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TILTING COHEN–MACAULAY REPRESENTATIONS

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Dedicated to the memory of Ragnar-Olaf Buchweitz

Abstract

This is a survey on recent developments in Cohen-Macaulay representations via tilting and cluster tilting theory. We explain triangle equivalences between the singularity categories of Gorenstein rings and the derived (or cluster) categories of finite dimensional algebras.

1 Introduction

The study of Cohen-Macaulay (CM) representations (Curtis and Reiner [1981], Yoshino [1990], Simson [1992], and Leuschke and Wiegand [2012]) is one of the active subjects in representation theory and commutative algebra. It has fruitful connections to singularity theory, algebraic geometry and physics. This article is a survey on recent developments in this subject.

The first half of this article is spent for background materials, which were never written in one place. In Section 2, we recall the notion of CM modules over Gorenstein rings, and put them into the standard framework of triangulated categories. This gives us powerful tools including Buchweitz's equivalence between the stable category $\underline{\mathrm{CM}}R$ and the singularity category, and Orlov's realization of the graded singularity category in the derived category, giving a surprising connection between CM modules and algebraic geometry. We also explain basic results including Auslander-Reiten duality stating that $\underline{\mathrm{CM}}R$ is a Calabi-Yau triangulated category for a Gorenstein isolated singularity R, and Gabriel's Theorem on quiver representations and its commutative counterpart due to Buchweitz-Greuel-Schreyer.

In Section 3, we give a brief introduction to tilting and cluster tilting. Tilting theory controls equivalences of derived categories, and played a central role in Cohen-Macaulay

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approximation theory around 1990 (Auslander and Buchweitz [1989] and Auslander and Reiten [1991]). The first main problem of this article is to find a tilting object in the stable category $\underline{\mathrm{CM}}^G R$ of a G-graded Gorenstein ring R. This is equivalent to find a triangle equivalence

$$\underline{\mathrm{CM}}^G R \simeq \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda) \tag{1-1}$$

with some ring Λ. It reveals a deep connection between rings R and Λ.

The notion of d-cluster tilting was introduced in higher Auslander-Reiten theory. A Gorenstein ring R is called d-CM-finite if there exists a d-cluster tilting object in CM R. This property is a natural generalization of CM-finiteness, and closely related to the existence of non-commutative crepant resolutions of Van den Bergh. On the other hand, the d-cluster category $\mathrm{C}_d(\Lambda)$ of a finite dimensional algebra Λ is a d-Calabi-Yau triangulated category containing a d-cluster tilting object, introduced in categorification of Fomin-Zelevinsky cluster algebras. The second main problem of this article is to find a triangle equivalence

$$\underline{\mathrm{CM}} R \simeq \mathrm{C}_d(\Lambda) \tag{1-2}$$

with some finite dimensional algebra Λ. This implies that R is d-CM-finite.

In the latter half of this article, we construct various triangle equivalences of the form (1-1) or (1-2). In Section 4, we explain results in Yamaura [2013] and Buchweitz, Iyama, and Yamaura [2018]. They assert that, for a large class of $\mathbb{Z}$-graded Gorenstein rings R in dimension 0 or 1, there exist triangle equivalences (1-1) for some algebras Λ.

There are no such general results in dimension greater than 1. Therefore in the main Sections 5 and 6 of this article, we concentrate on special classes of Gorenstein rings. In Section 5, we explain results on Gorenstein rings obtained from classical and higher preprojective algebras (Amiot, Iyama, and Reiten [2015], Iyama and Oppermann [2013], and Kimura [2018, 2016]). A crucial observation is that certain Calabi-Yau algebras are higher preprojective algebras and higher Auslander algebras at the same time. In Section 6, we explain results on CM modules on Geigle-Lenzing complete intersections and the derived categories of coherent sheaves on the associated stacks (Herschend, Iyama, Minamoto, and Oppermann [2014]). They are higher dimensional generalizations of weighted projective lines of Geigle-Lenzing.

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2 Preliminaries

2.1 Notations. We fix some conventions in this paper. All modules are right modules. The composition of $f: X \to Y$ and $g: Y \to Z$ is denoted by gf. For a ring Λ, we denote by $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules, by $\operatorname{proj}\Lambda$ the category of finitely generated projective Λ-modules, and by $\operatorname{gl.dim}\Lambda$ the global dimension of Λ. When Λ is G-graded, we denote by $\operatorname{mod}^G\Lambda$ and $\operatorname{proj}^G\Lambda$ the G-graded version, whose morphisms are degree preserving. We denote by k an arbitrary field unless otherwise specified, and by D the k-dual or Matlis dual over a base commutative ring.

2.2 Cohen-Macaulay modules. We start with the classical notion of Cohen-Macaulay modules over commutative rings (Bruns and Herzog [1993] and Matsumura [1989]).

Let R be a commutative noetherian ring. The *dimension* $\dim R$ of R is the supremum of integers $n \geq 0$ such that there exists a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals of R. The *dimension* $\dim M$ of $M \in \operatorname{mod} R$ is the dimension $\dim(R/\operatorname{ann} M)$ of the factor ring $R/\operatorname{ann} M$, where $\operatorname{ann} M$ is the annihilator of M.

The notion of depth is defined locally. Assume that R is a local ring with maximal ideal $\mathfrak{m}$ and $M \in \operatorname{mod} R$ is non-zero. An element $r \in \mathfrak{m}$ is called *M-regular* if the multiplication map $r: M \to M$ is injective. A sequence $r_1, \dots, r_n$ of elements in $\mathfrak{m}$ is called an *M-regular sequence* of length n if r_i is $(M/(r_1, \dots, r_{i-1})M)$-regular for all $1 \leq i \leq n$. The *depth* $\operatorname{depth} M$ of M is the supremum of the length of M-regular sequences. This is given by the simple formula

$$\operatorname{depth} M = \inf\{i \geq 0 \mid \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}.$$

The inequalities $\operatorname{depth} M \leq \dim M \leq \dim R$ hold. We call M (maximal) *Cohen-Macaulay* (or *CM*) if the equality $\operatorname{depth} M = \dim R$ holds or $M = 0$.

When R is not necessarily local, $M \in \operatorname{mod} R$ is called *CM* if $M_{\mathfrak{m}}$ is a CM $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R. The ring R is called *CM* if it is CM as an R-module. The ring R is called *Gorenstein* (resp. *regular*) if $R_{\mathfrak{m}}$ has finite injective dimension as an $R_{\mathfrak{m}}$-module (resp. $\operatorname{gl.dim} R_{\mathfrak{m}} < \infty$) for all maximal ideals $\mathfrak{m}$ of R. In this case, the injective (resp. global) dimension coincides with $\dim R_{\mathfrak{m}}$, but this is not true in the more general setting below. The following hierarchy is basic.

$$\text{Regular rings} \Longrightarrow \text{Gorenstein rings} \Longrightarrow \text{Cohen-Macaulay rings}$$

We will study CM modules over Gorenstein rings. Since we apply methods in representation theory, it is more reasonable to work in the following wider framework.

Definition 2.1 (Iwanaga [1979] and Enochs and Jenda [2000])**.** Let Λ be a (not necessarily commutative) noetherian ring, and $d \geq 0$ an integer. We call Λ (d-)*Iwanaga-Gorenstein* (or *Gorenstein*) if Λ has injective dimension at most d as a Λ-module, and also as a Λ^{op}-module.

Clearly, a commutative noetherian ring R is Iwanaga-Gorenstein if and only if it is Gorenstein and $\dim R < \infty$. Note that there are various definitions of non-commutative Gorenstein rings, e.g. Artin and Schelter [1987], Curtis and Reiner [1981], Fossum, Griffith, and Reiten [1975], Goto and Nishida [2002], and Iyama and Wemyss [2014]. Although Definition 2.1 is much weaker than them, it is sufficient for the aim of this paper.

Noetherian rings with finite global dimension are analogues of regular rings, and form special classes of Iwanaga-Gorenstein rings. The first class consists of *semisimple rings* (i.e. rings Λ with $\operatorname{gl.dim} \Lambda = 0$), which are products of matrix rings over division rings by Artin-Wedderburn Theorem. The next class consists of *hereditary rings* (i.e. rings Λ with $\operatorname{gl.dim} \Lambda \leq 1$), which are obtained from quivers.

Definition 2.2 (Assem, Simson, and Skowroński [2006])**.** A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of sets Q_0, Q_1 and maps $s, t \colon Q_1 \to Q_0$. We regard each element in Q_0 as a vertex, and $a \in Q_1$ as an arrow with source $s(a)$ and target $t(a)$. A *path* of length 0 is a vertex, and a *path* of length $\ell (\geq 1)$ is a sequence $a_1 a_2 \cdots a_\ell$ of arrows satisfying $t(a_i) = s(a_{i+1})$ for each $1 \leq i < \ell$.

For a field k, the *path algebra* kQ is defined as follows: It is a k-vector space with basis consisting of all paths on Q. For paths $p = a_1 \cdots a_\ell$ and $q = b_1 \cdots b_m$, we define $pq = a_1 \cdots a_\ell b_1 \cdots b_m$ if $t(a_\ell) = s(b_1)$, and $pq = 0$ otherwise.

Clearly $\dim_k(kQ)$ is finite if and only if Q is *acyclic* (that is, there are no paths p of positive length satisfying $s(p) = t(p)$).

Example 2.3. (a) (Assem, Simson, and Skowroński [ibid.]) The path algebra kQ of a finite quiver Q is hereditary. Conversely, any finite dimensional hereditary algebra over an algebraically closed field k is Morita equivalent to kQ for some acyclic quiver Q.

(b) A finite dimensional k-algebra Λ is 0-Iwanaga-Gorenstein if and only if Λ is *self-injective*, that is, $D\Lambda$ is projective as a Λ-module, or equivalently, as a Λ^{op}-module. For example, the group ring kG of a finite group G is self-injective.

(c) (Iyama and Wemyss [2014] and Curtis and Reiner [1981]) Let R be a CM local ring with canonical module ω and dimension d. An R-algebra Λ is called an *R-order* if it is CM as an R-module. Then an R-order Λ is d-Iwanaga-Gorenstein if and only if Λ is a *Gorenstein order*, i.e. $\operatorname{Hom}_R(\Lambda, \omega)$ is projective as a Λ-module, or equivalently, as a Λ^{op}-module.

An R-order Λ is called *non-singular* if $\operatorname{gl.dim} \Lambda = d$. They are classical objects for the case $d = 0, 1$ (Curtis and Reiner [1981]), and studied for $d = 2$ (Reiten and Van den Bergh [1989]). Non-singular orders are closely related to cluster tilting explained in Section 3.2.

2.3 The triangulated category of Cohen-Macaulay modules. CM modules can be defined naturally also for Iwanaga-Gorenstein rings.

Definition 2.4. Let Λ be an Iwanaga-Gorenstein ring. We call $M \in \operatorname{mod}\Lambda$ (maximal) *Cohen-Macaulay* (or *CM*) if $\operatorname{Ext}^i_\Lambda(M,\Lambda) = 0$ holds for all $i > 0$. We denote by $\operatorname{CM}\Lambda$ the category of CM Λ-modules.

We also deal with graded rings and modules. For an abelian group G and a G-graded Iwanaga-Gorenstein ring Λ, we denote by $\operatorname{CM}^G\Lambda$ the full subcategory of $\operatorname{mod}^G\Lambda$ consisting of all X which belong to $\operatorname{CM}\Lambda$ as ungraded Λ-modules.

When Λ is commutative Gorenstein, Definition 2.4 is one of the well-known equivalent conditions of CM modules. Note that, in a context of Gorenstein homological algebra (Auslander and Bridger [1969] and Enochs and Jenda [2000]), CM modules are also called Gorenstein projective, Gorenstein dimension zero, or totally reflexive.

Example 2.5. (a) Let Λ be a noetherian ring with $\operatorname{gl.dim}\Lambda < \infty$. Then $\operatorname{CM}\Lambda = \operatorname{proj}\Lambda$.

(b) Let Λ be a finite dimensional self-injective k-algebra. Then $\operatorname{CM}\Lambda = \operatorname{mod}\Lambda$.

(c) Let Λ be a Gorenstein R-order in Example 2.3(c). Then CM Λ-modules are precisely Λ-modules that are CM as R-modules.

We study the category $\operatorname{CM}^G\Lambda$ from the point of view of triangulated categories. We start with Quillen's exact categories (see Bühler [2010] for a more axiomatic definition).

Definition 2.6 (Happel [1988])**.** (a) An *exact category* is a full subcategory $\mathfrak{F}$ of an abelian category $\mathfrak{A}$ such that, for each exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathfrak{A}$ with $X, Z \in \mathfrak{F}$, we have $Y \in \mathfrak{F}$. In this case, we say that $X \in \mathfrak{F}$ is *projective* if $\operatorname{Ext}^1_{\mathfrak{A}}(X, \mathfrak{F}) = 0$ holds. Similarly we define *injective* objects in $\mathfrak{F}$.

(b) An exact category $\mathfrak{F}$ in $\mathfrak{A}$ is called *Frobenius* if:

- an object in $\mathfrak{F}$ is projective if and only if it is injective,
- any $X \in \mathfrak{F}$ admits exact sequences $0 \to Y \to P \to X \to 0$ and $0 \to X \to I \to Z \to 0$ in $\mathfrak{A}$ such that P and I are projective in $\mathfrak{F}$ and $Y, Z \in \mathfrak{F}$.

(c) The *stable category* $\underline{\mathfrak{F}}$ has the same objects as $\mathfrak{F}$, and the morphisms are given by $\underline{\operatorname{Hom}}_{\mathfrak{F}}(X,Y) = \operatorname{Hom}_{\mathfrak{F}}(X,Y)/P(X,Y)$, where $P(X,Y)$ is the subgroup consisting of morphisms which factor through projective objects in $\mathfrak{F}$.

Frobenius categories are ubiquitous in algebra. Here we give two examples.

Example 2.7. (a) For a G-graded Iwanaga-Gorenstein ring Λ, the category $\mathrm{CM}^G \Lambda$ of G-graded Cohen-Macaulay Λ-modules is a Frobenius category.

(b) For an additive category $\mathcal{A}$, the category $\mathrm{C}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ is a Frobenius category, whose stable category is the homotopy category $\mathrm{K}(\mathcal{A})$.

A *triangulated category* is a triple of an additive category $\mathcal{T}$, an autoequivalence $[1]: \mathcal{T} \to \mathcal{T}$ (called *suspension*) and a class of diagrams $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ (called *triangles*) satisfying a certain set of axioms. There are natural notions of functors and equivalences between triangulated categories, called *triangle functors* and *triangle equivalences*. For details, see e.g. Happel [1988] and Neeman [2001]. Typical examples of triangulated categories are given by the homotopy category $\mathrm{K}(\mathcal{A})$ of an additive category $\mathcal{A}$ and the derived category $\mathrm{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$.

A standard construction of triangulated categories is given by the following.

Theorem 2.8 (Happel [1988]). *The stable category $\underline{\mathcal{F}}$ of a Frobenius category $\mathcal{F}$ has a canonical structure of a triangulated category.*

Such a triangulated category is called *algebraic*. Note that the suspension functor $[1]$ of $\underline{\mathcal{F}}$ is given by the cosyzygy. Thus the i-th suspension $[i]$ is the i-th cosyzygy for $i \geq 0$, and the $(-i)$-th syzygy for $i < 0$. We omit other details.

As a summary, we obtain the following.

Corollary 2.9. *Let G be an abelian group and Λ a G-graded Iwanaga-Gorenstein ring. Then $\mathrm{CM}^G \Lambda$ is a Frobenius category, and therefore the stable category $\underline{\mathrm{CM}}^G \Lambda$ has a canonical structure of a triangulated category.*

We denote by $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}^G \Lambda)$ the bounded derived category of $\mathrm{mod}^G \Lambda$, and by $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}^G \Lambda)$ the bounded homotopy category of $\mathrm{proj}^G \Lambda$. We regard $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}^G \Lambda)$ as a thick subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}^G \Lambda)$. The *stable derived category* (Buchweitz [1987]) or the *singularity category* (Orlov [2009]) is defined as the Verdier quotient

$$\mathrm{D}_{\mathrm{sg}}^G(\Lambda) = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}^G \Lambda)/\mathrm{K}^{\mathrm{b}}(\mathrm{proj}^G \Lambda).$$

This is enhanced by the Frobenius category $\mathrm{CM}^G \Lambda$ as the following result shows.

Theorem 2.10 (Buchweitz [1987], Rickard [1989a], and Keller and Vossieck [1987]). *Let G be an abelian group and Λ a G-graded Iwanaga-Gorenstein ring. Then there is a triangle equivalence $\mathrm{D}_{\mathrm{sg}}^G(\Lambda) \simeq \underline{\mathrm{CM}}^G \Lambda$.*

Let us recall the following notion (Bruns and Herzog [1993]).

Definition 2.11. Let G be an abelian group and R a G-graded Gorenstein ring with $\dim R = d$ such that $R_0 = k$ is a field and $\bigoplus_{i \neq 0} R_i$ is an ideal of R. The *a-invariant* $a \in G$ (or *Gorenstein parameter* $-a \in G$) is an element satisfying $\operatorname{Ext}^d_R(k, R(a)) \simeq k$ in $\operatorname{mod}^{\mathbb{Z}} R$.

For a G-graded noetherian ring Λ, let

$$\operatorname{qgr} \Lambda = \operatorname{mod}^G \Lambda / \operatorname{mod}^G_0 \Lambda \tag{2-1}$$

be the Serre quotient of $\operatorname{mod}^G \Lambda$ by the subcategory $\operatorname{mod}^G_0 \Lambda$ of G-graded Λ-modules of finite length (Artin and Zhang [1994]). This is classical in projective geometry. In fact, for a $\mathbb{Z}$-graded commutative noetherian ring R generated in degree 1, $\operatorname{qgr} R$ is the category $\operatorname{coh} X$ of coherent sheaves on the scheme $X = \operatorname{Proj} R$ (Serre [1955]).

The following result realizes $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R)$ and $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R)$ inside of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}} R)$, where $\operatorname{mod}^{\geq n} R$ is the full subcategory of $\operatorname{mod}^{\mathbb{Z}} R$ consisting of all X satisfying $X = \bigoplus_{i \geq n} X_i$, and $(-)^*$ is the duality $\mathbf{R}\mathrm{Hom}_R(-, R) \colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}} R) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod}^{\mathbb{Z}} R)$.

Theorem 2.12 (Orlov [2009] and Iyama and Yang [2017])**.** *Let $R = \bigoplus_{i \geq 0} R_i$ be a $\mathbb{Z}$-graded Gorenstein ring such that R_0 is a field, and a the a-invariant of R.*

(a) *There is a triangle equivalence* $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}^{\geq 0} R) \cap \mathrm{D}^{\mathrm{b}}(\operatorname{mod}^{\geq 1} R)^* \simeq \mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R)$.

(b) *There is a triangle equivalence* $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}^{\geq 0} R) \cap \mathrm{D}^{\mathrm{b}}(\operatorname{mod}^{\geq a+1} R)^* \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R)$.

Therefore if $a = 0$, then $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R)$. If $a < 0$ (resp. $a > 0$), then there is a fully faithful triangle functor $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) \to \mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R)$ (resp. $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} R) \to \mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R)$). This gives a new connection between CM representations and algebraic geometry.

2.4 Representation theory. We start with Auslander-Reiten theory.

Let R be a commutative ring, and D the Matlis duality. A triangulated category $\mathcal{T}$ is called *R-linear* if each morphism set $\operatorname{Hom}_{\mathcal{T}}(X, Y)$ has an R-module structure and the composition $\operatorname{Hom}_{\mathcal{T}}(X, Y) \times \operatorname{Hom}_{\mathcal{T}}(Y, Z) \to \operatorname{Hom}_{\mathcal{T}}(X, Z)$ is R-bilinear. It is called *Hom-finite* if each morphism set has finite length as an R-module.

Definition 2.13 (Reiten and Van den Bergh [2002])**.** A *Serre functor* is an R-linear autoequivalence $\mathbb{S} \colon \mathcal{T} \to \mathcal{T}$ such that there exists a functorial isomorphism $\operatorname{Hom}_{\mathcal{T}}(X, Y) \simeq D \operatorname{Hom}_{\mathcal{T}}(Y, \mathbb{S}X)$ for any $X, Y \in \mathcal{T}$ (called *Auslander-Reiten duality* or *Serre duality*). The composition $\tau = \mathbb{S} \circ [-1]$ is called the *AR translation*.

For $d \in \mathbb{Z}$, we say that $\mathcal{T}$ is *d-Calabi-Yau* if $[d]$ gives a Serre functor of $\mathcal{T}$.

A typical example of a Serre functor is given by a smooth projective variety X over a field k. In this case, $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ has a Serre functor $- \otimes_X \omega[d]$, where ω is the canonical bundle of X and d is the dimension of X (Huybrechts [2006]).

Example 2.14 (Happel [1988] and Buchweitz, Iyama, and Yamaura [2018]). Let Λ be a finite dimensional k-algebra. Then $\mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$ has a Serre functor if and only if Λ is Iwanaga-Gorenstein, and $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$ has a Serre functor if and only if $\operatorname{gl.dim}\Lambda < \infty$. In both cases, the Serre functor is given by $\nu = - \overset{\mathbf{L}}{\otimes}_{\Lambda} (D\Lambda)$, and the AR translation is given by $\tau = \nu \circ [-1]$.

For AR theory of CM modules, we need the following notion.

Definition 2.15. Let R be a Gorenstein ring with $\dim R = d$. We denote by $\mathrm{CM}_0 R$ the full subcategory of $\mathrm{CM} R$ consisting of all X such that $X_{\mathfrak{p}} \in \operatorname{proj} R_{\mathfrak{p}}$ holds for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R_{\mathfrak{p}} < d$. When R is local, such an X is called *locally free on the punctured spectrum* (Yoshino [1990]). If R is G-graded, we denote by $\mathrm{CM}_0^G R$ the full subcategory of $\mathrm{CM}^G R$ consisting of all X which belong to $\mathrm{CM}_0 R$ as ungraded R-modules.

As before, $\mathrm{CM}_0^G R$ is a Frobenius category, and $\underline{\mathrm{CM}}_0^G R$ is a triangulated category. Note that $\mathrm{CM}_0 R = \mathrm{CM} R$ holds if and only if R satisfies Serre's (R_{d-1}) condition (i.e. $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R_{\mathfrak{p}} < d$). This means that R has at worst an isolated singularity if R is local.

The following is a fundamental theorem of CM representations.

Theorem 2.16 (Auslander [1978] and Auslander and Reiten [1987]). *Let R be a Gorenstein ring with* $\dim R = d$. *Then $\underline{\mathrm{CM}}_0 R$ is a $(d-1)$-Calabi-Yau triangulated category. If R is G-graded and has an a-invariant $a \in G$, then $\underline{\mathrm{CM}}_0^G R$ has a Serre functor $(a)[d-1]$.*

Let us introduce a key notion in Auslander-Reiten theory. We call an additive category $\mathcal{C}$ *Krull-Schmidt* if any object in $\mathcal{C}$ is isomorphic to a finite direct sum of objects whose endomorphism rings are local. We denote by $\operatorname{ind}\mathcal{C}$ the set of isomorphism classes of indecomposable objects in $\mathcal{C}$.

Definition 2.17 (Happel [1988]). Let $\mathcal{T}$ be a Krull-Schmidt triangulated category. We call a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathcal{T}$ an *almost split triangle* if:

- X and Z are indecomposable, and $h \neq 0$ (i.e. the triangle does not split).
- Any morphism $W \to Z$ which is not a split epimorphism factors through g.
- Any morphism $X \to W$ which is not a split monomorphism factors through f.

We say that $\mathcal{T}$ *has almost split triangles* if for any indecomposable object X (resp. Z), there is an almost split triangle $X \to Y \to Z \to X[1]$.

There is a close connection between almost split triangles and Serre functors.

Theorem 2.18 (Reiten and Van den Bergh [2002]). *Let $\mathcal{T}$ be an R-linear Hom-finite Krull-Schmidt triangulated category. Then $\mathcal{T}$ has a Serre functor if and only if $\mathcal{T}$ has almost split triangles. In this case, $X \simeq \tau Z$ holds in each almost split triangle $X \to Y \to Z \to X[1]$ in $\mathcal{T}$.*

When $\mathcal{T}$ has almost split triangles, one can define the *AR quiver* of $\mathcal{T}$, which has $\operatorname{ind} \mathcal{T}$ as the set of vertices. It describes the structure of $\mathcal{T}$ (see Happel [1988]). Similarly, *almost split sequences* and the AR quiver are defined for exact categories (Assem, Simson, and Skowroński [2006] and Leuschke and Wiegand [2012]).

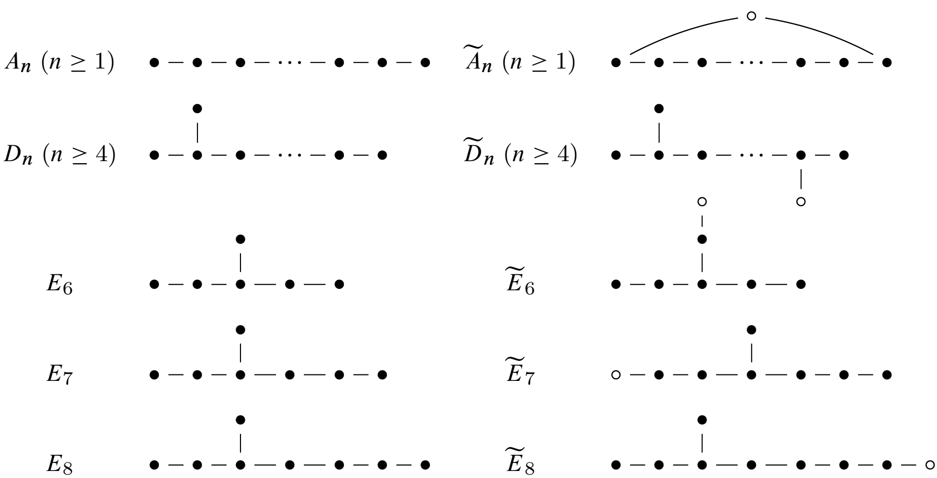
In the rest of this section, we discuss the following notion.

Definition 2.19. A finite dimensional k-algebra Λ is called *representation-finite* if $\operatorname{ind}(\operatorname{mod} \Lambda)$ is a finite set. It is also said to be *of finite representation type*.

The classification of representation-finite algebras was one of the main subjects in the 1980s. Here we recall only one theorem, and refer to Gabriel and Roĭter [1997] for further results.

A *Dynkin quiver* (resp. *extended Dynkin quiver*) is a quiver obtained by orienting each edge of one of the following diagrams A_n, D_n and E_n (resp. $\widetilde{A}_n$, $\widetilde{D}_n$ and $\widetilde{E}_n$).

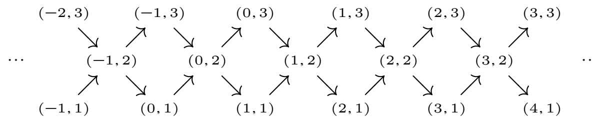
(2-2)



Now we are able to state Gabriel's Theorem below. For results in non-Dynkin case, we refer to Kac's theorem in Gabriel and Roĭter [ibid.].

Theorem 2.20 (Assem, Simson, and Skowroński [2006]). *Let Q be a connected acyclic quiver and k a field. Then kQ is representation-finite if and only if Q is Dynkin. In this case, there is a bijection between* $\operatorname{ind}(\operatorname{mod} kQ)$ *and the set Φ_+ of positive roots in the root system of Q.*

For a quiver Q, we define a new quiver $\mathbb{Z}Q$: The set of vertices is $\mathbb{Z} \times Q_0$. The arrows are $(\ell, a)\colon (\ell, s(a)) \to (\ell, t(a))$ and $(\ell, a^*)\colon (\ell, t(a)) \to (\ell+1, s(a))$ for each $\ell \in \mathbb{Z}$ and $a \in Q_1$. For example, if $Q = [1 \xrightarrow{a} 2 \xrightarrow{b} 3]$, then $\mathbb{Z}Q$ is as follows.



If the underlying graph Δ of Q is a tree, then $\mathbb{Z}Q$ depends only on Δ. Thus $\mathbb{Z}Q$ is written as $\mathbb{Z}\Delta$.

The AR quiver of $\mathrm{D^b}(\mathrm{mod}\, kQ)$ has a simple description (Happel [1988]).

Proposition 2.21 (Happel [ibid.])**.** (a) *Let Λ be a finite dimensional hereditary algebra. Then there is a bijection* $\mathrm{ind}(\mathrm{mod}\,\Lambda) \times \mathbb{Z} \to \mathrm{ind}\, \mathrm{D^b}(\mathrm{mod}\,\Lambda)$ *given by* $(X, i) \mapsto X[i]$.

(b) *For each Dynkin quiver Q, the AR quiver of* $\mathrm{D^b}(\mathrm{mod}\, kQ)$ *is* $\mathbb{Z}Q^{\mathrm{op}}$. *Moreover, the category* $\mathrm{D^b}(\mathrm{mod}\, kQ)$ *is presented by the quiver* $\mathbb{Z}Q^{\mathrm{op}}$ *with* mesh relations.

Note that $\mathbb{Z}Q$ has an automorphism τ given by $\tau(\ell, i) = (\ell-1, i)$ for $(\ell, i) \in \mathbb{Z} \times Q_0$, which corresponds to the AR translation.

Now we discuss CM-finiteness. For an additive category $\mathcal{C}$ and an object $M \in \mathcal{C}$, we denote by $\mathrm{add}\, M$ the full subcategory of $\mathcal{C}$ consisting of direct summands of finite direct sum of copies of M. We call M an *additive generator* of $\mathcal{C}$ if $\mathcal{C} = \mathrm{add}\, M$.

Definition 2.22. An Iwanaga-Gorenstein ring Λ is called *CM-finite* if $\mathrm{CM}\,\Lambda$ has an additive generator M. In this case, we call $\mathrm{End}_\Lambda(M)$ the *Auslander algebra*. When $\mathrm{CM}\,\Lambda$ is Krull-Schmidt, Λ is CM-finite if and only if $\mathrm{ind}(\mathrm{CM}\,\Lambda)$ is a finite set. It is also said to be *of finite CM type* or *representation-finite*.

Let us recall the classification of CM-finite Gorenstein rings given in the 1980s. Let k be an algebraically closed field of characteristic zero. A hypersurface $R = k[[x, y, z_2 \dots, z_d]]/(f)$ is called a *simple singularity* if

(2-3)
$$f = \begin{cases} x^{n+1} + y^2 + z_2^2 + \cdots + z_d^2 & A_n \\ x^{n-1} + xy^2 + z_2^2 + \cdots + z_d^2 & D_n \\ x^4 + y^3 + z_2^2 + \cdots + z_d^2 & E_6 \\ x^3y + y^3 + z_2^2 + \cdots + z_d^2 & E_7 \\ x^5 + y^3 + z_2^2 + \cdots + z_d^2 & E_8. \end{cases}$$

We are able to state the following result. We refer to Leuschke and Wiegand [2012] for results in positive characteristic.

Theorem 2.23 (Buchweitz, Greuel, and Schreyer [1987] and Knörrer [1987]). *Let R be a complete local Gorenstein ring containing the residue field k, which is an algebraically closed field of characteristic zero. Then R is CM-finite if and only if it is a simple singularity.*

We will see that tilting theory explains why Dynkin quivers appear in both Theorems 2.20 and 2.23 (see Example 4.5 and Corollary 5.2 below).

Now we describe the AR quivers of simple singularities. Recall that each quiver Q gives a new quiver $\mathbb{Z}Q$. For an automorphism ϕ of $\mathbb{Z}Q$, an orbit quiver $\mathbb{Z}Q/\phi$ is naturally defined. For example, $\mathbb{Z}Q/\tau$ is the *double* $\overline{Q}$ of Q obtained by adding an inverse arrow $a^*: j \to i$ for each arrow $a: i \to j$.

Proposition 2.24 (Yoshino [1990] and Dieterich and Wiedemann [1986]). *Let R be a simple singularity with* $\dim R = d$. *Then the AR quiver of* $\underline{\mathrm{CM}}R$ *is* $\mathbb{Z}\Delta/\phi$, *where* Δ *and* ϕ *are given as follows.*

(a) *If d is even, then Δ is the Dynkin diagram of the same type as R, and $\phi = \tau$.*

(b) *If d is odd, then Δ and ϕ are given as follows.*

R	A_{2n-1}	A_{2n}	D_{2n}	D_{2n+1}	E_6	E_7	E_8
Δ	D_{n+1}	A_{2n}	D_{2n}	A_{4n-1}	E_6	E_7	E_8
ϕ	$\tau\iota$	$\tau^{1/2}$	τ^2	$\tau\iota$	$\tau\iota$	τ^2	τ^2

Here ι is the involution of $\mathbb{Z}\Delta$ induced by the non-trivial involution of Δ, and $\tau^{1/2}$ is the automorphism of $\mathbb{Z}A_{2n}$ satisfying $(\tau^{1/2})^2 = \tau$.

In dimension 2, simple singularities (over a sufficiently large field) have an alternative description as invariant subrings. This enables us to draw the AR quiver of the category CM R systematically.

Example 2.25 (Auslander [1986] and Leuschke and Wiegand [2012]). Let $k[[u, v]]$ be a formal power series ring over a field k and G a finite subgroup of $\mathrm{SL}_2(k)$ such that $\#G$ is non-zero in k. Then CM S^G = add S holds, and the Auslander algebra $\mathrm{End}_{S^G}(S)$ is isomorphic to the *skew group ring* $S * G$. This is a free S-module with basis G, and the multiplication is given by $(sg)(s'g') = sg(s')gg'$ for $s, s' \in S$ and $g, g' \in G$. Thus the AR quiver of CM S^G coincides with the *Gabriel quiver* of $S * G$, and hence with the *McKay quiver* of G, which is the double of an extended Dynkin quiver. This is called *algebraic McKay correspondence*.

On the other hand, the dual graph of the exceptional curves in the minimal resolution X of the singularity Spec S^G is a Dynkin graph. This is called *geometric McKay correspondence*. There is a geometric construction of CM S^G-modules using X (Artin and Verdier [1985]), which is a prototype of non-commutative crepant resolutions (Van den Bergh [2004b,a]).

3 Tilting and cluster tilting

3.1 Tilting theory. Tilting theory is a Morita theory for triangulated categories. It has an origin in Bernstein-Gelfand-Ponomarev reflection for quiver representations, and established by works of Brenner-Butler, Happel-Ringel, Rickard, Keller and others (see e.g. Angeleri Hügel, Happel, and Krause [2007]). The class of silting objects was introduced to complete the class of tilting objects in the study of t-structures (Keller and Vossieck [1988]) and mutation (Aihara and Iyama [2012]).

Definition 3.1. Let $\mathcal{T}$ be a triangulated category. A full subcategory of $\mathcal{T}$ is *thick* if it is closed under cones, $[\pm 1]$ and direct summands. We call an object $T \in \mathcal{T}$ *tilting* (resp. *silting*) if $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ holds for all integers $i \neq 0$ (resp. $i > 0$), and the smallest thick subcategory of $\mathcal{T}$ containing T is $\mathcal{T}$.

The principal example of tilting objects appears in $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ for a ring Λ. It has a tilting object given by the stalk complex Λ concentrated in degree zero. Conversely, any triangulated category with a tilting object is triangle equivalent to $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ under mild assumptions (see Kimura [2016] for a detailed proof).

Theorem 3.2 (Keller [1994]). *Let $\mathcal{T}$ be an algebraic triangulated category and $T \in \mathcal{T}$ a tilting object. If $\mathcal{T}$ is idempotent complete, then there is a triangle equivalence $\mathcal{T} \simeq \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\mathrm{End}_{\mathcal{T}}(T))$ sending T to $\mathrm{End}_{\mathcal{T}}(T)$.*

As an application, one can deduce Rickard's fundamental Theorem (Rickard [1989b]), characterizing when two rings are derived equivalent in terms of tilting objects. Another application is the following converse of Proposition 2.21(b).

Example 3.3. Let $\mathcal{T}$ be a k-linear Hom-finite Krull-Schmidt algebraic triangulated category over an algebraically closed field k. If the AR quiver of $\mathcal{T}$ is $\mathbb{Z}Q$ for a Dynkin quiver Q, then $\mathcal{T}$ has a tilting object $T = \bigoplus_{i \in Q_0}(0, i)$ for $(0, i) \in \mathbb{Z} \times Q_0 = (\mathbb{Z}Q)_0 = \mathrm{ind}\,\mathcal{T}$. Thus there is a triangle equivalence $\mathcal{T} \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,kQ^{\mathrm{op}})$.

The following is the first main problem we will discuss in this paper.

Problem 3.4. *Find a G-graded Iwanaga-Gorenstein ring Λ such that there is a triangle equivalence $\underline{\mathrm{CM}}^G \Lambda \simeq \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Gamma)$ for some ring Γ. Equivalently (by Theorem 3.2), find a G-graded Iwanaga-Gorenstein ring Λ such that there is a tilting object in $\underline{\mathrm{CM}}^G \Lambda$.*

3.2 Cluster tilting and higher Auslander-Reiten theory. The notion of cluster tilting appeared naturally in a context of higher Auslander-Reiten theory (Iyama [2008]). It also played a central role in categorification of cluster algebras (Fomin and Zelevinsky [2002]) by using cluster categories, a new class of triangulated categories introduced in Buan, Marsh, Reineke, Reiten, and Todorov [2006], and preprojective algebras (Geiss, Leclerc, and Schröer [2013]). Here we explain only the minimum necessary background for the aim of this paper.

Let Λ be a finite dimensional k-algebra with $\mathrm{gl.dim}\,\Lambda \le d$. Then $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ has a Serre functor ν by Example 2.14. Using the *higher AR translation* $\nu_d := \nu \circ [-d]$ of $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$, the *orbit category* $\mathrm{C}^\circ_d(\Lambda) = \mathrm{D^b}(\mathrm{mod}\,\Lambda)/\nu_d$ is defined. It has the same objects as $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$, and the morphism space is given by

$$\mathrm{Hom}_{\mathrm{C}^\circ_d(\Lambda)}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}\,\Lambda)}(X, \nu_d^i(Y)),$$

where the composition is defined naturally. In general, $\mathrm{C}^\circ_d(\Lambda)$ does not have a natural structure of a triangulated category. The *d-cluster category* of Λ is a triangulated category $\mathrm{C}_d(\Lambda)$ containing $\mathrm{C}^\circ_d(\Lambda)$ as a full subcategory such that the composition $\mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{C}^\circ_d(\Lambda) \subset \mathrm{C}_d(\Lambda)$ is a triangle functor. It was constructed in Buan, Marsh, Reineke, Reiten, and Todorov [2006] for hereditary case where $\mathrm{C}_d(\Lambda) = \mathrm{C}^\circ_d(\Lambda)$ holds, and in Keller [2005, 2011], Amiot [2009], and Guo [2011] for general case by using a DG enhancement of $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$.

We say that Λ is *ν_d-finite* if $H^0(\nu_d^{-i}(\Lambda)) = 0$ holds for $i \gg 0$. This is automatic if $\mathrm{gl.dim}\,\Lambda < d$. In the hereditary case $d = 1$, Λ is ν_1-finite if and only if it is representation-finite. The following is a basic property of d-cluster categories.

Theorem 3.5 (Amiot [2009] and Guo [2011])**.** *Let Λ be a finite dimensional k-algebra with* $\mathrm{gl.dim}\,\Lambda \le d$. *Then Λ is ν_d-finite if and only if* $\mathrm{C}_d(\Lambda)$ *is Hom-finite. In this case,* $\mathrm{C}_d(\Lambda)$ *is a d-Calabi-Yau triangulated category.*

Thus, if Λ is ν_d-finite, then $\mathrm{C}_d(\Lambda)$ never has a tilting object. But the object Λ in $\mathrm{C}_d(\Lambda)$ still enjoys a similar property to tilting objects. Now we recall the following notion, introduced in Iyama [2007b] as a *maximal $(d-1)$-orthogonal* subcategory.

Definition 3.6 (Iyama [ibid.])**.** Let $\mathcal{T}$ be a triangulated or exact category and $d \ge 1$. We call a full subcategory $\mathcal{C}$ of $\mathcal{T}$ *d-cluster tilting* if $\mathcal{C}$ is a functorially finite subcategory of $\mathcal{T}$ such that

$$\begin{aligned}\mathcal{C} &= \{X \in \mathcal{T} \mid \forall i \in \{1,2,\dots,d-1\}\ \mathrm{Ext}^i_{\mathcal{T}}(\mathcal{C}, X) = 0\}\\ &= \{X \in \mathcal{T} \mid \forall i \in \{1,2,\dots,d-1\}\ \mathrm{Ext}^i_{\mathcal{T}}(X, \mathcal{C}) = 0\}.\end{aligned}$$

We call an object $T \in \mathcal{T}$ *d-cluster tilting* if $\mathrm{add}\,T$ is a d-cluster tilting subcategory.

If $\mathcal{T}$ has a Serre functor $\mathbb{S}$, then it is easy to show $(\mathbb{S}\circ[-d])(\mathcal{C}) = \mathcal{C}$. Thus it is natural in our setting $\mathcal{T} = \mathrm{D^b}(\mathrm{mod}\,\Lambda)$ to consider the full subcategory

$$\mathrm{U}_d(\Lambda) := \mathrm{add}\{\nu_d^i(\Lambda) \mid i \in \mathbb{Z}\} \subset \mathrm{D^b}(\mathrm{mod}\,\Lambda). \tag{3-1}$$

Equivalently, $\mathrm{U}_d(\Lambda) = \pi^{-1}(\operatorname{add} \pi\Lambda)$ for the functor $\pi : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \to \mathrm{C}_d(\Lambda)$. In the hereditary case $d = 1$, $\mathrm{U}_1(\Lambda) = \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$ holds if Λ is representation-finite, and otherwise $\mathrm{U}_1(\Lambda)$ is the connected component of the AR quiver of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$ containing Λ. This observation is generalized as follows.

Theorem 3.7 (Amiot [2009] and Iyama [2011]). *Let Λ be a finite dimensional k-algebra with* $\operatorname{gl.dim} \Lambda \le d$. *If Λ is ν_d-finite, then $\mathrm{C}_d(\Lambda)$ has a d-cluster tilting object Λ, and* $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$ *has a d-cluster tilting subcategory* $\mathrm{U}_d(\Lambda)$.

We define a full subcategory of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$ by

$$\mathrm{D}^{d\mathbb{Z}}(\operatorname{mod} \Lambda) = \{X \in \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \mid \forall i \in \mathbb{Z} \setminus d\mathbb{Z},\ H^i(X) = 0\}.$$

If $\operatorname{gl.dim} \Lambda \le d$, then any object in $\mathrm{D}^{d\mathbb{Z}}(\operatorname{mod} \Lambda)$ is isomorphic to a finite direct sum of $X[di]$ for some $X \in \operatorname{mod} \Lambda$ and $i \in \mathbb{Z}$. This generalizes Proposition 2.21(a) for hereditary algebras, and motivates the following definition.

Definition 3.8 (Herschend, Iyama, and Oppermann [2014]). Let $d \ge 1$. A finite dimensional k-algebra Λ is called *d-hereditary* if $\operatorname{gl.dim} \Lambda \le d$ and $\mathrm{U}_d(\Lambda) \subset \mathrm{D}^{d\mathbb{Z}}(\operatorname{mod} \Lambda)$.

It is clear that 1-hereditary algebras are precisely hereditary algebras. We have the following dichotomy of d-hereditary algebras.

Theorem 3.9 (Herschend, Iyama, and Oppermann [ibid.]). *Let Λ be a ring-indecomposable finite dimensional k-algebra with* $\operatorname{gl.dim} \Lambda \le d$. *Then Λ is d-hereditary if and only if either (i) or (ii) holds:*

(i) *There exists a d-cluster tilting object in* $\operatorname{mod} \Lambda$.

(ii) $\nu_d^{-i}(\Lambda) \in \operatorname{mod} \Lambda$ *holds for any* $i \ge 0$.

When $d = 1$, the above (i) holds if and only if Λ is representation-finite, and the above (ii) holds if and only if Λ is d-representation-infinite.

Definition 3.10. Let Λ be a d-hereditary algebra. We call Λ *d-representation-finite* if the above (i) holds, and *d-representation-infinite* if the above (ii) holds.

Example 3.11. (a) Let $\Lambda = kQ$ for a connected acyclic quiver Q. Then Λ is 1-representation-finite if Q is Dynkin, and 1-representation-infinite otherwise.

(b) Let X be a smooth projective variety with $\dim X = d$, and $T \in \operatorname{coh} X$ a tilting object in $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$. Then $\Lambda = \operatorname{End}_X(T)$ always satisfies $\operatorname{gl.dim} \Lambda \ge d$. If the equality holds, then Λ is d-representation-infinite (Buchweitz and Hille [2014]).

(c) There is a class of finite dimensional k-algebras called *Fano algebras* (Minamoto [2012] and Minamoto and Mori [2011]) in non-commutative algebraic geometry. So-called *extremely Fano algebras* Λ with $\operatorname{gl.dim}\Lambda = d$ are d-representation-infinite.

It is known that d-cluster tilting subcategories of a triangulated (resp. exact) category $\mathcal{T}$ enjoy various properties which should be regarded as higher analogs of those of $\mathcal{T}$. For example, they have *almost split* $(d+2)$*-angles* by Iyama and Yoshino [2008] (resp. d*-almost split sequences* by Iyama [2007b]), and structures of $(d+2)$*-angulated categories* by Geiss, Keller, and Oppermann [2013] (resp. d*-abelian categories* by Jasso [2016]). These motivate the following definition.

Definition 3.12 (cf. Definition 2.22)**.** An Iwanaga-Gorenstein ring Λ is called d*-CM-finite* if there exists a d-cluster tilting object M in $\mathrm{CM}\,\Lambda$. In this case, we call $\mathrm{End}_\Lambda(M)$ the d*-Auslander algebra* and $\underline{\mathrm{End}}_\Lambda(M)$ the *stable* d*-Auslander algebra*.

1-CM-finiteness coincides with classical CM-finiteness since 1-cluster tilting objects are precisely additive generators. d*-Auslander correspondence* gives a characterization of a certain nice class of algebras with finite global dimension as d-Auslander algebras (Iyama [2007a]). As a special case, it gives a connection with non-commutative crepant resolutions (NCCRs) of Van den Bergh [2004a]. Recall that a reflexive module M over a Gorenstein ring R gives an *NCCR* $\mathrm{End}_R(M)$ of R if $\mathrm{End}_R(M)$ is a non-singular R-order (see Example 2.3(c)).

Theorem 3.13 (Iyama [2007a])**.** *Let R be a Gorenstein ring with* $\dim R = d+1$*. Assume* $M \in \mathrm{CM}\,R$ *has R as a direct summand. Then M is a d-cluster tilting object in* $\mathrm{CM}\,R$ *if and only if M gives an NCCR of R and R satisfies Serre's (R_d) condition.*

The following generalizes Example 2.25.

Example 3.14 (Iyama [2007b] and Van den Bergh [2004a])**.** Let $S = k[[x_0, \dots, x_d]]$ be a formal power series ring and G a finite subgroup of $\mathrm{SL}_{d+1}(k)$ such that $\#G$ is non-zero in k. Then the S^G-module S gives an NCCR $\mathrm{End}_{S^G}(S) = S * G$ of S^G. If S^G has at worst an isolated singularity, then S is a d-cluster tilting object in $\mathrm{CM}\,S^G$, and hence S^G is d-CM-finite with the d-Auslander algebra $S * G$. As in Example 2.25, the quiver of $\operatorname{add} S$ coincides with the Gabriel quiver of $S * G$ and with the McKay quiver of G.

The following is the second main problem we will discuss in this paper.

Problem 3.15. *Find a d-CM-finite Iwanaga-Gorenstein ring. More strongly (by Theorem 3.7), find an Iwanaga-Gorenstein ring Λ such that there is a triangle equivalence* $\underline{\mathrm{CM}}\Lambda \simeq \mathrm{C}_d(\Gamma)$ *for some algebra* Γ.

We refer to Erdmann and Holm [2008] and Bergh [2014] for some necessary conditions for d-CM-finiteness. Besides results in this paper, a number of examples of NCCRs have been found, see e.g. Leuschke [2012], Wemyss [2016], and Špenko and Van den Bergh [2017] and references therein.

It is natural to ask how the notion of d-CM-finiteness is related to CM-tameness (e.g. Burban and Y. Drozd [2008]) and also the representation type of homogeneous coordinate rings of projective varieties (e.g. Faenzi and Malaspina [2017]).

4 Results in dimension 0 and 1

4.1 Dimension zero. In this subsection, we consider finite dimensional Iwanaga-Gorenstein algebras. We start with a classical result due to Happel [1988]. Let Λ be a finite dimensional k-algebra. The *trivial extension algebra* of Λ is $T(\Lambda) = \Lambda \oplus D\Lambda$, where the multiplication is given by $(\lambda, f)(\lambda', f') = (\lambda\lambda', \lambda f' + f\lambda')$ for $(\lambda, f), (\lambda', f') \in T(\Lambda)$. This is clearly a self-injective k-algebra, and has a $\mathbb{Z}$-grading given by $T(\Lambda)_0 = \Lambda$, $T(\Lambda)_1 = D\Lambda$ and $T(\Lambda)_i = 0$ for $i \neq 0, 1$.

Theorem 4.1 (Happel [ibid.]). *Let Λ be a finite dimensional k-algebra with* $\operatorname{gl.dim}\Lambda < \infty$. *Then $\underline{\operatorname{mod}}^{\mathbb{Z}} T(\Lambda)$ has a tilting object Λ such that $\underline{\operatorname{End}}^{\mathbb{Z}}_{T(\Lambda)}(\Lambda) \simeq \Lambda$, and there is a triangle equivalence*

$$\underline{\operatorname{mod}}^{\mathbb{Z}} T(\Lambda) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda). \tag{4-1}$$

As an application, it follows from Gabriel's Theorem 2.20 and covering theory that $T(kQ)$ is representation-finite for any Dynkin quiver Q. More generally, a large family of representation-finite self-injective algebras was constructed from Theorem 4.1. See a survey article (Skowroński [2006]).

Recently, Theorem 4.1 was generalized to a large class of $\mathbb{Z}$-graded self-injective algebras Λ. For $X \in \operatorname{mod}^{\mathbb{Z}} \Lambda$, let $X_{\geq 0} = \bigoplus_{i \geq 0} X_i$.

Theorem 4.2 (Yamaura [2013]). *Let $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ be a $\mathbb{Z}$-graded finite dimensional self-injective k-algebra such that* $\operatorname{gl.dim}\Lambda_0 < \infty$. *Then $\underline{\operatorname{mod}}^{\mathbb{Z}}\Lambda$ has a tilting object $T = \bigoplus_{i>0} \Lambda(i)_{\geq 0}$, and there is a triangle equivalence $\underline{\operatorname{mod}}^{\mathbb{Z}}\Lambda \simeq \mathrm{K}^{\mathrm{b}}(\operatorname{proj}\underline{\operatorname{End}}^{\mathbb{Z}}_{\Lambda}(T))$.*

If $\operatorname{soc}\Lambda \subset \Lambda_a$ for some $a \in \mathbb{Z}$, then $\underline{\operatorname{End}}^{\mathbb{Z}}_{\Lambda}(T)$ has a simple description

$$\underline{\operatorname{End}}^{\mathbb{Z}}_{\Lambda}(T) \simeq \begin{bmatrix} \Lambda_0 & 0 & \cdots & 0 & 0 \\ \Lambda_1 & \Lambda_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda_{a-2} & \Lambda_{a-3} & \cdots & \Lambda_0 & 0 \\ \Lambda_{a-1} & \Lambda_{a-2} & \cdots & \Lambda_1 & \Lambda_0 \end{bmatrix}.$$

For example, if $\Lambda = k[x]/(x^{a+1})$ with $\deg x = 1$, then $\underline{\mathrm{End}}_\Lambda^{\mathbb{Z}}(T)$ is the path algebra $k\mathbb{A}_a$ of the quiver of type A_a.

We end this subsection with posing the following open problem.

Problem 4.3. *Let $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ be a $\mathbb{Z}$-graded finite dimensional Iwanaga-Gorenstein algebra. When does $\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda$ have a tilting object?*

Recently, it was shown in Lu and Zhu [2017] and Kimura, Minamoto, and Yamaura [n.d.] independently that if $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ is a $\mathbb{Z}$-graded finite dimensional 1-Iwanaga-Gorenstein algebra satisfying $\mathrm{gl.dim}\,\Lambda_0 < \infty$, then the stable category $\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda$ has a silting object. We will see some other results in Section 5.4. We refer to Darpö and Iyama [2017] for some results on Problem 3.15.

4.2 Dimension one.

In this subsection, we consider a $\mathbb{Z}$-graded Gorenstein ring $R = \bigoplus_{i\geq 0} R_i$ with $\dim R = 1$ such that R_0 is a field. Let S be the multiplicative set of all homogeneous non-zerodivisors of R, and $K = RS^{-1}$ the $\mathbb{Z}$-graded total quotient ring. Then there exists a positive integer p such that $K(p) \simeq K$ as $\mathbb{Z}$-graded R-modules. In this setting, we have the following result (see Definitions 2.15 and 2.11 for $\mathrm{CM}_0^{\mathbb{Z}} R$ and the a-invariant).

Theorem 4.4 (Buchweitz, Iyama, and Yamaura [2018])**.** *Let $R = \bigoplus_{i\geq 0} R_i$ be a $\mathbb{Z}$-graded Gorenstein ring with $\dim R = 1$ such that R_0 is a field, and a the a-invariant of R.*

(a) *Assume $a \geq 0$. Then $\underline{\mathrm{CM}}_0^{\mathbb{Z}} R$ has a tilting object $T = \bigoplus_{i=1}^{a+p} R(i)_{\geq 0}$, and there is a triangle equivalence $\underline{\mathrm{CM}}_0^{\mathbb{Z}} R \simeq \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\underline{\mathrm{End}}_R^{\mathbb{Z}}(T))$.*

(b) *Assume $a < 0$. Then $\underline{\mathrm{CM}}_0^{\mathbb{Z}} R$ has a silting object $\bigoplus_{i=1}^{a+p} R(i)_{\geq 0}$. Moreover, it has a tilting object if and only if R is regular.*

An important tool in the proof is Theorem 2.12. The endomorphism algebra of T above has the following description.

$$\underline{\mathrm{End}}_R^{\mathbb{Z}}(T) = \begin{bmatrix} R_0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ R_1 & R_0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ R_{a-2} & R_{a-3} & \cdots & R_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ R_{a-1} & R_{a-2} & \cdots & R_1 & R_0 & 0 & 0 & \cdots & 0 & 0 \\ K_a & K_{a-1} & \cdots & K_2 & K_1 & K_0 & K_{-1} & \cdots & K_{2-p} & K_{1-p} \\ K_{a+1} & K_a & \cdots & K_3 & K_2 & K_1 & K_0 & \cdots & K_{3-p} & K_{2-p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{a+p-2} & K_{a+p-3} & \cdots & K_p & K_{p-1} & K_{p-2} & K_{p-3} & \cdots & K_0 & K_{-1} \\ K_{a+p-1} & K_{a+p-2} & \cdots & K_{p+1} & K_p & K_{p-1} & K_{p-2} & \cdots & K_1 & K_0 \end{bmatrix}.$$

As an application, we obtain the following graded version of Proposition 2.24(b).

Example 4.5. Let $R = k[x, y]/(f)$ be a simple singularity (2-3) with $\dim R = 1$ and the grading given by the list below. Then there is a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, kQ)$, where Q is the Dynkin quiver in the list below. In particular, the AR quiver of $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ is $\mathbb{Z}Q^{\mathrm{op}}$ (Araya [1999]).

R	A_{2n-1}	A_{2n}	D_{2n}	D_{2n+1}	E_6	E_7	E_8
$(\deg x, \deg y)$	$(1, n)$	$(2, 2n+1)$	$(1, n-1)$	$(2, 2n-1)$	$(3, 4)$	$(2, 3)$	$(3, 5)$
Q	D_{n+1}	A_{2n}	D_{2n}	A_{4n-1}	E_6	E_7	E_8

This gives a conceptual proof of the classical result that simple singularities in dimension 1 are CM-finite (Jacobinski [1967], J. A. Drozd and Roĭter [1967], and Greuel and Knörrer [1985]).

In the following special case, one can construct a different tilting object, whose endomorphism algebra is 2-representation-finite (Definition 3.10). This is closely related to the 2-cluster tilting object constructed in Burban, Iyama, Keller, and Reiten [2008].

Theorem 4.6 (Herschend and Iyama [n.d.]). *Let $R = k[x, y]/(f)$ be a hypersurface singularity with $f = f_1 f_2 \cdots f_n$ for linear forms f_i and $\deg x = \deg y = 1$. Assume that R is reduced.*

(a) *$\underline{\mathrm{CM}}^{\mathbb{Z}} R$ has a tilting object*

$$U = \bigoplus_{i=1}^{n} (k[x, y]/(f_1 f_2 \cdots f_i) \oplus k[x, y]/(f_1 f_2 \cdots f_i)(1))$$

(b) *$\underline{\mathrm{End}}_R^{\mathbb{Z}}(U)$ is a 2-representation-finite algebra. It is the Jacobian algebra of a certain quiver with potential.*

We refer to Demonet and Luo [2016], Jensen, King, and Su [2016], and Gelinas [2017] for other results in dimension one.

5 Preprojective algebras

5.1 Classical preprojective algebras. Preprojective algebras are widely studied objects with various applications, e.g. cluster algebras (Geiss, Leclerc, and Schröer [2013]), quantum groups (Kashiwara and Y. Saito [1997] and Lusztig [1991]), quiver varieties (Nakajima [1994]). Here we discuss a connection to CM representations.

Let Q be an acyclic quiver, and $\overline{Q}$ the double of Q obtained by adding an inverse arrow $a^*: j \to i$ for each arrow $a: i \to j$ in Q. The *preprojective algebra* of Q is the factor algebra of the path algebra $k\overline{Q}$ defined by

$$\Pi = k\overline{Q}/(\sum_{a \in Q_1} (aa^* - a^*a)). \tag{5-1}$$

We regard Π as a $\mathbb{Z}$-graded algebra by $\deg a = 0$ and $\deg a^* = 1$ for any $a \in Q_1$. Clearly $\Pi_0 = kQ$ holds. Moreover $\Pi_1 = \mathrm{Ext}^1_{kQ}(D(kQ), kQ)$ as a kQ-bimodule, and Π is isomorphic to the tensor algebra $T_{kQ}\,\mathrm{Ext}^1_{kQ}(D(kQ), kQ)$. Thus the kQ-module Π_i is isomorphic to the *preprojective kQ-module* $H^0(\tau^{-i}(kQ))$, where $\tau = \nu \circ [-1]$ is the AR translation. This is the reason why Π is called the preprojective algebra. Moreover, for the category $\mathrm{U}_1(kQ)$ defined in (3-1), there is an equivalence

$$\mathrm{U}_1(kQ) = \mathrm{add}\{\tau^{-i}(kQ) \mid i \in \mathbb{Z}\} \simeq \mathrm{proj}^{\mathbb{Z}}\,\Pi \tag{5-2}$$

given by $X \mapsto \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{U}_1(kQ)}(kQ, \tau^{-i}(X))$, which gives the following trichotomy.

Q	Dynkin	extended Dynkin	else
kQ	representation-finite	representation-tame	representation-wild
$\dim_k \Pi_i$	$\dim_k \Pi < \infty$	linear growth	exponential growth

It was known in 1980s that, Π in the extended Dynkin case has a close connection to simple singularities.

Theorem 5.1 (Auslander [1978], Geigle and Lenzing [1987, 1991], and Reiten and Van den Bergh [1989]). *Let Π be a preprojective algebra of an extended Dynkin quiver Q, e the vertex $\circ$ in* (2-2)*, and $R = e\Pi e$.*

(a) *R is a simple singularity $k[x, y, z]/(f)$ in dimension 2 with induced $\mathbb{Z}$-grading below, where p in type A_n is the number of clockwise arrows in Q. (Note that f coincides with* (2-3) *after a change of variables if k is sufficiently large.)*

Q, R	f	$(\deg x, \deg y, \deg z)$
A_n	$x^{n+1} - yz$	$(1, p, n+1-p)$
D_n	$x(y^2 + x^{\ell-1}y) + z^2$ if $n = 2\ell$ $x(y^2 + x^{\ell-1}z) + z^2$ if $n = 2\ell+1$	$(2, n-2, n-1)$
E_6	$x^2z + y^3 + z^2$	$(3, 4, 6)$
E_7	$x^3y + y^3 + z^2$	$(4, 6, 9)$
E_8	$x^5 + y^3 + z^2$	$(6, 10, 15)$

(b) *Πe is an additive generator of* CM R *and satisfies* $\mathrm{End}_R(\Pi e) = \Pi$. *Therefore R is CM-finite with an Auslander algebra* Π.

(c) Π *is Morita equivalent to the skew group ring* $k[u,v]*G$ *for a finite subgroup* G *of* $\mathrm{SL}_2(k)$ *if* k *is sufficiently large (cf. Example 2.25).*

By (b) and (5-2) above, there are equivalences $\mathrm{CM}^{\mathbb{Z}} R \simeq \mathrm{proj}^{\mathbb{Z}} \Pi \simeq \mathrm{U}_1(kQ) \subset \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, kQ)$. Thus the AR quivers of $\mathrm{CM}^{\mathbb{Z}} R$ and $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ are given by $\mathbb{Z}Q^{\mathrm{op}}$ and $\mathbb{Z}(Q^{\mathrm{op}}\backslash\{e\})$ respectively. Now the following result follows from Example 3.3.

Corollary 5.2. *Under the setting in Theorem 5.1, there is a triangle equivalence* $\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, kQ/(e))$.

Two other proofs were given in Kajiura, K. Saito, and A. Takahashi [2007], one uses explicit calculations of $\mathbb{Z}$-graded matrix factorizations, and the other uses Theorem 2.12. In Theorem 5.8 below, we deduce Corollary 5.2 from a general result on higher preprojective algebras. We refer to Kajiura, K. Saito, and A. Takahashi [2009] and Lenzing and de la Peña [2011] for results for some other hypersurfaces in dimension 2.

5.2 Higher preprojective algebras. There is a natural analog of preprojective algebras for finite dimensional algebras with finite global dimension.

Definition 5.3 (Iyama and Oppermann [2013]). Let Λ be a finite dimensional k-algebra with $\mathrm{gl.dim}\,\Lambda \le d$. We regard the highest extension $\mathrm{Ext}^d_\Lambda(D\Lambda,\Lambda)$ as a Λ-bimodule naturally, and define the $(d+1)$-*preprojective algebra* as the tensor algebra

$$\Pi_{d+1}(\Lambda) = T_\Lambda \mathrm{Ext}^d_\Lambda(D\Lambda,\Lambda).$$

This is the 0-th cohomology of the *Calabi-Yau completion* of Λ (Keller [2011]). For example, for an acyclic quiver Q, $\Pi_2(kQ)$ is the preprojective algebra (5-1).

The algebra $\Pi = \Pi_{d+1}(\Lambda)$ has an alternative description in terms of the higher AR translation $\nu_d = \nu \circ [-d]$ of $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$. The $\mathbb{Z}$-grading on Π is given by

$$\Pi_i = \mathrm{Ext}^d_\Lambda(D\Lambda,\Lambda)^{\otimes_\Lambda i} = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)}(\Lambda, \nu_d^{-i}(\Lambda))$$

for $i \ge 0$. Thus there is an isomorphism $\Pi \simeq \mathrm{End}_{\mathrm{C}_d(\Lambda)}(\Lambda)$ and an equivalence

$$\mathrm{U}_d(\Lambda) \simeq \mathrm{proj}^{\mathbb{Z}}\Pi \tag{5-3}$$

given by $X \mapsto \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{U}_d(\Lambda)}(\Lambda, \nu_d^{-i}(X))$. In particular, Π is finite dimensional if and only if Λ is ν_d-finite.

We see below that $\Pi_{d+1}(\Lambda)$ enjoys nice homological properties if Λ is d-hereditary.

Definition 5.4 (cf. Ginzburg [2006]). Let $\Gamma = \bigoplus_{i\ge 0}\Gamma_i$ be a $\mathbb{Z}$-graded k-algebra. We denote by $\Gamma^{\mathrm{e}} = \Gamma^{\mathrm{op}} \otimes_k \Gamma$ the enveloping algebra of Γ. We say that Γ is a d-*Calabi-Yau algebra of a-invariant* a (or *Gorenstein parameter* $-a$) if Γ belongs to $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}\Gamma^{\mathrm{e}})$ and $\mathbf{R}\mathrm{Hom}_{\Gamma^{\mathrm{e}}}(\Gamma,\Gamma^{\mathrm{e}})(a)[d] \simeq \Gamma$ holds in $\mathrm{D}(\mathrm{Mod}^{\mathbb{Z}}\Gamma^{\mathrm{e}})$.

For example, the $\mathbb{Z}$-graded polynomial algebra $k[x_1, \dots, x_d]$ with $\deg x_i = a_i$ is a d-Calabi-Yau algebra of a-invariant $-\sum_{i=1}^{n} a_i$.

Now we give a homological characterization of the $(d+1)$-preprojective algebras of d-representation-infinite algebras (Definition 3.10) as the explicit correspondence.

Theorem 5.5 (Keller [2011], Minamoto and Mori [2011], and Amiot, Iyama, and Reiten [2015])**.** *There exists a bijection between the set of isomorphism classes of d-representation-infinite algebras Λ and the set of isomorphism classes of $(d+1)$-Calabi-Yau algebras Γ of a-invariant -1. It is given by $\Lambda \mapsto \Pi_{d+1}(\Lambda)$ and $\Gamma \mapsto \Gamma_0$.*

Note that Γ above is usually non-noetherian. If Γ is right graded coherent, then for the category $\operatorname{qgr}\Gamma$ defined in (2-1), there is a triangle equivalence (Minamoto [2012])

$$\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{qgr}\Gamma). \tag{5-4}$$

Applying Theorem 5.5 for $d = 1, 2$, we obtain the following observations (see Van den Bergh [2015] for a structure theorem of (ungraded) Calabi-Yau algebras).

Example 5.6. Let k be an algebraically closed field.

(a) (cf. Bocklandt [2008]) 2-Calabi-Yau algebras of a-invariant -1 are precisely the preprojective algebras of disjoint unions of non-Dynkin quivers.

(b) (cf. Bocklandt [2008] and Herschend and Iyama [2011]) 3-Calabi-Yau algebras of a-invariant -1 are precisely the Jacobian algebras of quivers with 'good' potential with cuts.

The setting of our main result is the following.

Assumption 5.7. Let Γ be a $(d+1)$-Calabi-Yau algebras of a-invariant -1. Equivalently by Theorem 5.5, Γ is a $(d+1)$-preprojective algebra of some d-representation-infinite algebra. We assume that the following conditions hold for $\Lambda = \Gamma_0$.

(i) Γ is a noetherian ring, $e \in \Lambda$ is an idempotent and $\dim_k(\Gamma/(e)) < \infty$.

(ii) $e\Lambda(1-e) = 0$.

For example, let Q be an extended Dynkin quiver. If the vertex $\circ$ in (2-2) is a sink, then $\Gamma = \Pi_2(kQ)$ and $e = \circ$ satisfy Assumption 5.7 by Theorem 5.1.

Under Assumption 5.7(i), let $R = e\Gamma e$. Then R is a $(d+1)$-Iwanaga-Gorenstein ring, and the (Γ, R)-bimodule Γe plays an important role. It is a CM R-module, and gives a d-cluster tilting object in $\operatorname{CM} R$. Moreover the natural morphism $\Gamma \to \operatorname{End}_R(\Gamma e)$ is an isomorphism. Thus R is d-CM-finite and has a d-Auslander algebra Γ. The proof of these statements is parallel to Example 3.14.

Regarding Γe as a $\mathbb{Z}$-graded R-module, we consider the composition

$$F\colon \mathrm{D^b}(\operatorname{mod}\Lambda/(e)) \to \mathrm{D^b}(\operatorname{mod}\Lambda) \xrightarrow{-\overset{\mathbf{L}}{\otimes}_\Lambda \Gamma e} \mathrm{D^b}(\operatorname{mod}^{\mathbb{Z}} R) \to \underline{\mathrm{CM}}^{\mathbb{Z}} R,$$

where the first functor is induced from the surjective morphism $\Lambda \to \Lambda/(e)$, and the last functor is given by Theorem 2.10. Under Assumption 5.7(ii), F is shown to be a triangle equivalence. A crucial step is to show that F restricts to an equivalence $\mathrm{U}_d(\Lambda/(e)) \to \operatorname{add}\{\Gamma e(i) \mid i \in \mathbb{Z}\}$, which are d-cluster tilting subcategories of $\mathrm{D^b}(\operatorname{mod}\Lambda/(e))$ and $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ respectively (Theorem 3.7). Similarly, we obtain a triangle equivalence $\mathrm{C}_d(\Lambda/(e)) \simeq \underline{\mathrm{CM}} R$ by using universality of d-cluster categories (Keller [2005]). As a summary, we obtain the following results.

Theorem 5.8 (Amiot, Iyama, and Reiten [2015]). *Under Assumption 5.7(i), let $R = e\Gamma e$ and $\Lambda = \Gamma_0$.*

(a) *R is a $(d+1)$-Iwanaga-Gorenstein algebra, and Γe is a CM R-module.*

(b) *Γe is a d-cluster tilting object in* $\mathrm{CM}\, R$ *and satisfies* $\operatorname{End}_R(\Gamma e) = \Gamma$. *Thus R is d-CM-finite and has a d-Auslander algebra Γ*

(c) *If Assumption 5.7(ii) is satisfied, then there exist triangle equivalences*

$$\mathrm{D^b}(\operatorname{mod}\Lambda/(e)) \simeq \underline{\mathrm{CM}}^{\mathbb{Z}} R \;\; \textit{and} \;\; \mathrm{C}_d(\Lambda/(e)) \simeq \underline{\mathrm{CM}} R.$$

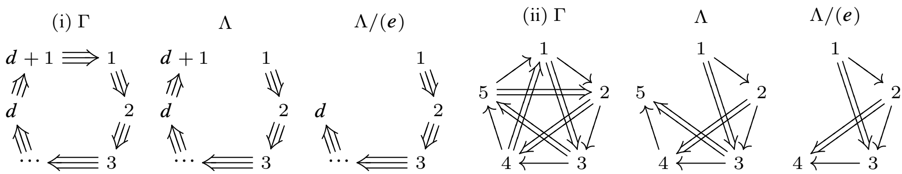
Similar triangle equivalences were given in de Völcsey and Van den Bergh [2016] and Kalck and Yang [2016] using different methods. There is a connection between (c) and (5-4) above via Theorem 2.12, see Amiot [2013].

In the case $d = 1$, the above (c) recovers Corollary 5.2 and a triangle equivalence $\underline{\mathrm{CM}} R \simeq \mathrm{C}_1(kQ/(e))$, which implies algebraic McKay correspondence in Example 2.25. Motivated by Example 3.14 and Theorem 5.1(c), we consider the following.

Example 5.9 (Amiot, Iyama, and Reiten [2015],Ueda [2008]). Let $S = k[x_0, \dots, x_d]$ be a polynomial algebra, and G a finite subgroup of $\mathrm{SL}_{d+1}(k)$. Then the skew group ring $\Gamma = S * G$ is a (ungraded) $(d+1)$-Calabi-Yau algebra. Assume that G is generated by the diagonal matrix $\operatorname{diag}(\zeta^{a_0}, \dots, \zeta^{a_d})$, where ζ is a primitive n-th root of unity and $0 \le a_j \le n-1$ for each j. Then Γ is presented by the McKay quiver of G, which has vertices $\mathbb{Z}/n\mathbb{Z}$, and arrows $x_j : i \to i + a_j$ for each i, j. Define a $\mathbb{Z}$-grading on Γ by $\deg(x_j : i \to i + a_j) = 0$ if $i < i + a_j$ as integers in $\{1, \dots, n\}$, and 1 otherwise. Then Γ is a $(d+1)$-Calabi-Yau algebra of a-invariant $-\sum_{0 \le j \le d} a_j/n$. Assume that this is -1, and let $e = e_n$. Then Assumption 5.7 is satisfied, and $e\Gamma e = S^G$ holds. Thus Theorem 5.8 gives triangle equivalences

$$\mathrm{D^b}(\operatorname{mod}\Lambda/(e)) \simeq \underline{\mathrm{CM}}^{\mathbb{Z}} S^G \;\; \text{and} \;\; \mathrm{C}_d(\Lambda/(e)) \simeq \underline{\mathrm{CM}} S^G.$$

Below we draw quivers for two cases (i) $n = d+1$ and $a_0 = \cdots = a_d = 1$, and (ii) $d = 2, n = 5$ and $(a_0, a_1, a_2) = (1, 2, 2)$.



In (i), $\Rrightarrow$ shows $d+1$ arrows, S^G is the Veronese subring $S^{(d+1)}$ and Λ is the Beilinson algebra. For $d = 2$, we recover the triangle equivalence $\mathrm{C}_2(kQ) \simeq \underline{\mathrm{CM}}S^G$ for $Q = [\,1 \Rrightarrow 2\,]$ given in Keller and Reiten [2008] and Keller, Murfet, and Van den Bergh [2011].

Note that similar triangle equivalences are given in Iyama and R. Takahashi [2013], Ueda [2012], and Mori and Ueyama [2016] for the skew group rings $S * G$ whose a-invariants are not equal to -1.

Example 5.10 (Dimer models)**.** Let G be a bipartite graph on a torus, and G_0 (resp. G_1, G_2) the set of vertices (resp. edges, faces) of G. We associate a quiver with potential (Q, W): The underlying graph of Q is the dual of the graph G, and faces of Q dual to white (resp. black) vertices are oriented clockwise (resp. anti-clockwise). Hence any vertex $v \in G_0$ corresponds to a cycle c_v of Q. Let $W = \sum_{v:\text{white}} c_v - \sum_{v:\text{black}} c_v$, and Γ the Jacobian algebra of (Q, W).

Under the assumption that G is *consistent*, Γ is a (ungraded) 3-Calabi-Yau algebra, and for any vertex e, $R = e\Gamma e$ is a Gorenstein toric singularity in dimension 3 (see Broomhead [2012] and Bocklandt [2012] and references therein). Using a perfect matching C on G, define a $\mathbb{Z}$-grading on Γ by $\deg a = 1$ for all $a \in C$ and $\deg a = 0$ otherwise. If both $\Gamma/(e)$ and $\Lambda = \Gamma_0$ are finite dimensional and $e\Lambda(1-e) = 0$ holds, then Theorem 5.8 gives triangle equivalences

$$\mathrm{D^b}(\mathrm{mod}\,\Lambda/(e)) \simeq \underline{\mathrm{CM}}^{\mathbb{Z}} R \quad \text{and} \quad \mathrm{C}_2(\Lambda/(e)) \simeq \underline{\mathrm{CM}} R.$$

5.3 d-representation-finite algebras. In this subsection, we study the $(d+1)$-preprojective algebras of d-representation-finite algebras. We start with the following basic properties.

Proposition 5.11 (Geiss, Leclerc, and Schröer [2006], Iyama [2011], and Iyama and Oppermann [2013])**.** *Let Λ be a d-representation-finite k-algebra and $\Pi = \Pi_{d+1}(\Lambda)$.*

(a) *Π is a $\mathbb{Z}$-graded finite dimensional self-injective k-algebra.*

(b) *$\underline{\mathrm{mod}}^{\mathbb{Z}}\Pi$ has a Serre functor $(-1)[d+1]$, and $\underline{\mathrm{mod}}\Pi$ is $(d+1)$-Calabi-Yau.*

(c) Π *is a (unique) d-cluster tilting object in* $\operatorname{mod}\Lambda$.

Now we give an explicit characterization of such Π.

Definition 5.12. Let $\Gamma = \bigoplus_{i\geq 0}\Gamma_i$ be a $\mathbb{Z}$-graded finite dimensional self-injective k-algebra. We denote by $\Gamma^{\mathrm{e}} = \Gamma^{\mathrm{op}}\otimes_k\Gamma$ the enveloping algebra of Γ. We say that Γ is a *stably d-Calabi-Yau algebra of a-invariant a* (or *Gorenstein parameter $-a$*) if $\mathbf{R}\mathrm{Hom}_{\Gamma^{\mathrm{e}}}(\Gamma,\Gamma^{\mathrm{e}})(a)[d]\simeq\Gamma$ in $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(\Gamma^{\mathrm{e}})$.

Now we give a homological characterization of the $(d+1)$-preprojective algebras of d-representation-finite algebras as the explicit correspondence.

Theorem 5.13 (Amiot and Oppermann [2014])**.** *There exists a bijection between the set of isomorphism classes of d-representation-finite algebras Λ and the set of isomorphism classes of stably $(d+1)$-Calabi-Yau self-injective algebras Γ of a-invariant -1. It is given by $\Lambda\mapsto\Pi_{d+1}(\Lambda)$ and $\Gamma\mapsto\Gamma_0$.*

Now let Λ be a d-representation-finite k-algebra, and $\Pi = \Pi_{d+1}(\Lambda)$. Let $\Gamma = \underline{\mathrm{End}}_{\Lambda}(\Pi)$ be the stable d-Auslander algebra of Λ. Then we have an equivalence

$$\mathrm{U}_d(\Lambda)\simeq\operatorname{proj}^{\mathbb{Z}}T(\Gamma) \tag{5-5}$$

of additive categories. Thus we have triangle equivalences

$$\underline{\operatorname{mod}}^{\mathbb{Z}}\Pi\overset{(5\text{-}3)}{\simeq}\underline{\operatorname{mod}}\mathrm{U}_d(\Lambda)\overset{(5\text{-}5)}{\simeq}\underline{\operatorname{mod}}^{\mathbb{Z}}T(\Gamma)\overset{(4\text{-}1)}{\simeq}\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Gamma).$$

By Proposition 5.11(b), the automorphism (-1) on $\underline{\operatorname{mod}}^{\mathbb{Z}}\Pi$ corresponds to ν_{d+1} on $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Gamma)$. Using universality of $(d+1)$-cluster categories (Keller [2005]), we obtain a triangle equivalence $\underline{\operatorname{mod}}\Pi\simeq\mathrm{C}_{d+1}(\Gamma)$. As a summary, we obtain the following.

Theorem 5.14 (Iyama and Oppermann [2013])**.** *Let Λ be a d-representation-finite k-algebra, $\Pi=\Pi_{d+1}(\Lambda)$, and $\Gamma=\underline{\mathrm{End}}_{\Lambda}(\Pi)$ the stable d-Auslander algebra of Λ. Then there exist triangle equivalences*

$$\underline{\operatorname{mod}}^{\mathbb{Z}}\Pi\simeq\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Gamma)\ \ \textit{and}\ \ \underline{\operatorname{mod}}\Pi\simeq\mathrm{C}_{d+1}(\Gamma).$$

Applying Theorem 5.14 for $d=1$, we obtain the following observations.

Example 5.15 (Amiot [2009] and Iyama and Oppermann [2013])**.** Let Π be the preprojective algebra of a Dynkin quiver Q, and Γ the stable Auslander algebra of kQ. Then there exist triangle equivalences

$$\underline{\operatorname{mod}}^{\mathbb{Z}}\Pi\simeq\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Gamma)\ \ \text{and}\ \ \underline{\operatorname{mod}}\Pi\simeq\mathrm{C}_2(\Gamma).$$

As an application, if a quiver Q' has the same underlying graph with Q, then the stable Auslander algebra Γ' of kQ' is derived equivalent to Γ since Π is common.

In the rest of this subsection, we discuss properties of $\Pi_{d+1}(\Lambda)$ for a more general class of Λ. We say that a finite dimensional k-algebra Λ with $\operatorname{gl.dim}\Lambda \leq d$ satisfies the *vosnex property* if Λ is ν_d-finite and satisfies $\operatorname{Hom}_{\mathrm{D^b}(\operatorname{mod}\Lambda)}(\mathrm{U}_d(\Lambda)[i], \mathrm{U}_d(\Lambda)) = 0$ for all $1 \leq i \leq d-2$. This is automatic if $d = 1, 2$ or Λ is d-representation-finite. In this case, the following generalization of Theorem 5.14 holds.

Theorem 5.16 (Iyama and Oppermann [2013]). *Let Λ be a finite dimensional k-algebra with* $\operatorname{gl.dim}\Lambda \leq d$ *satisfying the vosnex property. Then $\Pi = \Pi_{d+1}(\Lambda)$ is 1-Iwanaga-Gorenstein, $\Gamma = \underline{\operatorname{End}}_\Lambda(\Pi)$ satisfies* $\operatorname{gl.dim}\Gamma \leq d+1$, *and there exist triangle equivalences*

$$\underline{\mathrm{CM}}^{\mathbb{Z}}\Pi \simeq \mathrm{D^b}(\operatorname{mod}\Gamma) \quad \textit{and} \quad \underline{\mathrm{CM}}\Pi \simeq \mathrm{C}_{d+1}(\Gamma).$$

For more general Λ, we refer to Beligiannis [2015] for some properties of $\Pi_{d+1}(\Lambda)$.

5.4 Preprojective algebras and Coxeter groups. We discuss a family of finite dimensional k-algebras constructed from preprojective algebras and Coxeter groups.

Let Q be an acyclic quiver and Π the preprojective algebra of kQ. The *Coxeter group* of Q is generated by s_i with $i \in Q_0$, and the relations are the following.

- $s_i^2 = 1$ for all $i \in Q_0$.
- $s_i s_j = s_j s_i$ if there is no arrow between i and j in Q.
- $s_i s_j s_i = s_j s_i s_j$ if there is precisely one arrow between i and j in Q.

Let $w \in W$. An expression $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of w is called *reduced* if ℓ is minimal among all expressions of w. For $i \in Q_0$, let I_i be the two-sided ideal of Π generated by the idempotent $1 - e_i$. For a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, we define a two-sided ideal of Π by

$$I_w := I_{i_1} I_{i_2} \cdots I_{i_\ell}.$$

This is independent of the choice of the reduced expression of w. The corresponding factor algebra $\Pi_w := \Pi/I_w$ is a finite dimensional k-algebra. It enjoys the following remarkable properties.

Theorem 5.17 (Buan, Iyama, Reiten, and Scott [2009], Geiss, Leclerc, and Schröer [2007], and Amiot, Reiten, and Todorov [2011]). *Let $w \in W$.*

(a) *Π_w is a 1-Iwanaga-Gorenstein algebra.*

(b) *$\underline{\mathrm{CM}}\Pi_w$ is a 2-Calabi-Yau triangulated category.*

(c) *There exists a 2-cluster tilting object $\bigoplus_{j=1}^{\ell} e_{i_j} \Pi_{s_{i_j} \cdots s_{i_\ell}}$ in $\mathrm{CM}\,\Pi_w$.*

(d) *There exists a triangle equivalence* $\underline{\mathrm{CM}}\Pi_w \simeq \mathrm{C}_2(\Lambda)$ *for some algebra* Λ.

Therefore it is natural to expect that there exists a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}\Pi_w \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda')$ for some algebra Λ'. In fact, the following results are known, where we refer to Kimura [2018, 2016] for the definitions of *c-sortable*, *c-starting* and *c-ending*.

Theorem 5.18. *Let* $w = s_{i_1} \cdots s_{i_\ell}$ *be a reduced expression of* $w \in W$.

(a) *(Kimura [2018]) If w is c-sortable, then* $\underline{\mathrm{CM}}^{\mathbb{Z}}\Pi_w$ *has a tilting object* $\bigoplus_{i>0} \Pi_w(i)_{\geq 0}$.

(b) *(Kimura [2016])* $\underline{\mathrm{CM}}^{\mathbb{Z}}\Pi_w$ *has a silting object* $\bigoplus_{j=1}^{\ell} e_{i_j}\Pi_{s_{i_j}\cdots s_{i_\ell}}$. *This is a tilting object if the reduced expression is c-starting or c-ending.*

We end this section with posing the following natural question on 'higher cluster combinatorics' (e.g. Oppermann and Thomas [2012]), which will be related to derived equivalences of Calabi-Yau algebras since our I_w is a tilting object in $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Pi)$ if Q is non-Dynkin.

Problem 5.19. *Are there similar results to Theorems 5.17 and 5.18 for higher preprojective algebras? What kind of combinatorial structure will appear instead of the Coxeter groups?*

6 Geigle-Lenzing complete intersections

Weighted projective lines of Geigle and Lenzing [1987] are one of the basic objects in representation theory. For example, the simplest class of weighted projective lines gives us simple singularities in dimension 2 as certain Veronese subrings. We introduce a higher dimensional generalization of weighted projective lines following Herschend, Iyama, Minamoto, and Oppermann [2014].

6.1 Basic properties. For a field k and an integer $d \geq 1$, we consider a polynomial algebra $C = k[T_0, \ldots, T_d]$. For $n \geq 0$, let $\ell_1, \ldots, \ell_n$ be linear forms in C and $p_1, \ldots, p_n$ positive integers. For simplicity, we assume $p_i \geq 2$ for all i. Let

$$R = C[X_1, \ldots, X_n]/(X_i^{p_i} - \ell_i \mid 1 \leq i \leq n)$$

be the factor algebra of the polynomial algebra $C[X_1, \ldots, X_n]$, and

$$\mathbb{L} = \langle \vec{x}_1, \ldots, \vec{x}_n, \vec{c}\rangle / \langle p_i\vec{x}_i - \vec{c} \mid 1 \leq i \leq n\rangle.$$

the factor group of the free abelian group $\langle \vec{x}_1, \ldots, \vec{x}_n, \vec{c}\rangle$. Then $\mathbb{L}$ is an abelian group of rank 1 with torsion elements in general, and R is $\mathbb{L}$-graded by $\deg T_i = \vec{c}$ for all $0 \leq i \leq d$ and $\deg X_i = \vec{x}_i$ for all $1 \leq i \leq n$.

We call the pair $(R, \mathbb{L})$ a *Geigle-Lenzing (GL) complete intersection* if $\ell_1, \dots, \ell_n$ are in general position in the sense that each set of at most $d+1$ elements from $\ell_1, \dots, \ell_n$ is linearly independent. We give some basic properties.

Proposition 6.1. *Let $(R, \mathbb{L})$ be a GL complete intersection.*

(a) $X_1^{p_1} - \ell_1, \dots, X_n^{p_n} - \ell_n$ *is a $C[X_1, \dots, X_n]$-regular sequence.*

(b) *R is a complete intersection ring with* $\dim R = d+1$ *and has an a-invariant*

$$\vec{\omega} = (n-d-1)\vec{c} - \sum_{i=1}^{n} \vec{x}_i.$$

(c) *After a suitable linear transformation of variables $T_0, \dots, T_d$, we have*

$$R = \begin{cases} k[X_1, \dots, X_n, T_n, \dots, T_d] & \text{if } n \leq d+1, \\ k[X_1, \dots, X_n]/(X_i^{p_i} - \sum_{j=1}^{d+1} \lambda_{i,j-1} X_j^{p_j}) \mid d+2 \leq i \leq n) & \text{if } n \geq d+2. \end{cases}$$

(d) *R is regular if and only if R is a polynomial algebra if and only if $n \leq d+1$.*

(e) $\mathrm{CM}^{\mathbb{L}} R = \mathrm{CM}_0^{\mathbb{L}} R$ *holds, and $\underline{\mathrm{CM}}^{\mathbb{L}} R$ has a Serre functor $(\vec{\omega})[d]$ (Theorem 2.16).*

Let $\delta \colon \mathbb{L} \to \mathbb{Q}$ be a group homomorphism given by $\delta(\vec{x}_i) = \frac{1}{p_i}$ and $\delta(\vec{c}) = 1$. We consider the following trichotomy given by the sign of $\delta(\vec{\omega}) = n - d - 1 - \sum_{i=1}^{n} \frac{1}{p_i}$. For example, $(R, \mathbb{L})$ is Fano if $n \leq d+1$.

$\delta(\vec{\omega})$	< 0	$= 0$	> 0
$(R, \mathbb{L})$	Fano	Calabi-Yau	anti-Fano
$d = 1$	domestic	tubular	wild

In the classical case $d = 1$, the ring R has been studied in the context of weighted projective lines. The above trichotomy is given explicitly as follows.

- 5 types for domestic: $n \leq 2$, $(2, 2, p)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$.
- 4 types for tubular: $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ and $(2, 2, 2, 2)$.
- All other types are wild.

There is a close connection between domestic type and simple singularities. The following explains Corollary 5.2, where $R^{(\vec{\omega})} = \bigoplus_{i \in \mathbb{Z}} R_{i\vec{\omega}}$ is the Veronese subring.

Theorem 6.2 (Geigle and Lenzing [1991]). *If $(R, \mathbb{L})$ is domestic, then $R^{(\vec{\omega})}$ is a simple singularity $k[x, y, z]/(f)$ in dimension 2, and we have an equivalence $\mathrm{CM}^{\mathbb{L}} R \simeq \mathrm{CM}^{\mathbb{Z}} R^{(\vec{\omega})}$. The AR quiver is $\mathbb{Z}Q$, where Q is given by the following table.*

$(p_1, \dots, p_n)$	x	y	z	f	Q
(p, q)	X_1X_2	X_2^{p+q}	X_1^{p+q}	$x^{p+q} - yz$	$\widetilde{\mathbb{A}}_{p,q}$
$(2, 2, 2p)$	X_3^2	X_1^2	$X_1X_2X_3$	$x(y^2 + x^p y) + z^2$	$\widetilde{\mathbb{D}}_{2p+2}$
$(2, 2, 2p+1)$	X_3^2	X_1X_2	$X_1^2X_3$	$x(y^2 + x^p z) + z^2$	$\widetilde{\mathbb{D}}_{2p+3}$
$(2, 3, 3)$	X_1	X_2X_3	X_2^3	$x^2 z + y^3 + z^2$	$\widetilde{\mathbb{E}}_6$
$(2, 3, 4)$	X_2	X_3^2	X_1X_3	$x^3 y + y^3 + z^2$	$\widetilde{\mathbb{E}}_7$
$(2, 3, 5)$	X_3	X_2	X_1	$x^5 + y^3 + z^2$	$\widetilde{\mathbb{E}}_8$

6.2 Cohen-Macaulay representations. To study the category $\mathrm{CM}^{\mathbb{L}} R$, certain finite dimensional algebras play an important role. For a finite subset I of $\mathbb{L}$, let

$$A^I = \bigoplus_{\vec{x}, \vec{y} \in I} R_{\vec{x} - \vec{y}}.$$

We define the multiplication in A^I by $(r_{\vec{x},\vec{y}})_{\vec{x},\vec{y} \in I} \cdot (r'_{\vec{x},\vec{y}})_{\vec{x},\vec{y} \in I} = (\sum_{\vec{z} \in I} r_{\vec{x},\vec{z}} r'_{\vec{z},\vec{y}})_{\vec{x},\vec{y} \in I}$. Then A^I forms a finite dimensional k-algebra called the *I-canonical algebra*.

We define a partial order $\leq$ on $\mathbb{L}$ by writing $\vec{x} \leq \vec{y}$ if $\vec{y} - \vec{x}$ belongs to $\mathbb{L}_+$, where $\mathbb{L}_+$ is the submonoid of $\mathbb{L}$ generated by $\vec{c}$ and $\vec{x}_i$ for all i. For $\vec{x} \in \mathbb{L}$, let $[0, \vec{x}]$ be the interval in $\mathbb{L}$, and $A^{[0,\vec{x}]}$ the $[0, \vec{x}]$-canonical algebra. We call

$$A^{\mathrm{CM}} = A^{[0, d\vec{c} + 2\vec{\omega}]}$$

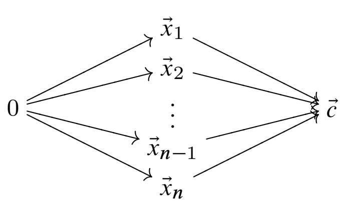
the *CM-canonical algebra*.

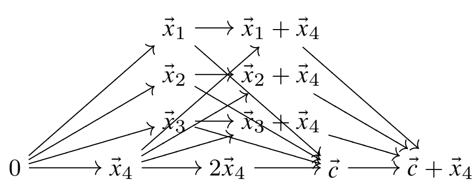
Example 6.3. The equality $d\vec{c} + 2\vec{\omega} = (n - d - 2)\vec{c} + \sum_{i=1}^{n} (p_i - 2)\vec{x}_i$ holds.

(a) If $n \leq d + 1$, then $A^{\mathrm{CM}} = 0$. If $n = d + 2$, then $A^{\mathrm{CM}} = \bigotimes_{i=1}^{n+2} k\mathbb{A}_{p_i - 1}$.

(b) If $n = d + 3$ and $p_i = 2$ for all i, then A^{CM} has the left quiver below.

(c) If $d = 1$, $n = 4$ and $(p_i)_{i=1}^{4} = (2, 2, 2, 3)$, then A^{CM} has the right quiver below.





The following is a main result in this section.

Theorem 6.4. *Let $(R, \mathbb{L})$ be a GL complete intersection. Then there is a triangle equivalence*

$$\underline{\mathrm{CM}}^{\mathbb{L}} R \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{CM}}).$$

In particular, $\underline{\mathrm{CM}}^{\mathbb{L}} R$ has a tilting object.

The case $n = d + 2$ was shown in Kussin, Lenzing, and Meltzer [2013] ($d = 1$) and Futaki and Ueda [2011]. An important tool in the proof is an $\mathbb{L}$-analogue of Theorem 2.12.

As an application, one can immediately obtain the following analogue of Theorem 2.23 by using the knowledge on A^{CM} in representation theory, where we call $(R, \mathbb{L})$ *CM-finite* if there are only finitely many isomorphism classes of indecomposable objects in $\mathrm{CM}^{\mathbb{L}} R$ up to degree shift (cf. Definition 2.22).

Corollary 6.5. *Let $(R, \mathbb{L})$ be a GL complete intersection. Then $(R, \mathbb{L})$ is CM-finite if and only if one of the following conditions hold.*

(i) $n \leq d + 1$.

(ii) $n = d + 2$, *and* $(p_1, \dots, p_n) = (2, \dots, 2, p_n)$, $(2, \dots, 2, 3, 3)$, $(2, \dots, 2, 3, 4)$ *or* $(2, \dots, 2, 3, 5)$ *up to permutation.*

We call a GL complete intersection $(R, \mathbb{L})$ *d-CM-finite* if there exists a d-cluster tilting subcategory $\mathcal{C}$ of $\mathrm{CM}^{\mathbb{L}} R$ such that there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{C}$ up to degree shift (cf. Definition 3.12). Now we discuss which GL complete intersections are d-CM-finite. Our Theorem 3.7 gives the following sufficient condition, where a tilting object is called *d-tilting* if the endomorphism algebra has global dimension at most d.

Proposition 6.6. *If $\underline{\mathrm{CM}}^{\mathbb{L}} R$ has a d-tilting object U, then $(R, \mathbb{L})$ is d-CM-finite and $\mathrm{CM}^{\mathbb{L}} R$ has the d-cluster tilting subcategory* $\mathrm{add}\{U(\ell\vec{\omega}), R(\vec{x}) \mid \ell \in \mathbb{Z}, \vec{x} \in \mathbb{L}\}$.

Therefore the following problem is of our interest.

Problem 6.7. *When does $\underline{\mathrm{CM}}^{\mathbb{L}} R$ have a d-tilting object? Equivalently, when is A^{CM} derived equivalent to an algebra Λ with* $\mathrm{gl.dim}\,\Lambda \leq d$?

Applying Tate's DG algebra resolutions (Tate [1957]), we can calculate $\mathrm{gl.dim}\, A^{\mathrm{CM}}$. Note that any element $\vec{x} \in \mathbb{L}$ can be written uniquely as $\vec{x} = a\vec{c} + \sum_{i=1}^{n} a_i \vec{x}_i$ for $a \in \mathbb{Z}$ and $0 \leq a_i \leq p_i - 1$, which is called the *normal form* of $\vec{x}$.

Theorem 6.8. (a) *Write $\vec{x} \in \mathbb{L}_+$ in normal form $\vec{x} = a\vec{c} + \sum_{i=1}^{n} a_i \vec{x}_i$. Then*

$$\mathrm{gl.dim}\, A^{[0,\vec{x}]} = \begin{cases} \min\{d+1, a + \#\{i \mid a_i \neq 0\}\} & \text{if } n \leq d+1, \\ 2a + \#\{i \mid a_i \neq 0\} & \text{if } n \geq d+2. \end{cases}$$

(b) *If* $n \geq d+2$*, then* A^{CM} *has global dimension* $2(n-d-2) + \#\{i \mid p_i \geq 3\}$.

We obtain the following examples from Theorem 6.8 and the fact that $k\mathbb{A}_2 \otimes_k k\mathbb{A}_m$ is derived equivalent to $k\mathbb{D}_4$ if $m=2$, $k\mathbb{E}_6$ if $m=3$, and $k\mathbb{E}_8$ if $m=4$.

Example 6.9. In the following cases, $\underline{\mathrm{CM}}^{\mathbb{L}} R$ has a d-tilting object.

(i) $n \leq d+1$.

(ii) $n = d+2 \geq 3$ and $(p_1, p_2, p_3) = (2,2,p_3)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$.

(iii) $n = d+2 \geq 4$ and $(p_1, p_2, p_3, p_4) = (3,3,p_3,p_4)$ with $p_3, p_4 \in \{3,4,5\}$.

(iv) $\#\{i \mid p_i = 2\} \geq 3(n-d)-4$.

The following gives a necessary condition for the existence of d-tilting object.

Proposition 6.10. *If* $\underline{\mathrm{CM}}^{\mathbb{L}} R$ *has a* d*-tilting object, then* $(R, \mathbb{L})$ *is Fano.*

Note that the converse is not true. For example, let $d=2$ and $(2,5,5,5)$. Then $(R,\mathbb{L})$ is Fano since $\delta(\vec{\omega}) = -\frac{1}{10}$. On the other hand, $A^{\mathrm{CM}} = \bigotimes_{i=1}^{3} k\mathbb{A}_4$ satisfies $\nu^5 = [9]$. One can show that A^{CM} is not derived equivalent to an algebra Λ with $\mathrm{gl.dim}\,\Lambda \leq 2$ by using the inequality $2(5-1) < 9$.

6.3 Geigle-Lenzing projective spaces. Let $(R,\mathbb{L})$ be a GL complete intersection. Recall that $\mathrm{mod}_0^{\mathbb{L}} R$ is the Serre subcategory of $\mathrm{mod}^{\mathbb{L}} R$ consisting of finite dimensional modules. We consider the quotient category

$$\mathrm{coh}\,\mathbb{X} = \mathrm{qgr}\,R = \mathrm{mod}^{\mathbb{L}} R / \mathrm{mod}_0^{\mathbb{L}} R.$$

We call objects in $\mathrm{coh}\,\mathbb{X}$ *coherent sheaves* on the *GL projective space* $\mathbb{X}$. We can regard $\mathbb{X}$ as the quotient stack $[(\mathrm{Spec}\,R \setminus \{R_+\}) / \mathrm{Spec}\,k[\mathbb{L}]]$ for $R_+ = \bigoplus_{\vec{x}>0} R_{\vec{x}}$. For example, if $n=0$, then $\mathbb{X}$ is the projective space $\mathbb{P}^d$.

We study the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbb{X})$, which is canonically triangle equivalent to the Verdier quotient $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{L}} R)/\mathrm{D}^{\mathrm{b}}(\mathrm{mod}_0^{\mathbb{L}} R)$. The duality $(-)^* = \mathbf{R}\mathrm{Hom}_R(-,R)\colon \mathrm{D}^{\mathrm{b}}($ $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{L}} R)$ induces a duality $(-)^*\colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbb{X}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbb{X})$. We define the category of *vector bundles* on $\mathbb{X}$ as

$$\mathrm{vect}\,\mathbb{X} = \mathrm{coh}\,\mathbb{X} \cap (\mathrm{coh}\,\mathbb{X})^*.$$

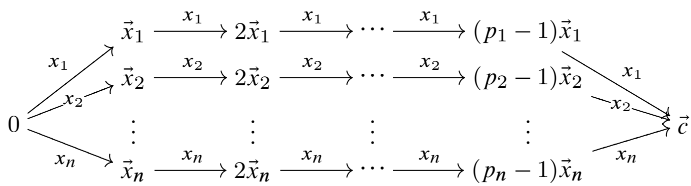
The composition $\mathrm{CM}^{\mathbb{L}} R \subset \mathrm{mod}^{\mathbb{L}} R \to \mathrm{coh}\,\mathbb{X}$ is fully faithful, and we can regard $\mathrm{CM}^{\mathbb{L}} R$ as a full subcategory of $\mathrm{vect}\,\mathbb{X}$. We have $\mathrm{CM}^{\mathbb{L}} R = \mathrm{vect}\,\mathbb{X}$ if $d=1$, but this is not the case if $d \geq 2$. In fact, we have equalities

$$\text{(6-1)}\qquad \begin{aligned}\mathrm{CM}^{\mathbb{L}} R &= \{X \in \operatorname{vect}\mathbb{X} \mid \forall \vec{x} \in \mathbb{L},\ 1 \leq i \leq d-1,\ \operatorname{Ext}^i_{\mathbb{X}}(\mathcal{O}(\vec{x}), X) = 0\} \\ &= \{X \in \operatorname{vect}\mathbb{X} \mid \forall \vec{x} \in \mathbb{L},\ 1 \leq i \leq d-1,\ \operatorname{Ext}^i_{\mathbb{X}}(X, \mathcal{O}(\vec{x})) = 0\}\end{aligned}$$

where $\mathcal{O}(\vec{x}) = R(\vec{x})$. Now we define the *$d$-canonical algebra* by

$$A^{\mathrm{ca}} = A^{[0, d\vec{c}]}.$$

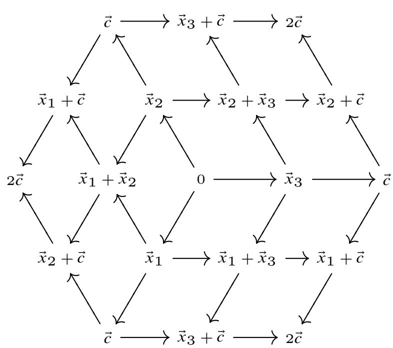
Example 6.11. (a) If $d = 1$, then A^{ca} is precisely the canonical algebra of Ringel [1984]. It is given by the following quiver with relations $x_i^{p_i} = \lambda_{i0} x_1^{p_1} + \lambda_{i1} x_2^{p_2}$ for any i with $3 \leq i \leq n$.

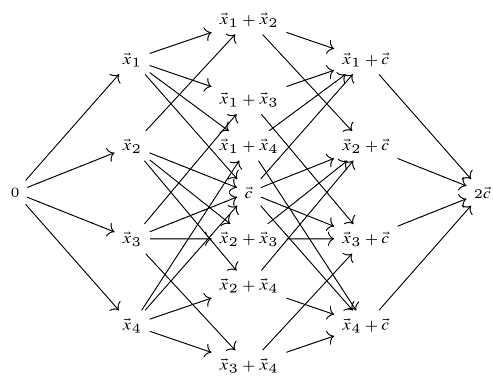


(b) If $n = 0$, then A^{ca} is the Beilinson algebra.

(c) If $d = 2$, $n = 3$ and $(p_i)_{i=1}^3 = (2, 2, 2)$, then A^{ca} has the left quiver below.

(d) If $d = 2$, $n = 4$ and $(p_i)_{i=1}^4 = (2, 2, 2, 2)$, then A^{ca} has the right quiver below.





As in the case of $\underline{\mathrm{CM}}^{\mathbb{L}} R$ and A^{CM}, we obtain the following results.

Theorem 6.12. *Let $\mathbb{X}$ be a GL projective space. Then there is a triangle equivalence*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X}) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{ca}}).$$

Moreover $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$ *has a tilting bundle* $\bigoplus_{\vec{x} \in [0, d\vec{c}]} \mathcal{O}(\vec{x})$.

Some cases were known before ($n = 0$ by Beilinson [1978], $d = 1$ by Geigle and Lenzing [1987], $n \leq d+1$ by Baer [1988], $n = d+2$ by Ishii and Ueda [2012]). An important tool in the proof is again an $\mathbb{L}$-analogue of Theorem 2.12.

We call $\mathbb{X}$ *vector bundle finite* (*VB-finite*) if there are only finitely many isomorphism classes of indecomposable objects in $\operatorname{vect}\mathbb{X}$ up to degree shift. There is a complete classification: $\mathbb{X}$ is VB-finite if and only if $d = 1$ and $\mathbb{X}$ is domestic.

We call $\mathbb{X}$ *d-VB-finite* if there exists a d-cluster tilting subcategory $\mathcal{C}$ of $\operatorname{vect}\mathbb{X}$ such that there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{C}$ up to degree shift. In the rest, we discuss which GL projective spaces are d-VB-finite. We start with the following relationship between d-cluster tilting subcategories of $\mathrm{CM}^{\mathbb{L}} R$ and $\operatorname{vect}\mathbb{X}$, which follows from (6-1).

Proposition 6.13. *The d-cluster-tilting subcategories of* $\mathrm{CM}^{\mathbb{L}} R$ *are precisely the d-cluster-tilting subcategories of* $\operatorname{vect}\mathbb{X}$ *containing* $\mathcal{O}(\vec{x})$ *for all* $\vec{x} \in \mathbb{L}$. *Therefore, if* $(R, \mathbb{L})$ *is d-CM-finite, then* $\mathbb{X}$ *is d-VB-finite.*

For example, if $n \leq d+1$, then $\mathrm{CM}^{\mathbb{L}} R = \operatorname{proj}^{\mathbb{L}} R$ is a d-cluster tilting subcategory of itself, and hence $\operatorname{vect}\mathbb{X}$ has a d-cluster tilting subcategory $\operatorname{add}\{\mathcal{O}(\vec{x}) \mid \vec{x} \in \mathbb{L}\}$. This implies Horrocks' splitting criterion for $\operatorname{vect}\mathbb{P}^d$ (Okonek, Schneider, and Spindler [1980]).

We give another sufficient condition for d-VB-finiteness. Recall that we call a tilting object V in $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$ *d-tilting* if $\operatorname{gl.dim}\operatorname{End}_{\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})}(V) \leq d$.

Proposition 6.14. *Let* $\mathbb{X}$ *be a GL projective space, and* V *a d-tilting object in* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$.

(a) *(cf. Example 3.11(b))* $\operatorname{gl.dim}\operatorname{End}_{\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})}(V) = d$ *holds. If* $V \in \operatorname{coh}\mathbb{X}$, *then* $\operatorname{End}_{\mathbb{X}}(T)$ *is a d-representation-infinite algebra.*

(b) *If* $V \in \operatorname{vect}\mathbb{X}$, *then* $\mathbb{X}$ *is d-VB-finite and* $\operatorname{vect}\mathbb{X}$ *has the d-cluster tilting subcategory* $\operatorname{add}\{V(\ell\vec{\omega}) \mid \ell \in \mathbb{Z}\}$.

Therefore it is natural to ask when $\mathbb{X}$ has a d-tilting bundle, or equivalently, when A^{ca} is derived equivalent to an algebra Λ with $\operatorname{gl.dim}\Lambda = d$. It follows from Theorem 6.8(a) that

$$\operatorname{gl.dim} A^{\mathrm{ca}} = \begin{cases} d & \text{if } n \leq d+1, \\ 2d & \text{if } n \geq d+2. \end{cases}$$

Thus, if $n \leq d+1$, then $\mathbb{X}$ has a d-tilting bundle. Using Example 6.9 and some general results on matrix factorizations, we have more examples.

Theorem 6.15. *In the following cases,* $\mathbb{X}$ *has a d-tilting bundle.*

(i) $n \leq d+1$.

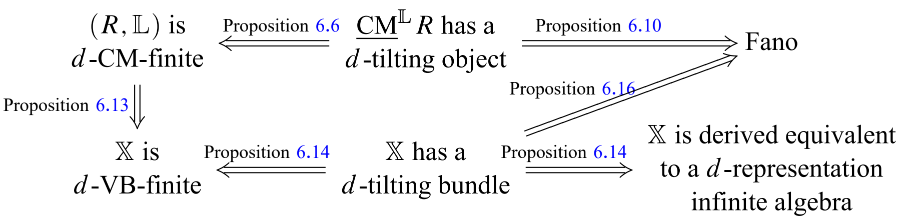
(ii) $n = d + 2 \geq 3$ *and* $(p_1, p_2, p_3) = (2, 2, p_3)$, $(2, 3, 3)$, $(2, 3, 4)$ *or* $(2, 3, 5)$.

(iii) $n = d + 2 \geq 4$ *and* $(p_1, p_2, p_3, p_4) = (3, 3, p_3, p_4)$ *with* $p_3, p_4 \in \{3, 4, 5\}$.

As in the previous subsection, we have the following necessary condition.

Proposition 6.16. *If* $\mathbb{X}$ *has a* d*-tilting bundle, then* $\mathbb{X}$ *is Fano.*

Some of our results in this section can be summarized as follows.



It is important to understand the precise relationship between these conditions. We refer to Chan [2017] and Buchweitz, Hille, and Iyama [n.d.] for results on existence of d-tilting bundles on more general varieties and stacks.

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