Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 1 (163–172)

ON NEGATIVE ALGEBRAIC K-GROUPS

Moritz Kerz

Abstract

We sketch a proof of Weibel's conjecture on the vanishing of negative algebraic K-groups and we explain an analog of this result for continuous K-theory of non-archimedean algebras.

1 Negative K-groups of schemes

For a scheme X Grothendieck introduced the K-group $K_0(X)$ in his study of the generalized Riemann–Roch theorem *Théorie des intersections et théorème de Riemann-Roch* [1971, Def. IV.2.2]. In case X has an ample family of line bundles one can describe $K_0(X)$ as the free abelian group generated by the locally free $\mathcal{O}_X$-modules $\mathcal{V}$ of finite type modulo the relation $[\mathcal{V}'] + [\mathcal{V}''] - [\mathcal{V}]$ for any short exact sequence

$$0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0,$$

see *Théorie des intersections et théorème de Riemann-Roch* [ibid., Sec. IV.2.9]. We denote by $X[t]$ resp. $X[t^{-1}]$ the scheme $X \times \mathbb{A}^1$ with parameter t resp. t^{-1} for the affine line $\mathbb{A}^1$, and we denote by $X[t, t^{-1}]$ the scheme $X \times \mathbb{G}_m$, where $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$. Bass successively defined negative algebraic K-groups of the scheme X (at least in the affine case) in degree $i < 0$ to be

$$K_i(X) = \operatorname{coker}\left[K_{i+1}(X[t]) \times K_{i+1}(X[t^{-1}]) \to K_{i+1}(X[t, t^{-1}])\right].$$

The two classical key properties, essentially due to Bass [1968], satisfied by these algebraic K-groups are the *Fundamental Theorem* and *Excision*.

The author is supported by the DFG through CRC 1085 *Higher Invariants* (Universität Regensburg).
MSC2010: primary 19D35; secondary 13D15.

Proposition 1 (Fundamental Theorem)**.** *For a quasi-compact, quasi-separated scheme X and $i \le 0$ there exists an exact sequence*

$$0 \to K_i(X) \to K_i(X[t]) \times K_i(X[t^{-1}]) \to K_i(X[t,t^{-1}]) \to K_{i-1}(X) \to 0$$

Furthermore, for a noetherian, regular scheme X we have $K_i(X) = 0$ for $i < 0$.

Proposition 2 (Excision)**.** *For a ring homomorphism $A \to A'$ and an ideal $I \subset A$ which maps isomorphically onto an ideal I' of A' the map $K_i(A, I) \to K_i(A', I')$ of relative K-groups is an isomorphism for $i \le 0$.*

Combining Proposition 2 with the Artin-Rees Lemma we get the following more geometric reformulation:

Corollary 3. *For a finite morphism of affine noetherian schemes $f : X' \to X$ and a closed immersion $Y \hookrightarrow X$ such that f is an isomorphism over $X \setminus Y$ the map $K_i(X, Y) \to K_i(X', Y')$ is an isomorphism for $i < 0$. Here $Y' = Y \times_X X'$.*

In general it is a hard problem to actually calculate the negative K-groups in concrete examples. One of the examples calculated in C. Weibel [2001, Sec. 6] reads:

Example 4. *For a field k and the normal surface $X = \operatorname{Spec} k[x, y, z]/(z^2 - x^3 - y^7)$ we have $K_{-1}(X) = k$ and $K_i(X) = 0$ for $i < -1$.*

In fact it is shown in C. Weibel [ibid.] that for a normal surface X we have $K_{-2}(X) = \mathbb{Z}^\rho$ and $K_i(X) = 0$ for $i < -2$, where ρ is the number of "loops" in the exceptional divisor of a resolution of singularities of X. We extend this calculation in Theorem 8 and Theorem 11 to higher dimensions.

For our results it is essential to understand in which sense we can extend Corollary 3 to global schemes. For this we have to study the non-connective algebraic K-theory spectrum $K(X)$ of a scheme X introduced in Thomason and Trobaugh [1990]. Its homotopy groups $K_i(X) = \pi_i K(X)$ for $i \le 0$ agree with the K-groups defined above.

As shown in Thomason and Trobaugh [ibid., Sec. 8], the functor K satisfies Zariski descent. More concretely, consider a noetherian scheme X of finite dimension and a closed subscheme $Y \hookrightarrow X$. Let $K(X, Y)$ be the homotopy fibre of $K(X) \to K(Y)$. Let $K_{i,(X,Y)}$ be the Zariski presheaf on X given by $U \mapsto \pi_i K(U, Y \cap U)$ and let $K^{\sim}_{i,(X,Y)}$ be its Zariski sheafification. There exists a convergent descent spectral sequence

$$E_2^{pq} = H^p(X, K^{\sim}_{-q,(X,Y)}) \Rightarrow K_{-p-q}(X, Y). \tag{1}$$

As a direct consequence of Corollary 3 and of Zariski descent we observe:

Corollary 5. *Let X be a noetherian scheme, let $Y \hookrightarrow X$ be a closed subscheme and let d be the dimension of the closure $\overline{X \setminus Y}$. Assume that $Y \hookrightarrow X$ is an isomorphism away from $\overline{X \setminus Y}$. Let $f : \tilde{X} \to X$ be a finite morphism which is an isomorphism over $X \setminus Y$. Set $E = f^{-1}(Y)$. Then the map $f^* : K_i(X, Y) \to K_i(\tilde{X}, E)$ is an isomorphism for $i < -d$.*

Remark 6. For $\tilde{X} = X_{\mathrm{red}}$ and $Y = \varnothing$ Corollary 5 can be refined to an isomorphism $K_i(X) \xrightarrow{\simeq} K_i(X_{\mathrm{red}})$ for $i \leq -\dim(X)$.

Proof. In order to prove Corollary 5 one compares the descent spectral sequence (1) with the corresponding descent spectral sequence

$$E_2^{pq} = H^p(X, (f_* K_{-q,(\tilde{X},E)})^\sim) \Rightarrow K_{-p-q}(\tilde{X}, E).$$

and one uses that

(i) $K^\sim_{i,(X,Y)} \to (f_* K_{i,(\tilde{X},E)})^\sim$ is an isomorphism for $i \leq 0$ by Corollary 3,

(ii) the sheaves $K^\sim_{i,(X,Y)}$ and $(f_* K_{i,(\tilde{X},E)})^\sim$ vanish away from $\overline{X \setminus Y}$.

Note that (ii) implies that $E_2^{pq} = 0$ for $p > d$ in both spectral sequences. □

2 Platification par éclatement

In this section we explain an application of *platification par éclatement* Raynaud and Gruson [1971, Sec. 5], which generalizes the vanishing result Kerz and Strunk [2017, Prop. 5]. The motivating picture one should keep in mind is that negative K-groups of Zariski-Riemann spaces vanish, since all coherent sheaves on Zariski-Riemann spaces have Tor-dimension ≤ 1.

Let X be a quasi-compact and quasi-separated scheme, let $Y \hookrightarrow X$ be a closed subscheme defined by an invertible ideal sheaf. Recall that an admissible blow-up of X (with respect to Y) is a blow up $\mathbf{Bl}_Z X \to X$ with center $Z \hookrightarrow X$ of finite presentation and set theoretically contained in Y, see Raynaud and Gruson [1971, Def. 5.1.3]. Also recall that the composition of admissible blow-ups is admissible Raynaud and Gruson [ibid., Lem. 5.1.4]. Let $X' \to X$ be a smooth morphism of finite presentation and set $Y' = Y \times_X X'$.

The following proposition is clear in case there exists a suitable resolution of singularities for X, in view of Proposition 1. We denote by $K_i(X \text{ on } Y)$ the K-theory of X with support on Y as in Thomason and Trobaugh [1990, Def. 6.4].

Proposition 7. *Assume that X' has an ample family of line bundles and assume that X is reduced. For $i < 0$ and $\gamma \in K_i(X' \text{ on } Y')$ there exists an admissible blow-up $\tilde{X} \to X$ such that the pullback of γ to $K_i(X' \times_X \tilde{X} \text{ on } Y' \times_X \tilde{X})$ vanishes.*

Proof. For simplicity of notation we assume that $X = X'$ throughout the proof.

By noetherian approximation, see Thomason and Trobaugh [1990, App. C], there exists a directed inverse system $(X_\alpha)_\alpha$ of schemes of finite type over $\mathbb{Z}$ with affine transition maps such that $X = \varprojlim_\alpha X_\alpha$. We may further assume that Y descends to a system of closed subschemes $Y_\alpha \hookrightarrow X_\alpha$ and that there exists $\gamma_\alpha \in K_i(X_\alpha \text{ on } Y_\alpha)$ pulling back to γ.

Under the assumption that we know Proposition 7 for noetherian schemes we can, for some fixed α, find a closed subscheme Z_α which is set theoretically contained in Y_α and such that the pullback of γ_α to $K_i(\mathbf{Bl}_{Z_\alpha} X_\alpha \text{ on } Y_\alpha \times_{X_\alpha} \mathbf{Bl}_{Z_\alpha} X)$ vanishes. Let $\tilde{X}$ be $\mathbf{Bl}_Z X$, where Z is the pullback of Z_α to X. In view of the commutative diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathbf{Bl}_{Y_\alpha} X_\alpha \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_\alpha \end{array}$$

the scheme $\tilde{X}$ satisfies the requested property of Proposition 7.

By what has been explained, we can assume without loss of generality that all schemes in Proposition 7 are noetherian. In view of Bass' definition of negative K-theory, discussed in Section 1, we see that $K_{-k}(X \text{ on } Y)$ is a quotient of $K_0(\mathbb{G}^k_{m,X} \text{ on } \mathbb{G}^k_{m,Y})$ for $k > 0$ in which elements induced from $K_0(\mathbb{A}^k_X \text{ on } \mathbb{A}^k_Y)$ vanish. However, combining Kerz and Strunk [2017, Lem. 6] and Thomason and Trobaugh [1990, Ex. 5.7] we see that the latter groups are generated by coherent $\mathcal{O}$-modules on $\mathbb{G}^k_{m,X}$ resp. $\mathbb{A}^k_X$ which have support over Y and have Tor-dimension ≤ 1 over X. So without loss of generality the given element γ is induced by such an $\mathcal{O}_{\mathbb{G}^k_{m,X}}$-module $\mathcal{V}$ (here $k = -i$).

Extend $\mathcal{V}$ to a coherent $\mathcal{O}_{\mathbb{A}^k_X}$-module $\overline{\mathcal{V}}$ with support over Y. Because of the existence of an ample family of line bundles there exists an exact sequence of coherent $\mathcal{O}_{\mathbb{A}^k_X}$-modules

$$0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to \overline{\mathcal{V}} \to 0$$

with $\mathcal{V}_2$ locally free. By Raynaud and Gruson [1971, Thm. 5.2.2] there exists an admissible blow-up $f : \tilde{X} \to X$ such that the strict transform of $\mathcal{V}_1$ along f is flat over X. This implies that the pullback $f^*\overline{\mathcal{V}}$ has Tor-dimension ≤ 1 over X. So the latter induces an element in $K_0(\mathbb{A}^k_{\tilde{X}})$ which induces $f^*(\gamma) \in K_i(\tilde{X})$ via the Bass construction. This shows the requested vanishing of $f^*(\gamma)$. □

3 Weibel's conjecture

In C. A. Weibel [1980, p. 2.9] Weibel conjectured that the following Theorem 8 holds.

Theorem 8. *For a noetherian scheme X of dimension $d < \infty$ we have:*

(i) $K_i(X) = 0$ *for* $i < -d$*,*

(ii) $K_i(X) \xrightarrow{\simeq} K_i(X[t_1, \dots, t_r])$ *is an isomorphism for* $i \leq -d$ *and any number of variables* r*.*

There have been various partial results on Weibel's conjecture during the past twenty years, in particular it was shown for varieties X in characteristic zero Cortiñas, Haesemeyer, Schlichting, and C. Weibel [2008]. A complete proof of Theorem 8 was first given in Kerz, Strunk, and Tamme [2018, Thm. B] based on a pro-descent result for algebraic K-theory of blow-ups. In this section we sketch a simplified and more direct version of that proof which does not use the excision theory for K-theory of simplicial rings, as developed in Kerz, Strunk, and Tamme [ibid., Sec. 4]. For simplicity we will stick to part (i) of Theorem 8 in the proof.

Remark 9. Almost verbatim the same argument as in the proof of Theorem 8 shows that the conclusion remains true with X replaced by a scheme X' which is smooth of finite type over a noetherian scheme of dimension $d < \infty$. This was observed in Sadhu [2017].

The essential observation is that using *derived schemes* and *derived blow-ups* one can show that the analog of Corollary 5 holds for blow-ups, see Proposition 10 below.

For the convenience of the reader we summarize some properties of derived schemes in the following. A derived scheme $\mathcal{X}$ is roughly speaking given by a topological space $|\mathcal{X}|$ together with a 'derived' sheaf of commutative simplicial rings $\mathcal{O}_{\mathcal{X}}$ on $|\mathcal{X}|$, see Lurie [2016, Sec. 1.1.5]. For a derived scheme $\mathcal{X}$ its topological space together with its sheaf of homotopy groups $\pi_0 \mathcal{O}_{\mathcal{X}}$ defines an ordinary scheme, which we denote $t\mathcal{X}$. The ∞-category of derived schemes has finite limits and t preserves finite limits.

For a quasi-compact, quasi-separated derived scheme $\mathcal{X}$ one can construct its associated stable ∞-category of perfect $\mathcal{O}_{\mathcal{X}}$-modules $\mathrm{Perf}(\mathcal{X})$, see Lurie [ibid., Sec. 9.6], and one can define the K-theory spectrum $K(\mathcal{X})$ as the non-connective K-theory spectrum of $\mathrm{Perf}(\mathcal{X})$ in the sense of Blumberg, Gepner, and Tabuada [2013, Sec. 9.1].

The two key properties about the K-theory of a derived scheme $\mathcal{X}$ that we need — and that are well-known to the experts — are:

(DK1) For a quasi-compact, quasi-separated derived scheme $\mathcal{X}$ and a finite covering $\mathcal{U}$ of $\mathcal{X}$ by quasi-compact open subschemes there is a descent spectral sequence

$$E_2^{pq} = \check{H}^p(\mathcal{U}, K_{-q,\mathcal{X}}) \Rightarrow K_{-p-q}(\mathcal{X}),$$

compare Clausen, Mathew, Naumann, and Noel [2016, App. A] and Thomason and Trobaugh [1990, Prop. 8.3].

(DK2) For $\mathcal{X}$ affine, that is $\mathcal{X}$ is the spectrum of a simplicial ring, the map $K_i(\mathcal{X}) \xrightarrow{\simeq} K_i(t\mathcal{X})$ is an isomorphism for $i \leq 1$, compare Blumberg, Gepner, and Tabuada [2013, Thm. 9.53] and Kerz, Strunk, and Tamme [2018, Thm. 2.16].

Putting properties (DK1) and (DK2) together yields:

(DK3) For a quasi-compact, separated derived scheme $\mathcal{X}$ which has a covering by $d+1$ affine open subschemes the maps $K_i(\mathcal{X}) \xrightarrow{\simeq} K_i(t\mathcal{X}) \xrightarrow{\simeq} K_i((t\mathcal{X})_{\mathrm{red}})$ are isomorphisms for $i \leq -d$.

Proposition 10. *Let $X = \operatorname{Spec} A$ be a noetherian local scheme, let $Y \hookrightarrow X$ be a closed subscheme. Set $d = \dim(X)$, $\tilde{X} = \mathbf{Bl}_Y X$ and $E = f^{-1}(Y)$. Then the map $f^*: K_i(X,Y) \to K_i(\tilde{X}, E)$ is an isomorphism for $i < -d$.*

Proof. Let $I \subset A$ be the ideal corresponding to Y. After replacing I by some power, we can assume that there exists a reduction of I generated by elements $a_0, \dots, a_r$ with $r < d$, see Huneke and Swanson [2006, Prop. 8.3.8]. Choose a noetherian ring A' together with a regular sequence $a'_0, \dots, a'_r \in A'$ whose image under a ring homomorphism $A' \to A$ is the sequence $a_0, \dots, a_r$. Set $X' = \operatorname{Spec} A'$ and $Y'(n) = \operatorname{Spec} A'/((a'_0)^{2^n}, \dots, (a'_r)^{2^n})$ for $n \geq 0$. The derived blow-up square

$$\begin{array}{ccc} \tilde{\mathcal{X}}(m) & \longleftarrow & \mathcal{E}(m,n) \\ \downarrow & & \downarrow \\ X & \longleftarrow & \mathcal{Y}(n) \end{array} \tag{2}$$

is defined as the derived pullback of the usual cartesian blow-up square

$$\begin{array}{ccc} \mathbf{Bl}_{Y'(m)} X' & \longleftarrow & E'(m,n) \\ \downarrow & & \downarrow \\ X' & \longleftarrow & Y'(n) \end{array}$$

According to a derived generalization Kerz, Strunk, and Tamme [2018, Thm. 3.7] of a descent result of Thomason [1993], the square (2) gives rise to an equivalence of relative K-theory spectra $K(X, \mathcal{Y}(n)) \xrightarrow{\simeq} K(\tilde{\mathcal{X}}(n), \mathcal{E}(n,n))$ for any $n \geq 0$.

By property (DK2) above, we know that

$$K_i(X, \mathcal{Y}(n)) \xrightarrow{\simeq} K_i(X, Y)$$

is an isomorphism for $i \leq 0, n \geq 0$ and by property (DK3) we know that

$$K_i(\tilde{\mathfrak{X}}(n), \mathcal{E}(n,n)) \xrightarrow{\simeq} K_i(t\tilde{\mathfrak{X}}(n), t\mathcal{E}(n,n))$$

is an isomorphism for $i < -d$, $n \geq 0$. Note that $\tilde{\mathfrak{X}}$ and $\mathcal{E}$ have affine coverings by $r+1 \leq d$ open subschemes.

Finally, we apply Corollary 5 to the cartesian square

$$\begin{array}{ccc} \tilde{X} & \longleftarrow & E(n) \\ \downarrow & & \downarrow \\ t\tilde{\mathfrak{X}}(m) & \longleftarrow & t\mathcal{E}(m,n) \end{array}$$

for n large depending on m, in which the vertical maps are finite by Huneke and Swanson [2006, Thm. 8.2.1]. We deduce that

$$\lim_n K_i(t\tilde{\mathfrak{X}}(m), t\mathcal{E}(m,n)) \xrightarrow{\simeq} \lim_n K_i(\tilde{X}, E(n))$$

is an isomorphism for $i < -d$ and $m \geq 0$. For $i < -d$ composing the isomorphisms

$$K_i(X,Y) \xrightarrow{\simeq} \lim_n K_i(X, \mathcal{Y}(n)) \xrightarrow{\simeq} \lim_n K_i(\tilde{\mathfrak{X}}(n), \mathcal{E}(n,n)) \xrightarrow{\simeq}$$
$$\lim_n K_i(t\tilde{\mathfrak{X}}(n), t\mathcal{E}(n,n)) \xrightarrow{\simeq} \lim_m \lim_n K_i(t\tilde{\mathfrak{X}}(m), t\mathcal{E}(m,n)) \xrightarrow{\simeq}$$
$$\lim_n K_i(\tilde{X}, E(n)) \xrightarrow{\simeq} K_i(\tilde{X}, E)$$

finishes the proof of Proposition 10. □

Proof of Theorem 8(i). For the proof we make an induction on $d = \dim(X)$. The case $d = 0$ is clear as then $K_i(X) \xrightarrow{\simeq} K_i(X_{\mathrm{red}})$ vanishes for $i < 0$ by Proposition 1. For the induction step we use the descent spectral sequence (1) in order to reduce to the case of a local scheme $X = \operatorname{Spec} A$, see Kerz, Strunk, and Tamme [2018, Prop. 6.1] for details. Since in the affine case $K_i(X) \xrightarrow{\simeq} K_i(X_{\mathrm{red}})$ is an isomorphism for $i \leq 0$, we can assume without loss of generality that X is reduced.

Fix $\gamma \in K_i(X)$ for some $i < -d$. Let $Y \hookrightarrow X$ be a closed subscheme defined by an invertible ideal sheaf such that $\gamma|_{X \setminus Y} = 0$. This means that γ can be lifted to an element $\gamma' \in K_i(X \text{ on } Y)$. By Proposition 7 there exists a blow-up $f : \tilde{X} \to X$ in a center $Z \hookrightarrow X$ which is set theoretically contained in Y such that the pullback of γ' along f vanishes, in particular $f^*(\gamma) = 0 \in K_i(\tilde{X})$.

Set $E = f^{-1}(Z)$ and consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} K_{i+1}(E) & \longrightarrow & K_i(\tilde{X}, E) & \longrightarrow & K_i(\tilde{X}) & \longrightarrow & K_i(E) \\ \uparrow & & \uparrow & & {\scriptstyle f^*}\uparrow & & \uparrow \\ K_{i+1}(Z) & \longrightarrow & K_i(X, Z) & \longrightarrow & K_i(X) & \longrightarrow & K_i(Z) \end{array} \tag{3}$$

As $\dim(Z), \dim(E) < d$ the K-groups in the outer corners of diagram (3) vanish by our induction assumption. The second vertical arrow in (3) is an isomorphism by Proposition 10. So the third vertical arrow is an isomorphism as well, which implies that $\gamma = 0$. □

While the negative K-groups $K_i(X)$ for $-d = -\dim(X) < i < 0$ can be quite hard to calculate, there is nice formula for $K_{-d}(X)$, which was shown in complete generality in Kerz, Strunk, and Tamme [2018, Cor. D] and previously for varieties in characteristic zero Cortiñas, Haesemeyer, Schlichting, and C. Weibel [2008].

Theorem 11. *For a noetherian scheme of dimension $d < \infty$ there is a canonical isomorphism*

$$K_{-d}(X) \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z}),$$

where on the right we take sheaf cohomology with respect to the cdh-topology on X.

4 Negative K-groups of affinoid algebras

In this section let k be a (non-discrete) non-archimedean complete field with ring of integers k°. By π we denote an element with absolute value $0 < |\pi| < 1$. For an affinoid algebra A over k, see Bosch [2014, Sec. 3.1], we write $A\langle t\rangle$ for the Tate algebra over A, which consists of those formal power series $a_0 + a_1 t + \cdots \in A[\![t]\!]$ with $\lim_{i\to\infty} a_i = 0$, similarly for $A\langle t^{-1}\rangle$ and $A\langle t, t^{-1}\rangle$. One defines non-positive continuous K-groups of A successively by $K_0^{\mathrm{cont}}(A) = K_0(A)$ and

$$K_i^{\mathrm{cont}}(A) = \operatorname{coker}\left[K_{i+1}^{\mathrm{cont}}(A\langle t\rangle) \times K_{i+1}^{\mathrm{cont}}(A\langle t^{-1}\rangle) \to K_{i+1}^{\mathrm{cont}}(A\langle t, t^{-1}\rangle)\right]$$

for $i < 0$. These negative continuous K-groups were defined and studied in Karoubi and Villamayor [1971, Sec. 7] and Calvo [1985]. They also coincide with the continuous pro-groups defined in Morrow [2016, Sec. 3] as is shown in Kerz, Saito, and Tamme [2018, Sec. 5].

We are about to show that the analog of Weibel's conjecture, i.e. Theorem 8, holds in the non-archimedean situation:

Theorem 12. *Assume that k is discretely valued. For an affinoid k-algebra A of dimension d we have:*

(i) $K_i^{\mathrm{cont}}(A) = 0$ *for* $i < -d$,

(ii) $K_i^{\mathrm{cont}}(A) \xrightarrow{\simeq} K_i^{\mathrm{cont}}(A\langle t_1, \dots, t_r\rangle)$ *is an isomorphism for* $i \le -d$ *and any number of variables* r.

I expect that the condition that k is discretely valued in Theorem 12 can be removed. Note that even for smooth affinoid algebras A the negative continuous K-groups do not necessarily vanish, as the following example shows.

Example 13. *Assume that the residue field of* k *has characteristic zero and let* $\pi \in k$ *be an element of absolute value* $0 < |\pi| < 1$. *For the affinoid algebra* $A = k\langle s,t\rangle/(t^2 - s^3 + s^2 - \pi)$ *we have* $K_{-1}^{\mathrm{cont}}(A) = \mathbb{Z}$.

The key fact for us is that for an admissible k°-algebra A_0 in the sense of Bosch [2014, Def. 7.3.3] with $A = A_0[1/\pi]$ we obtain an exact sequence, see Kerz, Saito, and Tamme [2018, Sec. 5],

$$(4) \quad K_0(A_0 \text{ on } (\pi)) \to K_0((A_0/(\pi))_{\mathrm{red}}) \to K_0^{\mathrm{cont}}(A) \to K_{-1}(A_0 \text{ on } (\pi)) \to \cdots .$$

Recall that an admissible k°-algebra A_0 is π-adically complete, topologically of finite type and π-torsion free.

The claim made in Example 13 follows from the exact sequence (4) by setting $A_0 = k^\circ\langle s,t\rangle/(t^2 - s^3 + s^2 - \pi)$ and using that in this case $K_i(A_0 \text{ on } (\pi)) = 0$ for $i < 0$ and that $K_{-1}((A_0/(\pi))_{\mathrm{red}}) = \mathbb{Z}$.

Proof of Theorem 12(i). We can assume without loss of generality that A is reduced. An admissible blow-up of $X = \operatorname{Spec} A_0$, where A_0 is an admissible k°-algebra as above, is defined as a blow-up in a center $Y \hookrightarrow X$ which is set theoretically contained in $\operatorname{Spec} A_0/(\pi)$. Let now $\tilde{X} = \mathbf{Bl}_Y X$ be such an admissible blow-up, $X_0 = (X \otimes_{A_0} A_0/(\pi))_{\mathrm{red}}$ and $\tilde{X}_0 = (\tilde{X} \otimes_{A_0} A_0/(\pi))_{\mathrm{red}}$. For $i < -d$ we obtain from Kerz, Saito, and Tamme [ibid., Prop. 5.8] the upper exact sequence in the commutative diagram

$$(5) \quad \begin{array}{ccccc} K_i(\tilde{X}_0) & \longrightarrow & K_i^{\mathrm{cont}}(A) & \longrightarrow & K_{i-1}(\tilde{X} \text{ on } (\pi)) \\ \uparrow & & \| & & \uparrow \\ K_i(X_0) & \longrightarrow & K_i^{\mathrm{cont}}(A) & \longrightarrow & K_{i-1}(X \text{ on } (\pi)) \end{array}$$

while the lower exact sequence is just part of (4). Both groups on the left of (5) vanish by Theorem 8 since $i < -d = -\dim(X_0) = -\dim(\tilde{X}_0)$. Consider an element $\alpha \in K_i^{\mathrm{cont}}(A)$. For its image $\alpha' \in K_{i-1}(X \text{ on } (\pi))$ we can use Proposition 7 in order to choose the admissible blow-up $\tilde{X}$ such that the pullback of α' to $K_{i-1}(\tilde{X} \text{ on } (\pi))$ vanishes. A diagram chase in (5) shows that α vanishes. □

The forthcoming PhD thesis of C. Dahlhausen will discuss the following conjecture, which is the non-archimedean analytic variant of Theorem 11.

Conjecture 14. *For an affinoid k-algebra A of dimension d there is an isomorphism*

$$K_{-d}^{\mathrm{cont}}(A) \cong H^d(\mathfrak{M}(A), \mathbb{Z}).$$

Here $\mathfrak{M}(A)$ is the Berkovich spectrum of multiplicative seminorms Berkovich [1990, Ch. 1].

Acknowledgments. I would like to thank H. Esnault, O. Gabber, M. Groechenig, M. Morrow, S. Saito, F. Strunk, G. Tamme and C. Weibel for helpful discussions.

References

Hyman Bass (1968). *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, pp. xx+762. MR: 0249491 (cit. on p. 163).

Vladimir G. Berkovich (1990). *Spectral theory and analytic geometry over non-Archimedean fields*. Vol. 33. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, pp. x+169. MR: 1070709 (cit. on p. 172).

Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada (2013). "A universal characterization of higher algebraic K-theory". *Geom. Topol.* 17.2, pp. 733–838. MR: 3070515 (cit. on pp. 167, 168).

Siegfried Bosch (2014). *Lectures on formal and rigid geometry*. Vol. 2105. Lecture Notes in Mathematics. Springer, Cham, pp. viii+254. MR: 3309387 (cit. on pp. 170, 171).

Adina Calvo (1985). "K-théorie des anneaux ultramétriques". *C. R. Acad. Sci. Paris Sér. I Math.* 300.14, pp. 459–462. MR: 788971 (cit. on p. 170).

Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel (June 2016). "Descent in algebraic K-theory and a conjecture of Ausoni-Rognes". arXiv: 1606.03328 (cit. on p. 168).

G. Cortiñas, C. Haesemeyer, M. Schlichting, and C. Weibel (2008). "Cyclic homology, cdh-cohomology and negative K-theory". *Ann. of Math. (2)* 167.2, pp. 549–573. MR: 2415380 (cit. on pp. 167, 170).

Craig Huneke and Irena Swanson (2006). *Integral closure of ideals, rings, and modules*. Vol. 336. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, pp. xiv+431. MR: 2266432 (cit. on pp. 168, 169).

Max Karoubi and Orlando Villamayor (1971). "K-théorie algébrique et K-théorie topologique. I". *Math. Scand.* 28, 265–307 (1972). MR: 0313360 (cit. on p. 170).

Moritz Kerz, Shuji Saito, and Georg Tamme (Feb. 2018). “K-theory of non-archimedean rings I”. arXiv: 1802.09819 (cit. on pp. 170, 171).

Moritz Kerz and Florian Strunk (2017). “On the vanishing of negative homotopy K-theory”. *J. Pure Appl. Algebra* 221.7, pp. 1641–1644. MR: 3614971 (cit. on pp. 165, 166).

Moritz Kerz, Florian Strunk, and Georg Tamme (2018). “Algebraic K-theory and descent for blow-ups”. *Invent. Math.* 211.2, pp. 523–577. MR: 3748313 (cit. on pp. 167–170).

Jacob Lurie (2016). “Spectral algebraic geometry”. Preprint (version dated October 13, 2016) (cit. on p. 167).

Matthew Morrow (Dec. 2016). “A historical overview of pro cdh descent in algebraic K-theory and its relation to rigid analytic varieties”. arXiv: 1612.00418 (cit. on p. 170).

Michel Raynaud and Laurent Gruson (1971). “Critères de platitude et de projectivité. Techniques de “platification” d’un module”. *Invent. Math.* 13, pp. 1–89. MR: 0308104 (cit. on pp. 165, 166).

Vivek Sadhu (Jan. 2017). “On the vanishing of Relative Negative K-theory”. arXiv: 1701.09059 (cit. on p. 167).

Théorie des intersections et théorème de Riemann-Roch (1971). Lecture Notes in Mathematics, Vol. 225. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre. Springer-Verlag, Berlin-New York, pp. xii+700. MR: 0354655 (cit. on p. 163).

R. W. Thomason (1993). “Les K-groupes d’un schéma éclaté et une formule d’intersection excédentaire”. *Invent. Math.* 112.1, pp. 195–215. MR: 1207482 (cit. on p. 168).

R. W. Thomason and Thomas Trobaugh (1990). “Higher algebraic K-theory of schemes and of derived categories”. In: *The Grothendieck Festschrift, Vol. III*. Vol. 88. Progr. Math. Birkhäuser Boston, Boston, MA, pp. 247–435. MR: 1106918 (cit. on pp. 164–166, 168).

Charles Weibel (2001). “The negative K-theory of normal surfaces”. *Duke Math. J.* 108.1, pp. 1–35. MR: 1831819 (cit. on p. 164).

Charles A. Weibel (1980). “K-theory and analytic isomorphisms”. *Invent. Math.* 61.2, pp. 177–197. MR: 590161 (cit. on p. 167).

Received 2017-12-08.

Moritz Kerz
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
Germany
moritz.kerz@mathematik.uni-regensburg.de