Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 4 (3263–3282)

ASYMPTOTIC ENUMERATION OF GRAPHS WITH GIVEN DEGREE SEQUENCE

Nicholas Wormald

Abstract

We survey results on counting graphs with given degree sequence, focusing on asymptotic results, and mentioning some of the applications of these results. The main recent development is the proof of a conjecture that facilitates access to the degree sequence of a random graph via a model incorporating independent binomial random variables. The basic method used in the proof was to examine the changes in the counting function when the degrees are perturbed. We compare with several previous uses of this type of method.

1 Introduction

We sometimes count objects in a class simply because they are there. This is especially true if they are abundantly occurring as mathematical objects (e.g. partitions, sets, or graphs with certain properties), and then we are often pleased if we obtain a simple formula. For example, the number of trees on n vertices is n^{n-2}. But we cannot hope for simple formulae in all cases, and even if we are extremely lucky and the formula is not very complicated, it may be hard to find or difficult to prove. Yet a formula is often useful in order to prove other things, and for such purposes we are frequently satisfied with an approximate or asymptotic formula. For instance, many results in probabilistic combinatorics (see e.g. Alon and Spencer [2000]) use such estimates.

The problems considered here involve graphs or, in an alternate guise, matrices. A non-negative integer $m \times n$ matrix A with row sums $\mathbf{r} = (r_1, \dots, r_m)$ and column sums $\mathbf{s} = (s_1, \dots, s_n)$ is equivalent to a bipartite multigraph G with vertex set $V_1 \cup V_2$ where $V_1 = \{u_1, \dots, u_m\}$ and $V_2 = \{v_1, \dots, v_n\}$. The (i, j) entry of the matrix is the multiplicity of the edge $u_i v_j$. Here A is the adjacency matrix of G. If A is 0-1 (binary) then G has no multiple edges and is thus a (simple) bipartite graph. If every row sum is 2, then A is the

Supported by Australian Laureate Fellowships grant FL120100125.
MSC2010: primary 05C80; secondary 05A16, 05C30, 60C05.

incidence matrix of a multigraph with degree sequence $\mathbf{s}$. If A is symmetric, then it is also the adjacency matrix of a pseudograph H (where loops and multiple edges are permitted). We can obtain H from G by identifying u_i with v_i, $i = 1, \ldots, n$. If A furthermore has zero diagonal, then H is a multigraph, and if is in addition 0-1, then H is a graph. For the bulk of this article, we discuss only the enumeration of graphs, bipartite graphs and multigraphs on a given set of vertices, without mentioning the immediate corollaries for matrices.

We focus on graphs with given degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$, where d_i is the degree of vertex i. For this purpose we normally implicitly assume that n is the number of vertices in the graphs concerned, and that they have vertex set $\{1, \ldots, n\}$. An early example of a formula concerning such graphs is from Moon [1970]: the number of trees with degree sequence $\mathbf{d}$ is precisely

$$\binom{n-2}{d_1 - 1, \ldots, d_n - 1} = \frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!}.$$

A graph is *d-regular* if its degree sequence is $(d, d, \ldots, d)$. No such neat formula is known for the number $g_{d,n}$ of d-regular graphs on n vertices, but we do have the asymptotic formula

$$g_{d,n} \sim \frac{(dn)! e^{-(d^2-1)/4}}{(dn/2)! 2^{dn/2} \prod d_i!} \tag{1-1}$$

as $n \to \infty$ with d fixed. Here $a \sim b$ means $a = b(1 + o(1))$, with $o()$ the Landau notation. This and many more developments are described in Section 2.

Counting graphs by degree sequence is strongly related to finding the distribution of the degree sequence of a random graph on n vertices in either of the two most common random graph models. In the model $\mathcal{G}(n, p)$, where edges occur independently and each with probability p, the degree of a vertex is distributed binomially as $\mathrm{Bin}(n-1, p)$, but the degrees of the vertices are not independent of each other. Bollobás [2001] devotes an early chapter to this topic. The model $\mathcal{G}(n, p)$ can be viewed as a mixture of the models $\mathcal{G}(n, m)$, where m is distributed binomially as $\mathrm{Bin}\big(n(n-1)/2, p\big)$. Here, $\mathcal{G}(n, m)$ has m edges selected from all $\binom{n}{2}$ possible positions uniformly at random. We use $g(\mathbf{d})$ to denote the number of graphs with degree sequence $\mathbf{d}$. Then the probability that a random graph $G \in \mathcal{G}(n, m)$ has degree sequence $\mathbf{d}$ can be evaluated precisely as $g(\mathbf{d}) / \binom{\binom{n}{2}}{m}$.

McKay and Wormald [1990a] made a conjecture (which has now been verified) on the number of graphs with degree sequence $\mathbf{d}$, for a wide-ranging choice of possible vectors $\mathbf{d}$ stated by Liebenau and Wormald [2017]. Let A_n and B_n be two sequences of probability spaces with the same underlying set for each n. Suppose that whenever the event H_n satisfies $\mathbb{P}(H_n) = n^{-O(1)}$ in either model, it is true that $\mathbb{P}_{A_n}(H_n) \sim \mathbb{P}_{B_n}(H_n)$. Then we say that A_n and B_n are *asymptotically quite equivalent* (a.q.e.). Also, throughout this paper we use $\omega(f(n))$ to denote a function of n such that $\omega(f(n))/f(n) \to \infty$ as $n \to \infty$. The conjecture from McKay and Wormald [1997] implies the following two propositions.

Let $\mathfrak{D}(\mathcal{G})$ denote the (random) degree sequence of a random graph $\mathcal{G}$. Define $\mathcal{B}_p(n)$ to be the random sequence consisting of n independent binomial variables $\mathrm{Bin}(n-1, p)$. We use $\mathcal{Q}|_F$ to denote the restriction of the probability space $\mathcal{Q}$ to an event F. The lack of specification of p in the following is due to the obvious fact that the restricted space $B_p(n)\ |_{\Sigma=2m}$ is independent of p. The restriction imposed on m in this proposition only excludes graphs with so few or so many edges that their degree sequence is excruciatingly boring.

Proposition 1.1. *Let Σ denote the sum of the components of the random vector $\mathcal{B}_p(n)$. Let $0 < p < 1$. Then $\mathfrak{D}(\mathcal{G}(n,m))$ and $B_p(n)\ |_{\Sigma=2m}$ are a.q.e. provided that* $\max\{m, \binom{n}{2} - m\} = \omega(\log n)$.

This gives a very appealing way to derive properties of the degree sequence of a random graph with n vertices and m edges: consider independent binomials as above with $p = 2m/n(n-1)$ and condition on the event $\Sigma = 2m$, which has (not very small) probability $\Theta(1/\sqrt{p(1-p)n^2})$. Even more appealing, it was also shown in McKay and Wormald [ibid.] that a statement like the above proposition would imply the following one.

Proposition 1.2. *Let Σ denote the sum of the components of the random vector $B_{\hat{p}}(n)$, where $\hat{p}$ is randomly chosen according to the normal distribution with mean p and variance $p(1-p)/n(n-1)$, truncated at 0 and 1. Then $\mathfrak{D}(\mathcal{G}(n, p))$ and $B_{\hat{p}}(n)\ |_{\Sigma \text{ is even}}$ are a.q.e. provided that $p(1-p) = \omega(\log^3 n/n^2)$.*

This second proposition gives even easier access to properties of the degree sequence of the random graph $\mathcal{G}(n, p)$, as conditioning on the parity of Σ is insignificant for many properties, and the effect of the random choice of $\hat{p}$ can be evaluated by integration. This was all made explicit in McKay and Wormald [ibid.], where there are a number of helper theorems to make it easy to transfer results from the independent binomial model $\mathcal{B}_p(n)$. It was also observed in McKay and Wormald [ibid.], from known asymptotic formulae, that the main conjecture (and hence also Propositions 1.1 and 1.2) holds when $p = o(1/\sqrt{n})$ or $p(1-p) > n/c \log n$. The gap between these ranges, where $p(1-p)$ is between roughly $1/\sqrt{n}$ and $1/\log n$, was only recently plugged (see Section 4.1), which proved the main conjecture of McKay and Wormald [1990a] in full.

This gives the following asymptotic formula for the number of graphs with given degree sequence, that holds provided the degrees are reasonably close to each other. The degree sequence of a random graph in $\mathcal{G}(n, p)$ with high probability falls into the range covered, for the p considered here. Given $\mathbf{d}$, let

$$d = \frac{1}{n}\sum_{i=1}^{n} d_i,$$

$$\mu = \mu(n) = d/(n-1)$$

and

$$\gamma_2 = (n-1)^{-2}\sum_{i=1}^{n}(d_i - d)^2.$$

Also, given a sequence $\mathbf{d}$ with even sum $2m$, let $P(\mathbf{d})$ denote the probability that $\mathbf{d}$ occurs in the model $B_p(n)\mid_{\Sigma=2m}$. Using Stirling's formula, we easily find

$$P(\mathbf{d}) \sim \sqrt{2}(\mu^{\mu}(1-\mu)^{1-\mu})^{n(n-1)/2}\prod\binom{n-1}{d_i}.$$

Proposition 1.3. *For some absolute constant $\epsilon > 0$,*

$$g(\mathbf{d}) \sim \exp\Big(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2}\Big)P(\mathbf{d}) \tag{1-2}$$

provided that dn is even, $\max_j |d_j - d| = o(n^{\epsilon}\min\{d, n-d-1\}^{1/2})$ *and* $n^2\min\{\mu, 1-\mu\} \to \infty$.

Note. Where we state asymptotics such as in (1-2) where we do not explicitly give $\mathbf{d}$ as a function of n, there are two possible interpretations. One is that we do, indeed, consider $\mathbf{d}$ a function of n. Then the limit easily makes sense, and the interpretation should be that this holds for any $\mathbf{d}(n)$ satisfying the given constraints. The other interpretation is that the asymptotic convergence should contain a bound that is uniform over all $\mathbf{d}$ under consideration. The first interpretation is the default, and two interpretations are easily seen to be equivalent when the permitted domain of $\mathbf{d}$ is suitably closed, such that one can consider the 'worst' sequence $\mathbf{d}(n)$ for each n.

The validity of this formula cannot possibly extend to very "eccentric" degree sequences, in particular certain non-graphical degree sequences (i.e. sequences for which no graph exists). Examples of degree sequences at the fringe of the formula's validity can be obtained as follows. Consider the degree sequence $\mathbf{d}$ with $n/2$ entries $d+x$ and $n/2$ of $d-x$, where $d = o(\sqrt{n})$ and $x \le d$. By expanding the formula in McKay and Wormald [1991a, Theorem 5.2] appropriately, we can check that $g(\mathbf{d})$ differs from the formula in (1-2) by a factor $\exp\big(x^6(2d^2 - x^2)/nd^5 + o(1)\big)$. Hence, for such d and x, (1-2) is correct iff $x = o(n^{1/6}d^{1/2})$, which is for all $x \le d$ when $d = o(n^{1/3})$, but not for larger d.

Formulae similar to (1-2) are also now known for bipartite graphs and loopless directed graphs; see Section 2.3.

The next section traces the development of results on this problem. Section 3 gives a description and observations on the basic approach used in Liebenau and Wormald [2017], and Section 4 discusses applications of the results (and methods). Some open problems are mentioned in the last section.

2 Results on enumeration of graphs by degrees

We focus on asymptotic results mainly because these formulae, for the problems of our concern, are much simpler than the corresponding known exact formulae. Due to the complexity of the exact formulae, they tend to be useless in proving results such as Propostions 1.1 and 1.2 and the applications in Section 4. For these, simple formulae are generally required, even if only approximate. The interest in limiting behaviour is thus prominent, as in many areas of mathematics.

2.1 Asymptotic results for graphs. Most of the following results come with explicit error bounds in the asymptotic approximations. To keep the description simple, we omit these error bounds, and similarly make little mention of a number of variations and extensions given in the papers quoted. The description in this section is basically in order of increasing maximum degree of the graphs being treated. This largely corresponds to chronological order, the main exception being for very dense graphs with degrees approximately cn.

Our story begins with Read [1958] thesis. Using Polya's cycle index theory and manipulation of generating functions, Read found a formula for the number of graphs with given degree sequence, from which he was able to obtain a simple asymptotic formula in the case of the number $g_3(n)$ of 3-regular graphs:

$$g_3(n) \sim \frac{(3n)!e^{-2}}{(3n/2)!288^{n/2}}. \tag{2-1}$$

(Here n is restricted to being even, as it is in all our formulae when the total degree parity condition forces it.)

Further progress on enumeration of regular graphs was stymied by the lack of an amenable approach. However, before long significant developments occurred in enumeration of $m \times n$ non-negative integer matrices, which in the case of 0-1 matrices correspond to bipartite graphs. Let $b(\mathbf{r}, \mathbf{s})$ denote the number of 0-1 matrices, of dimensions $m \times n$, with row sum vector $\mathbf{r}$ and column sum vector $\mathbf{s}$. We refer to the entries of a vector $\mathbf{r}$ as r_i, entries of $\mathbf{s}$ as s_i, and so on. We can assume these vectors have equal sums, and define for a vector $\mathbf{d}$

$$M_1(\mathbf{d}) = \sum_i d_i, \qquad \text{and in general} \qquad M_j(\mathbf{d}) = \sum_i [d_i]_j$$

where $[x]_j = x(x-1)\cdots(x-j+1)$. O'Neil [1969] showed that, as long as $m = n$ and some $\epsilon > 0$ satisfies $r_i, s_i \le (\log n)^{1/4-\epsilon}$ for all i,

$$b(\mathbf{r}, \mathbf{s}) \sim \frac{M_1! e^{-\alpha}}{\prod_{i=1}^m r_i! \prod_{i=1}^n s_i!} \tag{2-2}$$

as $M_1 \to \infty$, where $\alpha = M_2(\mathbf{r})M_2(\mathbf{s})/2M_1^2$ and $M_1 = M_1(\mathbf{r}) = M_1(\mathbf{s})$. Of course, in the corresponding bipartite graphs, M_1 is the number of edges and $\mathbf{r}$ and $\mathbf{s}$ are the degree sequences of the vertices in the two parts.

For fixed r, Everett and P. R. Stein [1971/72] gave a different proof for the r-regular case of (2-2), i.e. $r_i = s_i = r$ for all i (in particular this again requires $m = n$), using symmetric function theory. This is the bipartite version of (1-1).

A. Békéssy, P. Békéssy, and Komlós [1972] showed that (2-2) holds even for $m \neq n$, as long as there is a constant upper bound on maximum degree of the corresponding graphs, i.e. $\max_i r_i$ and $\max_i s_i$. They used the following model. Consider n buckets, and M_1 balls with r_i labelled i, $1 \le i \le m$. Distribute the balls at random in the buckets by starting with a random permutation of the balls, placing the first s_1 into bucket 1, the next s_2 into bucket 2, and so on. (The balls and buckets are all mutually distinguishable.) Each distribution corresponds to a matrix whose (i, j) entry is the number of balls labelled i falling into bucket j. In this model, it is easy to see that the 0-1 matrices with row sum vector $\mathbf{r}$ and column sum vector $\mathbf{s}$ are equiprobable. The number of permutations is $M_1!$, and the number of these corresponding to any one 0-1 matrix is the denominator of (2-2). Hence (2-2) follows once we show that the event that no entry is at least 2 has probability $e^{-\alpha+o(1)}$. This is done using an inclusion-exclusion technique, equivalent to applying Bonferroni's inequalities or Brun's sieve.

Mineev and Pavlov [1976] used more accurate analysis of the same model to show that (2-2) still holds with maxium degree $(\gamma \log n)^{1/4}$ for $\gamma < 2/3$, and a slightly more extended range in the regular case.

Bender [1974] used a model equivalent to that in A. Békéssy, P. Békéssy, and Komlós [1972] to obtain results for matrices with integer entries in the range $[0, \ldots, t]$, but with bounded row and column sums. He allowed some entries to be forced to be 0. This permits the diagonal to be forced to be 0 in the case $m = n$, hence giving a formula for the number of loopless digraphs with given in- and out-degree sequence.

About 20 years passed from Read's result (2-1) for the 3-regular case, before any advance was made in the case of non-bipartite graphs. In 1978 Bender and Canfield [1978] showed that the number $g(\mathbf{d})$, of graphs with degree sequence $\mathbf{d}$ with the maximum degree bounded, is given by

$$g(\mathbf{d}) \sim \frac{M_1! e^{-\alpha}}{(M_1/2)! 2^{M_1/2} \prod_{i=1}^n d_i!} \tag{2-3}$$

where $\alpha = M_2/2M_1 + M_2^2/4M_1^2$, and $M_j = M_j(\mathbf{d})$ for all statements about graphs. For this result, they used a model of involutions of a set of cardinality M_1 partitioned into blocks of sizes $d_1, d_2, \dots, d_n$. When there are no fixed points, we can regard the two elements in a 2-cycle of an involution as two balls of the same label, to recover the model of Békéssy et al. applied to the incidence matrix of graphs. (Bender and Canfield obtained other results for symmetric matrices with nonzero diagonal, in which the involutions are permitted to have fixed points.) We easily see that the number of such involutions corresponding to a given simple graph is precisely the denominator in (2-3), since the labels of the edges are immaterial, giving a factor $(M_1/2)!$ in addition to the considerations applied for the matrix counting in (2-2). The factor $e^{-\alpha}$ is shown to be asymptotically the probability that the graph obtained in the model is simple. We call this event $\mathcal{S}$.

Independently, in my PhD thesis Wormald [1978] I used the asymptotic results of A. Békéssy, P. Békéssy, and Komlós [1972] for bipartite graphs, to derive (2-3) for bounded degrees.

Bollobás [1980] gave the *configuration model*, in which d_i objects, commonly called half-edges, are assigned to each vertex i, and then paired up at random. Two paired half-edges form an edge joining the corresponding vertices. It is readily seen that conditioning on no loops or multiple edges gives a uniformly random graph. This is clearly equivalent to the earlier models, such as the involution model of Bender and Canfield, where each 2-cycle corresponds to two paired half-edges. The model was used in Bollobás [ibid.] to extend the validity of (2-3) to maximum degree $\sqrt{2\log n} - 1$, provided that a certain lower bound on the number of edges $(M_1/2)$ is satisfied. In place of the inclusion-exclusion based arguments in the earlier papers, Bollobás used the method of moments for Poisson random variables, which is essentially equivalent. Using this model, we can write

$$g(\mathbf{d}) = \frac{|\Phi| \mathbb{P}(\mathcal{S})}{\prod d_i!} \tag{2-4}$$

where Φ is the set of pairings in the model, with $|\Phi| = M_1!/\big((M_1/2)! 2^{M_1/2}\big)$.

Aside from enumeration results, Bollobás [1981], and then many others, found the configuration model a convenient starting point to prove properties of random graphs with given degrees.

Several further developments involved estimating $\mathbb{P}(\mathcal{S})$ for a wider range of degree sequences.

For the bipartite case, Bollobás and McKay [1986] extended the validity of the formula to cover $m \le n$ when maximum degree is at most $\log^{1/3} m$ and a certain lower bound on the number of edges is satisfied.

Again for the bipartite case, McKay [1984] then took a much bigger step, by introducing switchings as a technique for this problem. (See Section 3.1 for a detailed description of this method, including some subsequent developments mentioned below.) He obtained (2-2) under conditions that are a little complicated to state in general, but apply for all $r = o(n^{1/3})$ in the r-regular case (i.e. $r_i = s_i = r$ for all i, and $m = n$).

McKay [1985] applied the same technique to the graph case, with a similar restriction on degrees, again obtaining (2-3) with the same formula for α.

From this point onwards, the results for the two cases, graphs versus bipartite graphs, have generally been obtained more or less in tandem using the same methods, so we continue tracing only the graph case in detail.

It was evident that with considerably more effort, the switching approach should extend to higher degrees, but at the expense of much case analysis, which was enough of a deterrent to stifle such further development. Instead, after several years, McKay and Wormald [1991a] found a different version of switchings that enabled a much easier advance. The result was that for degree sequences with $\Delta = o(M_1^{1/3})$, (2-4) holds with

$$\mathbb{P}(\mathcal{S}) = \exp\Big(-\frac{M_2}{2M} - \frac{M_2^2}{4M^2} - \frac{M_2^2 M_3}{2M^4} + \frac{M_2^4}{4M^5} + \frac{M_3^2}{6M^3} + O\Big(\frac{\Delta^3}{M}\Big)\Big).$$

This covers the d-regular case for $d = o(\sqrt{n})$.

Janson [2009, 2014] was interested in characterising the degree sequences for which $\mathbb{P}(\mathcal{S}) \to 0$. He showed by analysing the configuration model for degree sequence $\mathbf{d}$, using the method of moments, that for $M_1 = \Theta(n)$,

$$\mathbb{P}(\mathcal{S}) \to 0 \quad \text{iff} \quad \sum d_i^2/n \to \infty,$$

and that for $M_2 = O(M_1)$ and $M_1 \to \infty$, we have the asymptotic formula

$$\mathbb{P}(\mathcal{S}) = \exp\Big(-\frac{1}{2}\sum \lambda_{ii} - \sum_{i<j} \big(\lambda_{ij} - \log(1+\lambda_{ij})\big)\Big) + o(1)$$

where $\lambda_{ij} = \sqrt{d_i(d_i-1)d_j(d_j-1)}/(2M_1)$. This was the first general result to apply to some sequences with maximum degree as large as $\sqrt{n}$.

Gao and Wormald [2016] analysed cases in which the configuration model produces edges of much higher multiplicity than previous studies, using a major extension of the switching method mentioned above. This resulted in an asymptotic formula for $g(\mathbf{d})$ when $\mathbf{d}$ satisfies some very complicated conditions. We describe one simple consequence. We say $(d_1, \ldots, d_n)$ is *power-law distribution-bounded with parameter* γ if there exists $C > 0$ such that the number of d_i taking value at least i is at most $Cn \sum_{j \geq i} j^{-\gamma}$ for all i and n. This condition restricts the maximum d_i to $O(n^{1/(\gamma-1)})$. Before Gao and Wormald [ibid.], there were no asymptotic enumeration formulae for such sequences when $\gamma \leq 3$. However, many naturally occurring networks seem to have power law degree sequences with $\gamma < 3$. From Gao and Wormald [ibid., Theorem 3] (corrected), we see that if $\mathbf{d}$ is a power-law distribution-bounded sequence with parameter $3 > \gamma > 1 + \sqrt{3} \approx 2.732$, then with Φ given just after (2-4),

$$g(\mathbf{d}) = \frac{|\Phi|}{\prod d_i!} \exp\left(-\frac{M_1}{2} + \frac{M_2}{2M_1} + \frac{3}{4} + \sum_{i<j} \log(1 + d_i d_j / M_1) + O\big(\xi\big) \right),$$

where $\xi = n^{(2+2\gamma-\gamma^2)/(\gamma-1)}$.

Recently, Burstein and Rubin [2015] presented an approach that would give a formula that is valid up to maximum degree $n^{1-\delta}$ for any fixed $\delta > 0$, using a finite amount of computation. We say a little more about this in Section 3.3.

Liebenau and Wormald [2017] introduced a new approach and "plugged the gap" in the formulae with the following result.

Theorem 2.1. *Let $\mu_0 > 0$ be a sufficiently small constant, and let $1/2 \leq \alpha < 3/5$. If $\mu = d/(n-1)$ satisfies $\mu \leq \mu_0$ and, for all fixed $K > 0$, $(\log n)^K/n = O(\mu)$, and $|d_i - d| \leq d^\alpha$ for all $i \in [n]$ then (1-2) holds (provided dn is even).*

Together with the previous results, this establishes Proposition 1.3 and hence Propositions 1.1 and 1.2. The method seems strong enough to cover all the results mentioned above, though some significant tinkering would need to be done to obtain the results for eccentric degree sequences in Janson [2009, 2014] and Gao and Wormald [2016].

At this point, we travel slightly back in time to consider results for dense graphs. Of course, extremely dense cases are covered by simply complementing the sparse cases above. All results for graphs of average degree comparable with n are based on extracting coefficients from the 'obvious' generating function:

(2-5)
$$g(\mathbf{d}) = [x_1^{d_1} \cdots x_n^{d_n}] \prod_{i<j} (1 + x_i x_j).$$

The generating function is derived by letting x_i mark the degree of vertex i, so the term $x_i x_j$ denotes the presence of the edge ij and the term 1 denotes its absence. Coefficients are extracted using Cauchy's integral formula, for multiple dimensions.

McKay and Wormald [1990a] evaluated the integrals to obtain the result of Proposition 1.3 in the case that $\mu(1-\mu) > c/\log n$ for fixed $c > 2/3$.

Barvinok and Hartigan [2013] used a similar approach, with different analysis, to obtain a result for a wider range of degrees when $\min\{\mu, 1-\mu\} = \Theta(1)$.

Independently and almost simultaneously with the completion of the proof of Proposition 1.3 in Liebenau and Wormald [2017], Isaev and McKay [2016] made an exciting new advance, by developing the theory of martingale concentration for complex martingales, which (amongst other things) enabled them to reproduce and extend the results in McKay and Wormald [1990a] and Barvinok and Hartigan [2013], by better analysis of the integrals involved. Moreover, they recently report (by private communication) being able to obtain a result, in a certain implicit form, that applies to a wide range of degree sequences with average degree at least n^a for any fixed $a > 0$. They have used this to show in particular that (1-2) is valid for the d-regular case with $d = \omega(n^{1/7})$.

2.2 Exact enumeration for graphs. We include here only a selection of results that bear some relation to our main topic of simple asymptotic formulae. These concern the number $g_d(n) = g(d, d, \ldots, d)$ of d-regular graphs on n vertices.

As mentioned above, Read [1960] found a formula for the number of graphs with given degree sequence. His approach was to count the incidence matrices of the graphs, i.e. 0-1 matrices with column sums vector $\mathbf{d}$ and all row sums 2. Pólya's Hauptsatz for enumeration under the action of the symmetric group was used to eliminate the distinction between matrices that are equivalent up to permuting the rows (edges). To eliminate multiple edges, Read used a version of the Hauptsatz in which the 'figures' are distinct. Unfortunately, this gives a very complicated formula, involving generating functions for which the extraction of coefficients is difficult. Nevertheless, in the case of $g_3(n)$, Read obtained a formula containing a single summation that he analysed to obtain (2-1).

One can also ask for simple recurrence relations. There are two direct uses of these: for efficient computation of the numbers, and also, as Read [1958] shows, the recurrence relation can be combined with an asymptotic formula to deduce an asymptotic series expansion. Read did this in the case of 3-regular graphs, finding the first few terms of a series in powers of n^{-1}. Read and Wormald [1980] found a similar recurrence for $g_4(n)$.

Goulden, Jackson, and Reilly [1983] considered the generating function (2-5), in the case of regular graphs, and obtained a different recurrence relation for $g_4(n)$.

Gessel (90) showed recurrence relations exist for $g_d(n)$ when d is fixed. (To be precise, he showed that the generating function for d-regular graphs is D-finite.) However, to our knowledge, these recurrences have not been found explicitly for any $d \geq 5$.

Chen and Louck [1999] obtained a formula for $g_3(n)$ by first getting a formula for matrices with row sums 3 and column sums 2 (and related problems) using a little symmetric function theory, and then using inclusion-exclusion to delete the multiple edges. The same method should work for $d > 3$ but would appear to get rapidly much more complicated.

2.3 Asymptotics for special types of graphs. Aside from Moon's formula for trees with given degree sequence mentioned in the introduction, many results are known on the asymptotic number of trees of some given variety with given degree sequence. We refer the interested reader to Drmota [2009].

As implied in Section 2, there have been further results on bipartite graphs, and a recent summary may be found in Liebenau and Wormald [2017]. In particular, that paper completes the proof of the analogue of Proposition 1.1 in the case that the two sides of the bipartite graph have reasonably similar cardinalities. See also Section 5.

2.4 Similar results for other structures. Greenhill and McKay [2013] obtained results for multigraphs analogous to the graph results of McKay and Wormald [1991a], for a similar range of degrees. Already in A. Békéssy, P. Békéssy, and Komlós [1972], the number of bipartite multigraphs with given degree sequence (with bounded maximum degree) was also obtained.

The results on graphs obtained in Liebenau and Wormald [2017] were accompanied by similar results on loopless directed graphs.

Some results on hypergraphs have been obtained my means similar to those discussed for graphs. For simplicity we omit these from the scope of this article.

A $k \times n$ Latin rectangle can be defined as an ordered set of k disjoint perfect matchings which partition the edges of a bipartite graph (i.e. a properly k-edge-coloured bipartite graph) on vertex sets $V_1 = \{1, \dots, n\}$ and $V_2 = \{n+1, \dots, 2n\}$. Asymptotic estimates of the numbers of these were obtained for ever-increasing k by Erdős and Kaplansky [1946], Yamamoto [1951], C. M. Stein [1978], culminating in the result of Godsil and McKay [1990] for $k = o(n^{6/7})$. In a recent preprint, Leckey, Liebenau, and Wormald [n.d.] reached $k = o(n/\log n)$ using the method described above for graph enumeration. See Section 3.3.

Kuperberg, Lovett, and Peled [2017] have a different probabilistic approach to enumeration of several other kinds of regular combinatorial structures such as orthogonal arrays, t-designs and certain regular hypergraphs.

3 The perturbation method

The author coined this term in Wormald [1996], to refer to enumeration methods based on comparing the number of structures with a given parameter set to the numbers of structures with slightly perturbed parameter sets. Overall, it can be expressed as estimating the ratio of probabilities of "adjacent" points in a discrete probability space. How to estimate the ratio depends on the application, and some examples are discussed below. It is relatively straightforward to use the information on ratios for adjacent points. We may use the following result, in which the probabilities actually occurring relate to $\mathcal{P}'$ and are compared with those in some "ideal" probability space $\mathcal{P}$. Here $\operatorname{diam}(F)$ is the diameter of F.

Lemma 3.1 (Liebenau and Wormald [2017]). *Let $\mathcal{P}$ and $\mathcal{P}'$ be probability spaces with the same underlying set Ω. Let F be a graph with vertex set $\mathcal{W} \subseteq \Omega$ such that $\mathbb{P}_{\mathcal{P}}(v), \mathbb{P}_{\mathcal{P}'}(v) > 0$ for all $v \in \mathcal{W}$. Suppose that $1 > \epsilon > 0$ with $\min\{\mathbb{P}_{\mathcal{P}}(\mathcal{W}), \mathbb{P}_{\mathcal{P}'}(\mathcal{W})\} \geq e^{-\epsilon}$, and that $C, \delta > 0$ satisfy, for every edge uv of F,*

$$\left| \log \frac{\mathbb{P}_{\mathcal{P}'}(u)}{\mathbb{P}_{\mathcal{P}'}(v)} - \log \frac{\mathbb{P}_{\mathcal{P}}(u)}{\mathbb{P}_{\mathcal{P}}(v)} \right| \leq C\delta. \tag{3-1}$$

Suppose further that $\operatorname{diam}(F) \leq r < \infty$. *Then for each $v \in \mathcal{W}$ we have*

$$|\log \mathbb{P}_{\mathcal{P}'}(v) - \log \mathbb{P}_{\mathcal{P}}(v)| \leq rC\delta + O(\epsilon).$$

The proof is entirely straightforward, using a telescoping product of ratios along a path joining u to v of length at most r.

Note: if the error function δ depends on u and v, it might be possible to take advantage of this and finish with a smaller error term than what is suggested by the lemma. In the applications so far, this would not give any significant gain.

Estimating the ratios of adjacent probabilities, and hence δ, is the main requirement for applying the method. We describe several different but related examples. The overall structure of the arguments in most cases was phrased differently from Lemma 3.1, but is essentially equivalent. In most cases, ratios of adjacent probabilities were found to approximate the ratio of corresponding probabilities of a Poisson distributed random variable. In such cases we may take $\Omega = \mathbb{N}$ and $\mathbb{P}_{\mathcal{P}}(i) = \mathbb{P}(X = i)$ for a given Poisson random variable X. The graph F then has edges $\{i, i+1\}$ for all i in some suitably defined set $\mathcal{W}$.

3.1 Switchings for pairings. Arguments involving switchings or similar concepts have been used in many places in combinatorics. An early example close to our topic is provided by the bounds on the probabilities of subgraphs of random graphs obtained by McKay [1981].

McKay [1985] applied switchings to estimate the probability of the event $\mathcal{S}$ (that a simple graph results) in the configuration model for degree sequence $\mathbf{d}$ described in Section 2.1. We can describe this as two rounds of the perturbation method, first estimating the number of pairings with no loops, and then, among those, the number with no double edge. (To be precise, even higher multiplicities were eliminated first.) Roughly, the double edge round was as follows. Let $\mathcal{C}_i$ denote set of pairings with i double edges. Take any pairing P in $\mathcal{C}_i$ for $i \geq 1$. To apply a *switching* to P, choose a random pair from a random double pair, call it ab, and any other random pair, call it cd, and then delete those two pairs and replace them by ac and bd. This is depicted in Figure 1. We can count the

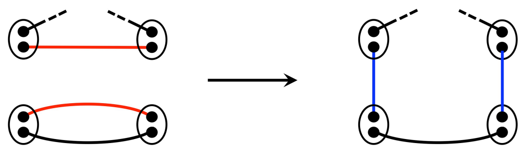

Figure 1: *Switching two pairs*

switchings that produce members of $\mathcal{C}_{i-1}$ in two ways: the number that can be applied to such a $P \in \mathcal{C}_i$, and the number that can *produce* a pairing $P' \in \mathcal{C}_{i-1}$. By estimating the ratio of these two numbers, McKay obtained an estimate of the ratio $\mathbb{P}(\mathcal{C}_i)/\mathbb{P}(\mathcal{C}_{i-1})$, in the model. In his argument, the ratios were used to estimate the ratios $\mathbb{P}(\mathcal{C}_i)/\mathbb{P}(\mathcal{C}_0)$, and the absolute sizes were obtained by arguing about the sum of these ratios. Alternatively and equivalently, we could apply Lemma 3.1 by lettin $\Omega = \mathbb{N}$ and assigning probabilities in $\mathcal{P}'$, according to the distribution of the random number, X, of double edges in a pairing. Then, with $\mathcal{P}$ defined as an appropriate Poisson random variable, we can deduce (3-1) for quite small values of δ and C. The set $\mathcal{W}$ is defined to be $[0, M_0]$ for a suitable constant M_0, and ϵ can be bounded by estimating $\mathbb{E}[X]_k$ for suitable large k.

To obtain sufficient accuracy, McKay introduced secondary switchings that enabled him to argue about the local structure of random elements of $\mathcal{C}_i$. This was needed because the number of reverse switchings depends heavily on the structure of P' mentioned above, though it is quite stable for typical P'. This enabled him to obtain the asymptotic formula in the case of d-regular graphs for $d = o(n^{1/3})$.

McKay and Wormald [1991a] introduced fancier switchings, involving more pairs, for which local structure had little effect on the number of reverse switchings, and hence directly gave accuracy comparable to the secondary switchings of McKay. Further secondary switchings produced results that, in the d-regular case, reached $d = o(\sqrt{n})$. An additional benefit of the fancier switchings was that they led to a useful uniform generator of d-regular graphs for $d = O(n^{1/3})$. (See Section 4.3.)

3.2 Relation to Stein's method. Stein's asymptotic formula for $k \times n$ Latin rectangles in C. M. Stein [1978] was based on Chen's method (which is elsewhere called the Stein-Chen or Chen-Stein method) for Poisson approximation, which was based in turn on a more general method of Stein for approximating a random variable. His basic strategy was the same as all results on this problem before Leckey, Liebenau, and Wormald [n.d.]: given $k-1$ rows of a rectangle, estimate the probability that a random new row is compatible with the given rows. Stein approached this as follows. Given the first $k-1$ rows, define X to be the number of column constraints that a random permutation, upon being inserted as the kth row, would violate. A permutation Π is associated with a random permutation Π' using a random pair (j_1, j_2) of distinct elements of $[n]$: $\Pi'(i)$ is defined as $\Pi(j_1)$ if $i = j_2$, $\Pi(j_2)$ if $i = j_1$, and $\Pi(i)$ otherwise. Then, with X' the number of column constraints violated by Π', C. M. Stein [1978, (3) in Section 2] is

$$\frac{\mathbb{P}(X = x+1)}{\mathbb{P}(X = x)} = \frac{\mathbb{P}(X' = x+1 \mid X = x)}{\mathbb{P}(X' = x \mid X = x+1)}.$$

This is clearly equivalent to restricting to pairs (j_1, j_2) where Π' violates one less column constraint than Π, which is a direct analogue of the switchings for pairings described above. Stein also uses secondary randomisation in C. M. Stein [ibid., (20), Section 2], which corresponds to the secondary switchings described above. However, he does not seem to use the analogue of the fancier switchings. On the other hand, analogues of the fancier switchings were applied to the problem of uniformly generating Latin rectangles by McKay and Wormald [1991b].

3.3 Iterated applications. In the ideal situation, we can start with a set of initial estimates for the ratios of adjacent probabilities, and then iteratively feed these estimates into equations that are derived from some operation, such as switchings, to improve the accuracy of the estimates.

We can view the equations as specifying an operator on functions which fixes the true ratios, except perhaps for a quantified error term. Normally we could then hope to apply standard concepts of fixed point analysis to show that the true ratios are close to a fixed point of this operator. Initial bounds are required on the true ratios, as they provide an initial "guess", and also simultaneously they can be used to guarantee that, essentially, this guess is in the domain of attraction of the correct fixed point of the operator.

Here are some examples.

(i) Nonexistence of subgraphs of $\mathcal{G}(n, p)$ and $\mathcal{G}(n, m)$.

Consider estimating the number $t_{n,m}$ of graphs with n vertices, m edges and no triangles (cycles of length 3). A. Frieze [1992] switched edges of subgraphs to different positions and then argued in a similar fashion to the argument for double edges in pairings described above. This gave an asymptotic formula for $t_{n,m}$, and similarly for other strictly balanced subgraphs, when $m = n^{1+\theta}$ for θ quite small. A result for $t_{n,m}$, with much larger θ was achieved by the author in Wormald [1996], using a different version of the perturbation method. Instead of moving edges to new positions, one of the basic operations considered was to add a triangle at a random position. This generally transforms the graph into one with an extra triangle, but sometimes two or more new ones can be created. The number of copies of two triangles sharing an edges is also recorded, as well as numbers of some other small clusters of triangles sharing edges. In this case vertices of F in Lemma 3.1 are vectors specifying the numbers of each type of cluster, and each vertex is adjacent to several others, being those adjacent in the integer lattice. The estimate for a ratio between adjacent vertices contains lower order terms involving similar ratios for other nearby vertices. The ratio for vectors adjacent in the vector component corresponding to a given cluster is estimated using a "switching" operation in which a copy of the corresponding cluster is added at random. Fixed points are not used explicitly, but estimates of lower and upper bounds on ratios are iterated. These arguments are complicated by a careful induction that yields the required initial bounds on the ratios. Stark and Wormald [2016] strengthened the method and also extended it to all strictly balanced subgraphs.

(ii) Degree switchings for counting graphs.

This is a rough outline of the argument in Liebenau and Wormald [2017]. Suppose we have a random graph G with degree sequence $\mathbf{d}-\mathbf{e}_b$ where $\mathbf{e}_v$ denotes the elementary unit vector with 1 in its v^{th} coordinate. Pick a random edge e incident with vertex a, and with v denoting the other end of e, remove e and add the edge bv. Let $B(a,b,\mathbf{d}-\mathbf{e}_b)$ denote the probability of the "bad" event that a loop or multiple edge is produced. If this event fails, the graph now has degree sequence $\mathbf{d}-\mathbf{e}_a$. We call this a *degree switching*. Simple counting shows that

$$R(a,b;\mathbf{d}) := \frac{g(\mathbf{d}-\mathbf{e}_a)}{g(\mathbf{d}-\mathbf{e}_b)} = \frac{d_a}{d_b}\cdot\frac{1-B(a,b,\mathbf{d}-\mathbf{e}_b)}{1-B(b,a,\mathbf{d}-\mathbf{e}_a)}. \tag{3-2}$$

Let $P_{av}(\mathbf{d})$ denote the probability that edge av occurs in a random graph with degree dequence $\mathbf{d}$. With a little work, we can express $B(a,b,\mathbf{d}-\mathbf{e}_b)$ using a combination of such probabilities, resulting in a formula for $R(a,b;\mathbf{d})$ in terms of the $P_{av}(\mathbf{d})$ for various a, v and $\mathbf{d}$.

By noting $d_a = \sum_v P_{av}$, we can also obtain

$$P_{av}(\mathbf{d}) = d_v\left(\sum_{b\in V\setminus\{a\}} R(b,a;\mathbf{d}-\mathbf{e}_v)\frac{1-P_{bv}(\mathbf{d}-\mathbf{e}_b-\mathbf{e}_v)}{1-P_{av}(\mathbf{d}-\mathbf{e}_a-\mathbf{e}_v)}\right)^{-1}$$

where $V=[n]$ is the set of all vertices. Iterating these two formulae produces a sequence of approximations to the functions P and R that converges, and it is convenient to consider, as described above, fixed points of the operators defined by these equations. Then the argument becomes one of proving that the limits of the convergent solutions are close to the fixed points of the operators, and that these are close to the true values of the ratios.

The required initial bounds on the true ratios were obtained in this case by a primitive analysis of a switching-type operation similar to that in Figure 1.

Lemma 3.1 was applied here, with Ω being the set of sequences on non-negative integers of sum $2m$. The probability of $\mathbf{d}$ in $\mathcal{P}'$ is proportional to $g(\mathbf{d})$, and in $\mathcal{P}$ is proportional to the target formula, i.e. the right hand side of (1-2).

(iii) A related approach for graphs.

While work on Liebenau and Wormald [2017] was under way, Burstein and Rubin [2015] (mentioned above) considered ratios of 'adjacent' degree sequences, and iterated their formulae to obtain higher accuracy ratios. Having estimated the ratios, they used them to compare $g(\mathbf{d})$ with some known value $g(\mathbf{d}')$. In terms of the ratios, they give a formula for $g(\mathbf{d})$, however it was not explicit enough in the general case to enable the derivation of results as simple as (1-2). In particular, they did not extend the range of validity of (1-2) past what was known at the time.

(iv) Counting Latin rectangles.

All results before Leckey, Liebenau, and Wormald [n.d.] considered adding a random row to a valid $(k-1)\times n$ Latin rectangle, and estimating the probability that the new row causes no conflicts with the previous rows. (The paper McKay and Wormald [1991b] is almost an exception, since all rows are considered at random, and switching operations were used to generate a random Latin rectangle. A similar anlysis on numbers would have yielded the asymptotic formula for $k = o(n^{1/3})$ which was already known.) The approach in Leckey, Liebenau, and Wormald [n.d.] is different. As mentioned above, enumerating $k \times n$ Latin rectangles is equivalent to counting k-regular bipartite graphs with vertex parts V_1 and V_2, both of cardinality n, which have been properly k-edge-coloured. Now consider all bipartite graphs with vertex parts V_1 and V_2, with n edges of each of k colours. The *colour-degree sequence* is the array of numbers d_{ij}, $i = 1,\dots,n$, $j = 1,\dots,k$ such that d_{ij} is the number of edges of colour j incident with vertex i. Then the problem becomes one of enumerating those graphs with the all-1's degree sequence. Equivalently, consider the probability of such a colour-degree sequence arising when an edge-coloured bipartite graph with n edges of each colour is chosen uniformly at random. The problem is simplified somewhat by restricting to those graphs in which the degrees on V_2 are all 1; there is a simple model for selecting a random graph subject to this condition. To solve it, degree-switchings are applied to the vertices in V_1, and the perturbation method is applied as in Lemma 3.1 with $\mathcal{P}$ being a certain multinomial probability distribution. The number of variables (colour-degrees) is much larger than the graph case discussed above, with the consequence that the argument "only" succeeds for $k = o(n/\log n)$.

4 Applications of the enumeration results

4.1 Models for joint degree distribution. It was shown in McKay and Wormald [1997] how to use Proposition 1.2 to transfer properties of a sequence of independent binomial variables to the degree sequence of the random graph $\mathcal{G}(n, p)$ (in the range of p for which the proposition holds). In a follow-up paper, the authors will use this to obtain results on the order statistics of the degree sequence, and also the distribution of the number of vertices of given degree, a problem discussed at some length by Barbour, Holst, and Janson [1992].

Similar models for bipartite graphs and loopless directed graphs have been investigated in the dense range by McKay and Skerman [2016], who observed that once we have the results on enumeration that are now supplied by Liebenau and Wormald [2017], these models can presumably be extended to all interesting ranges of density.

4.2 Subgraphs and properties of random graphs. The methods of counting graphs can frequently be modified to include certain edges as specified (or forbidden). This lets us estimate moments of random variables that count copies of given subgraphs. Armed with such formulae, if they are simple enough, we can derive properties of the subgraph counts of random graphs with given degrees.

Examples include perfect matchings in regular graphs (see Bollobás and McKay [1986]). Robinson and Wormald [1994] used this approach (and a new technique for analysing variance) to show that a random d-regular graph is highly likely to have a Hamilton cycle. Enumeration results were used similarly to prove various properties of random regular graphs of high degree by Krivelevich, Sudakov, Vu, and Wormald [2001]. There many other examples.

4.3 Random generation. The uniform generation of random graphs with given degree sequence, and related objects, has statistical uses (see Blitzstein and Diaconis [2010] for example). Methods and results of enumeration can be useful, or sometimes adapted, to this problem. See Wormald [1984] for examples with exact enumeration; the applicability of the configuration model and its earlier bipartite versions are obvious. McKay and Wormald [1990b] used switchings to generate random regular graphs uniformly, and this was extended by Gao and Wormald [2017].

4.4 Relations to graphs *without* specified degrees. Kim and Vu [2004] used asymptotic enumeration results, amongst other things, to show that a random d-regular graph is "sandwiched" in between two random graphs $\mathcal{G}(n, m)$ for two different values of m close to d, as long as d does not grow too slowly or too quickly. Their result was extended by

Dudek, A. Frieze, Ruciński, and Šileikis [2017] to larger d, which might have been easier had the results in Liebenau and Wormald [2017] been available.

One approach to obtaining results for graphs with a given number of edges is to prove results for given degree sequences and then essentially sum over all degree sequences. This was used for instance by Pittel and Wormald [2005] to obtain the distribution of the size of the 2-core in the random graphs $\mathcal{G}(n, p)$ and $\mathcal{G}(n, m)$. Asymptotic enumeration results are used heavily in such arguments.

5 Final questions

An obvious question that might soon be within reach, given the new methods arising in the past year or so, is to find necessary and sufficient conditions on $\mathbf{d}$ for (1-2) to hold.

In McKay [2010], McKay points out that the asymptotic formula for the number of bipartite graphs with given degree sequence is unknown when the two vertex parts have very different cardinalities. The method in Liebenau and Wormald [2017] goes some way towards alleviating this defect in our knowledge, but further work can still be done. In particular, Canfield and McKay [2005] suggest a formula of Good and Crook that might be valid to within a constant factor in all cases for the biregular case.

One problem that arose in some discussions with Boris Pittel and is still unsolved, is to construct a nice model for the degree sequence of a random connected graph with n vertices and m edges, in the sparse range, particularly when $m = \Theta(n)$.

The asymptotic number of acyclic digraphs with n vertices and cn edges is essentially unknown. Can we nevertheless find an asymptotic formula for the number of acyclic digraphs with given in- and out-degree sequence, with cn edges?

Acknowledgments. I would like to thank all my coauthors on this topic, and Brendan McKay in particular for highly productive collaborations at various stages of the work and also for checking an early draft of this article.

References

N. Alon and J. H. Spencer (2000). *The probabilistic method*. Second. Wiley-Interscience Series in Discrete Mathematics and Optimization. With an appendix on the life and work of Paul Erdős. Wiley-Interscience [John Wiley & Sons], New York, pp. xviii+301. MR: 1885388 (cit. on p. 3263).

A. D. Barbour, L. Holst, and S. Janson (1992). *Poisson approximation*. Vol. 2. Oxford Studies in Probability. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, pp. x+277. MR: 1163825 (cit. on p. 3279).

A. Barvinok and J. A. Hartigan (2013). "The number of graphs and a random graph with a given degree sequence". *Random Structures Algorithms* 42.3, pp. 301–348. MR: 3039682 (cit. on p. 3272).

A. Békéssy, P. Békéssy, and J. Komlós (1972). "Asymptotic enumeration of regular matrices". *Studia Sci. Math. Hungar.* 7, pp. 343–353. MR: 0335342 (cit. on pp. 3268, 3269, 3273).

E. A. Bender (1974). "The asymptotic number of non-negative integer matrices with given row and column sums". *Discrete Math.* 10, pp. 217–223. MR: 0389621 (cit. on p. 3268).

E. A. Bender and E. R. Canfield (1978). "The asymptotic number of labeled graphs with given degree sequences". *J. Combinatorial Theory Ser. A* 24.3, pp. 296–307. MR: 0505796 (cit. on p. 3268).

J. Blitzstein and P. Diaconis (2010). "A sequential importance sampling algorithm for generating random graphs with prescribed degrees". *Internet Math.* 6.4, pp. 489–522. MR: 2809836 (cit. on p. 3279).

B. Bollobás (1980). "A probabilistic proof of an asymptotic formula for the number of labelled regular graphs". *European J. Combin.* 1.4, pp. 311–316. MR: 595929 (cit. on p. 3269).

– (1981). "The independence ratio of regular graphs". *Proc. Amer. Math. Soc.* 83.2, pp. 433–436. MR: 624948 (cit. on p. 3269).

– (2001). *Random graphs*. Second. Vol. 73. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. xviii+498. MR: 1864966 (cit. on p. 3264).

B. Bollobás and B. D. McKay (1986). "The number of matchings in random regular graphs and bipartite graphs". *J. Combin. Theory Ser. B* 41.1, pp. 80–91. MR: 854605 (cit. on pp. 3270, 3279).

D. Burstein and J. Rubin (Nov. 2015). "Degree switching and partitioning for enumerating graphs to arbitrary orders of accuracy". arXiv: 1511.03738 (cit. on pp. 3271, 3278).

E. R. Canfield and B. D. McKay (2005). "Asymptotic enumeration of dense 0-1 matrices with equal row sums and equal column sums". *Electron. J. Combin.* 12, Research Paper 29, 31. MR: 2156683 (cit. on p. 3280).

W. Y. C. Chen and J. D. Louck (1999). "Enumeration of cubic graphs by inclusion-exclusion". *J. Combin. Theory Ser. A* 86.1, pp. 151–157. MR: 1682968 (cit. on p. 3272).

C. Cooper, A. Frieze, B. Reed, and O. Riordan (2002). "Random regular graphs of non-constant degree: independence and chromatic number". *Combin. Probab. Comput.* 11.4, pp. 323–341. MR: 1918719.

M. Drmota (2009). *Random trees*. An interplay between combinatorics and probability. SpringerWienNewYork, Vienna, pp. xviii+458. MR: 2484382 (cit. on p. 3273).

A. Dudek, A. Frieze, A. Ruciński, and M. Šileikis (2017). "Embedding the Erdős-Rényi hypergraph into the random regular hypergraph and Hamiltonicity". *J. Combin. Theory Ser. B* 122, pp. 719–740. MR: 3575225 (cit. on p. 3280).

P. Erdős and I. Kaplansky (1946). "The asymptotic number of Latin rectangles". *Amer. J. Math.* 68, pp. 230–236. MR: 0015356 (cit. on p. 3273).

C. J. Everett and P. R. Stein (1971/72). "The asymptotic number of integer stochastic matrices". *Discrete Math.* 1.1, pp. 55–72. MR: 0389620 (cit. on p. 3268).

A. Frieze (1992). "On small subgraphs of random graphs". In: *Random graphs, Vol. 2 (Poznań, 1989)*. Wiley-Intersci. Publ. Wiley, New York, pp. 67–90. MR: 1166608 (cit. on p. 3276).

A. M. Frieze and T. Łuczak (1992). "On the independence and chromatic numbers of random regular graphs". *J. Combin. Theory Ser. B* 54.1, pp. 123–132. MR: 1142268.

P. Gao, Y. Su, and N. Wormald (2012). "Induced subgraphs in sparse random graphs with given degree sequences". *European J. Combin.* 33.6, pp. 1142–1166. MR: 2904981.

P. Gao and N. Wormald (2016). "Enumeration of graphs with a heavy-tailed degree sequence". *Adv. Math.* 287, pp. 412–450. MR: 3422681 (cit. on pp. 3270, 3271).

– (2017). "Uniform generation of random regular graphs". *SIAM J. Comput.* 46.4, pp. 1395–1427. MR: 3686817 (cit. on p. 3279).

C. D. Godsil and B. D. McKay (1990). "Asymptotic enumeration of Latin rectangles". *J. Combin. Theory Ser. B* 48.1, pp. 19–44. MR: 1047551 (cit. on p. 3273).

I. P. Goulden, D. M. Jackson, and J. W. Reilly (1983). "The Hammond series of a symmetric function and its application to P-recursiveness". *SIAM J. Algebraic Discrete Methods* 4.2, pp. 179–193. MR: 699770 (cit. on p. 3272).

C. Greenhill and B. D. McKay (2013). "Asymptotic enumeration of sparse multigraphs with given degrees". *SIAM J. Discrete Math.* 27.4, pp. 2064–2089. MR: 3141749 (cit. on p. 3273).

M. Isaev and B. D. McKay (Apr. 2016). "Complex martingales and asymptotic enumeration". arXiv: 1604.08305 (cit. on p. 3272).

S. Janson (2009). "The probability that a random multigraph is simple". *Combin. Probab. Comput.* 18.1-2, pp. 205–225. MR: 2497380 (cit. on pp. 3270, 3271).

– (2014). "The probability that a random multigraph is simple. II". *J. Appl. Probab.* 51A.Celebrating 50 Years of The Applied Probability Trust, pp. 123–137. MR: 3317354 (cit. on pp. 3270, 3271).

J. H. Kim and V. H. Vu (2004). "Sandwiching random graphs: universality between random graph models". *Adv. Math.* 188.2, pp. 444–469. MR: 2087234 (cit. on p. 3279).

M. Krivelevich, B. Sudakov, V. H. Vu, and N. Wormald (2001). "Random regular graphs of high degree". *Random Structures Algorithms* 18.4, pp. 346–363. MR: 1839497 (cit. on p. 3279).

G. Kuperberg, S. Lovett, and R. Peled (2017). "Probabilistic existence of regular combinatorial structures". *Geom. Funct. Anal.* 27.4, pp. 919–972. MR: 3678505 (cit. on p. 3273).

K. Leckey, A. Liebenau, and N. Wormald (n.d.). *The asymptotic number of Latin rectangles*. Preprint (cit. on pp. 3273, 3276, 3278).

A. Liebenau and N. Wormald (Feb. 2017). "Asymptotic enumeration of graphs by degree sequence, and the degree sequence of a random graph". arXiv: 1702.08373 (cit. on pp. 3264, 3267, 3271–3274, 3277–3280).

B. D. McKay (1981). "Subgraphs of random graphs with specified degrees". *Congr. Numer.* 33, pp. 213–223. MR: 681916 (cit. on p. 3274).

– (1984). "Asymptotics for 0-1 matrices with prescribed line sums". In: *Enumeration and design (Waterloo, Ont., 1982)*. Academic Press, Toronto, ON, pp. 225–238. MR: 782316 (cit. on p. 3270).

– (1985). "Asymptotics for symmetric 0-1 matrices with prescribed row sums". *Ars Combin.* 19.A, pp. 15–25. MR: 790916 (cit. on pp. 3270, 3274).

– (2010). "Subgraphs of random graphs with specified degrees". In: *Proceedings of the International Congress of Mathematicians. Volume IV*. Hindustan Book Agency, New Delhi, pp. 2489–2501. MR: 2827981 (cit. on p. 3280).

– (2011). "Subgraphs of dense random graphs with specified degrees". *Combin. Probab. Comput.* 20.3, pp. 413–433. MR: 2784635.

B. D. McKay and F. Skerman (2016). "Degree sequences of random digraphs and bipartite graphs". *J. Comb.* 7.1, pp. 21–49. MR: 3436194 (cit. on p. 3279).

B. D. McKay and N. Wormald (1990a). "Asymptotic enumeration by degree sequence of graphs of high degree". *European J. Combin.* 11.6, pp. 565–580. MR: 1078713 (cit. on pp. 3264, 3265, 3271, 3272).

– (1990b). "Uniform generation of random regular graphs of moderate degree". *J. Algorithms* 11.1, pp. 52–67. MR: 1041166 (cit. on p. 3279).

– (1991a). "Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$". *Combinatorica* 11.4, pp. 369–382. MR: 1137769 (cit. on pp. 3266, 3270, 3273, 3275).

– (1991b). "Uniform generation of random Latin rectangles". *J. Combin. Math. Combin. Comput.* 9, pp. 179–186. MR: 1111853 (cit. on pp. 3276, 3278).

– (1997). "The degree sequence of a random graph. I. The models". *Random Structures Algorithms* 11.2, pp. 97–117. MR: 1610253 (cit. on pp. 3265, 3279).

M. P. Mineev and A. I. Pavlov (1976). "The number of $(0,1)$-matrices with given sums over the rows and columns". *Dokl. Akad. Nauk SSSR* 230.2, pp. 271–274. MR: 0414381 (cit. on p. 3268).

J. W. Moon (1970). *Counting labelled trees*. Vol. 1969. From lectures delivered to the Twelfth Biennial Seminar of the Canadian Mathematical Congress (Vancouver).

Canadian Mathematical Congress, Montreal, Que., pp. x+113. MR: 0274333 (cit. on p. 3264).

P. E. O'Neil (1969). "Asymptotics and random matrices with row-sum and column-sum restrictions". *Bull. Amer. Math. Soc.* 75, pp. 1276–1282. MR: 0257116 (cit. on p. 3268).

B. Pittel and N. Wormald (2005). "Counting connected graphs inside-out". *J. Combin. Theory Ser. B* 93.2, pp. 127–172. MR: 2117934 (cit. on p. 3280).

R. C. Read (1958). "Some enumeration problems in graph theory". PhD thesis. University of London (University College of the West Indies) (cit. on pp. 3267, 3272).

– (1960). "The enumeration of locally restricted graphs. II". *J. London Math. Soc.* 35, pp. 344–351. MR: 0140443 (cit. on p. 3272).

R. C. Read and N. Wormald (1980). "Number of labeled 4-regular graphs". *J. Graph Theory* 4.2, pp. 203–212. MR: 570354 (cit. on p. 3272).

R. W. Robinson and N. Wormald (1994). "Almost all regular graphs are Hamiltonian". *Random Structures Algorithms* 5.2, pp. 363–374. MR: 1262985 (cit. on p. 3279).

D. Stark and N. Wormald (Aug. 2016). "The probability of nonexistence of a subgraph in a moderately sparse random graph". arXiv: 1608.05193 (cit. on p. 3277).

C. M. Stein (1978). "Asymptotic evaluation of the number of Latin rectangles". *J. Combin. Theory Ser. A* 25.1, pp. 38–49. MR: 499035 (cit. on pp. 3273, 3275, 3276).

N. Wormald (1978). "Some Problems in the Enumeration of Labelled Graphs". PhD thesis (cit. on p. 3269).

– (1984). "Generating random regular graphs". *J. Algorithms* 5.2, pp. 247–280. MR: 744493 (cit. on p. 3279).

– (1996). "The perturbation method and triangle-free random graphs". In: *Proceedings of the Seventh International Conference on Random Structures and Algorithms (Atlanta, GA, 1995)*. Vol. 9. 1-2, pp. 253–270. MR: 1611697 (cit. on pp. 3273, 3276).

K. Yamamoto (1951). "On the asymptotic number of Latin rectangles". *Jap. J. Math.* 21, 113–119 (1952). MR: 0051203 (cit. on p. 3273).

Received 2017-12-04.

NICHOLAS WORMALD
SCHOOL OF MATHEMATICAL SCIENCES
MONASH UNIVERSITY
AUSTRALIA
nick.wormald@monash.edu