

# ON NEGATIVE ALGEBRAIC $K$ -GROUPS

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## Abstract

We sketch a proof of Weibel’s conjecture on the vanishing of negative algebraic  $K$ -groups and we explain an analog of this result for continuous  $K$ -theory of non-archimedean algebras.

## 1 Negative $K$ -groups of schemes

For a scheme  $X$  Grothendieck introduced the  $K$ -group  $K_0(X)$  in his study of the generalized Riemann–Roch theorem in [Berthelot, Grothendieck, and Illusie \[1971, Def. IV.2.2\]](#). In case  $X$  has an ample family of line bundles one can describe  $K_0(X)$  as the free abelian group generated by the locally free  $\mathcal{O}_X$ -modules  $\mathcal{V}$  of finite type modulo the relation  $[\mathcal{V}'] + [\mathcal{V}'' ] - [\mathcal{V}]$  for any short exact sequence

$$0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0,$$

see [Berthelot, Grothendieck, and Illusie \[ibid., Sec. IV.2.9\]](#). We denote by  $X[t]$  resp.  $X[t^{-1}]$  the scheme  $X \times \mathbb{A}^1$  with parameter  $t$  resp.  $t^{-1}$  for the affine line  $\mathbb{A}^1$ , and we denote by  $X[t, t^{-1}]$  the scheme  $X \times \mathbb{G}_m$ , where  $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ . Bass successively defined negative algebraic  $K$ -groups of the scheme  $X$  (at least in the affine case) in degree  $i < 0$  to be

$$K_i(X) = \operatorname{coker} [K_{i+1}(X[t]) \times K_{i+1}(X[t^{-1}]) \rightarrow K_{i+1}(X[t, t^{-1}])].$$

The two classical key properties, essentially due to [Bass \[1968\]](#), satisfied by these algebraic  $K$ -groups are the *Fundamental Theorem* and *Excision*.

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**Proposition 1** (Fundamental Theorem). *For a quasi-compact, quasi-separated scheme  $X$  and  $i \leq 0$  there exists an exact sequence*

$$0 \rightarrow K_i(X) \rightarrow K_i(X[t]) \times K_i(X[t^{-1}]) \rightarrow K_i(X[t, t^{-1}]) \rightarrow K_{i-1}(X) \rightarrow 0$$

Furthermore, for a noetherian, regular scheme  $X$  we have  $K_i(X) = 0$  for  $i < 0$ .

**Proposition 2** (Excision). *For a ring homomorphism  $A \rightarrow A'$  and an ideal  $I \subset A$  which maps isomorphically onto an ideal  $I'$  of  $A'$  the map  $K_i(A, I) \rightarrow K_i(A', I')$  of relative  $K$ -groups is an isomorphism for  $i \leq 0$ .*

Combining [Proposition 2](#) with the Artin-Rees Lemma we get the following more geometric reformulation:

**Corollary 3.** *For a finite morphism of affine noetherian schemes  $f : X' \rightarrow X$  and a closed immersion  $Y \hookrightarrow X$  such that  $f$  is an isomorphism over  $X \setminus Y$  the map  $K_i(X, Y) \rightarrow K_i(X', Y')$  is an isomorphism for  $i < 0$ . Here  $Y' = Y \times_X X'$ .*

In general it is a hard problem to actually calculate the negative  $K$ -groups in concrete examples. One of the examples calculated in [C. Weibel \[2001, Sec. 6\]](#) reads:

**Example 4.** *For a field  $k$  and the normal surface  $X = \text{Spec } k[x, y, z]/(z^2 - x^3 - y^7)$  we have  $K_{-1}(X) = k$  and  $K_i(X) = 0$  for  $i < -1$ .*

In fact it is shown in [C. Weibel \[ibid.\]](#) that for a normal surface  $X$  we have  $K_{-2}(X) = \mathbb{Z}^\rho$  and  $K_i(X) = 0$  for  $i < -2$ , where  $\rho$  is the number of “loops” in the exceptional divisor of a resolution of singularities of  $X$ . We extend this calculation in [Theorem 8](#) and [Theorem 11](#) to higher dimensions.

For our results it is essential to understand in which sense we can extend [Corollary 3](#) to global schemes. For this we have to study the non-connective algebraic  $K$ -theory spectrum  $K(X)$  of a scheme  $X$  introduced in [Thomason and Trobaugh \[1990\]](#). Its homotopy groups  $K_i(X) = \pi_i K(X)$  for  $i \leq 0$  agree with the  $K$ -groups defined above.

As shown in [Thomason and Trobaugh \[ibid., Sec. 8\]](#), the functor  $K$  satisfies Zariski descent. More concretely, consider a noetherian scheme  $X$  of finite dimension and a closed subscheme  $Y \hookrightarrow X$ . Let  $K(X, Y)$  be the homotopy fibre of  $K(X) \rightarrow K(Y)$ . Let  $K_{i,(X,Y)}$  be the Zariski presheaf on  $X$  given by  $U \mapsto \pi_i K(U, Y \cap U)$  and let  $K_{i,(X,Y)}^\sim$  be its Zariski sheafification. There exists a convergent descent spectral sequence

$$(1) \quad E_2^{p,q} = H^p(X, K_{-q,(X,Y)}^\sim) \Rightarrow K_{-p-q}(X, Y).$$

As a direct consequence of [Corollary 3](#) and of Zariski descent we observe:

**Corollary 5.** *Let  $X$  be a noetherian scheme, let  $Y \hookrightarrow X$  be a closed subscheme and let  $d$  be the dimension of the closure  $\overline{X \setminus Y}$ . Assume that  $Y \hookrightarrow X$  is an isomorphism away from  $\overline{X \setminus Y}$ . Let  $f : \tilde{X} \rightarrow X$  be a finite morphism which is an isomorphism over  $X \setminus Y$ . Set  $E = f^{-1}(Y)$ . Then the map  $f^* : K_i(X, Y) \rightarrow K_i(\tilde{X}, E)$  is an isomorphism for  $i < -d$ .*

**Remark 6.** For  $\tilde{X} = X_{\text{red}}$  and  $Y = \emptyset$  [Corollary 5](#) can be refined to an isomorphism  $K_i(X) \xrightarrow{\cong} K_i(X_{\text{red}})$  for  $i \leq -\dim(X)$ .

*Proof.* In order to prove [Corollary 5](#) one compares the descent spectral sequence (1) with the corresponding descent spectral sequence

$$E_2^{p,q} = H^p(X, (f_* K_{-q, (\tilde{X}, E)})^\sim) \Rightarrow K_{-p-q}(\tilde{X}, E).$$

and one uses that

- (i)  $K_{i, (X, Y)}^\sim \rightarrow (f_* K_{i, (\tilde{X}, E)})^\sim$  is an isomorphism for  $i \leq 0$  by [Corollary 3](#),
- (ii) the sheaves  $K_{i, (X, Y)}^\sim$  and  $(f_* K_{i, (\tilde{X}, E)})^\sim$  vanish away from  $\overline{X \setminus Y}$ .

Note that (ii) implies that  $E_2^{p,q} = 0$  for  $p > d$  in both spectral sequences. □

## 2 Platisation par éclatement

In this section we explain an application of *platisation par éclatement* [Raynaud and Gruson \[1971, Sec. 5\]](#), which generalizes the vanishing result [Kerz and Strunk \[2017, Prop. 5\]](#). The motivating picture one should keep in mind is that negative  $K$ -groups of Zariski-Riemann spaces vanish, since all coherent sheaves on Zariski-Riemann spaces have Tor-dimension  $\leq 1$ .

Let  $X$  be a quasi-compact and quasi-separated scheme, let  $Y \hookrightarrow X$  be a closed subscheme defined by an invertible ideal sheaf. Recall that an admissible blow-up of  $X$  (with respect to  $Y$ ) is a blow up  $\mathbf{Bl}_Z X \rightarrow X$  with center  $Z \hookrightarrow X$  of finite presentation and set theoretically contained in  $Y$ , see [Raynaud and Gruson \[1971, Def. 5.1.3\]](#). Also recall that the composition of admissible blow-ups is admissible [Raynaud and Gruson \[ibid., Lem. 5.1.4\]](#). Let  $X' \rightarrow X$  be a smooth morphism of finite presentation and set  $Y' = Y \times_X X'$ .

The following proposition is clear in case there exists a suitable resolution of singularities for  $X$ , in view of [Proposition 1](#). We denote by  $K_i(X \text{ on } Y)$  the  $K$ -theory of  $X$  with support on  $Y$  as in [Thomason and Trobaugh \[1990, Def. 6.4\]](#).

**Proposition 7.** *Assume that  $X'$  has an ample family of line bundles and assume that  $X$  is reduced. For  $i < 0$  and  $\gamma \in K_i(X'$  on  $Y')$  there exists an admissible blow-up  $\tilde{X} \rightarrow X$  such that the pullback of  $\gamma$  to  $K_i(X' \times_X \tilde{X}$  on  $Y' \times_X \tilde{X})$  vanishes.*

*Proof.* For simplicity of notation we assume that  $X = X'$  throughout the proof.

By noetherian approximation, see [Thomason and Trobaugh \[1990, App. C\]](#), there exists a directed inverse system  $(X_\alpha)_\alpha$  of schemes of finite type over  $\mathbb{Z}$  with affine transition maps such that  $X = \varprojlim_\alpha X_\alpha$ . We may further assume that  $Y$  descends to a system of closed subschemes  $Y_\alpha \hookrightarrow X_\alpha$  and that there exists  $\gamma_\alpha \in K_i(X_\alpha$  on  $Y_\alpha)$  pulling back to  $\gamma$ .

Under the assumption that we know [Proposition 7](#) for noetherian schemes we can, for some fixed  $\alpha$ , find a closed subscheme  $Z_\alpha$  which is set theoretically contained in  $Y_\alpha$  and such that the pullback of  $\gamma_\alpha$  to  $K_i(\mathbf{B}\mathbf{I}_{Z_\alpha} X_\alpha$  on  $Y_\alpha \times_{X_\alpha} \mathbf{B}\mathbf{I}_{Z_\alpha} X)$  vanishes. Let  $\tilde{X}$  be  $\mathbf{B}\mathbf{I}_Z X$ , where  $Z$  is the pullback of  $Z_\alpha$  to  $X$ . In view of the commutative diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathbf{B}\mathbf{I}_{Y_\alpha} X_\alpha \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_\alpha \end{array}$$

the scheme  $\tilde{X}$  satisfies the requested property of [Proposition 7](#).

By what has been explained, we can assume without loss of generality that all schemes in [Proposition 7](#) are noetherian. In view of Bass' definition of negative  $K$ -theory, discussed in [Section 1](#), we see that  $K_{-k}(X$  on  $Y)$  is a quotient of  $K_0(\mathbb{G}_{m,X}^k$  on  $\mathbb{G}_{m,Y}^k)$  for  $k > 0$  in which elements induced from  $K_0(\mathbb{A}_X^k$  on  $\mathbb{A}_Y^k)$  vanish. However, combining [Kerz and Strunk \[2017, Lem. 6\]](#) and [Thomason and Trobaugh \[1990, Ex. 5.7\]](#) we see that the latter groups are generated by coherent  $\mathcal{O}$ -modules on  $\mathbb{G}_{m,X}^k$  resp.  $\mathbb{A}_X^k$  which have support over  $Y$  and have Tor-dimension  $\leq 1$  over  $X$ . So without loss of generality the given element  $\gamma$  is induced by such an  $\mathcal{O}_{\mathbb{G}_{m,X}^k}$ -module  $\mathcal{U}$  (here  $k = -i$ ).

Extend  $\mathcal{U}$  to a coherent  $\mathcal{O}_{\mathbb{A}_X^k}$ -module  $\overline{\mathcal{U}}$  with support over  $Y$ . Because of the existence of an ample family of line bundles there exists an exact sequence of coherent  $\mathcal{O}_{\mathbb{A}_X^k}$ -modules

$$0 \rightarrow \mathcal{V}_1 \rightarrow \mathcal{V}_2 \rightarrow \overline{\mathcal{U}} \rightarrow 0$$

with  $\mathcal{V}_2$  locally free. By [Raynaud and Gruson \[1971, Thm. 5.2.2\]](#) there exists an admissible blow-up  $f : \tilde{X} \rightarrow X$  such that the strict transform of  $\mathcal{V}_1$  along  $f$  is flat over  $X$ . This implies that the pullback  $f^*\overline{\mathcal{U}}$  has Tor-dimension  $\leq 1$  over  $X$ . So the latter induces an element in  $K_0(\mathbb{A}_{\tilde{X}}^k)$  which induces  $f^*(\gamma) \in K_i(\tilde{X})$  via the Bass construction. This shows the requested vanishing of  $f^*(\gamma)$ .

□

### 3 Weibel’s conjecture

In C. A. Weibel [1980, p. 2.9] Weibel conjectured that the following [Theorem 8](#) holds.

**Theorem 8.** *For a noetherian scheme  $X$  of dimension  $d < \infty$  we have:*

- (i)  $K_i(X) = 0$  for  $i < -d$ ,
- (ii)  $K_i(X) \xrightarrow{\simeq} K_i(X[t_1, \dots, t_r])$  is an isomorphism for  $i \leq -d$  and any number of variables  $r$ .

There have been various partial results on Weibel’s conjecture during the past twenty years, in particular it was shown for varieties  $X$  in characteristic zero [Cortiñas, Haese-meyer, Schlichting, and C. Weibel \[2008\]](#). A complete proof of [Theorem 8](#) was first given in [Kerz, Strunk, and Tamme \[2018, Thm. B\]](#) based on a pro-descent result for algebraic  $K$ -theory of blow-ups. In this section we sketch a simplified and more direct version of that proof which does not use the excision theory for  $K$ -theory of simplicial rings, as developed in [Kerz, Strunk, and Tamme \[ibid., Sec. 4\]](#). For simplicity we will stick to part (i) of [Theorem 8](#) in the proof.

**Remark 9.** Almost verbatim the same argument as in the proof of [Theorem 8](#) shows that the conclusion remains true with  $X$  replaced by a scheme  $X'$  which is smooth of finite type over a noetherian scheme of dimension  $d < \infty$ . This was observed in [Sadhu \[2017\]](#).

The essential observation is that using *derived schemes* and *derived blow-ups* one can show that the analog of [Corollary 5](#) holds for blow-ups, see [Proposition 10](#) below.

For the convenience of the reader we summarize some properties of derived schemes in the following. A derived scheme  $\mathcal{X}$  is roughly speaking given by a topological space  $|\mathcal{X}|$  together with a ‘derived’ sheaf of commutative simplicial rings  $\mathcal{O}_{\mathcal{X}}$  on  $|\mathcal{X}|$ , see [Lurie \[2016, Sec. 1.1.5\]](#). For a derived scheme  $\mathcal{X}$  its topological space together with its sheaf of homotopy groups  $\pi_0 \mathcal{O}_{\mathcal{X}}$  defines an ordinary scheme, which we denote  $t\mathcal{X}$ . The  $\infty$ -category of derived schemes has finite limits and  $t$  preserves finite limits.

For a quasi-compact, quasi-separated derived scheme  $\mathcal{X}$  one can construct its associated stable  $\infty$ -category of perfect  $\mathcal{O}_{\mathcal{X}}$ -modules  $\text{Perf}(\mathcal{X})$ , see [Lurie \[ibid., Sec. 9.6\]](#), and one can define the  $K$ -theory spectrum  $K(\mathcal{X})$  as the non-connective  $K$ -theory spectrum of  $\text{Perf}(\mathcal{X})$  in the sense of [Blumberg, Gepner, and Tabuada \[2013, Sec. 9.1\]](#).

The two key properties about the  $K$ -theory of a derived scheme  $\mathcal{X}$  that we need — and that are well-known to the experts — are:

- (DK1) For a quasi-compact, quasi-separated derived scheme  $\mathcal{X}$  and a finite covering  $\mathcal{U}$  of  $\mathcal{X}$  by quasi-compact open subschemes there is a descent spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, K_{-q, \mathcal{X}}) \Rightarrow K_{-p-q}(\mathcal{X}),$$

compare [Clausen, Mathew, Naumann, and Noel \[2016, App. A\]](#) and [Thomason and Trobaugh \[1990, Prop. 8.3\]](#).

(DK2) For  $\mathcal{X}$  affine, that is  $\mathcal{X}$  is the spectrum of a simplicial ring, the map  $K_i(\mathcal{X}) \xrightarrow{\cong} K_i(t\mathcal{X})$  is an isomorphism for  $i \leq 1$ , compare [Blumberg, Gepner, and Tabuada \[2013, Thm. 9.53\]](#) and [Kerz, Strunk, and Tamme \[2018, Thm. 2.16\]](#).

Putting properties (DK1) and (DK2) together yields:

(DK3) For a quasi-compact, separated derived scheme  $\mathcal{X}$  which has a covering by  $d + 1$  affine open subschemes the maps  $K_i(\mathcal{X}) \xrightarrow{\cong} K_i(t\mathcal{X}) \xrightarrow{\cong} K_i((t\mathcal{X})_{\text{red}})$  are isomorphisms for  $i \leq -d$ .

**Proposition 10.** *Let  $X = \text{Spec } A$  be a noetherian local scheme, let  $Y \hookrightarrow X$  be a closed subscheme. Set  $d = \dim(X)$ ,  $\tilde{X} = \mathbf{Bl}_Y X$  and  $E = f^{-1}(Y)$ . Then the map  $f^* : K_i(X, Y) \rightarrow K_i(\tilde{X}, E)$  is an isomorphism for  $i < -d$ .*

*Proof.* Let  $I \subset A$  be the ideal corresponding to  $Y$ . After replacing  $I$  by some power, we can assume that there exists a reduction of  $I$  generated by elements  $a_0, \dots, a_r$  with  $r < d$ , see [Huneke and Swanson \[2006, Prop. 8.3.8\]](#). Choose a noetherian ring  $A'$  together with a regular sequence  $a'_0, \dots, a'_r \in A'$  whose image under a ring homomorphism  $A' \rightarrow A$  is the sequence  $a_0, \dots, a_r$ . Set  $X' = \text{Spec } A'$  and  $Y'(n) = \text{Spec } A' / ((a'_0)^{2^n}, \dots, (a'_r)^{2^n})$  for  $n \geq 0$ . The derived blow-up square

$$(2) \quad \begin{array}{ccc} \tilde{X}(m) & \longleftarrow & \mathcal{E}(m, n) \\ \downarrow & & \downarrow \\ X & \longleftarrow & \mathcal{Y}(n) \end{array}$$

is defined as the derived pullback of the usual cartesian blow-up square

$$\begin{array}{ccc} \mathbf{Bl}_{Y'(m)} X' & \longleftarrow & E'(m, n) \\ \downarrow & & \downarrow \\ X' & \longleftarrow & Y'(n) \end{array}$$

According to a derived generalization [Kerz, Strunk, and Tamme \[2018, Thm. 3.7\]](#) of a descent result of [Thomason \[1993\]](#), the square (2) gives rise to an equivalence of relative  $K$ -theory spectra  $K(X, \mathcal{Y}(n)) \xrightarrow{\cong} K(\tilde{X}(n), \mathcal{E}(n, n))$  for any  $n \geq 0$ .

By property (DK2) above, we know that

$$K_i(X, \mathcal{Y}(n)) \xrightarrow{\cong} K_i(X, Y)$$

is an isomorphism for  $i \leq 0, n \geq 0$  and by property (DK3) we know that

$$K_i(\tilde{\mathcal{X}}(n), \mathcal{E}(n, n)) \xrightarrow{\cong} K_i(t\tilde{\mathcal{X}}(n), t\mathcal{E}(n, n))$$

is an isomorphism for  $i < -d, n \geq 0$ . Note that  $\tilde{\mathcal{X}}$  and  $\mathcal{E}$  have affine coverings by  $r + 1 \leq d$  open subschemes.

Finally, we apply [Corollary 5](#) to the cartesian square

$$\begin{array}{ccc} \tilde{X} & \longleftarrow & E(n) \\ \downarrow & & \downarrow \\ t\tilde{\mathcal{X}}(m) & \longleftarrow & t\mathcal{E}(m, n) \end{array}$$

for  $n$  large depending on  $m$ , in which the vertical maps are finite by [Huneke and Swanson \[2006, Thm. 8.2.1\]](#). We deduce that

$$\lim_n K_i(t\tilde{\mathcal{X}}(m), t\mathcal{E}(m, n)) \xrightarrow{\cong} \lim_n K_i(\tilde{X}, E(n))$$

is an isomorphism for  $i < -d$  and  $m \geq 0$ . For  $i < -d$  composing the isomorphisms

$$\begin{aligned} K_i(X, Y) &\xrightarrow{\cong} \lim_n K_i(X, \mathcal{Y}(n)) \xrightarrow{\cong} \lim_n K_i(\tilde{\mathcal{X}}(n), \mathcal{E}(n, n)) \xrightarrow{\cong} \\ &\lim_n K_i(t\tilde{\mathcal{X}}(n), t\mathcal{E}(n, n)) \xrightarrow{\cong} \lim_m \lim_n K_i(t\tilde{\mathcal{X}}(m), t\mathcal{E}(m, n)) \xrightarrow{\cong} \\ &\lim_n K_i(\tilde{X}, E(n)) \xrightarrow{\cong} K_i(\tilde{X}, E) \end{aligned}$$

finishes the proof of [Proposition 10](#). □

*Proof of [Theorem 8\(i\)](#).* For the proof we make an induction on  $d = \dim(X)$ . The case  $d = 0$  is clear as then  $K_i(X) \xrightarrow{\cong} K_i(X_{\text{red}})$  vanishes for  $i < 0$  by [Proposition 1](#). For the induction step we use the descent spectral sequence (1) in order to reduce to the case of a local scheme  $X = \text{Spec } A$ , see [Kerz, Strunk, and Tamme \[2018, Prop. 6.1\]](#) for details. Since in the affine case  $K_i(X) \xrightarrow{\cong} K_i(X_{\text{red}})$  is an isomorphism for  $i \leq 0$ , we can assume without loss of generality that  $X$  is reduced.

Fix  $\gamma \in K_i(X)$  for some  $i < -d$ . Let  $Y \hookrightarrow X$  be a closed subscheme defined by an invertible ideal sheaf such that  $\gamma|_{\mathcal{O}_Y} = 0$ . This means that  $\gamma$  can be lifted to an element  $\gamma' \in K_i(X \text{ on } Y)$ . By [Proposition 7](#) there exists a blow-up  $f : \tilde{X} \rightarrow X$  in a center  $Z \hookrightarrow X$  which is set theoretically contained in  $Y$  such that the pullback of  $\gamma'$  along  $f$  vanishes, in particular  $f^*(\gamma) = 0 \in K_i(\tilde{X})$ .

Set  $E = f^{-1}(Z)$  and consider the commutative diagram with exact rows

$$(3) \quad \begin{array}{ccccccc} K_{i+1}(E) & \longrightarrow & K_i(\tilde{X}, E) & \longrightarrow & K_i(\tilde{X}) & \longrightarrow & K_i(E) \\ & & \uparrow & & \uparrow & & \uparrow \\ & & K_{i+1}(Z) & \longrightarrow & K_i(X, Z) & \longrightarrow & K_i(X) & \longrightarrow & K_i(Z) \end{array}$$

As  $\dim(Z), \dim(E) < d$  the  $K$ -groups in the outer corners of diagram (3) vanish by our induction assumption. The second vertical arrow in (3) is an isomorphism by [Proposition 10](#). So the third vertical arrow is an isomorphism as well, which implies that  $\gamma = 0$ .  $\square$

While the negative  $K$ -groups  $K_i(X)$  for  $-d = -\dim(X) < i < 0$  can be quite hard to calculate, there is nice formula for  $K_{-d}(X)$ , which was shown in complete generality in [Kerz, Strunk, and Tamme \[2018, Cor. D\]](#) and previously for varieties in characteristic zero [Cortiñas, Haesemeyer, Schlichting, and C. Weibel \[2008\]](#).

**Theorem 11.** *For a noetherian scheme of dimension  $d < \infty$  there is a canonical isomorphism*

$$K_{-d}(X) = H_{\text{cdh}}^d(X, \mathbb{Z}),$$

where on the right we take sheaf cohomology with respect to the cdh-topology on  $X$ .

## 4 Negative $K$ -groups of affinoid algebras

In this section let  $k$  be a (non-discrete) non-archimedean complete field with ring of integers  $k^\circ$ . By  $\pi$  we denote an element with absolute value  $0 < |\pi| < 1$ . For an affinoid algebra  $A$  over  $k$ , see [Bosch \[2014, Sec. 3.1\]](#), we write  $A\langle t \rangle$  for the Tate algebra over  $A$ , which consists of those formal power series  $a_0 + a_1 t + \dots \in A[[t]]$  with  $\lim_{i \rightarrow \infty} a_i = 0$ , similarly for  $A\langle t^{-1} \rangle$  and  $A\langle t, t^{-1} \rangle$ . One defines non-positive continuous  $K$ -groups of  $A$  successively by  $K_0^{\text{cont}}(A) = K_0(A)$  and

$$K_i^{\text{cont}}(A) = \text{coker} [K_{i+1}^{\text{cont}}(A\langle t \rangle) \times K_{i+1}^{\text{cont}}(A\langle t^{-1} \rangle) \rightarrow K_{i+1}^{\text{cont}}(A\langle t, t^{-1} \rangle)]$$

for  $i < 0$ . These negative continuous  $K$ -groups were defined and studied in [Karoubi and Villamayor \[1971, Sec. 7\]](#) and [Calvo \[1985\]](#). They also coincide with the continuous groups defined in [Morrow \[2016, Sec. 3\]](#) as is shown in [Kerz, Saito, and Tamme \[2018, Sec. 5\]](#).

We are about to show that the analog of Weibel's conjecture, i.e. [Theorem 8](#), holds in the non-archimedean situation:

**Theorem 12.** *Assume that  $k$  is discretely valued. For an affinoid  $k$ -algebra  $A$  of dimension  $d$  we have:*

- (i)  $K_i^{\text{cont}}(A) = 0$  for  $i < -d$ ,
- (ii)  $K_i^{\text{cont}}(A) \xrightarrow{\cong} K_i^{\text{cont}}(A\langle t_1, \dots, t_r \rangle)$  is an isomorphism for  $i \leq -d$  and any number of variables  $r$ .

I expect that the condition that  $k$  is discretely valued in [Theorem 12](#) can be removed. Note that even for smooth affinoid algebras  $A$  the negative continuous  $K$ -groups do not necessarily vanish, as the following example shows.

**Example 13.** *Assume that the residue field of  $k$  has characteristic zero and let  $\pi \in k$  be an element of absolute value  $0 < |\pi| < 1$ . For the affinoid algebra  $A = k\langle s, t \rangle / (t^2 - s^3 + s^2 - \pi)$  we have  $K_{-1}^{\text{cont}}(A) = \mathbb{Z}$ .*

The key fact for us is that for an admissible  $k^\circ$ -algebra  $A_0$  in the sense of [Bosch \[2014, Def. 7.3.3\]](#) with  $A = A_0[1/\pi]$  we obtain an exact sequence, see [Kerz, Saito, and Tamme \[2018, Sec. 5\]](#),

$$(4) \quad K_0(A_0 \text{ on } (\pi)) \rightarrow K_0((A_0/(\pi))_{\text{red}}) \rightarrow K_0^{\text{cont}}(A) \rightarrow K_{-1}(A_0 \text{ on } (\pi)) \rightarrow \cdots$$

Recall that an admissible  $k^\circ$ -algebra  $A_0$  is  $\pi$ -adically complete, topologically of finite type and  $\pi$ -torsion free.

The claim made in [Example 13](#) follows from the exact sequence (4) by setting  $A_0 = k^\circ\langle s, t \rangle / (t^2 - s^3 + s^2 - \pi)$  and using that in this case  $K_i(A_0 \text{ on } (\pi)) = 0$  for  $i < 0$  and that  $K_{-1}((A_0/(\pi))_{\text{red}}) = \mathbb{Z}$ .

*Proof of [Theorem 12\(i\)](#).* We can assume without loss of generality that  $A$  is reduced. An admissible blow-up of  $X = \text{Spec } A_0$ , where  $A_0$  is an admissible  $k^\circ$ -algebra as above, is defined as a blow-up in a center  $Y \hookrightarrow X$  which is set theoretically contained in  $\text{Spec } A_0/(\pi)$ . Let now  $\tilde{X} = \mathbf{Bl}_Y X$  be such an admissible blow-up,  $X_0 = (X \otimes_{A_0} A_0/(\pi))_{\text{red}}$  and  $\tilde{X}_0 = (\tilde{X} \otimes_{A_0} A_0/(\pi))_{\text{red}}$ . For  $i < -d$  we obtain from [Kerz, Saito, and Tamme \[ibid., Prop. 5.8\]](#) the upper exact sequence in the commutative diagram

$$(5) \quad \begin{array}{ccccc} K_i(\tilde{X}_0) & \longrightarrow & K_i^{\text{cont}}(A) & \longrightarrow & K_{i-1}(\tilde{X} \text{ on } (\pi)) \\ \uparrow & & \parallel & & \uparrow \\ K_i(X_0) & \longrightarrow & K_i^{\text{cont}}(A) & \longrightarrow & K_{i-1}(X \text{ on } (\pi)) \end{array}$$

while the lower exact sequence is just part of (4). Both groups on the left of (5) vanish by [Theorem 8](#) since  $i < -d = -\dim(X_0) = -\dim(\tilde{X}_0)$ . Consider an element  $\alpha \in K_i^{\text{cont}}(A)$ . For its image  $\alpha' \in K_{i-1}(X \text{ on } (\pi))$  we can use [Proposition 7](#) in order to choose

the admissible blow-up  $\tilde{X}$  such that the pullback of  $\alpha'$  to  $K_{i-1}(\tilde{X}$  on  $(\pi))$  vanishes. A diagram chase in (5) shows that  $\alpha$  vanishes.  $\square$

The forthcoming PhD thesis of C. Dahlhausen will discuss the following conjecture, which is the non-archimedean analytic variant of [Theorem 11](#).

**Conjecture 14.** *For an affinoid  $k$ -algebra  $A$  of dimension  $d$  there is an isomorphism*

$$K_{-d}^{\text{cont}}(A) \cong H^d(\mathfrak{M}(A), \mathbb{Z}).$$

Here  $\mathfrak{M}(A)$  is the Berkovich spectrum of multiplicative seminorms [Berkovich \[1990, Ch. 1\]](#).

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