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SHARP SPHERE PACKINGS

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Abstract

In this talk we will speak about recent progress on the sphere packing problem. The packing problem can be formulated for a wide class of metric spaces equipped with a measure. An interesting feature of this optimization problem is that a slight change of parameters (such as the dimension of the space or radius of the spheres) can dramatically change the properties of optimal configurations. We will focus on those cases when the solution of the packing problem is particularly simple. Namely, we say that a packing problem is sharp if its density attains the so-called linear programming bound. Several such configurations have been known for a long time and we have recently proved that the E_8 lattice sphere packing in $\mathbb{R}^8$ and the Leech lattice packing in $\mathbb{R}^{24}$ are sharp. Moreover, we will discuss common unusual properties of shared by such configurations and outline possible applications to Fourier analysis.

1 Introduction

The classical sphere packing problem asks for the densest possible configuration of non-overlapping equal balls in the three dimensional Euclidean space. This natural and even naive question remained open for several centuries and has driven a lot of research in geometry, combinatorics and optimization. The complete proof of the sphere packing problem was given by T. Hales in 1998 Hales [2005].

A similar question can be asked for Euclidean spaces of dimensions other then three or for spaces with other geometries, such as a sphere, a projective space, or the Hamming space. The packing problem is not only an exciting mathematical puzzle, it also plays a role in computer science and signal processing as a mathematical model of the error correcting codes.

In this paper we will focus on the upper bounds for the sphere packing densities. There exist different methods for proving such bounds. One conceptually simple and still rather

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powerful approach is the linear programming. We are particularly interested in those packing problems, which can be completely solved by this method. We will call such arrangements of balls the *sharp packings*.

The sharp packings have many interesting properties. In particular, the distribution of pairwise distances between the centers of sharply packed spheres gives rise to summation and interpolation formulas. In the last section of this paper we will discuss a new interpolation formula for the Schwartz functions on the real line.

2 Linear programming bounds for sphere packings in metric spaces

Let (M, dist) be a metric space equipped with a measure μ. For $x \in M$ and $r > 0$ we denote by $B(x,r)$ the open ball with center x and radius r. Let X be a discrete subset of M such that $\mathrm{dist}(x,y) \geq 2r$ for any distinct $x, y \in X$. Then the set $\mathcal{P} := \cup_{x\in X} B(x,r)$ is a *sphere packing* in M. We define the *density* of $\mathcal{P}$ as

$$\Delta_{\mathcal{P}} := \sup_{x_0\in M} \limsup_{R\to\infty} \frac{\mu(\mathcal{P}\cap B(x_0,R))}{\mu(B(x_0,R))}.$$

Our goal is to search for densest possible configurations and to prove upper bounds on the packing density.

The *linear programming* is a powerful and simple method to prove upper bounds for the packing problems. This technique was successfully applied to obtain upper bounds in a wide range of discrete optimization problems such as error-correcting codes Delsarte [1972], equal weight quadrature formulas Delsarte, Goethals, and Seidel [1977], and spherical codes Kabatiansky and Levenshtein [1978] and Pfender and Ziegler [2004]. In this section we explain the idea behind this method, consider several examples, and discuss the limitations of this approach.

The essence of the linear programming method is the replacement of a complicated geometrical optimization problem by a simpler convex optimization problem.

We say that a function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is *geometrically positive* (with respect to a metric space M) if

$$\sum_{x,y\in Y} g(\mathrm{dist}(x,y)) \geq 0$$

for all finite subsets $Y \subset M$.

We can obtain an upper bound for the packing density by solving the following convex optimization problem. For simplicity we assume that M is compact.

Lemma 2.1. *Fix $r > 0$. Let $g_r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function and c_0 be a positive constant such that*

(i) $g_r - c_0$ is geometrically positive
(ii) $g_r(t) \leq 0$ for all $t \in [2r, \infty)$. Then any packing of balls of radius r in M has cardinality at most

$$\frac{g_r(0)}{c_0}.$$

Proof. Let $X \subset M$ be a subset such that $\text{dist}(x, y) \geq 2r$ for any pair of distinct points $x, y \in X$. Then condition (i) implies

$$\sum_{x,y \in X} g_r(\text{dist}(x, y)) = \sum_{x,y \in X} (g_r(\text{dist(x, y)}) - c_0) + |X|^2 c_0 \geq |X|^2 c_0.$$

On the other hand, by condition (ii)

$$\sum_{x,y \in X} g_r(\text{dist}(x, y)) = |X|\, g_r(0) + \sum_{\substack{x,y \in X \\ x \neq y}} g_r(\text{dist}(x, y)) \leq |X|\, g_r(0).$$

Hence, we arrive at

$$|X| \leq \frac{g_r(0)}{c_0}.$$

□

Unfortunately, the description of the cone of geometrically positive functions is usually a very difficult problem. Therefore, we will consider a smaller cone, the cone of so-called positive-definite functions. A function $p : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is *positive definite* (with respect to a metric space M) if

$$\sum_{x,y \in Y} w_x\, w_y\, p(\text{dist}(x, y)) \geq 0$$

for all finite subsets $Y \subset M$ and all collections of real weights $\{w_y\}_{y \in Y}$. For metric spaces M with a big isometry group the cone of positive definite functions has a simple description in terms of representation theory.

Theorem 2.2. *(Bocher 1941) Let G be a topological group acting continuously on a topological space M. For every G-invariant positive-definite kernel $p : M \times M \to \mathbb{C}$, there exists a unitary representation V of G and a continuous, G-equivariant map $\phi : M \to V$ such that $p(x, y) = \langle \phi(x), \phi(y) \rangle$ for all $x, y \in M$.*

For example, the following theorem characterizes positive definite functions on the standard sphere $S^{d-1} = \{x \in \mathbb{R}^d \mid \|x\|^2 = 1\}$. Let $P_k^d(t)$ denote the degree k ultraspherical (i.e. Gegenbauer) polynomial, normalized with $P_k^d(1) = 1$. These polynomials are orthogonal with respect to the measure $(1 - t^2)^{(d-3)/2}\, dt$ on $[-1, 1]$.

Theorem 2.3. *(Schoenberg [1942]) A function $g : [0,2] \to \mathbb{R}$ is positive definite with respect to the sphere S^{d-1} if and only if*

$$g(s) = \sum_{k=0}^{\infty} c_k P_k^d(1 - \frac{1}{2}s^2)$$

where $c_k \geq 0$.

H. Cohn and N. Elkies have applied the linear programming technique to the sphere packing problem in Euclidean space Cohn and Elkies [2003]. This problem is rather subtle since the Euclidean space in non-compact and the Lebesgue measure of the whole space is not finite.

Let us setup some notations in order to formulate the main result of Cohn and Elkies [ibid.]. The *Fourier transform* of an L^1 function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\mathfrak{F}(f)(y) = \widehat{f}(y) := \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot y}\, dx, \quad y \in \mathbb{R}^d$$

where $x \cdot y = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x-y\|^2$ is the standard scalar product in $\mathbb{R}^d$. A C^∞ function $f : \mathbb{R}^d \to \mathbb{C}$ is called a *Schwartz function* if it tends to zero as $\|x\| \to \infty$ faster then any inverse power of $\|x\|$, and the same holds for all partial derivatives of f. The set of all Schwartz functions is called the *Schwartz space*. The Fourier transform is an automorphism of this space. We will also need the following wider class of functions. We say that a function $f : \mathbb{R}^d \to \mathbb{C}$ is *admissible* if there is a constant $\delta > 0$ such that $|f(x)|$ and $|\widehat{f}(x)|$ are bounded above by a constant times $(1+|x|)^{-d-\delta}$. The following theorem is the key result of Cohn and Elkies [ibid.]:

Theorem 2.4 (Cohn and Elkies [ibid.])**.** *Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is an admissible function, $r_0 \in \mathbb{R}_{>0}$ and they satisfy:*

(2-1) $$f(x) \leq 0 \text{ for } \|x\| \geq r_0,$$

(2-2) $$\widehat{f}(x) \geq 0 \text{ for all } x \in \mathbb{R}^d$$

and

(2-3) $$f(0) = \widehat{f}(0) = 1.$$

Then the density of d-dimensional sphere packings is bounded above by

$$\frac{\pi^{\frac{d}{2}} r_0^d}{2^d\, \Gamma(\frac{d}{2}+1)} = \frac{f(0)}{\widehat{f}(0)} \cdot \operatorname{Vol} B_d(0, \frac{r_0}{2}).$$

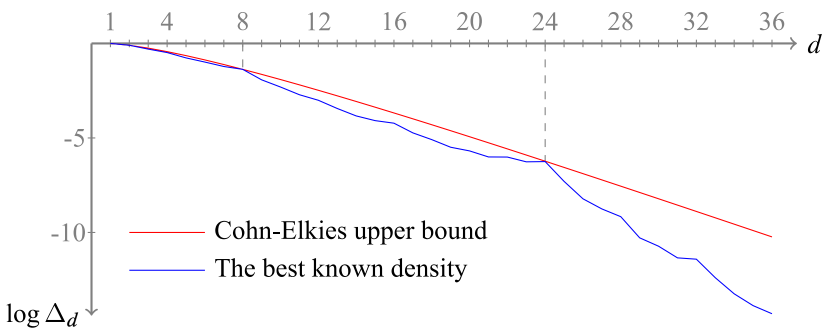


Figure 1: Upper and lower bounds for Δ_d

The *sphere packing constant* Δ_d is the supremum of all densities of sphere packings in $\mathbb{R}^d$.

Cohn and Elkies have numerically applied Theorem 2.4 to the sphere packing constant in dimensions from 1 to 36. The numerical results obtained in Cohn and Elkies [ibid.] are illustrated in Figure 1. The red line represents an upper bound obtained from Theorem 2.4 and the blue line shows the density of the best known configuration in each dimension.

3 Sharp linear programming bounds

A natural question is whether the linear programming bounds can be sharp. As we have relaxed our original optimization problem, we do not expect sharp bounds in general. However, we know several examples when the linear programming technique provides a complete solution to the optimization problem.

A beautiful example is the computation of the kissing number in dimensions 8 and 24. We recall, that the *kissing number* $K(d)$ is the maximal number of "blue" spheres that can touch a "red" sphere of the same size in d-dimensional Euclidean space. It was proven by Odlyzko and Sloane [1979] and independently by Levenshtein [1979] that $K(8) = 240$ and $K(24) = 196560$. The proof of this result is based on the linear programming method. Let us consider the kissing problem in dimension 8 in more detail. The kissing configuration can be described as follows. Consider 112 vectors of type $(0^6, \pm 2^2)$ that is, with 2 non-zero coordinates, which are ± 2 and 128 vectors of type $(\pm 1^8)$ with an even number of positive components. All the $122 + 128 = 240$ vectors have length $2\sqrt{2}$. The minimum distance between these vectors also equals $2\sqrt{2}$. Therefore, they form a kissing configuration. The only missing step is a construction of a suitable positive definite function p_8. Consider the following polynomial on $[-1,1]$:

$$p_8(t) := (t+1)^2\,(t+\frac{1}{2})^2\,t^2\,(t-\frac{1}{2}).$$

The coefficients c_k of the expansion of p_8 in Gegenbauer polynomials are all non-negative and $p_8(1)/c_0 = 240$.

The dimensions 8 and 24 are also special for the sphere packing problem in the Euclidean space. On the Figure 1 we can see that the blue line representing a lower bound on the sphere packing constant and the red line representing Cohn-Elkies bound come very close together at the dimensions 8 and 24. In Cohn and Elkies [2003] Cohn and Elkies proved the following estimates

Theorem 3.1 (Cohn and Elkies [ibid.]). *We have*

$$\Delta_8 \leq 1.00016\,\Delta_{E_8},$$

$$\Delta_{24} \leq 1.019\,\Delta_{\Lambda_{24}}.$$

Here Δ_{E_8} denotes the density of the E_8-lattice packing in $\mathbb{R}^8$ and $\Delta_{\Lambda_{24}}$ denotes the density of the Leech lattice packing in $\mathbb{R}^{24}$.

It is proven in M. S. Viazovska [2017] and Cohn, Kumar, Miller, Radchenko, and M. Viazovska [2017] that the Cohn-Elkies linear programming bounds are indeed sharp in these dimensions. The sphere packing problem in dimensions 8 and 24 will be discussed in more detail in the next section.

At the end of this section, we would like to mention several packing problems for which the numerical linear programming bounds are extremely close to the known lower bounds, however the question whether these bounds are sharp is still open. The first example, is the packing of equal disks in dimension 2. The packing problem itself has been solved long time ago Thue [1910], Fejes [1943] by a geometric method. The numerical results of Cohn and Elkies [2003] suggest that the linear programming bound is also sharp in this case, however the exact solution is not known yet.

There is a numerical evidence that the packing problem can be solved by linear programming also for other convex center symmetric bodies in $\mathbb{R}^2$. H. Cohn and G. Minton have numerically studied the packings with translates 2-dimensional of L_p-balls using linear programming bounds proven in Cohn and Elkies [ibid., Theorem B.1]. Recall, that for $p > 0$ a p-ball in $\mathbb{R}^d$ is the set of points $x = (x_1, \dots, x_d)$ such that

$$|x_1|^p + \ldots + |x_d|^p \leq 1.$$

Cohn and Minton conjecture that the resulting bounds are sharp. Thanks to a theorem proven by L. Fejes Tóth we know that the optimal packing of congruent convex center symmetric bodies in $\mathbb{R}^2$ is always a lattice packing. An open question is whether this result can be proven by linear programming.

Finally, interesting numerical results has been obtained for translative packings of L_p-balls in $\mathbb{R}^3$. On Figure 2 we plot the upper and lower bounds for such packings computed in Dostert [2017]. We know that the classical sphere packing problem in dimension 3 can not be solved by linear programming. However, for L_p-balls with parameter p in the interval $(1.2, 1.4)$ the lower and upper bounds come extremely close. So there is a hope that these bounds are sharp for some values of p.

4 The sphere packing problem in dimensions 8 and 24

In this section we will consider the sphere packing problem in the Euclidean spaces of dimensions 8 and 24.

It the 8-dimensional Euclidean space there exists a highly structured configuration – the E_8 lattice, which we have already mentioned in Section 3. The E_8-lattice $\Lambda_8 \subset \mathbb{R}^8$ is given by

$$\Lambda_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 | \; \textstyle\sum_{i=1}^8 x_i \equiv 0 \pmod 2\}.$$

Λ_8 is the unique even, unimodular lattice of rank 8. The minimal distance between two points in Λ_8 is $\sqrt{2}$. The E_8-lattice sphere packing is the packing of unit balls with centers at $\sqrt{2}\Lambda_8$.

The following theorem implies that the optimality of the E_8-lattice sphere packing can be proven by the Cohn-Elkies method.

Theorem 4.1. *(M. S. Viazovska [2017]) There exists a radial Schwartz function $f_{E_8} : \mathbb{R}^8 \to \mathbb{R}$ which satisfies:*

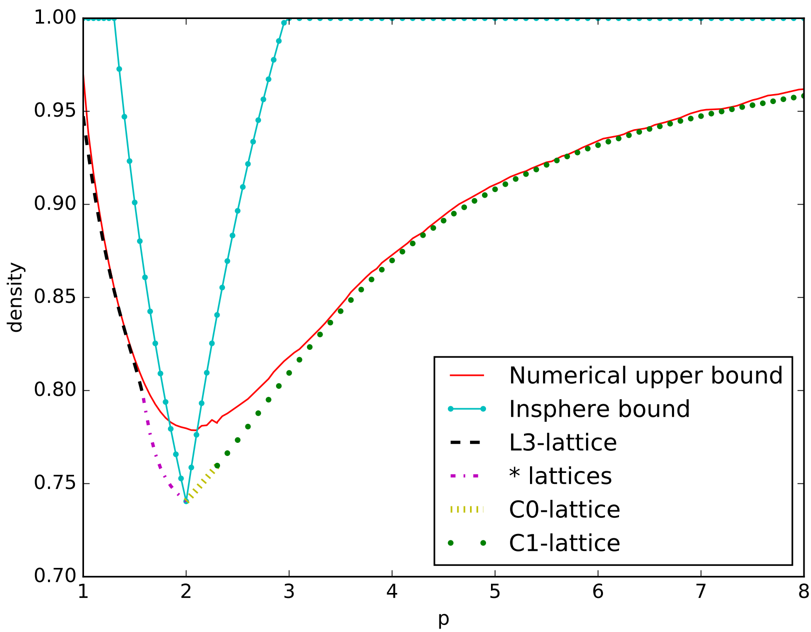
$$f_{E_8}(x) \leq 0 \textit{ for } \|x\| \geq \sqrt{2}$$
$$\widehat{f}_{E_8}(x) \geq 0 \textit{ for all } x \in \mathbb{R}^8$$
$$f_{E_8}(0) = \widehat{f}_{E_8}(0) = 1.$$

An immediate corollary of Theorems 2.4 and 4.1.

Theorem 4.2. *No packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the E_8 lattice packing. Therefore $\Delta_8 = \frac{\pi^4}{384} \approx 0.25367$.*

Also in dimension 24 there exists a lattice with unusually tight structure. The Leech lattice Λ_{24} is an even, unimodular lattice of rank 24. The minimal distance between two points in Λ_{24} is 2, and it is the only even, unimodular lattice of rank 24 with this property. The Leech lattice sphere packing is the packing of unit balls with centers at Λ_{24}. The

Figure 2: Lower and upper bounds for the density of translative packings of p-balls in $\mathbb{R}^3$ computed in Dostert [2017].



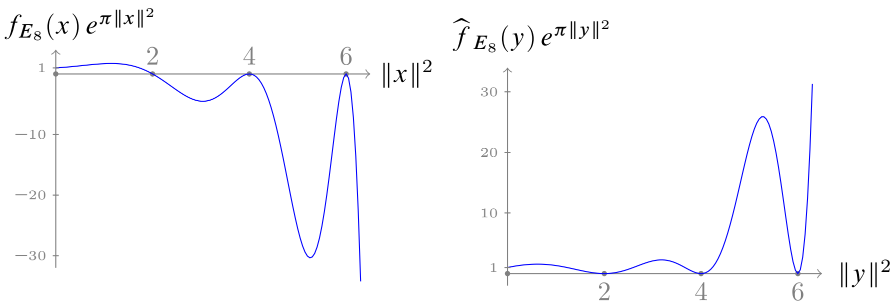


Figure 3: Plot of the functions f_{E_8} and $\widehat{f}_{E_8}$

optimality of this packing also has been proved by the Cohn-Elkies linear programming method.

Theorem 4.3. *(Cohn, Kumar, Miller, Radchenko, and M. Viazovska [2017]) There exists a radial Schwartz function* $f_{\Lambda_{24}} : \mathbb{R}^{24} \to \mathbb{R}$ *which satisfies:*

$$f_{\Lambda_{24}}(x) \leq 0 \text{ for } \|x\| \geq 2$$
$$\widehat{f}_{\Lambda_{24}}(x) \geq 0 \text{ for all } x \in \mathbb{R}^{24}$$
$$f_{\Lambda_{24}}(0) = \widehat{f}_{\Lambda_{24}}(0) = 1.$$

This result immediately implies

Theorem 4.4. *(Cohn, Kumar, Miller, Radchenko, and M. Viazovska [ibid.]) No packing of unit balls in the Euclidean space* $\mathbb{R}^{24}$ *has density greater than that of the Leech lattice packing. Therefore* $\Delta_{24} = \frac{\pi^{12}}{12!} \approx 0.00193$.

Remarks:

1. Without loss of generality we may assume that f_{E_8} is radial.

2. By the Poisson summation formula we have

$$f_{E_8}(0) \geq \sum_{\ell \in \Lambda_8} f_{E_8}(\ell) = \sum_{\ell \in \Lambda_8} \widehat{f}_{E_8}(\ell) \geq \widehat{f}_{E_8}(0).$$

This can happen only if $f_{E_8}(\sqrt{2n}) = \widehat{f}_{E_8}(\sqrt{2n}) = 0$ for all $n \in \mathbb{Z}_{>0}$.

5 Fourier interpolation

The idea behind our construction of f_{E_8} and $f_{\Lambda_{24}}$ is the hypothesis that a radial Schwartz function p can be uniquely reconstructed from the values

$$\{p(\sqrt{2n}), p'(\sqrt{2n}), \widehat{p}(\sqrt{2n}), \widehat{p}'(\sqrt{2n})\}_{n=0}^{\infty}$$

The proof of this statement is a goal an ongoing project of the author in collaboration with H. Cohn, A. Kumar, S. D. Miller, and D. Radchenko.

In this section we will present a simpler first degree interpolation formula of this type.

Theorem 5.1. *(Radchenko, Viazovska Radchenko and M. Viazovska [2017]) There exists a collection of Schwartz functions $b_0, a_n\colon \mathbb{R} \to \mathbb{R}$ with the property that for any Schwartz function $p\colon \mathbb{R} \to \mathbb{R}$ and any $x \in \mathbb{R}$ we have*

$$\begin{aligned} p(x) =& c_0(x)\, p'(0) + \sum_{n\in\mathbb{Z}} a_n(x)\, p(\operatorname{sign}(n)\sqrt{|n|}) \\ &+ \widehat{c}_0(x)\, p'(0) + \sum_{n\in\mathbb{Z}} \widehat{a}_n(x) \widehat{p}(\operatorname{sign}(n)\sqrt{|n|}), \end{aligned} \tag{5-1}$$

where the right-hand side converges absolutely.

Moreover, we can describe all possible collections of values of a Schwartz function at the points $\{\pm\sqrt{n}\}_{n=0}^{\infty}$.

Denote by $\mathfrak{s}$ the vector space of all rapidly decaying sequences of real numbers, i.e., sequences $(x_n)_{n\geq 0}$ such that for all $k > 0$ we have $n^k x_n \to 0, n \to \infty$.

We denote by $\mathcal{S}$ the space of Schwartz functions on $\mathbb{R}$. Consider the map $\Psi\colon \mathcal{S} \to \mathbb{R}^2 \oplus \mathfrak{s} \oplus \mathfrak{s}$ given by

$$\Psi(p) = \left(p'(0), \widehat{p}'(0), (p(\operatorname{sign}(n)\sqrt{|n|}))_{n\in\mathbb{Z}}, (\widehat{p}(\operatorname{sign}(n)\sqrt{|n|}))_{n\in\mathbb{Z}}\right).$$

Theorem 5.2. *(Radchenko, Viazovska Radchenko and M. Viazovska [ibid.]) The map Ψ is an isomorphism between the space of Schwartz functions and the vector space* $\ker L \subset$ *$\mathbb{R}^2 \oplus \mathfrak{s} \oplus \mathfrak{s}$, where $L\colon \mathbb{R}^2 \oplus \mathfrak{s} \oplus \mathfrak{s} \to \mathbb{R}^2$ is the linear map*

$$\begin{aligned} L :& (x_0', y_0', (x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}) \mapsto \\ &\Bigg(\sum_{n\in\mathbb{Z}} x_{n^2} - \sum_{n\in\mathbb{Z}} y_{n^2}, \\ &2x_0' + \sum_{n\in\mathbb{Z}} \operatorname{sign}(n)\frac{r_3(|n|)\, x_n}{\sqrt{|n|}} - 2iy_0' - \sum_{n\in\mathbb{Z}} i\, \operatorname{sign}(n)\frac{r_3(|n|) y_n}{\sqrt{|n|}}\Bigg). \end{aligned}$$

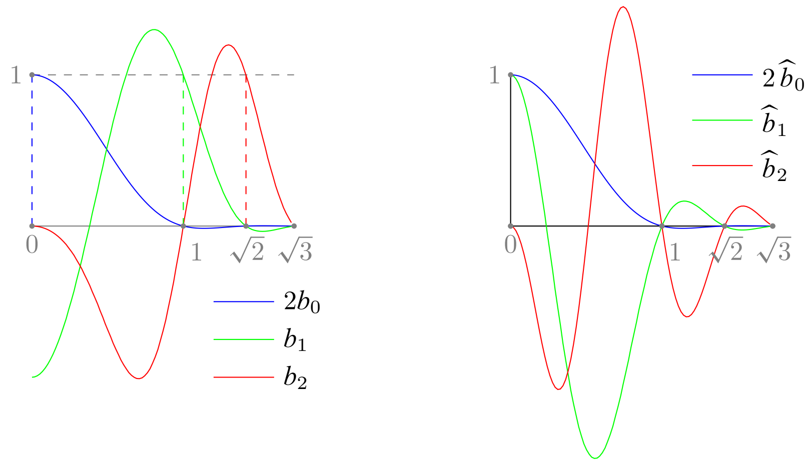


Figure 4: Plots of $b_n(x) := a_n(x) + a_n(-x)$ and $\widehat{b}_n$ for $n = 0, 1, 2$.

Also Theorem 5.1 allows us to construct an unusual family of discrete measures on the real line. A *cristalline measure* on $\mathbb{R}^d$ is a tempered distribution μ such that μ and $\widehat{\mu}$ are both charges with locally finite support. A simplest example of a cristalline measure is the *Dirac comb*

$$\mu_{\text{Dirac}} = \sum_{n \in \mathbb{Z}} \delta_n.$$

Recently, Lev and Olevskii [2015] have proven that crystalline measures with uniformally discrete support and spectrum (the support of the Fourier transform) can be obtained from the Dirac comb by dilations, shifts, multiplication on exponentials, and taking linear combinations.

Theorem 5.3. *(Lev and Olevskii [ibid.]) Let μ be a crystalline measure on $\mathbb{R}$ with uniformally discrete support and spectrum. Then the support of μ is contained in a finite union of translates of a certain lattice L. Moreover, μ is of the form*

$$\mu = \sum_{j=1}^{N} P_j \sum_{\lambda \in L + \theta_j} \delta_\lambda$$

where θ_j, $j = 1, \dots, N$ are real numbers and P_j, $j = 1, \dots, N$ are trigonometric polynomials.

The interpolation formula implies that there exists a continuous family of *exotic* crystalline measures

$$\mu_x := \delta_x + \delta_{-x} - \sum_{n=0}^{\infty} b_n(x)\,(\delta_{\sqrt{n}} + \delta_{-\sqrt{n}}).$$

Let us briefly explain our strategy for the construction of the interpolating basis a_n, c_0 introduced in Theorem 5.1. We will separately consider the odd and even components of the Schwartz functions. We set $b_n(x) = a_n(x) + a_n(-x)$. Then the symmetry implies $b_n = b_{-n}$.

Let us consider the generating series formed by the functions $\{b_n\}_{n=0}^{\infty}$ and their Fourier transforms. For $x \in \mathbb{R}$ and a complex number τ with $\Im(\tau) > 0$ we define

$$F(x,\tau) := \sum_{n=0}^{\infty} b_n(x)\, e^{\pi i n \tau}$$

$$\widetilde{F}(x,\tau) := \sum_{n=0}^{\infty} \widehat{b}_n(x)\, e^{\pi i n \tau}.$$

We will show that these two functions satisfy a functional equation with respect to the variable τ. Indeed, the interpolation formula interpolation formula (5-1) applied to the Gaussian $e^{\pi i x^2 \tau}$ gives

$$e^{\pi i x^2 \tau} = F(x,\tau) + \frac{1}{\sqrt{-i\tau}}\widetilde{F}(x, \frac{-1}{\tau}).$$

In Radchenko and M. Viazovska [2017] we solve this functional equation using the theory of modular integrals. A similar idea also leads to the constuction of functions f_{E_8} and $f_{\Lambda_{24}}$ in Theorems 4.1 and 4.3.

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