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INVERSE PROBLEMS FOR LINEAR AND NON-LINEAR HYPERBOLIC EQUATIONS

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Abstract

We consider inverse problems for hyperbolic equations and systems and the solutions of these problems based on the focusing of waves. Several inverse problems for linear equations can be solved using control theory. When the coefficients of the modelling equation are unknown, the construction of the point sources requires solving blind control problems. For non-linear equations we consider a new artificial point source method that applies the non-linear interaction of waves to create microlocal points sources inside the unknown medium. The novel feature of this method is that it utilizes the non-linearity as a tool in imaging, instead of considering it as a difficult perturbation of the system. To demonstrate the method, we consider the non-linear wave equation and the coupled Einstein and scalar field equations.

1 Introduction

One of the simplest models for waves is the linear hyperbolic equation

$$\partial_t^2 u(t,x) - c(x)^2 \Delta u(t,x) = 0 \quad \text{in } \mathbb{R} \times \Omega$$

where $\Omega \subset \mathbb{R}^n$ and $c(x)$ is the wave speed. This equation models e.g. acoustic waves. In inverse problems one has access to measurements of waves (the solutions $u(t,x)$) on the boundary, or in a subset of the domain Ω, and one aims to determine unknown coefficients (e.g., $c(x)$) in the interior of the domain.

In particular, we will consider *anisotropic* materials, where the wave speed depends on the direction of propagation. This means that the scalar wave speed $c(x)$, where $x =$

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$(x^1, x^2, \ldots, x^n) \in \Omega$, is replaced by a positive definite symmetric matrix $(g^{jk}(x))_{j,k=1}^n$, and the wave equation takes for example the form

$$\frac{\partial^2}{\partial t^2} u(t,x) - \sum_{j,k=1}^{n} g^{jk}(x) \frac{\partial^2 u}{\partial x^j \partial x^k}(t,x) = 0. \tag{1}$$

Anisotropic materials appear frequently in applications such as in seismic imaging, where one wishes to determine the interior structure of the Earth by making various measurements of waves on its surface.

It is convenient to interpret the anisotropic wave speed (g^{jk}) as the inverse of a Riemannian metric, thus modelling the medium as a *Riemannian manifold*. This is due to fact that if $\Psi : \Omega \to \Omega$ is a diffeomorphism such that $\Psi|_{\partial\Omega} = Id$ (and for an equation of the form (1) it is also assumed to be volume preserving in $\Omega \subset \mathbb{R}^n$), then all boundary measurements for the metric g and the pull-forward metric $\Psi^* g$ coincide. Thus to prove uniqueness results for inverse problems, one has to consider properties that are invariant in diffeomorphisms and try to reconstruct those uniquely, for example, to show that an underlying manifold structure can be uniquely determined. In practice, the inverse problem in a subset of the Euclidean space is solved in two steps. The first is to reconstruct the underlying manifold structure. The second step is to find an embedding of the constructed manifold in the Euclidean space using additional a priori information. In this paper we conspace-timeate on the first step.

2 Inverse problems for linear equations

In this section we review the classical results for Gel'fand inverse problems Gelfand [1954] for linear scalar wave equations. Note that these results require that the coefficients of the equation, or at least the leading order coefficients, are time independent. In addition, it is required that the associated operator is selfadjoint or that it satisfies strong geometrical assumptions, for example that all geodesics exit the domain at a given time. In Section 4 we show how these results can be obtained using a focusing of waves that produces point sources inside the unknown medium. In Sections 3 and 5 we consider inverse problems non-linear hyperbolic equations and systems, and consider the recently developed artificial point source method based on the non-linear interaction of waves.

Let (N, g) be an n-dimensional Riemannian manifold and consider the wave equation

$$\begin{aligned} &\partial_t^2 u(t,x) - \Delta_g u(t,x) = 0 \quad \text{in } (0,\infty) \times N, \\ &\partial_\nu u|_{\mathbb{R}_+ \times \partial N} = f, \quad u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \end{aligned} \tag{2}$$

where Δ_g is the Laplace–Beltrami operator corresponding to a smooth time-independent Riemannian metric g on N. In coordinates $(x_j)_{j=1}^n$ this operator has the representation

$$\Delta_g u = \sum_{j,k=1}^{n} \det(g)^{-1/2} \frac{\partial}{\partial x^j} \left(\det(g)^{1/2} g^{jk} \frac{\partial}{\partial x^k} u \right),$$

where $g(x) = [g_{jk}(x)]_{j,k=1}^n$, $\det(g) = \det(g_{jk}(x))$ and $[g^{jk}]_{j,k=1}^n = g(x)^{-1}$.

The solution of (2), corresponding to the boundary value f (which is interpreted as a boundary source), is denoted by $u^f = u^f(t,x)$.

Let us assume that the boundary ∂N is known. The inverse problem is to reconstruct the manifold N and the metric g when we are given the set

$$\{(f, u^f|_{\mathbb{R}_+ \times \partial N}) : \ f \in C_0^\infty(\mathbb{R}_+ \times \partial N)\},$$

that is, the Cauchy data of solutions corresponding to all possible boundary sources $f \in C_0^\infty(\mathbb{R}_+ \times \partial N)$. This data is equivalent to the *response operator*

$$\Lambda_{N,g} : f \mapsto u^f|_{\mathbb{R}_+ \times \partial N}, \tag{3}$$

which is also called the *Neumann-to-Dirichlet map*. Physically, $\Lambda_{N,g} f$ describes the measurement of the medium response to any applied boundary source f. In 1990s, the combination of Belishev's and Kurylev's boundary control method Belishev and Y. V. Kurylev [1992] and Tataru's unique continuation theorem Tataru [1995] gave a solution to the inverse problem of determining the isometry type of a Riemannian manifold (N, g) with given boundary ∂N and the Neumann-to-Dirichlet map $\Lambda_{N,g}$.

Theorem 2.1 (Belishev and Y. V. Kurylev [1992] and Tataru [1995])**.** *Let (N_1, g_1) and (N_2, g_2) be compact smooth Riemannian manifolds with boundary. Assume that there is a diffeomorphism $\Phi : \partial N_1 \to \partial N_2$ such that*

$$\Phi^*(\Lambda_{N_1,g_1} f) = \Lambda_{N_2,g_2}(\Phi^* f), \quad \textit{for all } f \in C_0^\infty(\mathbb{R}_+ \times \partial N_1). \tag{4}$$

Then (N_1, g_1) and (N_2, g_2) are isometric Riemannian manifolds.

Above, $\Phi^* f$ is the pull-back of f in Φ. Theorem 2.1 can be used to prove the uniqueness of other inverse problems. Katchalov, Kurylev, Mandache, and the author showed in Katchalov, Y. Kurylev, Lassas, and Mandache [2004] the equivalence of spectral inverse problems with several different measurements, that in particular implies the following result.

Theorem 2.2 (Katchalov, Y. Kurylev, Lassas, and Mandache [ibid.])**.** *Let ∂N be given. Then the Neumann-to-Dirichet map $\Lambda : \partial_\nu u|_{\mathbb{R}_+ \times \partial N} \mapsto u|_{\mathbb{R}_+ \times \partial N}$, for heat equation $(\partial_t -$*

$\Delta_g)u = 0$, *or for the Schrödinger equation* $(i\partial_t - \Delta_g)u = 0$, *with vanishing initial data* $u|_{t=0} = 0$ *determine the Neumann-to-Dirichet map for the wave equation, and therefore, the manifold* (N, g) *up to an isometry.*

The stability of the solutions of the above inverse problems have been analyzed in Anderson, Katsuda, Y. Kurylev, Lassas, and Taylor [2004], Bao and Zhang [2014], Bosi, Y. Kurylev, and Lassas [2017], and P. Stefanov and G. Uhlmann [2005].

Without making strong assumptions about the geometry of the manifold, the existing uniqueness results for linear hyperbolic equations with vanishing initial data are limited to equations whose coefficients are time independent or real analytic in time (see e.g. Anderson, Katsuda, Y. Kurylev, Lassas, and Taylor [2004], Belishev and Y. V. Kurylev [1992], Eskin [2017], Katchalov, Y. Kurylev, and Lassas [2001], Y. Kurylev, Oksanen, and Paternain [n.d.], and Oksanen [2013]). The reason for this is that these results are based on Tataru's unique continuation theorem Tataru [1995]. This sharp unique continuation result does not work for general wave equations whose coefficients are not real analytic in time, as shown by Alinhac [1983]. Alternatively, one can study inverse problems for hyperbolic equations by using the Fourier transform in the time variable and reducing the problem to an inverse boundary spectral problem for an elliptic equation. Note that this also requires that the coefficients are time independent. The obtained inverse spectral problems (see e.g. A. Nachman, Sylvester, and G. Uhlmann [1988]) can be solved using the complex geometrical optics introduced in Sylvester and G. Uhlmann [1987].

Open Problem 1: Do the boundary ∂N and the Neumann-to-Dirichlet map for a wave equation $\square_g u = 0$ determine the coefficient $g^{jk}(t,x)$ that depends on variables t and x?

In many applications, waves can not be detected on the part of the boundary where sources are applied, that is, one is given only a restricted Neumann-to-Dirichlet map. Next we consider such problems.

We say that (2) is exactly controllable from $\Gamma_1 \subset \partial N$ if there is $T > 0$ such that the map

$$\mathcal{U} : L^2((0,T) \times \Gamma_1) \to L^2(N) \times H^{-1}(N), \qquad \mathcal{U}(f) = (u^f(T), \partial_t u^f(T)) \tag{5}$$

is surjective. In 1992, Bardos, Lebeau, and Rauch gave a sufficient geometric condition for exact controllability and showed that this condition is also close to being necessary Bardos, Lebeau, and Rauch [1992]. Roughly speaking, this geometric controllability condition requires that all geodesics (that reflect from the boundary) in the domain N intersect transversally to the set Γ_1 before time T.

Under the geometric controllability condition the inverse problem with a restricted Neumann-to-Dirichlet map can be solved using exact controllability results Lassas and Oksanen [2014]. However, in the general setting the following problem is open.

Open Problem 2: Assume that we are given open subsets $\Gamma_1, \Gamma_2 \subset \partial N$, such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \varnothing$, and the restricted Neumann-to-Dirichlet map $\Lambda_{\Gamma_1,\Gamma_2} : f \mapsto u^f|_{\mathbb{R}_+ \times \Gamma_2}$ defined for functions $f \in C_0^\infty(\mathbb{R}_+ \times \Gamma_1)$. Do these data determine (N, g) up to an isometry?

Similarly, systems having terms causing energy absorption can be considered when the geometric controllability condition is valid Y. Kurylev and Lassas [2000], but the following problem is open.

Open Problem 3: Consider Equation (2) where the Laplace operator Δ_g is replaced by a non-selfadjoint operator, for example, the wave equation of the form $(\partial_t^2 - \Delta_g + q(x))u(t,x) = 0$, where $q(x)$ is complex valued. Do the boundary ∂N and Neumann-to-Dirichlet map $\Lambda_{N,g,q}$ for this equation determine (N, g) and $q(x)$ up to an isometry?

The boundary control method has been used to solve inverse problems for some hyperbolic systems of equations, e.g. for Maxwell and Dirac equations, see Y. Kurylev, Lassas, and Somersalo [2006] in the special cases when the wave velocity is independent of polarization.

Open Problem 4: Consider a hyperbolic system of equations where the velocity of waves depends on the polarisation, such as elastic equations or Maxwell's equations in anisotropic medium. Do ∂N and the response operator defined on the boundary determine the system up to a diffeomorphism?

3 Inverse problems for non-linear equations

The present theory of inverse problems has largely been confined to the case of linear equations. For the few existing results on non-linear equations (e.g. Isakov [1993], Salo and Zhong [2012], and Sun and G. Uhlmann [1997]) the non-linearity is an obstruction rather than a helpful feature.

Below, we consider inverse problems for non-linear hyperbolic equations and use non-linearity as a tool to solve the problems. This enables us to solve inverse problems for non-linear equations for which the corresponding problems for linear equations are still unsolved (e.g. when the coefficients depend on the time variable or are complex valued, cf. Open Problems 1 and 2). Below, we will first consider scalar wave-equation with simple quadratic non-linearity. Later we consider inverse problems for the Einstein equations that can be solved using the non-linear interaction of gravitational waves and matter field waves. The inverse problems for the Einstein equations (in particular the passive problems

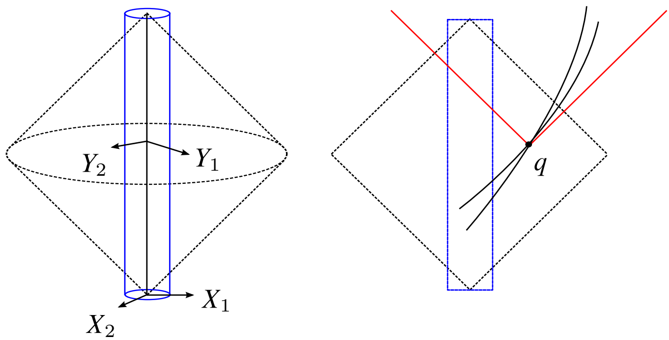

Figure 1: *Left.* The setting of Theorems 3.1 and 5.1. The solid black line depicts the time-like geodesic μ and the blue cylinder is its neighbourhood where measurements are made. The dashed double cone is the set $I(p^-, p^+)$ which properties are reconstructed from the data. In Theorem 5.1 we use the frame Y_1, Y_2, Y_3 moving along μ to define the Fermi coordinates in the blue cylinder (the third direction is suppressed in the picture). *Right.* A schematic picture of the proof of Theorem 3.1. Geodesics, depicted as black curves, that are sent from the neighbourhood of μ intersect at the point $q \in I(p^-, p^+)$. We consider (four) distorted plane waves that propagate near the geodesics that interact at the point q and produce propagating singularities (in red), analogous to those generated by a point source at q.

considered below) could be applied in the gravitational astronomy initiated by the direct detection of gravitational waves B. P. Abbott et al. [2016].

3.1 Notation. Let (M, g) be a $(1+3)$-dimensional time-oriented Lorentzian manifold of signature $(-,+,+,+)$. Let $q \in M$. The set of future pointing *light-like* vectors at q is defined by

$$L_q^+ M = \{\theta \in T_q M \setminus 0 : \ g(\theta, \theta) = 0, \ \theta \text{ is future-pointing}\}.$$

A vector $\theta \in T_q M$ is *time-like* if $g(\theta, \theta) < 0$ and *space-like* if $g(\theta, \theta) > 0$. *Causal vectors* are the collection of time-like and light-like vectors, and a curve γ is time-like (light-like, causal, future-pointing) if the tangent vectors $\dot{\gamma}$ are time-like (light-like, causal, future-pointing).

For $p, q \in M$, the notation $p \ll q$ means that p, q can be joined by a future-pointing time-like curve. The *chronological future* and *past* of $p \in M$ are

$$I^+(p) = \{q \in M \ : \ p \ll q\}, \quad I^-(p) = \{q \in M \ : \ q \ll p\}.$$

To emphasise the Lorentzian structure of (M, g) we sometimes write $I^{\pm}_{M,g}(p) = I^{\pm}(p)$. We will denote throughout the paper

$$I(p,q) = I^+(p) \cap I^-(q). \tag{6}$$

A time-oriented Lorentzian manifold (M, g) is *globally hyperbolic* if there are no closed causal paths in M, and for any $p, q \in M$ the set $J(p,q)$ is compact. The set $J(p,q)$ is defined analogously to $I(p,q)$ but with the partial order $p \ll q$ replaced by $p \leq q$, meaning that p and q can be joined by a future-pointing causal curve or $p = q$. According to Bernal and Sánchez [2005], a globally hyperbolic manifold is isometric to the product manifold $\mathbb{R} \times N$ with the Lorentzian metric given by

$$g = -\beta(t, y)dt^2 + \kappa(t, y), \tag{7}$$

where $\beta : \mathbb{R} \times N \to \mathbb{R}_+$ and κ is a Riemannian metric on N depending on t.

3.2 Active measurements. Let (M, g) be a 4-dimensional globally hyperbolic Lorentzian manifold and assume, without loss of generality, that $M = \mathbb{R} \times N$ with a metric of the form (7). Let $t_0 > 0$ and consider the semilinear wave equation

$$\Box_g u(x) + a(x)u(x)^2 = f(x), \quad \text{for } x \in (-\infty, t_0) \times N, \tag{8}$$

$$u = 0, \;\; f = 0, \quad \text{in } (-\infty, 0) \times N. \tag{9}$$

Here $a \in C^\infty(M)$ is a nowhere vanishing function that may be complex valued, and

$$\Box_g u = \sum_{j,k=0}^{n} |\det(g)|^{-1/2} \frac{\partial}{\partial x^j}\left(|\det(g)|^{1/2} g^{jk} \frac{\partial}{\partial x^k} u \right).$$

Let $\mu \subset (0, t_0) \times N$ be a time-like curve and V be its open neighbourhood. The solution of (8)–(9) exists when the source f is supported in V and satisfies $\|f\|_{C^k(\overline{V})} < \varepsilon$, where $k \in \mathbb{Z}_+$ is sufficiently large and $\varepsilon > 0$ is sufficiently small. For such sources f we define the measurement operator

$$L_V : f \mapsto u|_V. \tag{10}$$

Note L_V is equivalent to its graph that is given by the data set

$$\mathfrak{D}_{L_V} = \{(u|_V, f) \; : \; u \text{ and } f \text{ satisfy (8),(9)}, \; f \in C^k_0(V), \; \|f\|_{C^k(\overline{V})} < \varepsilon\}. \tag{11}$$

Theorem 3.1 (Y. Kurylev, Lassas, and G. Uhlmann [2014]). *Let (M, g) be a globally hyperbolic 4-dimensional Lorentzian manifold. Let μ be a time-like path containing p^+*

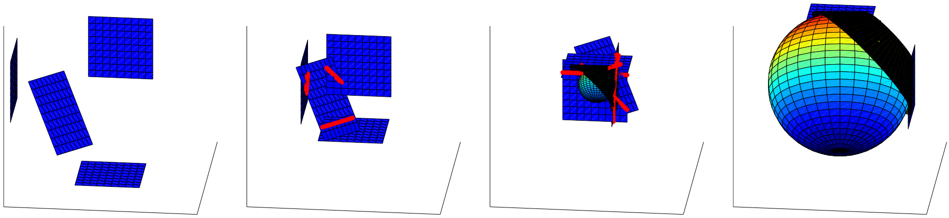

Figure 2: Four plane waves propagate in space. When the planes intersect, the non-linearity of the hyperbolic system produces new waves. *Left:* Plane waves before interacting. *Middle left:* The two-wave interactions (red line segments) appear but do not cause singularities propagating to new directions. *Middle right and Right:* All plane waves have intersected and new waves have appeared. The three-wave interactions cause conic waves (the black surface). Only one such wave is shown in the figure. The interaction of four waves causes a microlocal point source that sends a spherical wave in all future light-like directions.

and p^-. Let $V \subset M$ be a neighborhood of μ and let $a : M \to \mathbb{R}$ be a nowhere vanishing C^∞-smooth function. Then $(V, g|_V)$ and the measurement operator L_V determine the topology, differentiable structure and the conformal class of the metric g in the double cone $I_{M,g}(p^-, p^+)$.

When M has a significant Ricci-flat part, Theorem 3.1 can be strengthened.

Corollary 3.2. *Assume that (M, g) and V satisfy the conditions of Theorem 3.1. Moreover, assume that $W \subset I_{M,g}(p^-, p^+)$ is Ricci-flat and all topological components of W intersect V. Then the metric tensor g is determined in W uniquely.*

The proof of Theorem 3.1 uses the results on the inverse problem for passive measurements for point sources, described below, and the non-linear interaction of waves having conormal singularities. There are many results on such non-linear interaction, starting with the studies of Bony [1986], R. Melrose and Ritter [1985], Holt [1995]. However, these studies differ from the proof of Theorem 3.1 in that they assumed that the geometrical setting of the interacting singularities, and in particular the locations and types of caustics, is known a priori. In inverse problems we study waves on an unknown manifold, so we do not know the underlying geometry and, therefore, the location of the singularities of the waves. For example, the waves can have caustics that may even be of an unstable type.

Theorem 3.1 only concerns the recovery of the conformal type of the metric. The recovery of all coefficients up to a natural gauge transformation has in some special cases been considered (in Lassas, G. Uhlmann, and Wang [n.d.] and Wang and Zhou [2016]), but for general equations both the compete recovery of all coefficients and the stable solvability of the inverse problem are open questions.

Open Problem 5 (Recovery of all coefficients for non-linear wave equation): Assume that we are given a time-like path μ, its neighborhood $V \subset M$, and the map $L_V : f \mapsto u|_V$ for the non-linear equation $\square_g u + B(x, D)u + a(x)u(x)^2 = f$, defined for small sources f supported in V, where $B(x, D)$ is a first order differential operator. Is it possible to construct the metric tensor g and the operator $B(x, D)$ in $I(p^-, p^+)$ up to a local gauge transformation?

Open Problem 6 (Stability of the inverse problem for non-linear wave equation): Assume that we are given a time-like curve μ, its neigbhorhood $V \subset M$, the map L_V with an error, and $p^-, p^+ \in \mu$. Is it possible to construct the set $I(p^-, p^+)$ and the metric g in $I(p^-, p^+)$ with an error that can be estimated in terms of the geometric bounds for M and the error in the given data?

For certain inverse problems for linear wave equations the essential features of several measurements can be packed in a single measurement Helin, Lassas, and Oksanen [2014] and Helin, Lassas, Oksanen, and Saksala [2016]. The corresponding problem for non-linear equations is open.

Open Problem 7 (Single measurement inverse problem for non-linear wave equations): Can we construct a source f such that the set V and the measurement $L_V f$ uniquely determine $I(p^-, p^+)$ and the metric g on $I(p^-, p^+)$?

3.3 Passive measurements.

The earliest light observation set is an idealized notion of measurements of light coming from a point source.

Definition 3.3. *Let M be a Lorentzian manifold, $V \subset M$ be open, and $q \in M$. The light observation set of $q \in M$ in V is*

$$\mathcal{P}_V(q) = \{\gamma_{q,\xi}(t) \in M \ : \ t \geq 0, \ \xi \in L^+_q M\} \cap V,$$

where $\gamma_{q,\xi}$ denotes the geodesic emanating from q to the direction ξ. The earliest light observation set of $q \in M$ in V is

$$\mathcal{E}_V(q) = \{x \in \mathcal{P}_V(q) : \text{ there are no } y \in \mathcal{P}_V(q) \text{ such that } y \ll x \text{ in } (V, g)\}.$$

The set $\mathcal{P}_V(q)$ can be viewed as a model of a measurement where light emitted by a point source at q is recorded in V. As gravitational wave packets propagate at the speed of light, $\mathcal{P}_V(q)$ could also correspond to an observation where a gravitational wave is generated at q and detected in V. The set $\mathcal{E}_V(q)$ is related to the distance difference functions used in Riemannian geometry.

Definition 3.4. *Let N be a Riemannian manifold with the distance function $dist_N(x,y)$ and let $U \subset N$ be an open set. The distance difference function in the observation set U corresponding to a point $x \in N$ is*

$$D_x : U \times U \to \mathbb{R}, \quad D_x(z_1,z_2) := dist_N(z_1,x) - dist_N(z_2,x). \tag{12}$$

Consider a Riemannian manifold where the distance between two points is the travel time of waves between these points. When *a spontaneous point source* produces a wave at some unknown point $x \in N$, at some unknown time $t \in \mathbb{R}$, the produced wave is observed at the point $z \in U$ at time $T_{t,x}(z) = \mathrm{dist}_N(z,x) + t$. These observation times at two points $z_1, z_2 \in U$ determine the distance difference function by

$$D_x(z_1,z_2) = T_{t,x}(z_1) - T_{t,x}(z_2) = \mathrm{dist}_N(z_1,x) - \mathrm{dist}_N(z_2,x).$$

Physically, this function corresponds to the difference in times when the wave produced by a point source at (t,x) is observed at z_1 and z_2.

When $M = \mathbb{R} \times N$ is the Lorentzian manifold given by the product metric of N and $(\mathbb{R}, -dt^2)$, the earliest light observation set corresponding to a point $q = (t_0, x_0)$ and $V = \mathbb{R} \times U$, where $x \in N$ and $t_0 \in \mathbb{R}$, is given by

$$\mathcal{E}_V(q) = \{(t,y) \in \mathbb{R} \times U : \ \mathrm{dist}_N(y,x_0) = t - t_0\}.$$

Similarly, the earliest light observation set $\mathcal{E}_V(q)$ corresponding to $q = (t_0, x_0)$ determines the distance difference function D_{x_0} by

$$D_{x_0}(z_1,z_2) = t_1 - t_2, \quad \text{if } \exists\, t_1, t_2 \in \mathbb{R} \text{ such that } (t_1,z_1), (t_2,z_2) \in \mathcal{E}_V(q). \tag{13}$$

The following theorem says, roughly speaking, that observations of a large number of point sources in a region W determine the structure of the spacetime in W, up to a conformal factor.

Theorem 3.5 (Y. Kurylev, Lassas, and G. Uhlmann [2014]). *Let (M_j, g_j), where $j = 1,2$, be two open globally hyperbolic Lorentzian manifolds of dimension $1+n$, $n \geq 2$. Let $\mu_j : [0,1] \to M_j$ be a future-pointing time-like path, let $V_j \subset M_j$ be a neighbourhood of $\mu_j([0,1])$, and let $W_j \subset I^-_{M_j,g_j}(\mu_j(1)) \setminus I^-_{M_j,g_j}(\mu_j(0))$ be open and relatively compact, $j = 1,2$. Assume that there is a conformal diffeomorphism $\phi : V_1 \to V_2$ such that $\phi(\mu_1(s)) = \mu_2(s)$, $s \in [0,1]$, and*

$$\{\phi(\mathcal{E}_{V_1}(q)) : \ q \in W_1\} = \{\mathcal{E}_{V_2}(q) : \ q \in W_2\}.$$

Then there is a diffeomorphism $\Psi : W_1 \to W_2$ and a strictly positive function $\alpha \in C^\infty(W_1)$ such that $\Psi^* g_2 = \alpha g_1$ and $\Psi|_{V_1 \cap W_1} = \phi$.

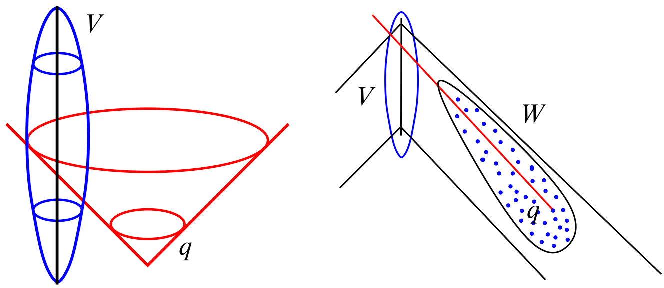

Figure 3: *Left.* When there are no cut points, the earliest light observation set $\mathcal{E}_V(q)$ is the intersection of the cone and the open set V. The cone is the union of future-pointing light-like geodesics from q, and the ellipsoid depicts V. *Right.* The setting of Theorem 3.5. The domain $W \subset M$ (with a black boundary) contains several points sources and a light ray from the point $q \in W$ reaches the observation set V (with a blue boundary).

In the Riemannian case, the whole metric can be determined under conditions described in the following theorem.

Theorem 3.6 (Lassas and Saksala [2015]). *Let (N, g) be a connected Riemannian manifold without boundary, that is either complete or compact, of the dimension $n \geq 2$. Let $W \subset N$ be a compact set with non-empty complement $U = M \setminus W$. Then the pair $(U, g|_U)$ and the distance difference functions $\{D_x \in C(U \times U) : x \in W\}$ uniquely determine the manifold (N, g) up to an isometry.*

A classical distance function representation of a compact Riemannian manifold N is the Kuratowskii embedding, $\mathcal{K} : x \mapsto \mathrm{dist}_N(x, \cdot)$, from N to the space of the continuous functions $C(N)$ on it. The mapping $\mathcal{K} : N \to C(N)$ is an isometry so that $\mathcal{K}(N)$ is an isometric representation of N in a vector space $C(N)$. Next we consider a similar embedding that is applicable for inverse problems.

Let $x \in N$ and define a function $D_x : \overline{U} \times \overline{U} \to \mathbb{R}$ by formula (12). Let $\mathfrak{D} : N \to C(\overline{U} \times \overline{U})$ be given by $\mathfrak{D}(x) = D_x$. Theorem 3.6 implies that the set $\mathfrak{D}(N) = \{D_x : x \in N\}$ can be considered as an embedded image of the manifold (N, g) in the space $C(\overline{U} \times \overline{U})$ in the embedding $x \mapsto D_x$. Thus, $\mathfrak{D}(N)$ can be considered as a representation of the manifold N, given in terms of the distance difference functions, and we call it the *distance difference representation* of the manifold of N in $C(\overline{U} \times \overline{U})$.

The embedding $\mathfrak{D}$ is different to the above embedding $\mathcal{K}$ in the following way that makes it important for inverse problems: With $\mathfrak{D}$ one does not need to know a priori the set N in order to consider the function space $C(\overline{U} \times \overline{U})$ where we can embed N. Indeed, when the observation set U is given, we can determine the topological properties of N

by constructing the set $\mathfrak{D}(N)$ that is homeomorphic to N, and then consider $\mathfrak{D}(N)$ as a "copy" of the unknown manifold N embedded in the known function space $C(\overline{U} \times \overline{U})$.

4 Ideas for proofs and reconstruction methods

4.1 The focusing of waves for linear equations. Let $u^f(t,x)$ denote the solution of the hyperbolic Equation (2), let $\Lambda = \Lambda_{N,g}$ be the Neumann-to-Dirichlet map for the Equation (2), and let dS_g denote the Riemannian volume measure on the manifold $(\partial N, g_{\partial N})$. We start with the Blagovestchenskii identity BlagoveščenskiÄ [1969] (see also Katchalov, Y. Kurylev, and Lassas [2001]) which states that the inner product of waves at any time can be computed from boundary data.

Lemma 4.1. *Let $f, h \in C_0^\infty(\mathbb{R}_+ \times \partial N)$ and $T > 0$. Then*

$$\langle u^f(T), u^h(T)\rangle_{L^2(N)} = \int_N u^f(T,x)u^h(T,x)\, dV_g(x) = \tag{14}$$
$$= \frac{1}{2}\int_L \int_{\partial M} (f(t,x)(\Lambda h)(s,x) - (\Lambda f)(t,x)h(s,x))\, dS_g(x)dtds,$$

where dV_g is the volume measure on the Riemannian manifold (N,g) and $L = \{(s,t) \in (\mathbb{R}_+)^2 : 0 \le t+s \le 2T,\ t < s\}$. A similar formula can be written to compute $\langle u^f(T), 1\rangle_{L^2(N)}$ in terms of f, $(\partial N, dS_g)$, and Λ.

We also need an approximate controllability result that is based on the following fundamental unique continuation theorem of Tataru [1995].

Theorem 4.2. *Let $u(t,x)$ solve the wave equation $\partial_t^2 u - \Delta_g u = 0$ in $N \times \mathbb{R}$ and $u|_{(0,2T_1)\times\Gamma} = 0$ and $\partial_\nu u|_{(0,2T_1)\times\Gamma} = 0$, where $\Gamma \subset \partial N$ is open and non-empty. Then $u(t,x) = 0$ in K_{Γ,T_1}, where*

$$K_{\Gamma,T_1} = \{(t,x) \in \mathbb{R}\times N :\ dist_N(x,\Gamma) < T_1 - |t-T_1|\}$$

is the double cone of influence.

The quantitative stability results for Tataru-type unique continuation have recently been obtained by Bosi, Kurylev, and the author, Bosi, Y. Kurylev, and Lassas [2016], and by Laurent and Léautaud Laurent and Léautaud [2015]. Theorem 4.2 gives rise to the following approximate controllability result:

Corollary 4.3. *For any open $\Gamma \subset \partial N$ and $T_1 > 0$,*

$$cl_{L^2(N)}\{u^f(T_1,\cdot) : f \in C_0^\infty((0,T_1)\times\Gamma)\} = L^2(N(\Gamma,T_1)).$$

Here $N(\Gamma,T_1) = \{x \in N : dist_N(x,\Gamma) < T_1\}$ is the domain of influence of Γ at time T_1, cl denotes the closure, and $L^2(N(\Gamma,T_1)) = \{v \in L^2(N) :\ supp\,(v) \subset cl(N(\Gamma,T_1))\}$.

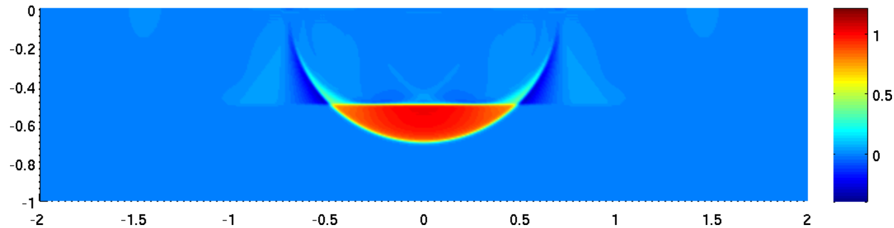

Figure 4: The numerical simulation on family of waves $u^{f_{\varepsilon,\alpha}}$ that focus to a point as $\alpha \to 0$ and $\varepsilon \to 0$, by de Hoop, Kepley, and Oksanen [2016]. The figure shows the wave $u^{f_{\varepsilon,\alpha}}(x,T)$ at the time $t = T$ in the rectangle $x \in [-2,0] \times [-2,2]$ that is concentrated in the neigborhood $A'_\varepsilon \setminus A''_\varepsilon$ of the point x_1. Waves are controlled by a boundary source supported on the top of the rectangle and $f_{\varepsilon,\alpha}$ is constructed using the local Neumann-to-Dirichlet map.

4.1.1 Blind control problems. The inverse problem for the linear wave Equation (2) can be solved by using blind control problems. To consider this approach, we consider first an example of such a control problem.

Example 1: The blind deconvolution problem. The problem is to determine unknown functions f and g when the convolution $m = g * f$ is given. Naturally, this problem has no unique solution. In practical settings when a priori assumptions about f and g are given, one can approach the problem by solving a regularised problem, for example finding (f,g) minimizing $\|f*g-m\|^2_{L^2(\mathbb{R})}+R(f,g)$ where $R(f,g) = \alpha(\|f\|^2_X+\|g\|^2_Y)$ and X and Y are suitable Banach spaces, e.g. Sobolev spaces, and $\alpha > 0$ is a regularization parameter (see e.g. Mueller and Siltanen [2012]).

Below we consider a blind control problem for a wave equation on a compact manifold N. Our aim is to find a boundary source f that produces a wave $u^f(t,x)$ solving the wave equation with metric g such that at time $t = T$ the value of the wave, $u^f(T,x)$, is close to a function $m(x)$. When the domain N and the metric g on it are known, this is a traditional control problem. We consider a blind control problem when the metric g is unknown and we only know ∂N and the map Λ. Below we are particularly interested in the case when $m(x) = \chi_A(x)$ is the indicator function of a set $A = A(z_1, z_2, \dots, z_J; T_0, T_1, T_2, \dots, T_J) \subset N$,

$$A = \{x \in N : \operatorname{dist}_N(x, \partial N) < T_0\} \cup \bigcup_{j=1}^{J} B_N(z_j, T_j), \tag{15}$$

where points $z_j \in \partial N$, $j = 1, 2, \dots, J$, and values $T_j \in (0, T)$ are given and $B_N(z_j, T_j)$ are the balls of the manifold N with the centre z_j and radius T_j. We consider the mimimization problem

$$\min_{f \in Y_A} \|u^f(T) - 1\|^2_{L^2(N)} + \alpha \|f\|^2_{L^2([0,T]\times\partial N)} \tag{16}$$

where $Y_A \subset L^2([0, T] \times \partial N)$ is the space of functions $f(t, x)$, supported in the union of the sets $[T - T_0, T] \times \partial N$ and $\bigcup_{j=1}^J \{(t, z) \in [0, T] \times \partial N : t > T_j - \mathrm{dist}_{\partial M}(z, z_j)\}$, and $\alpha > 0$ is a small regularisation parameter.

When $\alpha \to 0$, it follows from Bingham, Y. Kurylev, Lassas, and Siltanen [2008] and de Hoop, Kepley, and Oksanen [2016] that the solutions f_α of the minimization Equation (16) satisfy

$$\lim_{\alpha \to 0} u^{f_\alpha}(T) = \chi_A \quad \text{in } L^2(N). \tag{17}$$

Moreover, a modification of the minimization Equation (16) (see Dahl, Kirpichnikova, and Lassas [2009]) has the solution $\tilde{f}_\alpha$ such that

$$\lim_{\alpha \to 0} u^{\tilde{f}_\alpha}(T) = \chi_A \quad \text{and} \quad \lim_{\alpha \to 0} \partial_t u^{\tilde{f}_\alpha}(T) = 0, \tag{18}$$

where limits take place in $L^2(N)$.

Using Lemma 4.1 we can solve the above minimization Equation (16) when we do not know the metric g in the manifold N but only the boundary measurements given in terms of the Dirichlet-to-Neumann map $\Lambda_{N,g}$. By (17), this means that the solutions of the minimization Equation (16) are approximate solutions for a blind control problem. We emphasise that one does not need to assume that the wave equation has an exact controllability property to consider this control problem.

Let $z_1 \in \partial N$ and ν be the unit interior normal of ∂N, and define the cut-locus function as

$$\tau_{\partial N}(z_1) = \sup\{s > 0 : \ \mathrm{dist}_N(\gamma_{z_1,\nu}(T_1), \partial N) = s\}.$$

When $T_1 \in (0, T)$ satisfies $T_1 < \tau_{\partial N}(z_1)$, the geodesic $\gamma_{z_1,\nu}([0, T_1])$ is the shortest curve connecting $x_1 = \gamma_{z_1,\nu}(T_1)$ to the boundary ∂N. For $\varepsilon > 0$, let

$$A'_\varepsilon = \{x \in N : \ \mathrm{dist}_N(x, \partial N) < T_1 - \varepsilon\} \cup B_N(z_1, T_1 + \varepsilon),$$
$$A''_\varepsilon = \{x \in N : \ \mathrm{dist}_N(x, \partial N) < T_1 - \varepsilon\}$$

be sets of the form (15). Then the interior of $A'_\varepsilon \setminus A''_\varepsilon$ is a small neighbourhood of x_1. Let $f'_{\varepsilon,\alpha}$ and $f''_{\varepsilon,\alpha}$ be the solutions of the minimization problems (16) with objective functions $\chi_{A'_\varepsilon}$ and $\chi_{A''_\varepsilon}$, respectively. When $\alpha > 0$ is small, (17) implies that the boundary source

$f_{\varepsilon,\alpha} = f'_{\varepsilon,\alpha} - f''_{\varepsilon,\alpha}$ produces a wave $u^{f_{\varepsilon,\alpha}}(t,x)$ such that $u^{f_{\varepsilon,\alpha}}(T,x)$ is concentrated in the set $A'_\varepsilon \setminus A''_\varepsilon$. Further, when $\varepsilon \to 0$, the set $A'_\varepsilon \setminus A''_\varepsilon$ tends to the point x_1.

Numerical methods to constructing the family of focused waves, $u^{f_{\varepsilon,\alpha}}(T,x)$, by solving blind control problems similar to (17) have been developed by M. de Hoop, P. Kepley, and L. Oksanen de Hoop, Kepley, and Oksanen [2016] (see Fig. 4).

As discussed above, the minimization Equation (17) can be modified –see Dahl, Kirpichnikova, and Lassas [2009] and (18)– so that their solutions are boundary sources $\tilde{f}_{\varepsilon,\alpha} \in L^2([0,T] \times \partial N)$ that produce waves $u^{\tilde{f}_{\varepsilon,\alpha}}(t,x)$ for which the pair $(u^{\tilde{f}_{\varepsilon,\alpha}}(T,x), \partial_t u^{\tilde{f}_{\varepsilon,\alpha}}(T,x))$ is concentrated near point x_1. Moreover, when the sources are multiplied by a factor $c_\varepsilon = 1/\mathrm{vol}\,(A'_\varepsilon \setminus A''_\varepsilon)$, we have, in sense of distributions,

$$\lim_{\varepsilon\to 0}\lim_{\alpha\to 0} (u^{c_\varepsilon \tilde{f}_{\varepsilon,\alpha}}(T,x), \partial_t u^{c_\varepsilon \tilde{f}_{\varepsilon,\alpha}}(T,x)) = (\delta_{x_1}, 0),$$

where $\delta_{x_1} \in \mathcal{D}'(N)$ is the delta distribution supported at x_1. This implies that the wave $u^{\tilde{f}_{\varepsilon,\alpha}}(t,x)$ is at times $t > T$ close to the time derivative of Green's function $G(t,x;T,x_1)$ corresponding to the point source $\delta_{x_1}(x)\delta(t-T)$ at (T,x_1). Furthermore, the boundary observations of the time derivative $\partial_t G(t,x;T,x_1)$ determine the boundary values of Green's function $G(t,x;T,x_1)$.

For convex manifolds the boundary observations of the above Green's function determine the distance difference function D_{x_1} corresponding to the point x_1, see (13). For general manifolds, the distance difference function D_{x_1} can be constructed by computing the L^2-norms of the waves $u^{f_\alpha}(T,x)$, where f_α solve the minimization Equation (16) with different sets A of the form (15), see Bingham, Y. Kurylev, Lassas, and Siltanen [2008]. When $T > \mathrm{diam}\,(M)/2$, the above focusing of waves, that creates a point source, can be replicated for arbitrary point $x_1 \in N$. Assuming that manifold N is a subset of a compact or closed manifold $\widetilde{N}$ and that we know the exterior $\widetilde{N} \setminus N$ and the metric on this set, Theorem 3.6 implies that the collection of the distance difference functions $\mathcal{D}(N) = \{D_{x_1} : x_1 \in N\}$ determine the isometry type of the Riemannian manifold (N,g). A similar construction of manifold (N,g) can also be made when we are not given the exterior $\widetilde{N} \setminus N$ but when we are given only ∂N and Λ (see Katchalov, Y. Kurylev, and Lassas [2001]).

4.2 Non-linear equations and artificial point sources. Below we consider the non-linear wave and the main ideas used to prove Theorem 3.1.

Let $f = \epsilon h$, $\epsilon > 0$, and write an asymptotic expansion of the solution u of (8),

$$u = \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \epsilon^4 w_4 + \mathcal{O}(\epsilon^5),$$

where

$$w_1 = \Box_g^{-1} h, \quad w_2 = -\Box_g^{-1}(w_1 \cdot w_1), \quad w_3 = -2\Box_g^{-1}(w_1 \cdot w_2), \tag{19}$$
$$w_4 = -\Box_g^{-1}(a w_2 \cdot w_2) - 2\Box_g^{-1}(a w_1 \cdot w_3).$$

We say, for example, that w_3 results from the interaction of w_1 and w_2, and consider such interactions in general.

Let us consider for the moment $\mathbb{R}^4$ with the Minkowski metric g. We can choose in $\mathbb{R}^4$ coordinates x^j, $j = 1, 2, 3, 4$, such that the hyperplanes $K_j = \{x^j = 0\}$ are light-like, that is, $T_p K_j$ contains a light-like vector for all $p \in \mathbb{R}^4$. The plane waves $u_j(x) = (x^j)_+^m$, where $m > 0$, are solutions to the wave equation $\Box_g u = 0$. They are singular on the hyperplanes K_j, in fact, they are conormal distributions in $I^{-m-1}(N^* K_j)$ (see Greenleaf and G. Uhlmann [1993] and R. B. Melrose and G. A. Uhlmann [1979]).

The proof of Theorem 3.1 is based on an analysis of the interaction of four waves. Analogously to (19), the derivative $u^{(4)} = \partial_{\epsilon_1}\partial_{\epsilon_2}\partial_{\epsilon_3}\partial_{\epsilon_4} u_{\vec{\epsilon}}\big|_{\vec{\epsilon}=0}$ of the solution $u_{\vec{\epsilon}}$ of (8)–(9) with the source

$$f_{\vec{\epsilon}}(x) = \sum_{j=1}^{4} \epsilon_j f_j(x), \quad \vec{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4),$$

is a linear combination of terms such as

$$\widetilde{w}_4 = \Box_g^{-1}(S_{1234}), \quad S_{1234} = u_4 \Box_g^{-1}(u_3 \Box_g^{-1}(u_2 u_1)). \tag{20}$$

Moreover, a suitable choice of f_j, $j = 1, 2, 3, 4$, guarantees that the term (20) dominates the other terms in $u^{(4)}$. For example, two waves u_1 and u_2 are singular on hyperplanes K_1 and K_2, respectively, and these singularities interact on $K_1 \cap K_2$. The interaction of the three waves u_1, u_2, and u_3 happens on the intersection $K_{123} = K_1 \cap K_2 \cap K_3$ which is a line. As $N^* K_{123}$ contains light-like directions that are not in union, $N^* K_1 \cup N^* K_2 \cup N^* K_3$, this interaction produces interesting singularities that start to propagate. These singularities correspond to the black conic wave in Fig. 3. Finally, singularities of all four waves u_j, $j = 1, 2, 3, 4$ interact at the point $\{q\} = \bigcap_{j=1}^4 K_j$. The singularities from the point q propagate along the light cone emanating from this point and with suitably chosen sources f_j the wave $u^{(4)}$ is singular on the light cone $\mathcal{L}(q)$. Thus S_{1234} can be considered as a microlocal point source that sends similar singularities in all directions as a point source located at the point q, and these singularities are observed in the set V. The singularities caused by the interactions of three waves produce artefacts that need be removed from the analysis. In this way, we see that the non-linear interaction of waves gives us the intersection of the light cone $\mathcal{L}(q)$ and the observation domain V. The above-described microlocal point source S_{1234} can be produced at an arbitrary point q in the future of the set V, and hence we can determine the earliest light observation sets $\mathcal{E}_V(q)$ for any such point. After letting q vary in $I(p^-, p^+)$, we apply Theorem 3.5 to recover the topology, differentiable structure, and the conformal class of g in $I(p^-, p^+)$.

5 Einstein-matter field equations

Einstein's equations for a Lorentzian metric $g = (g_{jk})$ are

$$\mathrm{Ein}(g) = T,$$

where $\mathrm{Ein}_{jk}(g) = \mathrm{Ric}_{jk}(g) - \frac{1}{2}(g^{pq}\mathrm{Ric}_{pq}(g))g_{jk}$. Here Ric denotes the Ricci tensor, $g^{-1} = (g^{pq})$ and $T = (T_{jk})$ is the stress-energy tensor. In vacuum $T = 0$. Einstein's equations coupled with scalar fields $\phi = (\phi_l)$, $l = 1, 2, \dots, L$, and a source $\mathfrak{F} = (\mathfrak{F}^1, \mathfrak{F}^2)$ are

$$\mathrm{Ein}(g) = T, \quad T = \mathbb{T}(g,\phi) + \mathfrak{F}^1, \tag{21}$$

$$\square_g \phi_l - \partial_{\phi_l}\mathcal{V}_l(x,\phi) = \mathfrak{F}^2_l, \quad l = 1, 2, ..., L. \tag{22}$$

Here $\mathfrak{F} = (\mathfrak{F}^1, \mathfrak{F}^2_1, \dots, \mathfrak{F}^2_L)$ models a source in active measurements, see Section 5.1. The standard coupling $\mathbb{T} = (\mathbb{T}_{jk})$ of g and ϕ is given by

$$\mathbb{T}_{jk}(g,\phi) = \sum_{l=1}^{L}\left(\partial_j\phi_l\partial_k\phi_l - \frac{1}{2}g_{jk}g^{pq}\partial_p\phi_l\partial_q\phi_l - \mathcal{V}_l(x,\phi)g_{jk}\right),$$

the potentials $\mathcal{V}_l$ are smooth functions $M \times \mathbb{R}^L \to \mathbb{R}$.

Below, we consider the case when M is 4-dimensional. We say that $(M, \widehat{g})$ and $\widehat{\phi}$ are the background spacetime and scalar fields if they are C^∞-smooth, satisfy (21)–(22) with $\mathfrak{F} = 0$ and $(M, \widehat{g})$ is globally hyperbolic. Again, we write M in the form $M = \mathbb{R} \times N$. We will consider equations (21)–(22) with the initial conditions

$$g = \widehat{g},\ \phi = \widehat{\phi},\ \mathfrak{F} = 0, \quad \text{in } (-\infty, 0) \times N. \tag{23}$$

The source $\mathfrak{F}$ can not be arbitrary since the Bianchi identities imply that $\mathrm{div}_g\mathrm{Ein}(g) = 0$, whence the stress energy tensor T needs to satisfy the conservation law

$$\mathrm{div}_g T = 0. \tag{24}$$

This again implies the compatibility condition

$$\mathrm{div}_g \mathfrak{F}^1 + \sum_{l=1}^{L}\mathfrak{F}^2_l \nabla\phi_l = 0. \tag{25}$$

In local coordinates, the divergence is $\text{div}_g T = \nabla_p(g^{pj}T_{jk})$, $k = 1,2,3,4$, where ∇ is the covariant derivative with respect to g. The conservation law (24) for Einstein's equations dictates, roughly speaking, that any source in the equation must take energy from some fields in order to increase energy in other fields.

Observe that in the system (21)–(22) the metric of the spacetime begins to change as soon as $\mathfrak{F}$ becomes non-zero, and that the system is invariant with respect to diffeomorphisms. We model an active measurement by factoring out the diffeomorphism invariance by using Fermi coordinates.

Let $\widehat{g}$ and $\widehat{\phi}$ be a background spacetime and scalar fields, and g be a close to $\widehat{g}$. We recall that $M = \mathbb{R} \times N$. Let $p \in \{0\} \times N$, and let $\xi \in T_pM$ be time-like. Define $\mu_g(s) = \gamma_{p,\xi}(s)$ to be the geodesic with respect to g satisfying $\mu(0) = p$ and $\dot{\mu}(0) = \xi$. Let X_j, $j = 0,1,2,3$, be a basis of T_pM, with $X_0 = \xi$, and consider the following Fermi coordinates Φ_g,

$$\Phi_g(s, y^1, y^2, y^3) = \exp_{\mu_g(s)}(y^j Y_j), \quad \Phi_g : V \to M,$$

where Y_j is the parallel transport of X_j along μ_g, $j = 1,2,3$. Here the parallel transport and the exponential map exp are with respect to g, and $V = (0,1) \times B$ where B is a ball centered at the origin in $\mathbb{R}^3$. We suppose that B is small enough so that the Fermi coordinates are well-defined with metric $\widehat{g}$ in $\overline{V}$. Below, we denote the Fermi coordinates of $(M, \widehat{g})$ by $\Phi = \Phi_{\widehat{g}}$.

Let $t_0 > 0$ and consider a Lorentzian metric g on $(-\infty, t_0) \times N$ such that the corresponding Fermi coordinates $\Phi_g : V \to \mathbb{R} \times N$ are well-defined. We define the data set similar to (11),

$$\begin{aligned}\mathfrak{D} = \{(\Phi_g^* g|_V, \Phi_g^*\phi|_V, \Phi_g^*\mathfrak{F}|_V) :& (g,\phi,\mathfrak{F}) \text{ satisfies (21),(22),(23)}, \\ & \mathfrak{F} \in C_0^k(\Phi_g(V)),\ \|\mathfrak{F}\|_{C^k} < \varepsilon\},\end{aligned}$$

where Φ_g^* is the pullback under Φ_g, k is large enough, and $\varepsilon > 0$ is small enough.

Theorem 5.1 (Y. Kurylev, Lassas, Oksanen, and G. Uhlmann [2014]). *Let (M, g) be a globally hyperbolic 4-dimensional Lorentzian manifold. Let $\mu_{\widehat{g}}([0,1])$ be a time-like geodesic and let $p^- = \mu(0)$ and $p^+ = \mu(1)$. Suppose $L \geq 4$, and we have the non-degeneracy condition*

$$(\partial_j \widehat{\phi}_l)_{j,l=1}^4 \text{ is invertible at all points in } \overline{\Phi(V)}. \tag{26}$$

Then the data set $\mathfrak{D}$ determines the topology, differentiable structure and conformal class of the metric $\widehat{g}$ in the double cone $I(p^-, p^+)$ in $(M, \widehat{g})$.

Analogous results for inverse problem for the Einstein-Maxwell system with vacuum background metric are considered in Lassas, G. Uhlmann, and Wang [2017].

5.1 More on active measurements. Recall that the source $\mathcal{F}$ must satisfy the compatibility condition (25). In particular, the set of allowed sources $\mathcal{F}$ depends on the solution (g, ϕ) of the system (21)–(22). Due to this difficulty we use a construction that we call an adaptive source for the scalar fields. Consider the following special case of (21),(22),(23),

$$\begin{aligned} &\mathrm{Ein}(g) = T, \quad T = F^1 + \mathbb{T}(g, \phi), \\ &\square_g \phi - \partial_\phi \mathcal{V}(\phi) = F^2 + \mathcal{S}(g, \phi, \nabla\phi, F, \nabla F), \\ &g = \widehat{g}, \quad \phi = \widehat{\phi}, \quad \text{in } (-\infty, 0) \times N. \end{aligned} \tag{27}$$

Here $F = (F^1, F^2)$ are primary sources and $\mathcal{S}(g, \phi, \nabla\phi, F, \nabla F)$ is the secondary source function that vanishes outside the support of the primary source F and adapts to values of the sources F and fields (g, ϕ). The secondary source functions can be considered as an abstract model for the measurement devices that one uses to implement the sources. When (26) is valid, functions $\mathcal{S}$ can be constructed so that the conservation law (24) is valid for all sufficiently small F (see Y. Kurylev, Lassas, Oksanen, and G. Uhlmann [2014]).

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