) < (1, \theta_1) \qquad \textit{or} \qquad (1, \theta_1) < (p, q) < (\theta_0, 1)$$

then there is a periodic orbit $P \subset S^3 \setminus (P_0 \cup P_1)$ such that $p = \operatorname{link}(P, P_0)$ and $q = \operatorname{link}(P, P_1)$.

The main tools in the proof are the contact homology theory introduced by Momin [2011] and the intersection theory of punctured holomorphic curves in dimension four developed by Siefring [2011].

Another source of motivation for Theorem 6.2 is a result due to Angenent [2005] which we now recall. It concerns geodesic flows on Riemannian two-spheres. Let g be a Riemannian metric on S^2, and let $\gamma : \mathbb{R} \to S^2$ be a closed geodesic of length L parametrized with unit speed. In particular $\gamma(t)$ is L-periodic. Jacobi fields along γ are characterized by the second order ODE $y''(t) = -K(\gamma(t))y(t)$ where K denotes the Gaussian curvature. Given a (non-trivial) solution $y(t)$ we can write $y'(t) + iy(t) = r(t)e^{i\theta(t)}$ in polar coordinates. The Poincaré inverse rotation number of γ is defined as

$$\rho(\gamma) = \frac{L}{2\pi} \lim_{t\to+\infty} \frac{\theta(t)}{t} \tag{17}$$

The special case of the results from Angenent [ibid.] that we would like to emphasize concerns the case when γ is simple, that is, $\gamma|_{[0,L)}$ is injective. Denote by $n(t)$ a normal vector along $\gamma(t)$. Given relatively prime integers $p, q \neq 0$ and $\epsilon > 0$ small, a (p, q)-satellite about γ is the equivalence class of the immersion $\alpha_\epsilon : \mathbb{R}/\mathbb{Z} \to S^2$

$$\alpha_\epsilon(t) = \exp_{\gamma(qtL)}\left(\epsilon \sin(2\pi pt)\, n(qtL)\right). \tag{18}$$

Two immersions are equivalent if they are homotopic through immersions, but self-tangencies and tangencies with γ are not allowed.

Theorem 6.3 (Angenent [2005]). *If a rational number p'/q' strictly between $\rho(\gamma)$ and 1 is written in lowest terms then there exists a closed geodesic which is a (p',q')-satellite about γ.*

Let us explain the connection between theorems 6.2 and 6.3. The unit tangent bundle $T^1S^2 = \{v \in TS^2 \mid g(v,v) = 1\}$ admits a contact form λ_g whose Reeb flow coincides with the geodesic flow. It is given by the restriction to T^1S^2 of the pull-back of the tautological 1-form on T^*S^2 by the associated Legendre transform. The L-periodic orbits $\dot\gamma(t)$ and $-\dot\gamma(-t)$ form a link l_γ on T^1S^2 transverse to the contact structure $\ker\lambda_g$. There exists a double cover $S^3 \to T^1S^2$ that pulls back the Reeb flow of λ_g to a Reeb flow on (S^3,ξ_0). Moreover, it pulls back the link l_γ to a Hopf link consisting of periodic orbits $P_0 \cup P_1$ just like in the statement of Theorem 6.2. Note that $\rho(\gamma) \neq 1$ forces the vectors $(\theta_0 = 2\rho(\gamma) - 1, 1)$ and $(1, \theta_1 = 2\rho(\gamma) - 1)$ to span a non-empty sector. Then Theorem 6.2 captures the contractible (p',q')-satellites of Theorem 6.3 up to homotopy, and a refinement for Reeb flows on the standard $\mathbb{R}P^3$ (Hryniewicz, Momin, and Salomão [2015, Theorem 1.9]) captures all the (p',q')-satellites of Theorem 6.3 up to homotopy. Of course, we do not hope to capture geodesics up to equivalence of satellites because Theorem 6.2 deals with more general flows than those dealt by Theorem 6.3. For instance, it handles non-reversible Finsler geodesic flows with a pair of closed geodesics homotopic to a pair of embedded loops through immersions without positive tangencies. In particular, it covers reversible Finsler metrics with a simple closed geodesic.

Finally, a pair of closed Reeb orbits forming a Hopf link is not known to exist in general for a Reeb flow on (S^3,ξ_0). Each of its components is unknotted, transverse to ξ_0 and has self-linking number -1; we refer to such a closed curve as a Hopf fiber. The existence of at least one closed Reeb orbit on (S^3,ξ_0) which is a Hopf fiber is proved in Hofer, Wysocki, and Zehnder [1996b]; this is a difficult result. If P is a nondegenerate closed orbit which bounds a disk-like global surface of section then P is a Hopf fiber and its rotation number is > 1. Moreover, a fixed point of the first return map, assured by Brouwer's translation theorem, determines a closed orbit P' which forms a Hopf link with P. One may ask whether every closed orbit which is a Hopf fiber and has rotation number > 1 admits another closed orbit forming together a Hopf link. In that direction we have the following result which may be seen as a version of Brouwer's translation theorem for Reeb flows on (S^3,ξ_0).

Theorem 6.4 (Hryniewicz, Momin, and Salomão [n.d.]). *Assume that a Reeb flow on (S^3,ξ_0) admits a closed Reeb orbit P which is a Hopf fiber. If the transverse rotation number $\rho(P)$ belongs to $(1,+\infty) \setminus \left\{1 + \frac{1}{k} : k \in \mathbb{N}\right\}$ then there exists a closed orbit P' simply linked to P.*

The closed orbit P' in Theorem 6.4 is not even known to be unknotted.

A Closed geodesics on a Riemannian two-sphere

The purpose of this appendix is to describe the steps of a new proof of the existence of infinitely many closed geodesics on any Riemannian two-sphere. The argument is based on a combination of Angenent's theorem (Theorem 6.3) and the work of Hingston [1993], it serves as an alternative to the classical proof that combines results of Victor Bangert and John Franks. We recommend Oancea [2015] for an account of the closed geodesic problem on Riemannian manifolds.

Theorem A.1 (Bangert [1993] and Franks [1992]). *Every Riemannian metric on S^2 admits infinitely many closed geodesics.*

We start with a remark from Hingston [1993]. The space of embedded loops in the two-sphere carries a 3-dimensional homology class modulo short loops. One can use Grayson's curve shortening flow to run a min-max argument over this class and obtain a special simple closed geodesic γ_*. The crucial fact here is that Grayson's curve shortening flow preserves the property of being embedded. The sum of the Morse index and the nullity of γ_* is larger than or equal to 3, in particular $\rho(\gamma_*) \geq 1$.

If $\rho(\gamma_*) = 1$ (Hingston's non-rotating case) then γ_* is a very special critical point of the energy functional. The growth of Morse indices under iterations of γ_* follows a specific pattern. Index plus nullity of γ_* is equal to 3, and if γ_* is isolated then its local homology is non-trivial in degree 3. If every iterate of γ_* is isolated then the analysis of Hingston [ibid.] shows that there are infinitely many closed geodesics. If some iterate γ_* is not isolated then already there are infinitely many closed geodesics. Hence we are left with the case $\rho(\gamma_*) > 1$, which is covered by Theorem 6.3. Theorem A.1 is proved. The case $\rho(\gamma_*) > 1$ is handled independently by Theorem 6.2.

The work of Hingston [ibid.] triggered many developments, including a proof of the Conley conjecture for standard symplectic tori in Hingston [2009]. In Ginzburg [2010] used Floer homology and Hingston's methods to prove the Conley conjecture for aspherical symplectic manifolds.

References

Alberto Abbondandolo, Barney Bramham, Umberto L. Hryniewicz, and Pedro A. S. Salomão (2017a). "A systolic inequality for geodesic flows on the two-sphere". *Math. Ann.* 367.1-2, pp. 701–753. MR: 3606452 (cit. on pp. 968, 970).

Alberto Abbondandolo, Barney Bramham, Umberto L. Hryniewicz, and Pedro A. S. Salomão (Sept. 2017b). "Contact forms with large systolic ratio in dimension three". To appear in *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*. arXiv: 1709.01621 (cit. on pp. 968, 972).

– (Oct. 2017c). "Systolic ratio, index of closed orbits and convexity for tight contact forms on the three-sphere". arXiv: 1710.06193 (cit. on pp. 968, 973).

– (2018). "Sharp systolic inequalities for Reeb flows on the three-sphere". *Invent. Math.* 211.2, pp. 687–778. MR: 3748316 (cit. on pp. 968, 972).

Peter Albers, Joel W. Fish, Urs Frauenfelder, Helmut Hofer, and Otto van Koert (2012). "Global surfaces of section in the planar restricted 3-body problem". *Arch. Ration. Mech. Anal.* 204.1, pp. 273–284. MR: 2898741 (cit. on p. 975).

Peter Albers, Joel W. Fish, Urs Frauenfelder, and Otto van Koert (2013). "The Conley-Zehnder indices of the rotating Kepler problem". *Math. Proc. Cambridge Philos. Soc.* 154.2, pp. 243–260. MR: 3021812 (cit. on p. 976).

Peter Albers, Urs Frauenfelder, Otto van Koert, and Gabriel P. Paternain (2012). "Contact geometry of the restricted three-body problem". *Comm. Pure Appl. Math.* 65.2, pp. 229–263. MR: 2855545 (cit. on p. 976).

J. C. Álvarez Paiva and F. Balacheff (2014). "Contact geometry and isosystolic inequalities". *Geom. Funct. Anal.* 24.2, pp. 648–669. MR: 3192037 (cit. on p. 969).

Sigurd B. Angenent (2005). "Curve shortening and the topology of closed geodesics on surfaces". *Ann. of Math. (2)* 162.3, pp. 1187–1241. MR: 2179729 (cit. on pp. 981, 982).

Florent Balacheff (2006). "Sur la systole de la sphère au voisinage de la métrique standard". *Geom. Dedicata* 121, pp. 61–71. MR: 2276235 (cit. on p. 969).

– (2010). "A local optimal diastolic inequality on the two-sphere". *J. Topol. Anal.* 2.1, pp. 109–121. MR: 2646992 (cit. on p. 969).

Victor Bangert (1993). "On the existence of closed geodesics on two-spheres". *Internat. J. Math.* 4.1, pp. 1–10. MR: 1209957 (cit. on pp. 960, 983).

George D Birkhoff (1914). "The restricted problem of three bodies". *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 39.1, pp. 265–334 (cit. on pp. 974, 975).

George D. Birkhoff (1913). "Proof of Poincaré's geometric theorem". *Trans. Amer. Math. Soc.* 14.1, pp. 14–22. MR: 1500933 (cit. on p. 980).

– (1966). *Dynamical systems*. With an addendum by Jurgen Moser. American Mathematical Society Colloquium Publications, Vol. IX. American Mathematical Society, Providence, R.I., pp. xii+305. MR: 0209095 (cit. on p. 961).

F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder (2003). "Compactness results in symplectic field theory". *Geom. Topol.* 7, pp. 799–888. MR: 2026549.

B. Bramham and H. Hofer (2012). "First steps towards a symplectic dynamics". In: *Surveys in differential geometry. Vol. XVII*. Vol. 17. Surv. Differ. Geom. Int. Press, Boston, MA, pp. 127–177. MR: 3076060 (cit. on p. 960).

Barney Bramham (2015a). "Periodic approximations of irrational pseudo-rotations using pseudoholomorphic curves". *Ann. of Math. (Second Series)* 181.3, pp. 1033–1086. MR: 3296820 (cit. on p. 979).

– (2015b). "Pseudo-rotations with sufficiently Liouvillean rotation number are C^0-rigid". *Invent. Math.* 199.2, pp. 561–580. MR: 3302121 (cit. on p. 979).

Christopher B. Croke (1988). "Area and the length of the shortest closed geodesic". *J. Differential Geom.* 27.1, pp. 1–21. MR: 918453 (cit. on p. 968).

Joel W. Fish and Richard Siefring (Nov. 2013). "Connected sums and finite energy foliations I: Contact connected sums". arXiv: 1311.4221 (cit. on p. 979).

John Franks (1992). "Geodesics on S^2 and periodic points of annulus homeomorphisms". *Invent. Math.* 108.2, pp. 403–418. MR: 1161099 (cit. on pp. 960, 964, 983).

David Fried (1982). "The geometry of cross sections to flows". *Topology* 21.4, pp. 353–371. MR: 670741 (cit. on p. 961).

Hansjörg Geiges (2008). *An introduction to contact topology*. Vol. 109. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. xvi+440. MR: 2397738 (cit. on p. 962).

Étienne Ghys (2009). "Right-handed vector fields & the Lorenz attractor". *Jpn. J. Math.* 4.1, pp. 47–61. MR: 2491282 (cit. on p. 961).

Viktor L. Ginzburg (2010). "The Conley conjecture". *Ann. of Math. (2)* 172.2, pp. 1127–1180. MR: 2680488 (cit. on p. 983).

Emmanuel Giroux (2002). "Géométrie de contact: de la dimension trois vers les dimensions supérieures". In: *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*. Higher Ed. Press, Beijing, pp. 405–414. MR: 1957051 (cit. on p. 965).

Mikhael Gromov (1983). "Filling Riemannian manifolds". *J. Differential Geom.* 18.1, pp. 1–147. MR: 697984 (cit. on p. 968).

– (1996). "Systoles and intersystolic inequalities". In: *Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992)*. Vol. 1. Sémin. Congr. Soc. Math. France, Paris, pp. 291–362. MR: 1427763.

Adam Harris and Gabriel P. Paternain (2008). "Dynamically convex Finsler metrics and J-holomorphic embedding of asymptotic cylinders". *Ann. Global Anal. Geom.* 34.2, pp. 115–134. MR: 2425525 (cit. on pp. 966, 967).

Nancy Hingston (1993). "On the growth of the number of closed geodesics on the two-sphere". *Internat. Math. Res. Notices* 9, pp. 253–262. MR: 1240637 (cit. on pp. 960, 983).

– (2009). "Subharmonic solutions of Hamiltonian equations on tori". *Ann. of Math. (2)* 170.2, pp. 529–560. MR: 2552101 (cit. on p. 983).

H. Hofer (1993). "Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three". *Invent. Math.* 114.3, pp. 515–563. MR: 1244912 (cit. on p. 960).

H. Hofer, K. Wysocki, and E. Zehnder (1995a). "A characterisation of the tight three-sphere". *Duke Math. J.* 81.1. A celebration of John F. Nash, Jr., 159–226 (1996). MR: 1381975 (cit. on pp. 966, 967, 978).

– (1995b). "Properties of pseudo-holomorphic curves in symplectisations. II. Embedding controls and algebraic invariants". *Geom. Funct. Anal.* 5.2, pp. 270–328. MR: 1334869.

– (1996a). "Properties of pseudoholomorphic curves in symplectisations. I. Asymptotics". *Ann. Inst. H. Poincaré Anal. Non Linéaire* 13.3, pp. 337–379. MR: 1395676.

– (1996b). "Unknotted periodic orbits for Reeb flows on the three-sphere". *Topol. Methods Nonlinear Anal.* 7.2, pp. 219–244. MR: 1481697 (cit. on pp. 963, 982).

– (1998). "The dynamics on three-dimensional strictly convex energy surfaces". *Ann. of Math. (2)* 148.1, pp. 197–289. MR: 1652928 (cit. on pp. 961, 963, 964, 973, 975, 978, 980).

– (1999a). "A characterization of the tight 3-sphere. II". *Comm. Pure Appl. Math.* 52.9, pp. 1139–1177. MR: 1692144 (cit. on pp. 966, 967, 973, 978).

– (1999b). "Properties of pseudoholomorphic curves in symplectizations. III. Fredholm theory". In: *Topics in nonlinear analysis*. Vol. 35. Progr. Nonlinear Differential Equations Appl. Birkhäuser, Basel, pp. 381–475. MR: 1725579.

– (2003). "Finite energy foliations of tight three-spheres and Hamiltonian dynamics". *Ann. of Math. (2)* 157.1, pp. 125–255. MR: 1954266 (cit. on pp. 960, 962, 977–979).

Helmut Hofer and Eduard Zehnder (1994). *Symplectic invariants and Hamiltonian dynamics*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, pp. xiv+341. MR: 1306732 (cit. on p. 970).

Umberto L. Hryniewicz (2012). "Fast finite-energy planes in symplectizations and applications". *Trans. Amer. Math. Soc.* 364.4, pp. 1859–1931. MR: 2869194 (cit. on pp. 962, 963, 978).

– (2014). "Systems of global surfaces of section for dynamically convex Reeb flows on the 3-sphere". *J. Symplectic Geom.* 12.4, pp. 791–862. MR: 3333029 (cit. on pp. 962, 963, 978).

Umberto L. Hryniewicz, Joan E. Licata, and Pedro A. S. Salomão (2015). "A dynamical characterization of universally tight lens spaces". *Proc. Lond. Math. Soc. (3)* 110.1, pp. 213–269. MR: 3299604 (cit. on pp. 964, 967, 978).

Umberto L. Hryniewicz, A. Momin, and Pedro A. S. Salomão (n.d.). "On simply linked Reeb orbits in the tight three-sphere". In preparation (cit. on p. 982).

Umberto L. Hryniewicz, Al Momin, and Pedro A. S. Salomão (2015). “A Poincaré-Birkhoff theorem for tight Reeb flows on S^3”. *Invent. Math.* 199.2, pp. 333–422. MR: 3302117 (cit. on pp. 960, 981, 982).

Umberto L. Hryniewicz and Pedro A. S. Salomão (2011). “On the existence of disk-like global sections for Reeb flows on the tight 3-sphere”. *Duke Math. J.* 160.3, pp. 415–465. MR: 2852366 (cit. on pp. 964, 978).

– (2013). “Global properties of tight Reeb flows with applications to Finsler geodesic flows on S^2”. *Math. Proc. Cambridge Philos. Soc.* 154.1, pp. 1–27. MR: 3002580 (cit. on p. 967).

– (2016). “Elliptic bindings for dynamically convex Reeb flows on the real projective three-space”. *Calc. Var. Partial Differential Equations* 55.2, Art. 43, 57. MR: 3485982 (cit. on p. 966).

Umberto L. Hryniewicz, Pedro A. S. Salomão, and K. Wysocki (n.d.). “Genus zero global surfaces of section for Reeb flows in dimension three”. In preparation (cit. on pp. 965, 966).

Al Momin (2011). “Contact homology of orbit complements and implied existence”. *J. Modern Dyn.* 5.4, pp. 409–472 (cit. on p. 981).

Alexandru Oancea (2015). “Morse theory, closed geodesics, and the homology of free loop spaces”. In: *Free loop spaces in geometry and topology*. Vol. 24. IRMA Lect. Math. Theor. Phys. With an appendix by Umberto Hryniewicz. Eur. Math. Soc., Zürich, pp. 67–109. MR: 3444362 (cit. on p. 983).

Naiara V. de Paulo and Pedro A. S. Salomão (n.d.). “On the multiplicity of periodic orbits and homoclinics near critical energy levels of Hamiltonian systems in $\mathbb{R}^4$”. To appear in *Transactions of the Amer. Math. Soc.* (cit. on p. 979).

– (2018). “Systems of transversal sections near critical energy levels of Hamiltonians systems in $\mathbb{R}^4$”. *Memoirs of the Amer. Math. Soc.* 252.1202 (cit. on p. 979).

Henri Poincaré (1912). “Sur un théorème de géométrie”. *Rend. Circ. Mat. Palermo* 33, pp. 375–407 (cit. on pp. 959, 980).

Hans-Bert Rademacher (2007). “Existence of closed geodesics on positively curved Finsler manifolds”. *Ergodic Theory Dynam. Systems* 27.3, pp. 957–969. MR: 2322187 (cit. on p. 966).

Pedro A. S. Salomão (2003). “Convex energy levels of Hamiltonian systems”. *Qual. Theory Dyn. Syst.* 4.2, 439–457 (2004). MR: 2129729 (cit. on p. 979).

Sol Schwartzman (1957). “Asymptotic cycles”. *Ann. of Math. (2)* 66, pp. 270–284. MR: 0088720 (cit. on p. 961).

Richard Siefring (2011). “Intersection theory of punctured pseudoholomorphic curves”. *Geom. Topol.* 15.4, pp. 2351–2457. MR: 2862160 (cit. on pp. 965, 981).

Dennis Sullivan (1976). “Cycles for the dynamical study of foliated manifolds and complex manifolds”. *Invent. Math.* 36, pp. 225–255. MR: 0433464 (cit. on p. 961).

Claude Viterbo (2000). "Metric and isoperimetric problems in symplectic geometry". *J. Amer. Math. Soc.* 13.2, pp. 411–431. MR: 1750956 (cit. on pp. 968, 970).

Received 2018-02-07.

Pedro A. S. Salomão
psalomao@ime.usp.br

Umberto L. Hryniewicz
umbertolh@gmail.com