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A PANORAMA OF SINGULAR SPDES

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Abstract

I will review the setting and some of the recent results in the field of singular stochastic partial differential equations (SSPDEs). Since Hairer's invention of regularity structures this field has experienced a rapid development. SSPDEs are non-linear equations with random and irregular source terms which make them ill-posed in classical sense. Their study involves a tight interplay between stochastic analysis, analysis of PDEs (including paradifferential calculus) and algebra.

1 Introduction

This contribution aims to give an overview of the recent developments at the interface between stochastic analysis and PDE theory where a series of new tools have been put in place to analyse certain classes of stochastic PDEs (SPDEs) whose rigorous understanding was, until recently, very limited. Typically these equations are non-linear and the randomness quite ill behaved from the point of view of standard functional spaces. In the following I will use the generic term *singular stochastic PDEs* (SSPDEs) to denote these equations.

The interplay between the algebraic structure of the equations, the irregular behaviour of the randomness and the weak topologies needed to handle such behaviour provide a fertile ground where new point of views have been developed and old tools put into work in new ways Hairer [2014], Gubinelli, Imkeller, and Perkowski [2015], Otto and Weber [2016], Kupiainen [2016], Bailleul and Bernicot [2016a], Bruned, Hairer, and Zambotti [2016], Chandra and Hairer [2016], and Bruned, Chandra, Chevyrev, and Hairer [2017].

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2 Ways of describing a function

2.1 From ODEs to rough differential equations. The simpler setting we can discuss is that of an ordinary differential equation perturbed by a random function in a non-linear way. Consider the Cauchy problem for $y : \mathbb{R}_+ \to \mathbb{R}^d$,

$$(1) \qquad \begin{cases} \dot{y}(t) = \varepsilon^{1/2} f(y(t))\eta(t), & t > 0, \\ y(0) = y_0 \in \mathbb{R}^d \end{cases}$$

where the dot denotes time derivative, $f : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^d)$ is a family of smooth vector fields in $\mathbb{R}^d$ ($\mathcal{L}(\mathbb{R}^n; \mathbb{R}^d)$ are the linear maps form $\mathbb{R}^n$ to $\mathbb{R}^d$), $\eta : \mathbb{R}_+ \to \mathbb{R}^n$ is a smooth centered $\mathbb{R}^m$–valued Gaussian random function and $\varepsilon > 0$ a small parameter.

If we are interested in the $\varepsilon \to 0$ limit of this equation we would better rescale it to see some interesting dynamics going on. In term of the rescaled variable $y_\varepsilon(t) = y(t/\varepsilon)$ the equation has the form

$$(2) \qquad \begin{cases} \dot{y}_\varepsilon(t) = f(y_\varepsilon(t))\eta_\varepsilon(t), & t > 0, \\ y_\varepsilon(0) = y_0 \in \mathbb{R}^d \end{cases}$$

where $\eta_\varepsilon(t) = \varepsilon^{-1/2}\eta(t/\varepsilon)$. If we assume that η is stationary, has fast decaying correlations (e.g. exponentially fast) and independent components, then we can prove that η_ε converges in law to a white noise ξ, that is the Gaussian random distribution with covariance given by

$$\mathbb{E}[\xi(t)\xi(s)] = \delta(t - s).$$

This convergence takes place as random elements of the Hölder–Besov space[1] $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}$ for any $\alpha < -1/2$. Alternatively, and more in line with classical probability theory, one could look at the integral function $x_\varepsilon(t) = \int_0^t \eta_\varepsilon(s)\mathrm{d}s$ and conclude that it converges in the Hölder topology $\mathcal{C}^{\alpha+1}$ to the Brownian motion.

This procedure is reminiscent of homogenisation E [2011] but while there there are essentially only two (or a finite number) of scales which play a fundamental role here all the scales remain coupled also after the passage to the limit. Indeed one would now like to argue that the solution y_ε of Equation (2) converges to the solution z of the ODE

$$(3) \qquad \begin{cases} \dot{z}(t) = f(z(t))\xi(t), & t > 0, \\ z(0) = y_0 \in \mathbb{R}^d. \end{cases}$$

where ξ is the white noise on $\mathbb{R}$. However we quickly realise that this equation is not well posed. Indeed we cannot hope better regularity for z than $\mathcal{C}^{1+\alpha}$ (e.g. in the simple

[1] The choice of this space is not canonical for this convergence in law but will fit our intended applications, other choices do not lead to substantial improvements in the arguments below.

setting where f is constant) and in this situation the pointwise product of $f(z)$ (still a $\mathcal{C}^{1+\alpha}$ function) and the distribution ξ of regularity $\mathcal{C}^\alpha$ is not a well defined operation.

Remark 2.1. *That this is not only a technical difficulty can be understood easily by considering the following example. Take* $f_\varepsilon(t) = \varepsilon^{1/2}\sin(t/\varepsilon)$ *and* $g_\varepsilon(t) = \varepsilon^{-1/2}\sin(t/\varepsilon)$. *Then for any* $\alpha < -1/2$, $f_\varepsilon \to 0$ *in* $\mathcal{C}^{\alpha+1}$ *and* $g_\varepsilon \to 0$ *in* $\mathcal{C}^\alpha$ *but* $h_\varepsilon(t) := f_\varepsilon(t)g_\varepsilon(t) = \sin^2(t/\varepsilon) = 1 - \cos(2t/\varepsilon)/2$ *and* $h_\varepsilon \to 1$ *in* $\mathcal{C}^{2\alpha+1}$. *We see that the product cannot be extended continuously in* $\mathcal{C}^{\alpha+1} \times \mathcal{C}^\alpha$ *as we would need to have a robust meaning for Equation (3) in the framework of Hölder–Besov spaces.*

This difficulty has been realised quite early in stochastic analysis and is at the origin of the invention of stochastic calculus by Itō (and the independent work of Dœblin) and has shaped ever since the study of stochastic processes, see e.g. Watanabe and Ikeda [1981] and Revuz and Yor [2004]. Itō's approach give a meaning to (3) by prescribing a certain preferred approximation scheme (the forward Riemman sum) to the integral version of the r.h.s. of the equation. The resulting Itō integral comes with estimates which are at the core of stochastic integration theory. However the Itō integral in *not* the right description for the limiting Equation (3). Indeed Wong and Zakai proved that the limit is given by another interpretation of the product, that provided by the Stratonovich integral.

2.2 Reconstruction of a coherent germ. From a strictly analytic viewpoint, without resorting to probabilistic techniques, Equation (3) should stand for a description of the possible limit points of the sequence $(y_\varepsilon)_\varepsilon$. Compactness arguments should provide methods to prove limits exists along subsequence and under nice conditions we would hope to be able to prove that there is only such a limit point, settling the problem of the convergence of the whole sequence. This is the standard approach. Usually in PDE theory some refined methods are put in place in order to establish some sort of compactness (e.g. convexity, maximum principle, concentrated or compensated compactness, Young measures), others maybe are needed to show uniqueness (e.g. entropy solutions, energy estimates, viscosity solutions), see e.g. Evans [1990]. But here we are stuck at a more primitive level, we do not have an effective description to start with, and this inability comes (as is fast to realise) with the inability to obtain useful and general apriori bounds for the compactness step.

With the aim of identifying such a description we could think of constructing a good local approximations of the function z. Around a given time $s > 0$ we imagine that z has the behaviour obtained by freezing the vectorfield f at that given time: write the ODE in the weak form and assume that for any smooth, compactly supported $\varphi : \mathbb{R}_+ \to \mathbb{R}$ we have

$$\left| \int_{\mathbb{R}} \varphi_s^\lambda(t)(\dot z(t) - f(z(s))\xi(t))\mathrm{d}t \right| \leqslant \lambda^\gamma N(\varphi) \tag{4}$$

where $\varphi_s^\lambda(t) := \lambda^{-1}\varphi((t-s)/\lambda)$ is a rescaled and centred version of φ, $N(\varphi), \gamma > 0$ constants not depending on λ or s (we let $z(t) = y_0$ and $\xi(t) = 0$ if $t < 0$). If ξ is a smooth function then this bound implies Equation (3) since $\varphi^\lambda \to \delta$ as $\lambda \to 0$, so this description of z is as detailed as the ODE. On the other hand it has the fundamental advantage of having *decoupled* the product $f(z(t))\xi(t)$ into $f(z(s))\xi(t)$. Equation (4) makes sense even when ξ is a distribution and in particular in the case of the white noise we are looking after. That this description is quite powerful is witnessed by Hairer's *reconstruction theorem* Hairer [2014].

Theorem 2.2 (Hairer's reconstruction). *Let $\gamma > 0$, $\alpha \in \mathbb{R}$ and $G = (G_x)_{x\in\mathbb{R}^d}$ be a family of distributions in $\mathcal{S}'(\mathbb{R}^d)$ such that there exists a constant $L(G)$ and a constant $N(\varphi)$ for which*

$$\sup_x |G_x(\varphi_x^\lambda)| \leqslant \lambda^{-\alpha} N(\varphi) L(G), \qquad \lambda \in (0,1) \tag{5}$$

$$|G_y(\varphi_x^\lambda) - G_x(\varphi_x^\lambda)| \leqslant \lambda^\gamma N(\varphi) L(G) P(|x-y|/\lambda), \qquad \lambda \in (0,1) \tag{6}$$

where P is a continuous function with at most polynomial growth. Then there exists a universal constant C_γ and a unique distribution $g = \mathcal{R}(G) \in \mathcal{S}'(\mathbb{R}^d)$, the reconstruction of G, such that

$$|g(\varphi_x^\lambda) - G_x(\varphi_x^\lambda)| \leqslant C_\gamma \lambda^\gamma N(\varphi) L(G), \qquad \lambda \in (0,1).$$

For the sake of the exposition I simplified a bit the setting and gave a slightly different formulation of this results which can be appreciated independently of other details of Hairer's theory of regularity structures Hairer [ibid.]. We give here an idea of proof, without pretension to make it fully rigorous. We call the family G a *germ* and the quantity in the l.h.s. of Equation (6) the *coherence* of the germ G. The theorem states a relation between coherent germs and distributions.

Proof. (sketch) *Uniqueness.* Assume $g, \tilde{g}$ are two possible reconstructions of G, then

$$|g(\varphi_x^\lambda) - \tilde{g}(\varphi_x^\lambda)| \lesssim \lambda^\gamma.$$

For any given test function $\psi \in \mathcal{S}(\mathbb{R}^d)$ we let $T_\lambda\psi(y) = \int \psi(x)\varphi_x^\lambda(y)\mathrm{d}x$. Then $T_\lambda\psi \to \psi$ in $\mathcal{S}(\mathbb{R}^d)$ and $g = \tilde{g}$ since

$$|(g-\tilde{g})(\psi)| = \lim_{\lambda\to 0} |(g-\tilde{g})(T_\lambda\psi)| \lesssim \liminf_{\lambda\to 0} \int |g(\varphi_x^\lambda) - \tilde{g}(\varphi_x^\lambda)||\psi(x)|\mathrm{d}x = 0.$$

Existence. We follow an idea of Otto and Weber Otto and Weber [2016]. Introduce the heat semigroup to perform a multiscale decomposition. Let $T_i = P_{2^{-i}}$ where $(P_t)_t$ is the heat kernel, then $T_{i+1}T_{i+1} = T_i$ for $i \geqslant 0$. Let

$$\mathcal{R}_N G(x) := \int_{y,z} T_N(x-y)T_N(y-z)G_y(z),$$

where $\int_{y,z}$ denotes the integral in (y,z) over $\mathbb{R}^d \times \mathbb{R}^d$. Note that if $G_y(x) = f(x)$ for some Schwartz distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ then $\mathcal{R}_N G(x) = \int_{y,z} T_N(x-y)T_N(y-z)f(z) = f(T_{N+1}(x-\cdot)) \to f(x)$ in $\mathcal{S}'(\mathbb{R}^d)$. In general, in order to control the limit $\mathcal{R} = \lim_{N\to\infty} \mathcal{R}_N$ for more general germs we look at

$$(\mathcal{R}_{n+1}G - \mathcal{R}_n G)(x) := \mathcal{A}_n G(x) + \mathcal{B}_n G(x),$$

where

$$\begin{aligned} \mathcal{A}_n G(x) &:= \textstyle\int_{y,z}(T_{n+1}-T_n)(x-y)(T_n+T_{n+1})(y-z)G_y(z), \\ \mathcal{B}_n G(x) &:= \textstyle\int_{y,z,r} T_{n+1}(x-r)T_{n+1}(r-y)T_{n+1}(y-z)(G_y(z)-G_r(z)). \end{aligned}$$

Using (6) the terms $\mathcal{B}_n G(x)$ can be estimated by $|\mathcal{B}_n G(x)| \lesssim 2^{-n\gamma} N(\varphi)L(G)$, and they can be resummed over n since $\gamma > 0$. The terms $\mathcal{A}_n G$ are localized at scale 2^{-n} thanks to the factor $(T_{n+1}-T_n)$ and they behave as "orthogonal" contributions: once tested agains a test function ψ they can be estimated as $|\mathcal{A}_n G(\psi)| \lesssim \|(T_{n+1}-T_n)\psi\|_{L^1} 2^{-\alpha n} N(\varphi)L(G)$, thanks to the Equation (5). From we deduce that

$$\sum_n |\mathcal{A}_n G(\psi)| \lesssim N(\varphi)L(G) \sum_n \|(T_{n+1}-T_n)\psi\|_{L^1} 2^{-\alpha n} \lesssim N(\varphi)L(G)\|\psi\|_{B^{\alpha}_{1,1}},$$

where $B^{\alpha}_{1,1}$ is the Besov space with norm $\|\psi\|_{B^{\alpha}_{1,1}} = \sum_{n\geqslant 1} 2^{-\alpha n}\|\Delta_n \psi\|_{L^1}$. From these observations is easy to deduce that $\mathcal{R}_N G \to \mathcal{R}G$ as a distribution. In order to identify $\mathcal{R}G$ we observe that, for fixed $L > 0$,

$$\mathcal{R}_L G(x) - \mathcal{R}_L G_h(x) = \int_{y,z} T_L(x-y)T_L(y-z)(G_y(z)-G_h(z)),$$

and if $|x-h| \simeq 2^{-L}$ we have $|\mathcal{R}_L G(x) - \mathcal{R}_L G_h(x)| \lesssim 2^{-L\gamma}$, while

$$\begin{aligned} \mathcal{R}G(x) - G_h(x) &= \mathcal{R}(G-G_h)(x) \\ &= \mathcal{R}_L(G-G_h)(x) + \sum_{n>L} \mathcal{A}_n(G-G_h)(x) + \sum_{n>L} \mathcal{B}_n G(x) \end{aligned}$$

It is not difficult to estimate $|\mathcal{A}_n(G-G_h)(\psi_h^{2^{-L}})| \lesssim 2^{-n\gamma}$ and finally deduce that $|(\mathcal{R}G - G_h)(\psi_h^{2^{-L}})| \lesssim 2^{-L\gamma}$. ∎

Let us go back to our equation. Let $G_s(t) = f(z(s))\xi(t)$ be our germ and consider its coherence:

$$G_u(\varphi_s^\lambda) - G_s(\varphi_s^\lambda) = \int_t \varphi^\lambda(t-s)(f(z(u)) - f(z(s)))\xi(t) = (f(z(u)) - f(z(s)))\xi(\varphi_s^\lambda).$$

Assuming that $z \in \mathcal{C}^{\alpha+1}$ and that $\xi \in \mathcal{C}^\alpha$ we have, for some polynomially growing P,

$$|G_u(\varphi_s^\lambda) - G_s(\varphi_s^\lambda)| \lesssim \lambda^{2\alpha+1} \|z\|_{\mathcal{C}^{\alpha+1}} \|\xi\|_{\mathcal{C}^\alpha} P((u-s)/\lambda).$$

We see that if $\gamma = 2\alpha - 1 > 0$ we can meet the conditions of Theorem 2.2. In this case the ODE can be replaced by the formulation (4) and the resulting theory coincides with the theory of differential equations build upon the Young integral Young [1936] and P. K. Friz and Hairer [2014].

This is not yet enough for us. White noise restrict the allowed values for α in the range $\alpha < 1/2$ and in this case $2\alpha - 1 < 0$. In this case the description is not precise enough to uniquely determine the distribution $\dot z(t)$ using only the assumption $z \in \mathcal{C}^\alpha$. Going back to the ODE and thinking about a Taylor expansion for the r.h.s. we come up with a refined description of the solution given by the new germ:

$$G_s(t) = f(z(s))\xi(t) + f'(z(s))f(z(s)) \int_s^t \xi(u)\mathrm{d}u,$$

where we denoted f' the gradient of f. Its coherence is given by (we let $f_2(z) = f'(z)f(z)$)

$$\begin{aligned} G_u(\varphi_s^\lambda) - G_s(\varphi_s^\lambda) &= \left[f(z(u)) - f(z(s)) - f_2(z(s)) \textstyle\int_s^u \xi(r)\mathrm{d}r\right] \xi(\varphi_s^\lambda) \\ &\quad + [f_2(z(u)) - f_2(z(s))] \left[\textstyle\int_t \varphi^\lambda(t-s) \left(\int_s^t \xi(r)\mathrm{d}r\right) \xi(t)\right]. \end{aligned}$$

In order to meet the conditions of the reconstruction theorem we can require

$$\left|\int_t \varphi^\lambda(t-s) \left(\int_s^t \xi(r)\mathrm{d}r\right) \xi(t)\right| \lesssim \lambda^{2\alpha+1}, \tag{7}$$

and

$$\left|f(z(u)) - f(z(s)) - f_2(z(s)) \int_s^u \xi(r)\mathrm{d}r\right| \lesssim \lambda^{2\alpha+2}, \tag{8}$$

from which we see that $|G_u(\varphi_s^\lambda) - G_s(\varphi_s^\lambda)| \lesssim \lambda^{3\alpha+2}$. Provided $\alpha > 3/2$ we can reconstruct in a unique way a distribution g from this germ and verify the equation $\dot z = g$

(at least in the weak sense). Equation (7) is a condition on ξ, Equation (8) one on z. In particular, by Taylor expansion, this latter holds if the bound

$$(9)\qquad \left| z(u) - z(s) - f(z(s)) \int_s^u \xi(r)\mathrm{d}r \right| \lesssim \lambda^{2\alpha+2},$$

holds for z. This is a refinement of the Hölder assumption $z \in \mathcal{C}^{\alpha+1}$. Building on these basic observation is possible to develop a complete well–posedness theory showing that, provided $\alpha > -3/2$ there is a continuous map $\Phi : \Xi \mapsto z$ taking the germ

$$\Xi_s(t) = (\Xi_s^{(1)}(t), \Xi_s^{(2)}(t)) = \left(\xi(t), \left(\int_s^t \xi(r)\mathrm{d}r \right) \xi(t) \right)$$

satisfying $|\Xi_s^{(1)}(\varphi_s^\lambda)| \lesssim \lambda^\alpha$ and $|\Xi_s^{(2)}(\varphi_s^\lambda)| \lesssim \lambda^{2\alpha+1}$ to the unique Hölder function z satisfying the relation

$$\left| \int_{\mathbb{R}} \varphi_s^\lambda(t) \left(\dot{z}(t) - f(z(s))\xi(t) - f_2(z(s)) \int_s^t \xi(r)\mathrm{d}r \right) \mathrm{d}t \right| \lesssim \lambda^{3\alpha+2}.$$

The original difficulties are here not completely solved, indeed the germ $\Xi^{(2)}$ is not apriori well defined given that it contains a pointwise product between the distribution ξ and the function $\int_s^{\cdot} \xi(r)\mathrm{d}r$. However in this new perspective we have accomplished a major step: restricting the difficulty to a well defined quantity which can be analysed from the point of view of stochastic analysis without any reference to the ODE problem and its non-linearity.

The map Φ is called Itō–Lyons map P. K. Friz and Hairer [2014]. Its regularity and the fact that it provides an extension of the solution map for the classical ODE allows to control the limit of the Equation (2) as $\varepsilon \to 0$. Indeed *provided* we can show that the germ

$$(10)\qquad \Xi_s^\varepsilon(t) = \left(\eta_\varepsilon(t), \left(\int_s^t \eta_\varepsilon(r)\mathrm{d}r \right) \eta(t) \right)$$

converges in the appropriate topology to Ξ then we can conclude that $y_\varepsilon = \Phi(\Xi^\varepsilon) \to \Phi(\Xi) = z$.

In this generalisation however there is a catch. The limiting problem is now defined in terms of a more complex object Ξ than the original white noise ξ. Any two sequences $(\eta_\varepsilon)_\varepsilon$ and $(\tilde{\eta}_\varepsilon)_\varepsilon$ approximating ξ and lifted into germs Ξ^ε and $\tilde{\Xi}^\varepsilon$ can converge to two limits Ξ and $\tilde{\Xi}$ for which $\Xi^{(1)} = \tilde{\Xi}^{(1)} = \xi$ but $\Xi^{(2)} \neq \tilde{\Xi}^{(2)}$. In this case the corresponding solutions y_ε and $\tilde{y}_\varepsilon$ to Equation (2) will in general converge to different limits $z = \Phi(\Xi)$, $\tilde{z} = \Phi(\tilde{\Xi})$.

2.3 From ODEs to PDEs. We have given an outlook of the use of Hairer's reconstruction theorem in the analysis of a controlled ODE. The Itō–Lyons map has been invented and initially studied by Lyons T. Lyons [1998], T. Lyons and Qian [2002], T. J. Lyons, Caruana, and Lévy [2007], and P. K. Friz and Hairer [2014] and is at the base of the *theory of rough paths* (RPT). Lyons' theory goes far beyond to the limit $\alpha > -3/2$ down to any $\alpha > -1$. This full range corresponds to the *subcritical* regime where scaling dictates that the noise is a perturbation of the first order differential operator ∂_t. The reformulation given here is essentially that introduced in the context of RPT by Davie Davie [2007] and here reshaped in the language of Theorem 2.2. Functions satisfying conditions like eq. (9) are called *controlled paths* in RPT and were introduced in Gubinelli [2004] in order to provide a nice analytical setting for a fixed point argument leading to the Itō–Lyons map and moreover to decouple the reconstruction of germs from the construction of the fixpoint. In the case of ODEs one can avoid the use of the reconstruction theorem by using the *sewing lemma* Gubinelli [2004], Feyel and De La Pradelle [2006], and Feyel, de La Pradelle, and Mokobodzki [2008]:

Lemma 2.3 (Sewing lemma)**.** *Let $\gamma > 0$. Let $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ a function such that*

$$|G(s,t) - G(s,u) - G(u,t)| \leqslant L_G |t-s|^{1+\gamma}, \qquad s \leqslant u \leqslant t.$$

for some $L_G > 0$. Then there exists a unique function $g : \mathbb{R}_+ \to \mathbb{R}$ such that

$$|g(t) - g(s) - G(s,t)| \leqslant C_\gamma L_G |t-s|^{1+\gamma}$$

for a universal constant C_γ.

Sewing and reconstruction are not equivalent. The sewing lemma combines in one operation the reconstruction operated by Theorem 2.2 and the integration needed to pass from $\dot{z}$ and z.

Hairer's regularity structure theory Hairer [2014] and P. K. Friz and Hairer [2014] builds over Theorem 2.2 a vast generalisation of Lyons' rough path theory and provides a solution theory for a large class of subcritical parabolic SPDEs. A recent series of three other papers Bruned, Hairer, and Zambotti [2016], Chandra and Hairer [2016], and Bruned, Chandra, Chevyrev, and Hairer [2017] complete the construction of this theory by "automatizing" the lifting of all the structures needed to deal with the various aspects of the solution theory for a generic singular SPDE: the construction of the appropriate model and regularity structure, the stochastic estimates and the identification of a suitable class of regular equations which possess limits (i.e. that can be *renormalized*).

2.4 Paraproducts and the paracontrolled Ansatz. In Gubinelli, Imkeller, and Perkowski [2015] an alternative approach has been introduced to handle the difficult product in (3) by decoupling it according to a multiscale decomposition. Write

$$f(z(t))\xi(t) = \sum_{n,m\geqslant -1} \int_{s,r} K_n(t-s)K_m(t-r)f(z(s))\xi(r),$$

where we let $(K_n)_{n\geqslant -1}$ to be kernels of Littlewood–Paley (LP) type which provide a resolution of a given distribution into "blocks" with specific frequency localization. See Bahouri, Chemin, and Danchin [2011] and Gubinelli and Perkowski [2015] for details on LP decomposition, Besov spaces and for the paraproduct estimates discussed below. Writing Δ_n for the operator of convolution with the kernel K_n we can decompose the product of two distributions g, h as above into three contributions according to the case where $n \leqslant m-1$, $|n-m| \leqslant 1$ and $n \geqslant m+1$:

$$gh = g \otimes h + g \circledcirc h + g \oslash h,$$

where we let

$$g \otimes h = h \oslash g := \sum_{n<m-1} (\Delta_n g)(\Delta_n h), \quad g \circledcirc h := \sum_{|n-m|\leqslant 1} (\Delta_n g)(\Delta_n h).$$

These operators are well behaved in several function spaces. For illustrative purpose we will use them mainly in the Hölder–Besov spaces $\mathscr{C}^\alpha = B^\alpha_{\infty,\infty}$ but other choices are possible. The LP decomposition can be chosen in such a way that these operators can be extended to bilinear bounded operators in Hölder–Besov spaces according to the following estimates:

$$\begin{aligned} \|g \otimes h\|_{\mathscr{C}^\alpha} &\lesssim \|g\|_{L^\infty}\|h\|_{\mathscr{C}^\alpha}, & \alpha \in \mathbb{R}, \\ \|g \otimes h\|_{\mathscr{C}^{\alpha+\beta}} &\lesssim \|g\|_{\mathscr{C}^\beta}\|h\|_{\mathscr{C}^\alpha}, & \alpha \in \mathbb{R}, \beta < 0, \\ \|g \circledcirc h\|_{\mathscr{C}^{\alpha+\beta}} &\lesssim \|g\|_{\mathscr{C}^\beta}\|h\|_{\mathscr{C}^\alpha}, & \alpha+\beta > 0. \end{aligned}$$

We see that the *resonant product* $g \circledcirc h$ is defined only for functions whose sum of regularities is positive while the *paraproduct* $g \otimes h$ is always well defined. Another key observation is that the paraproduct does not improve the regularity of its r.h.s. while the resonant product (when it is well defined) improves the regularity of its factor of lower regularity.

Paraproducts and related operations were introduced by Bony and Meyer Bony [1981] and Meyer [1981] for the use in the regularity theory of fully–nonlinear hyperbolic equation. It is not the aim of the present exposition to cover the vast literature these ideas generated, which includes the calculus of paradifferential operators. The reader can refer to Bahouri, Chemin, and Danchin [2011], Metivier [2008], Taylor [2000], Tao [2006], and Alinhac and Gérard [1991] for some expositions on the results and applications of these tools to PDEs.

An basic result in the theory of paraproducts is Bony's *paralinearization* Bony [1981], Bahouri, Chemin, and Danchin [2011], and Gubinelli, Imkeller, and Perkowski [2015] which in our context reads

$$z \in \mathscr{C}^{2\alpha} \mapsto R_f(z) := f(z) - f'(z) \olessthan z \in \mathscr{C}^{2\alpha}, \qquad \alpha > 0. \tag{11}$$

Moreover we have a commutator lemma proved in Gubinelli, Imkeller, and Perkowski [2015].

Lemma 2.4. *If $\beta + \gamma + \delta > 0$ there exists a bounded trilinear operator $Q : \mathscr{C}^\beta \times \mathscr{C}^\gamma \times \mathscr{C}^\delta \to \mathscr{C}^{\gamma+\delta}$ such that for smooth function g, h, l*

$$Q(g, h, l) = (g \olessthan h) \circledcirc l - g(h \circledcirc l).$$

Going back to our ODE we can expand its r.h.s. as

$$\dot{z} = f(z)\xi = f(z) \olessthan \xi + f(z) \ogreaterthan \xi + f(z) \circledcirc \xi.$$

If we assume that $\xi \in \mathscr{C}^\alpha$ and $z \in \mathscr{C}^{1+\alpha}$ then $f(z) \olessthan \xi \in \mathscr{C}^\alpha$ and $f(z) \circledast \xi \in \mathscr{C}^{2\alpha+1}$ (at least when $2\alpha + 1 > 0$). The key idea is to perform a change of variables to encode the heuristic that the more irregular contribution to $\dot{z}$ comes from the paraproduct $f(z) \olessthan \xi$. We formulate a *paracontrolled Ansatz* by introducing a new unknown $z^\sharp \in \mathscr{C}^{2\alpha+2}$ such that

$$z = f(z) \olessthan X + z^\sharp, s \tag{12}$$

where X solves the equation $\dot{X} = \xi$. Doing so gives

$$\dot{z}^\sharp = \dot{z} - f(z) \olessthan \dot{X} - (\partial_t f(z)) \olessthan X = f(z) \circledcirc \xi + f(z) \ogreaterthan \xi - (\partial_t f(z)) \olessthan X.$$

From the *paralinearization* (11) and the commutator Lemma 2.4 follows that

$$f(z) \circledcirc \xi = f'(z)(z \circledcirc \xi) + Q(f'(z), z, \xi) + R_f(z) \circledcirc \xi,$$

in the sense that the difference $f(z) \circledcirc \xi - f'(z)(z \circledcirc \xi)$ is well defined as soon as $3\alpha + 2 > 0$. Recalling (12) we can further simplify this expression into

$$f(z) \circledcirc \xi = f'(z) f(z)(X \circledcirc \xi) + f'(z) Q(f(z), X, \xi) + f'(z)(z^\sharp \circledcirc \xi) + \\ + Q(f'(z), z, \xi) + R_f(z) \circledcirc \xi$$

Finally our original ODE is transformed into the following equation for $z^\sharp$:

$$\dot{z}^\sharp = f_2(z)(X \circledcirc \xi) + \Psi(z, z^\sharp, \xi) \tag{13}$$

where we collected into Ψ all the less interesting contributions which are well under control (as the reader can check) assuming $z, z^\sharp, \xi$ have regularities $\alpha + 1, 2\alpha + 2, \alpha$.

The problematic term $X \circledcirc \xi$ here plays the role of the term (7) in the rough path approach. The paracontrolled Ansatz (12), the role of the Equation (9). If we assume that $X \circledcirc \xi \in \mathcal{C}^{2\alpha+1}$ (as scaling considerations and Equation (7) suggests) then Equation (13) is a well defined differential equation (non-local, with some low order paradifferential terms) which can be solved for $z^\sharp \in \mathcal{C}^{2\alpha+2}$. Technically, in order for (13) to be an equation for $z^\sharp \in \mathcal{C}^{2\alpha+2}$ we need to solve for z in (12) or to consider as unknown the system $(z, z^\sharp)$. Both approaches are possible provided small modifications are introduced in the considerations above. For more details Gubinelli, Imkeller, and Perkowski [2015] and Gubinelli and Perkowski [2015].

As a consequence we can identify the Itō–Lyons map Ψ as the map going from the *enhanced noise* $\Xi = (\xi, X \circledcirc \xi) \in \mathcal{C}^\alpha \times \mathcal{C}^{2\alpha+1}$ to the solution $z \in \mathcal{C}^{\alpha+1}$ via $Z = (z, z^\sharp) \in \mathcal{C}^{\alpha+1} \times \mathcal{C}^{2\alpha+2}$. As before this solution maps agrees with the solution of the ODE whenever ξ is smooth and can be used to control the limit $y_\varepsilon \to z$ provided we can prove that the enhanced noise $J(\eta_\varepsilon) := (\eta_\varepsilon, (\partial_t^{-1}\eta_\varepsilon) \circledcirc \eta_\varepsilon)$ converges in $\mathcal{C}^\alpha \times \mathcal{C}^{2\alpha+1}$ as $\varepsilon \to 0$ (recall that the assumption $3\alpha + 2 > 0$ is in force here).

We record these basic relations into the following diagram:

$$
\begin{array}{cccccccc}
 & \eta_\varepsilon & \overset{J}{\longleftrightarrow} & \Xi_\varepsilon & \overset{\Psi}{\longmapsto} & Y_\varepsilon = (y_\varepsilon, y_\varepsilon^\sharp) & \longmapsto & y_\varepsilon \\
\varepsilon \to 0 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \xi & \longleftarrow\!\!\shortmid & \Xi & \overset{\Psi}{\longmapsto} & Z = (z, z^\sharp) & \longmapsto & z \\
 & \Cap & & \Cap & & \Cap & & \Cap \\
 & \mathcal{C}^\alpha & & \mathcal{C}^\alpha \times \mathcal{C}^{2\alpha+1} & & \mathcal{C}^{\alpha+1} \times \mathcal{C}^{2\alpha+2} & & \mathcal{C}^{\alpha+1}
\end{array}
$$

The paracontrolled Ansatz transforms a problem of singular SPDEs into well–posed PDE problem featuring some paradifferential operators. The major drawback is that certain equations are out of reach for this technique, essentially because we understand quite poorly a systematic paradifferential development of generic non-linearities beyond the first order. Higher–order paralinearization has been investigated long ago by Chemin Chemin [1988a,b] and some higher–order commutators introduced in the work of Bailleul and Bernicot Bailleul and Bernicot [2016b,a] but the technical advantage over regularity structures tends to be less evident.

It should be remarked that the core of all these approaches (regularity structures, rough paths theory or paracontrolled distributions) lies in three basic steps:

a) Transform the original equation into a well-posed analytical problem either via lifting into regularity structures (that is constructing and manipulating local germs as in

Sect. 2.2) or performing a change of variable by removing some leading order paradifferential term (like in Sect. 2.4);

b) Analyze the resulting systems in terms of a finite family of basic non-linear functionals Ξ of the given data (which could be stochastic or not), their are called, according to the approach used, *enhanced noise* (in the paracontrolled approach) or *model* (in Hairer's regularity structures) or *rough path* (in Lyons' RPT theory);

c) Construct the associated solution (Itō–Lyons) map Ψ and determine the relevant topologies on the enhanced noise Ξ with respect to which Ψ has nice continuity properties.

These three steps provide the analytical backbone around which other considerations can be developed. For example, in problems related to scaling limits like those described by Equation (2), one is led to study the probabilistic convergence of lifts $\Xi_\varepsilon = J(\eta_\varepsilon)$ of smooth random fields η_ε to limiting enhanced noises Ξ in the appropriate topology. This convergence will carry on to solutions of Equation (2) via the continuous solution map Ψ.

2.5 Ambiguities. Even in the situation where the approximation η_ε converges towards a smooth object θ but only in a very weak topology (like $\mathcal{C}^\alpha$ in the setting described above, with $\alpha \in (-3/2, -1/2)$), it is *not true* that the solutions y_ε converge to the solution of the ODE

$$\dot{z} = f(z)\theta.$$

Indeed if we assume that $\Xi = \lim_\varepsilon \Xi_\varepsilon$ exists we should have $\Xi^1 = \theta$ but in general $\Theta = J(\theta) \neq \Xi$. Going back to the definition of the solution map we find out that if we let $\sigma = \Theta^{(2)} - \Xi^{(2)}$ we have $y_\varepsilon \to z = \Psi(\Xi)$ where z satisfies

$$\dot{z} = f(z)\theta + f_2(z)\sigma.$$

A correction term appears in the formulation of the limiting problem, a relic of the limiting procedure. This phenomenon has been studied in stochastic analysis McShane [1972] and Sussmann [1991], in rough path theory for ODEs P. Friz and Oberhauser [2009] and P. Friz, Gassiat, and T. Lyons [2015] but also in relation to some SPDEs Hairer and Maas [2012]. Under certain conditions one can have $\theta = 0$ and $\sigma \neq 0$. In this case the final result is a form of stochastic homogenisation and, from the point of view of the techniques we discuss here, has been considered for certain SPDEs in Hairer, Pardoux, and Piatnitski [2013].

2.6 Other approaches. Other possible frameworks for the analysis of singular SPDEs have been developed recently. Bailleul and Bernicot Bailleul and Bernicot [2016a,b] introduced a semigroup approach to paraproducts with the aim of extending the paracontrolled calculus to manifolds via invariant constructions. They also investigated higher order versions of the paracontrolled calculus as we already remarked. Kupiainen Kupiainen [2016] and Kupiainen and Marcozzi [2017] introduced a renormalization group approach where the solution is described at every scale by an *effective* equation which do not possess any singularity. The main task of the analysis is to construct these effective description satisfying recursive equations. Finally Otto and Weber Otto and Weber [2016] use a semigroup approach to decouple the singular products and identify a suitable family of stochastic objects playing the role of *enhanced noise*. In their approach the necessary Schauder estimates are derived via an extension of the Krylov–Safanov kernel–free method and this allows them to treat certain classes of quasi-linear equations. Following their pioneering work Bailleul, Debussche and Hofmanová Bailleul, Debussche, and Hofmanová [2016] used the paradifferential Ansatz to solve quasi-linear equations. A key idea of Otto and Weber approach is the introduction of a parametric family of enhanced noises which take into account the modulation of the parabolic regularisation effects given by the quasi-linear nature of the equation. Transporting this idea in the paracontrolled framework Furlan and the author Furlan and Gubinelli [2016] introduced a non–linear paraproduct and related operator which allows to cover the results of Otto and Weber in the framework of paracontrolled distributions.

3 Weak universality

One motivation for the study of SSPDE is the phenomenon of *weak universality*. This term refers to the fact that the large scale behaviour of certain classes of random PDEs or other Markovian random fields with small non–linearities or small noise depends on very few details of the exact model under consideration and that it can be described by singular SPDEs. I will illustrate this phenomenon describing recent results of Hairer and Quastel Hairer and J. Quastel [2015] on the convergence of a large class of $1+1$ interface growth models to the Kardar–Parisi–Zhang (KPZ) equation.

3.1 The Hairer–Quastel universality result. Consider a continuous growth model Halpin-Healy and Zhang [1995] given by an height function $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ solving the equation

$$\partial_t h = \partial_x^2 h + \sigma F(\partial_x h) + \delta\eta, \tag{14}$$

where ∂_t, ∂_x denote time and space derivatives, σ, δ are parameters, η is a smooth space–time Gaussian process and F an even polynomial. The various contributions in the r.h.s accounts for various phenomena: smoothing of the surface ($\partial_x^2 h$) (e.g. due to thermal fluctuations), lateral growth mechanism ($F(\partial_x h)$) and microscopic fluctuations in the growth rate (η). There are two interesting regimes in this equation: according to whether the non-linearity dominates or the noise dominates the behaviour at scales of order 1:

a) *Intermediate disorder regime* ($\sigma = 1$ and $\delta \ll 1$) : the noise is small. In this case we let $\varepsilon = \delta^2$ and consider the rescaled field $\tilde{h}_\varepsilon(t,x) = h(t/\varepsilon^2, x/\varepsilon)$ which satisfies

$$\partial_t \tilde{h}_\varepsilon = \partial_x^2 \tilde{h}_\varepsilon + \varepsilon^{-2} F(\varepsilon \partial_x \tilde{h}_\varepsilon) + \eta_\varepsilon, \tag{15}$$

where $\eta_\varepsilon(t,x) = \varepsilon^{-3/2}\eta(t/\varepsilon^2, x/\varepsilon)$. Formal Taylor expansion of the non-linear term gives

$$\partial_t \tilde{h}_\varepsilon = \partial_x^2 \tilde{h}_\varepsilon + \varepsilon^{-2} F(0) + \frac{1}{2} F''(0)(\partial_x \tilde{h}_\varepsilon)^2 + \mathcal{O}(\varepsilon^2 (\partial_x \tilde{h}_\varepsilon)^4) + \eta_\varepsilon; \tag{16}$$

b) *Weak asymmetry* ($\delta = 1$ and $\sigma \ll 1$) : the non–linearity is small. We let $\varepsilon = \sigma^2$ and consider $\tilde{h}_\varepsilon(t,x) = \varepsilon^{1/2} h(t/\varepsilon^2, x/\varepsilon)$ which satisfies

$$\partial_t \tilde{h}_\varepsilon = \partial_x^2 \tilde{h}_\varepsilon + \varepsilon^{-1} F(\varepsilon^{1/2} \partial_x \tilde{h}_\varepsilon) + \eta_\varepsilon, \tag{17}$$

where η_ε is define as in the intermediate disorder regime and Taylor expansion gives now

$$\partial_t \tilde{h}_\varepsilon = \partial_x^2 \tilde{h}_\varepsilon + \varepsilon^{-1} F(0) + \frac{1}{2} F''(0)(\partial_x \tilde{h}_\varepsilon)^2 + \mathcal{O}(\varepsilon (\partial_x \tilde{h}_\varepsilon)^4) + \eta_\varepsilon. \tag{18}$$

The parameter ε has been chosen as a measure of the microscopic spatial scale. The random field η_ε converges (under appropriate conditions on the covariance of η) to the space–time white noise $\xi = \xi(t,x)$ with covariance

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y).$$

In both regimes and as $\varepsilon \to 0$, one would like to argue formally that there are constant c_ε, λ such that the random field $h_\varepsilon(t,x) = \tilde{h}_\varepsilon(t,x) - c_\varepsilon t$ converges to the solution of the Kardar–Parisi–Zhang Kardar, Parisi, and Zhang [1986] equation

$$\partial_t h = \partial_x^2 h + \frac{\lambda}{2}(\partial_x h)^2 + \xi. \tag{19}$$

Unfortunately these heuristic considerations do not stand up to further scrutiny. First the Taylor approximations turn out to be partially justified in the intermediate disorder regime but not in the weak asymmetric one, second, and more importantly the KPZ equation is strongly ill posed since the presence of the space time white noise imposes a very weak regularity on h which makes the nonlinear term not well defined.

3.2 KPZ universality. These problems has been open for very long time since the original work of Kardar, Parisi and Zhang Kardar, Parisi, and Zhang [ibid.] in the '80 where they introduced the equation to describe the universality class of one dimensional growth models. Their hypothesis was that a large class of models featuring the basic mechanisms at work in Equation (14) must show characteristic *universal* large scale properties. This conjecture is mathematically quite open, even if there have been recent important progress to prove rigorously the existence of this *KPZ universality class*. For an introduction to the mathematical literature the reader can consult the contribution of Quastel to the 2014 ICM J. D. Quastel [2014] or the lecture notes of Corwin Corwin [2012].

The universal object behind this universality class, the *KPZ fixpoint* has been described recently by Matetski, Quastel and Remenik Matetski, J. Quastel, and Remenik [2016] via exact formulas for its finite dimensional marginals. The KPZ Equation (19) itself does *not* corresponds to this fixpoint. Kardar Parisi and Zhang introduced their equation as one of many possible models whose large scale properties were universal. In this respect the KPZ fixpoint is the large scale limit of the KPZ equation. Some rigorous results are available which partially confirm this conjecture Spohn [2011], Amir, Corwin, and J. Quastel [2011], Balázs, J. Quastel, and Seppäläinen [2011], and Borodin and Corwin [2014]. The large scale limit of the KPZ equation should correspond to a vanishing viscosity and vanishing noise limit, in the precise form

$$\partial_t H_\rho = \rho \partial_x^2 H_\rho + \frac{1}{2}(\partial_x H_\rho)^2 + \rho^{1/2}\xi.$$

where $\rho \to 0$ J. Quastel [2012]. In this regime the function H_ρ should converge to the random field $\mathcal{H}$ described in Matetski, J. Quastel, and Remenik [2016].

Weak universality of KPZ stands for the fact that the KPZ Equation (19) itself can be understood as a common limit to many models under the more restrictive conditions we discussed before, namely weak asymmetry or intermediate disorder. The first mathematical result in this direction is due to Bertini and Giacomin Bertini and Giacomin [1997] in 1997. They showed that the integrated density field h_ε of a weakly asymmetric version of the exclusion process on $\mathbb{Z}$ converges upon rescaling and appropriate recentering the "solution" of the KPZ equation. As we already observed the KPZ equation is a singular SPDE which is not classically well-posed. What Bertini and Giacomin really did was to prove the convergence of the field $\phi_\varepsilon = \exp(h_\varepsilon)$ to the unique positive solution ϕ of the stochastic heat equation (SHE)

$$\partial_t \phi = \partial_x^2 \phi + \phi \xi \qquad (20)$$

where the product $\phi\xi$ is understood via Itō stochastic calculus. The SHE is a standard SPDE which can be solved via standard tools (see e.g. the classic lecture notes of Walsh Walsh [1986] for the solution theory in bounded domain). This exponential transformation to a linear PDE is called Hopf–Cole transformation and the convergence results of Bertini and Giacomin justifies the fact that the *correct* notion of solution to the KPZ Equation (19) should have the property that $\phi = \exp(h)$ satisfies the SHE, this is called the Hopf–Cole solution.

3.3 A notion of solution for KPZ. After the Bertini and Giacomin convergence result there were available a candidate solution (the Hopf–Cole solution) but not an equation yet! The situation remained unclear until Hairer Hairer [2013] used the ideas and the tools of rough path theory to formulate the KPZ equation as a well–posed SPDE. This first work was instrumental to the development of the far reaching theory of regularity structures and inspired the construction of alternative theories, like that of paracontrolled distributions.

I will sketch the solution theory for the KPZ equation in terms of paracontrolled distributions as described in Gubinelli and Perkowski [2017b]. We proceed like in the analysis of the ODE by transforming the problem in order to obtain a formulation which is amenable to standard techniques. In this respect we consider the model equation

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \frac{\lambda}{2}(\partial_x h_\varepsilon)^2 + \xi_\varepsilon,$$

where ξ_ε is a smooth approximation of the white noise and for technical reasons we consider it on the periodic domain $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. This equation has smooth local solution (fixing some nice initial condition), however as $\varepsilon \to 0$ we loose all the useful estimates since $\xi_\varepsilon \to \xi$ only in as a space–time distribution (with parabolic regularity $-3/2$).

We split the unknown h_ε into four components as $h_\varepsilon = X_\varepsilon + Y_\varepsilon + Z_\varepsilon + H_\varepsilon$ and let $X_\varepsilon, Y_\varepsilon, Z_\varepsilon$ be solutions of

$$\mathcal{L} X_\varepsilon = \xi_\varepsilon,\ \mathcal{L} Y_\varepsilon = \frac{\lambda}{2}(\partial_x X_\varepsilon)^2,\ \mathcal{L} Z_\varepsilon = \frac{\lambda}{2}(\partial_x Y_\varepsilon)^2 + \lambda \partial_x X_\varepsilon \partial_x Y_\varepsilon + \lambda \partial_x (X_\varepsilon + Y_\varepsilon)\partial_x Z_\varepsilon$$

where $\mathcal{L} = \partial_t - \Delta$. Then the function H_ε solves

$$\mathcal{L} H_\varepsilon = \frac{\lambda}{2}(\partial_x H_\varepsilon)^2 + \frac{\lambda}{2}(\partial_x Z_\varepsilon)^2 + \lambda \partial_x (X_\varepsilon + Y_\varepsilon + Z_\varepsilon)\partial_x H_\varepsilon. \tag{21}$$

This transformation isolates the most singular contributions in the equation into the functions $X_\varepsilon, Y_\varepsilon, Z_\varepsilon$ which depends in an explicit fashion on the underlying noise ξ. The regularisation properties of the heat semigroup allows to prove that X_ε is uniformly in $C_t \mathcal{C}^{1/2-\kappa}$. In the following κ denotes some arbitrarily small positive constant and $C_t \mathcal{C}^\alpha$ denotes the space of continuous functions of time with values in $\mathcal{C}^\alpha$. It can also be shown that $(\partial_x X_\varepsilon)^2 \to +\infty$ as $\varepsilon \to 0$ (almost surely) and that there exists a constant $C_\varepsilon \to +\infty$ such that $\llbracket (\partial_x X_\varepsilon)^2 \rrbracket = (\partial_x X_\varepsilon)^2 - C_\varepsilon$ converges to a well defined random field in $C_t \mathcal{C}^{-1-\kappa}$. Here the notation $\llbracket (\partial_x X_\varepsilon)^2 \rrbracket$ stands for the Wick product Janson [1997].

This is an hint that our original formulation was not quite correct. In order to hope for some well defined limit, we should *renormalize* the equation and consider instead

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \frac{\lambda}{2}[(\partial_x h_\varepsilon)^2 - C_\varepsilon] + \xi_\varepsilon,$$

and accordingly redefine Y_ε as the solution to the equation

$$\mathcal{L} Y_\varepsilon = \frac{\lambda}{2}[(\partial_x X_\varepsilon)^2 - C_\varepsilon] = \frac{\lambda}{2}[\![(\partial_x X_\varepsilon)^2]\!].$$

After these changes one can show that Y_ε converges in $C_t\mathcal{C}^{1-\kappa}$. Similar problems arise with the non-linear terms in the definition of Z_ε. A priori other renormalizations are expected whenever we try to multiply terms whose sums of regularities is not strictly positive. In the following I will assume that these renormalization have been performed by a modification in the equation for Z_ε in such a way that Z_ε has a limit in $C_t\mathcal{C}^{3/2-\kappa}$ and that the Equation (21) maintains the same form. The reader interested in the details of the precise renormalization procedure needed here can refer to the original paper of Hairer Hairer [2013] or to Hairer [2014] and Gubinelli and Perkowski [2017b]. At this point it seems that Equation (21) could be used to get uniform estimates for H_ε in $C_t\mathcal{C}^{3/2-\kappa}$, however a crucial difficulty still remains, due to the product $\partial_x X_\varepsilon \partial_x H_\varepsilon$. The sum of regularities is barely negative: $-1/2-\kappa$ for the factor $\partial_x X_\varepsilon$ and $1/2-\kappa$ for $\partial_x H_\varepsilon$. Note that

$$\mathcal{L} H_\varepsilon = \lambda \partial_x X_\varepsilon \circledcirc\!\!\!\!\!<\; \partial_x H_\varepsilon + \lambda \partial_x X_\varepsilon \circledcirc \partial_x H_\varepsilon + \cdots$$

where from now on the dots $(\cdots)$ means terms of higher regularity which do not pose problems. Taking into account this expansion we introduce the paracontrolled Ansatz

$$H_\varepsilon = \lambda Q_\varepsilon \circledcirc\!\!\!\!\!<\; \partial_x H_\varepsilon + H_\varepsilon^\sharp,$$

where $\mathcal{L} Q_\varepsilon = \partial_x X_\varepsilon$. Using the approach described in Section 2.4 we can verify that $H_\varepsilon^\sharp$ solves a parabolic equation of the form

$$\mathcal{L} H_\varepsilon^\sharp = \lambda^2(\partial_x H_\varepsilon)(\partial_x X_\varepsilon \circledcirc \partial_x Q_\varepsilon) + \cdots$$

for which well–posedness holds provided we give an *off line* definition of $\partial_x X_\varepsilon \circledcirc \partial_x Q_\varepsilon$ as usual by now. At the end of the analysis one obtain, locally in time, a continuous solution map

$$\Psi : \Xi_\varepsilon := (X_\varepsilon, Y_\varepsilon, Z_\varepsilon, \partial_x X_\varepsilon \circledcirc \partial_x Q_\varepsilon) \mapsto (H_\varepsilon, H_\varepsilon^\sharp)$$

which allows to pass to the limit for $(h_\varepsilon)_\varepsilon$ as $\varepsilon \to 0$ and obtain a random field $h \in C_t\mathcal{C}^{1/2-\kappa}$ provided we show that

$$\Xi_\varepsilon = (X_\varepsilon, Y_\varepsilon, Z_\varepsilon, \partial_x X_\varepsilon \circledcirc \partial_x Q_\varepsilon) \to \Xi,$$

in the appropriate topology. The limit random field h satisfies an equation which formally can be written as (recall that $C_\varepsilon \to +\infty$!)

$$\partial_t h = \partial_x^2 h + \frac{\lambda}{2}[(\partial_x h)^2 - \infty] + \xi. \tag{22}$$

By itself this equation is purely formal. We have to resort to a description of h based on our analysis above to make it precise. We know that

$$h = X + Y + Z + \lambda Q \circledcirc \partial_x H + H^\sharp \tag{23}$$

where $\Psi(\Xi) = (H, H^\sharp)$ and then we have

$$[(\partial_x h)^2 - \infty] = \lim_\varepsilon [(\partial_x h_\varepsilon)^2 - C_\varepsilon] = [\![(\partial_x X)^2]\!] + 2\partial_x X \partial_x (Y+Z) + 2(\partial_x Y)^2 + 2(\partial_x Z)^2$$

$$+ 2\partial_x (X + Y + Z)\partial_x(\lambda Q \circledcirc \partial_x H + H^\sharp) + (\lambda \partial_x (Q \circledcirc \partial_x H) + \partial_x H^\sharp)^2,$$

where all the objects in the r.h.s. are well defined. In particular, from the limiting procedure we see that we can understand the product $\partial_x X \partial_x (Q \circledcirc \partial_x H)$ via the commutator lemma as

$$\begin{aligned}\partial_x X \partial_x (Q \circledcirc \partial_x H) &= \partial_x X (\partial_x Q \circledcirc \partial_x H) + \cdots \\ &= \partial_x X \circledcirc (\partial_x Q \circledcirc \partial_x H) + \partial_x H (\partial_x X \circledcirc \partial_x Q) + \cdots .\end{aligned}$$

This representation gives a well defined meaning to the r.h.s. of the Equation (22) for all the functions of the form (23) for any choice of $(H, H^\sharp) \in C_t \mathcal{C}^{1/2-\kappa} \times C_t \mathcal{C}^{3/2-\kappa}$, not necessarily satisfying the equation. However remark that this definition of $[(\partial_x h)^2 - \infty]$ depends heavily on the enhancement Ξ which, as we have already seen above, cannot be in general determined by the noise ξ but carries information about the limiting procedure.

3.4 Convergence to KPZ for the growth model. We now have a description for a candidate limit to the Equation (15) or (17). We will stick to the weakly asymmetric regime (17) since the intermediate noise regime (15) can be treated with a similar but easier approach. The result of Hairer and Quastel is the following (Theorem 1.2 in Hairer and J. Quastel [2015]).

Theorem 3.1. *Let F be an even polinomial and $(\tilde h_\varepsilon)_\varepsilon$ a sequence of solutions of*

$$\partial_t \tilde h_\varepsilon = \partial_x^2 \tilde h_\varepsilon + \varepsilon^{-1} F(\varepsilon^{1/2} \partial_x \tilde h_\varepsilon) + \xi^{(\varepsilon)}.$$

where $\xi^{(\varepsilon)}$ is a regularization of the space–time white noise via a nice smoothing kernel ρ at scale ε, namely $\xi^{(\varepsilon)} = \rho_\varepsilon * \xi$ with $\rho_\varepsilon(t,x) = \varepsilon^{-3/2}\rho(t/\varepsilon^2, x/\varepsilon)$. Let C_0 be the constant

$$C_0 = \int\int (\partial_x P * \rho)(t,x) \mathrm{d}t \, \mathrm{d}x \tag{24}$$

where P is the heat kernel on $\mathbb{T}$, moreover let μ_{C_0} be the Gaussian measure on $\mathbb{R}$ with variance C_0 and define the constants

$$\lambda = \int F''(x)\mu_{C_0}(\mathrm{d}x), \qquad v = \int F(x)\mu_{C_0}(\mathrm{d}x). \tag{25}$$

Then there exists a further constant c such that random field

$$h_\varepsilon(t,x) = \tilde{h}_\varepsilon(t,x) - (v/\varepsilon + c)t \tag{26}$$

converges in law to the Hopf–Cole solution of the KPZ equation.

Let us remark that the Hopf–Cole transformation which was the key tool in Bertini and Giacomin analysis of the weakly asymmetric exclusion process is not applicable here, despite the fact that the theorem can be formulated in terms of Hopf–Cole solution. Indeed one can try to perform the change of variables $\phi_\varepsilon = \exp(\tilde{h}_\varepsilon)$ but the resulting equation for ϕ_ε is as difficult as the original equation.

In the rest of this section we give some ideas on how the estimates needed to establish Theorem 3.1 can be obtained in the paracontrolled setting described above. By performing the transformation (26) we see that h_ε is a solution to

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \varepsilon^{-1}[F(\varepsilon^{1/2}\partial_x h_\varepsilon) - v] + \eta_\varepsilon.$$

The naive approach of expanding the non-linearity around 0 does not really work since soon one realizes that there are no useful estimates for $\partial_x \tilde{h}_\varepsilon$ in L^∞. Even for the linear equation

$$\partial_t \tilde{X}_\varepsilon = \partial_x^2 \tilde{X}_\varepsilon + \eta_\varepsilon.$$

the best one can have (from stochastic considerations) is $\|\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-\kappa}$ for some arbitrarily small κ. We can however mimic the paracontrolled decomposition and let $h_\varepsilon = \tilde{X}_\varepsilon + \tilde{Y}_\varepsilon + \tilde{Z}_\varepsilon + \tilde{H}_\varepsilon$ where $\tilde{Y}_\varepsilon, \tilde{Z}_\varepsilon, \tilde{H}_\varepsilon$ will be fixed below. Now we have the possibility to expand the non-linearity around the solution $\tilde{X}_\varepsilon$ of the linear equation, giving

$$\varepsilon^{-1}F(\varepsilon^{1/2}\partial_x h_\varepsilon) = \varepsilon^{-1}[F(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon) - v] + \varepsilon^{-1/2}F'(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon)\partial_x(\tilde{Y}_\varepsilon + \tilde{Z}_\varepsilon + \tilde{H}_\varepsilon)$$

$$+\frac{1}{2}F''(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon)[\partial_x(\tilde{Y}_\varepsilon + \tilde{Z}_\varepsilon + \tilde{H}_\varepsilon)]^2 + \mathcal{O}(\varepsilon^{1/2}F'''(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon)[\partial_x(\tilde{Y}_\varepsilon + \tilde{Z}_\varepsilon + \tilde{H}_\varepsilon)]^3)$$

The terms $\tilde{Y}_\varepsilon + \tilde{Z}_\varepsilon + \tilde{H}_\varepsilon$ will behave better than $\tilde{X}_\varepsilon$ and the Taylor remainder is now negligible in the limit thanks to the factor $\varepsilon^{1/2}$. The other terms can be cast in a form very similar to that used for the KPZ equation by letting

$$\Lambda_\varepsilon = F''(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon), \quad \partial_x \hat{X}_\varepsilon = \varepsilon^{-1/2}F'(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon), \quad \mathcal{L}\tilde{Y}_\varepsilon = \varepsilon^{-1}[F(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon) - v],$$

$$\mathcal{L}\tilde{Z}_\varepsilon = \partial_x \hat{X}_\varepsilon \partial_x(\tilde{Y}_\varepsilon + \tilde{Z}_\varepsilon) + \frac{1}{2}\Lambda_\varepsilon(\partial_x \tilde{Y}_\varepsilon)^2 + \frac{1}{2}\Lambda_\varepsilon \partial_x \tilde{Y}_\varepsilon \partial_x \tilde{Z}_\varepsilon.$$

With these definition the equation for $\tilde{H}_\varepsilon$ becomes

$$\mathcal{L}\tilde{H}_\varepsilon = \left[\partial_x \hat{X}_\varepsilon + \frac{1}{2}\Lambda_\varepsilon(\partial_x \tilde{Y}_\varepsilon + \partial_x \tilde{Z}_\varepsilon)\right]\partial_x \tilde{H}_\varepsilon + \frac{1}{2}\Lambda_\varepsilon(\partial_x \tilde{H}_\varepsilon)^2 + \frac{1}{2}\Lambda_\varepsilon(\partial_x \tilde{Z}_\varepsilon)^2$$
$$+\mathcal{O}(\varepsilon^{1/2}F'''(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon)[\partial_x(\tilde{Y}_\varepsilon + \tilde{Z}_\varepsilon + \tilde{H}_\varepsilon)]^3).$$

Comparing this equation with Equation (21) one can argue that the convergence can be proven if we are able to show that

$$(\Lambda_\varepsilon, \tilde{X}_\varepsilon, \partial_x \hat{X}_\varepsilon, \tilde{Y}_\varepsilon, \tilde{Z}_\varepsilon) \to (\lambda, X, \lambda\partial_x X, Y, Z)$$

and some other relations coming from the paracontrolled Ansatz, needed to control the product $\partial_x \hat{X}_\varepsilon \partial_x \tilde{H}_\varepsilon$. All these conditions could be in principle be established via a tour de force of intricate computations involving Gaussian random fields. See Furlan and Gubinelli [2017] for similar estimate for weak universality in reaction diffusion equations via paracontrolled analysis or Hairer and Quastel Hairer and J. Quastel [2015] for the estimation of the stochastic terms in regularity structures.

Let us highlight the role of the constants λ, v defined in (25). The constant λ is the limit of the random field Λ_ε, indeed $\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon(t,x)$ is a Gaussian random variable whose asymptotic variance do not go to zero and converges to C_0 defined in Equation (24). From this is natural to deduce that the average of $\Lambda_\varepsilon(t,x)$ converges to

$$\mathbb{E}\Lambda_\varepsilon(t,x) = \mathbb{E}F''(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon(t,x)) \to \int F''(x)\mu_{C_0}(\mathrm{d}x) = \lambda.$$

Fluctuations around this average go to zero in $C_t\mathcal{C}^{-\kappa}$. As for v, its role is to center the random field $\varepsilon^{-1}F(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon)$ so that its average is zero. Stochastic analysis then shows that

$$\varepsilon^{-1}[F(\varepsilon^{1/2}\partial_x \tilde{X}_\varepsilon) - v] \to \frac{\lambda}{2}(\partial_x X)$$

in $C_t\mathcal{C}^{-1-\kappa}$ as $\varepsilon \to 0$.

3.5 Other weak universality results. Weak universality results in the context of KPZ equation have been proven using a variety of techniques. Discrete versions of the Hopf–Cole transformations allow to tackle the limit from the point of view of the SHE and prove weak universality for certain classes of weakly asymmetric exclusion processes Bertini and Giacomin [1997], Amir, Corwin, and J. Quastel [2011], Borodin and Corwin [2014], and Corwin and Tsai [2017] and for the free energy of directed random polymers in the intermediate disorder regime Alberts, Khanin, and J. Quastel [2010].

The KPZ equation strictly speaking does not have an invariant probability measure but it has an invariant measure given by the distribution of a two-sided geometric Brownian motion with a height shift given by Lebesgue measure Funaki and J. Quastel [2015]. Based on this invariant measure and on a stationary martingale problem formulation, Gonçalves and Jara Gonçalves and Jara [2014] introduced another notion of solution which they called *energy solution of KPZ*. This allowed to prove convergence to energy solutions for a large class of particle system for which the Hopf–Cole strategy was unavailable Gonçalves, Jara, and Sethuraman [2015], Blondel, Gonçalves, and Simon [2016], and Diehl, Gubinelli, and Perkowski [2017]. In Gubinelli and Jara [2013] this notion was refined and in Gubinelli and Perkowski [2017a] it has been shown to identify a unique solution which essentially coincide with the Hopf–Cole solution.

Weak universality has been investigated also in the context of reaction diffusion equations in $d = 2, 3$ dimensions in Shen and Weber [2016], Mourrat and Weber [2017a], Hairer and Xu [2016], Shen and Xu [2017], and Furlan and Gubinelli [2017], for $d = 2$ diffusion in random environment in Chouk, Gairing, and Perkowski [2017] and for the non-linear wave equation with additive noise in $d = 2$ dimensions Gubinelli, Koch, and Oh [2017].

4 Stochastic quantisation in three dimensions

The dynamical Φ^4_3 model has been the first serious application of regularity structures Hairer [2014]. This model corresponds formally to the SPDE

$$\partial_t \varphi - \Delta\varphi + \varphi^3 - \infty\varphi = \xi \tag{27}$$

in $\mathbb{T}^3$, where ξ is space–time white noise and Δ the Laplacian on $\mathbb{T}^3$. This equation is also called stochastic quantisation equation (SQE) for a $3d$ scalar field with quadratic interaction. It can be understood as the weak universal limit of certain reaction diffusion equations (see Sect. 3.5) or as a stochastic dynamics which is reversible with respect to the Φ^4_3 Euclidean quantum field theory. This latter object can be described formally as the probability measure μ given by

$$\mu(\mathrm{d}\phi) = Z^{-1} \exp\left[-\int_{\mathbb{T}^3} (\phi(x)^4 - \infty\phi(x)^2)\mathrm{d}x\right] \mu_0(\mathrm{d}\phi)$$

where μ_0 is the Gaussian measure on $\mathcal{S}'(\mathbb{T}^3)$ with covariance $(1-\Delta)^{-1}$. This formulation is formal since μ is not absolutely continuous wrt. μ_0 and has to be understood rigorously via a limiting procedure involving a regularised exponent in the exponential (the interaction). The construction of this measure has been one of the major successes of constructive QFT Glimm [1968], Glimm and Jaffe [1973], and Feldman [1974] and is considered one serious toy model to test constructive renormalization procedures ever since Rivasseau [1991], Benfatto, Cassandro, Gallavotti, Nicoló, Olivieri, Presutti, and Scacciatelli [1980, 1978], and Gallavotti [1985]. One of the simplest construction of this measure (still quite non-trivial) is given in Brydges, Fröhlich, and Sokal [1983].

The dynamical model is inspired by the idea of *stochastic quantization* introduced by Parisi and Wu Parisi and Wu [1981]: *define* the measure μ by constructing a stochastic dynamics evolving in a fictious additional time variable.

Stochastic quantisation has various unexpected advantages. Physically it provides a way to introduce a regularisation without breaking some fundamental symmetries of the model being studied (for example the gauge symmetry Jona-Lasinio and Parrinello [1988], Bertini, Jona-Lasinio, and Parrinello [1993], and Jona-Lasinio and Parrinello [1990]). Mathematically it provides a solid ground where to attempt a controlled perturbation theory (as we will see below). Indeed in the equation the random field ξ is exactly Gaussian while under the measure μ one can identify such "free" fields only resorting to renormalization group ideas Gallavotti [1985].

Stochastic quantisation has been rigorously studied in two dimensions by Jona–Lasinio and Mitter Jona-Lasinio and Mitter [1985], by Albeverio and Rœckner Albeverio and Röckner [1991] and by Da Prato and Debussche Da Prato and Debussche [2003]. In three dimensions solutions φ are distributions living in $C_t\mathscr{C}^{-1/2-\kappa}(\mathbb{T}^3)$. The definition of the non-linear terms is highly nontrivial and, unlike the KPZ equation, cannot be attacked with RP techniques. Hairer's solution of this problem (locally in time) showed the power and fexibility of these new methods. A bit later Catellier and Chouk Catellier and Chouk [2013] described an equivalent solution theory for (27) using a paracontrolled Ansatz. See also Mourrat, Weber, and Xu [2016] for a simplified approach to the construction of the stochastic terms. Kupiainen Kupiainen and Marcozzi [2017] showed that Wilsonian renormalization group can be adapted to deal with stochastic PDEs and provided yet another solution theory.

Important results on the SQE are those of Mourrat and Weber Mourrat and Weber [2016] which were able to extend the solution theory (using paracontrolled distributions) globally in time (but still on the torus $\mathbb{T}^3$) with a tour de force of estimates. They were able to leverage the strong drift given by the cubic term to show that solutions of the SQE "comes down from infinity" (that is, they forget the initial condition) in finite time. This opens the way to a rigorous implementation of *real* stochastic quatisation by attempting to prove that whatever the initial condition, solutions of (27) converges to μ as time goes to infinity. The problem is now quite well understood in the $d = 2$ case where Mourrat and Weber proved global space–time existence (i.e. in the full plane) for the dynamics Mourrat and Weber [2017b] and where we now have a quite good understanding of the spectral gap and the exponential convergence to equilibrium of the dynamics Tsatsoulis and Weber [2016]. The $d = 3$ case is less understood, due to the more intricate solution theory.

However the work of Mourrat and Weber shows that on the microscopic scale the dynamics is essentially dominated by perturbation theory around the linear equation and that the relevant non–linear features can be taken into account by the large scales. This decomposition is analogous to the approach put in place in constructive QFT to handle the "large field problem" Benfatto, Cassandro, Gallavotti, Nicoló, Olivieri, Presutti, and Scacciatelli [1980]. Dirichlet form description of the SQE has been investigated in $d = 2, 3$ R. Zhu and X. Zhu [2017] and Röckner, R. Zhu, and X. Zhu [2017a,b].

5 Other results

Many other results have been obtained in the last few year for other type of SSPDEs. In this section I will list some of the more interesting.

- Systems of KPZ–like equations have been studied by Funaki and Hoshino Funaki and Hoshino [2016].

- The dynamic version of the Sine–Gordon model in $d = 2$ has been studied by Hairer and Shen Hairer and Shen [2016]. In this model the regularity of the stochastic terms depend on the value of a parameter and singularities have relation with the phase transition of the $2d$ Coulomb gas.

- The techniques introduced to handle SSPDE can also be used to study unbounded operators which are formally not well defined. Allez and Chouk Allez and Chouk [2015] studied the Anderson Hamiltonian in $d = 2$, i.e. the unbounded operator on $L^2(\mathbb{T}^2)$ given by $H = -\Delta + \xi$ where ξ is a white noise in $\mathbb{T}^2$. They observe that the domain of this operator can be described quite effectively via a paracontrolled Ansatz. Cannizzaro and Chouk Cannizzaro and Chouk [2015] introduced a singular martingale problem to describe the law of a diffusion with a random singular drift. The generator of the martingale problem is a formal object which has to be understood via paracontrolled calculus.

- A complex Ginzburg–Landau model has been studied by Hoshino Hoshino, Inahama, and Naganuma [2017] and Hoshino [2017] using the ideas of Mourrat and Weber to obtain global in time solutions.

- Non-linear dispersive and hyperbolic singular SPDEs have been studied by Debussche and Weber Debussche and Weber [2016] on the torus $\mathbb{T}^2$ and then extended to the full space Debussche and Martin [2017] by Debussche and Martin. A nonlinear hyperbolic wave equation in $\mathbb{T}^2$ has been studied by Oh, Koch and the author in Gubinelli, Koch, and Oh [2017].

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