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STUDY AND RESEARCH PATHS: A MODEL FOR INQUIRY

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Abstract

This paper presents a line of research in didactics of mathematics developed during the past decade within the Anthropological Theory of the Didactic around what we call *study and research paths* (SRPs). SRPs are initially proposed as a study format based on the inquiry of open questions, which can be implemented at all educational levels, from pre-school to university, including teacher education and professional development. Additionally, they provide a general schema for analysing any kind of teaching and learning process, by especially pointing out the more or less explicit questions that lead the study process and the way new knowledge is built or introduced to elaborate answers to these questions. Current research on SRPs focuses on their *didactic ecology*, defined as the set of conditions required to generally implement SRPs at different educational levels, together with the constraints that hinder their development and dissemination.

1 Delimiting a unit of analysis

1.1 The anthropological theory of the didactic. Mathematics education—or *didactics of mathematics*, as we prefer to call it in many countries—is still a young field of research and comprises different approaches that do not always share their main assumptions or goals. The research here presented corresponds to the Anthropological Theory of the Didactic (ATD), a framework whose main creator, Yves Chevallard, received the ICMI Hans Freudenthal Medal in 2009 in recognition of the foundation and development of "a very original, fruitful and influential research programme in mathematics education". To begin with, I will briefly explain how the ATD defines and delimits the object of study of didactics and the type of research questions that are primarily raised.

From the perspective of the ATD, the aim of didactics as a science is to elucidate the mechanisms by which, in a given society, knowledge is diffused within institutions and among persons. The conception of knowledge adopted is very broad. It embraces what

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is usually considered as knowledge as such, in the sense of theoretical elaborations or constructions or, according to the dictionary, the "sum of what is known" in a given domain or discipline, close to what the French mathematician Georges Bouligand (1889-1979) defined as *syntheses*, which "keep track of new problems and assembles results known to coordinate an inventory of methods and operations" Bouligand [1957, p. 139]. And it also includes the practical dimension of knowledge, the know-how that supports all kind of human activities.

Knowledge, in both the theoretical and practical sense, is modelled in the ATD through the notion of *praxeology*. The term is formed by a combination of *praxis*—the know-how or ways of doing—and *logos*—an organised discourse about the praxis. The praxis and the logos blocks of a praxeology are in turn made up of two distinct elements: *types of tasks* and *techniques* to carry them out, for the praxis; a *technology* or discourse about the technique, and a *theory* or justification of the technology, for the logos.

One of the main postulates of the ATD is that any kind of human activity, as well as the knowledge (in the broad sense) derived from it, can be described in terms of praxeologies. Therefore, group theory or complex analysis are praxeologies, made up of elaborated theoretical discourses that describe, justify and structure a wide array of problems and techniques. However, there also exist more humble praxeologies that are activated, for instance, when we wash dishes, ride a bike or give a lecture. Many of the praxeologies people enact are difficult to describe: they consist of informal techniques that do not always have a name and include poorly organised descriptions and justifications, based on implicit assumptions and concepts. The situation is a little different in the case of sciences or academic disciplines, since a great collective effort is regularly made to make them explicit, by describing the methods used, especially to test them and share them with the community; by specifying their main assumptions and organising them coherently; by defining and structuring the notions that constitute these assumptions, the results gathered and the methods used to produce them, that will soon become new assumptions to put to use.

The need to disseminate praxeologies clearly contributes to developing them, by enriching their description (logos) and by assembling amalgams of praxeologies to build new better organised bodies of knowledge. It also helps developing their praxis to adapt it to new situations and new users. Didactics as a research field is mainly concerned by the study of "the conditions and constraints under which praxeologies start to live, migrate, change, operate, perish, disappear, be reborn, etc. within human groups" Chevallard [2007].

The dissemination of praxeologies takes place through what we call *didactic systems*. A didactic system is a tern $S(X, Y, \wp)$ which is formed any time a person or a group of persons Y (the teachers) does or do something to help a group of persons X (the students) to learn a given body of knowledge or praxeology $\wp$. X and Y can be reduced to single persons x and y, which can also coincide, thus forming an auto-didactic system $S(x, x, \wp)$.

The questioning about the delimitation, composition and origin of $\wp$—the knowledge or praxeology to be studied—is a core problem in didactics, and it leads to what we call the *epistemological or praxeological dimension* of the problem. The *dynamic* of didactic systems—what X and Y do to make it evolve—and the *conditions and constraints* that enable and hinder this dynamic are also important dimensions at the centre of the didactic questioning. They correspond to what we call the *economy* and the *ecology* of didactic systems.

1.2 The scale of levels of didactic codeterminacy. Didactic systems do not exist in a vacuum. In order to facilitate the analysis of their ecology, we consider a *scale of levels of didactic codeterminacy* Chevallard [2002]. The higher levels of the scale correspond to the conditions and constraints related to the general way of organising teaching and learning processes (Fig. 1). The level of *pedagogies* comprises everything X and Y do for the didactic system to run that does not depend on the particular praxeology $\wp$ at play. For instance, many of the instructional formats that are usually proposed to improve university teaching practices (for instance, "interactive lectures", "cooperative learning", "discovery learning", "participative tutorials", etc.) are defined independently of the precise content that is to be taught and learn and can thus be located at the pedagogical level. Their specification to a given content is then left under the teachers' own responsibility, even if it is not always a trivial affair...

The level of *schools* includes all the infrastructures provided by educational institutions to organise didactic systems and help them run: organisations of groups of teachers and students, structures in courses and modules, physical and virtual spaces, time schedules, final exam obligations, access to knowledge resources and experts, accreditations, etc. Depending on the school systems and traditions, some pedagogical resources will be easier to develop than others and, therefore, some types of praxeologies will be easier to disseminate than others.

The levels situated at the higher end of the scale include the way teaching and learning processes are conceived and managed in *societies* or, when shared by different societies,

in *civilisations*. The scale ends at the most general level, the level of *humanity* (Fig. 1).

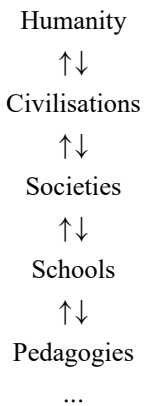

Figure 1. Scale of levels of didactic codeterminacy - Higher levels

In our societies, teaching and learning processes are mainly organised according to what has been called the *paradigm of visiting works* Chevallard [2015]. In this paradigm, instructional processes are determined by the selection of a set of works or praxeological organisations—a curriculum—that students are asked to "visit" under the guidance of the teacher. The visit includes learning what those works are made of, which their main elements are and how they can be used, for instance to solve some given sets of problems—usually called "applications". It not only comprises becoming aware of their existence, but also acknowledging their importance as historical productions.

In this paradigm, the selection of praxeologies that form a curriculum leads to specific knowledge organisations, which can vary depending on the society, school institution and historical period considered, but remain relatively stable over long periods of time. Think, for instance, in a first year university course of Calculus or Linear Algebra, or in the teaching of equations in secondary school. The lower levels of the scale of didactic codeterminacy take these structures into account by distinguishing different "sizes" of the praxeological organisations: *disciplines, sectors, domains, themes and questions* (Fig. 2). Thus, when a didactic system is formed in a regular school setting, the question about the delimitation and composition of $\wp$ is answered internally: $\wp$ corresponds to this or that type of tasks, or to this or that theme, domain, sector or discipline, which, in turn, belongs to (or is composed of) these themes, domains, sectors, etc.

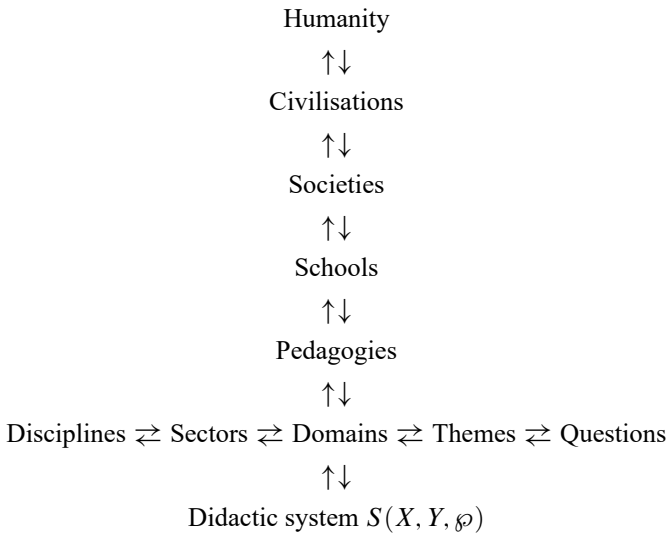

Figure 2. Scale of codeterminacy in the paradigm of visiting works

The scale of didactic codeterminacy is first of all a tool for researchers in didactics to question the reality they aim at studying. Its main utility is to enlarge our vision towards certain empirical fields that are traditionally kept outside the didacticians' perspective and are thus taken for granted. In effect, a great amount of research in didactics that focuses on the levels of the questions or the themes that are taught and learnt, rarely questions the specific structuring of disciplines, sectors or domains these questions or themes belong to. Therefore, many of the conditions and constraints that come from the lower levels remain hidden, as if they were not part of the problems addressed. Let us take a single example. Many studies about the teaching and learning of negative numbers assume that this work belongs to the school arithmetical domain, where numbers are introduced as measures of quantities—directional quantities, in this case. The possibility of introducing integers within the algebraic domain, for instance as necessary tools to give coherence to the work with equations and formulas Cid [2015], is rarely considered, mainly because this situation does not correspond to the current school structuring of the mathematical content. It is important for research in didactics to question the way mathematics is organised and considered at the different levels of the scale of didactic codeterminacy. At this respect, didacticians have to assume a different perspective from the one of the teachers who are asked to teach a body of knowledge that is already organised into sectors, domains and even themes. On the contrary, didactic analyses and interventions should dare approach

the higher levels of the scale, in spite of the important methodological difficulties they might encounter.

A first step in this direction is the study of what we call the *mathematical praxeological models prevailing in educational institutions* regarding the content to be taught: how is mathematics conceived, why is it structured in these sectors and domains, what is the role of such and such themes or domains in the whole organisation, which are and which could be their raisons d'être. To avoid taking the vision proposed by the educational institution for granted, researchers in didactics had to build their own *praxeological reference models* of the mathematical works that are taught and learnt, if they want to be able to question them, as a first step towards proposing possible alternatives Bosch and Gascón [2006]. The consideration of integers as algebraic entities is a good example of these reference praxeological models: once it is specified as an alternative mathematical organisation (compared to the school ones), it can be used to point out different didactic phenomena that explain many difficulties encountered by secondary school teachers and students in relation to negative numbers—and to algebra Cid [2015].

Questioning the prevailing epistemological models in educational institutions is at the core of research in didactics of mathematics in the perspective proposed by the theory of didactic situations Brousseau [1997], which is at the origin of the ATD. The evolution of a didactic system $S(X, Y, \wp)$ undoubtedly depends on what X and Y can do, but also on *how the praxeology* $\wp$ *is delimited, conceived, considered, imported, used, legitimised* in school institutions—as well as at the other higher levels of codeterminacy. We talk about *didactic phenomena* to refer to all regular facts occurring in teaching and learning processes that are specific of the content. The frontier between *didactic* and *pedagogical* phenomena is not always clear and many teaching or learning difficulties—like the students' lack of motivation, for instance—are sometimes approached as *pedagogical* phenomena when they are of a clear *didactic* nature. Didactic research tries to overcome the frontiers between the pedagogical proposals that rarely take into account the specificity of the content to be taught and learnt, and the specific levels related to the structuring of this content. All in all, and in order to be operational, any general pedagogical change—such as the incorporation of competencies as a key tool to define educational objectives—have to be developed on close interaction with the specific praxeological organisations that are at the centre of didactic systems.

2 A change of paradigm: from "visiting works" to "questioning the world"

The paradigm of visiting works bases its legitimacy on the social importance given to the selected bodies of knowledge that constitute the curriculum. These are supposed to

have been chosen because of their utility in the future life of students, but this utility is more assumed than proved, since there is rarely the need in making it explicit. The main objective of the study process is defined by getting to know the selected praxeologies and being able to activate them—up to a given extent—, the raison d'être of these praxeologies, not only the reasons for learning them but also their reasons for existence, can remain in the shadow or simply be delayed, presented as something that will appear later on—if it does.

This does not mean that the paradigm of visiting works necessarily implies "transmissive" instructional formats, where the teacher presents or depicts a given body of knowledge for the students to acquire it and try to apply it through a given set of activities. It is also compatible with student-centred instructional formats or even with inquiry or problem-based learning. However, in these cases, the choice and role of the activities, problems or inquiries remains subordinate to the construction of the praxeologies.

This traditional way to disseminate mathematical knowledge is based on the transmission of *syntheses*—in the sense of Bouligand. The way mathematical knowledge has been selected and organised for schools, is structured as an already finished product, which includes precise terminology to describe its main notions, results and techniques. However, it leaves little room for the questions that motivated or could motivate their construction. This situation leads to what Chevallard [2015] has called the *monumentalisation of curriculum*, where each selected mathematical work appears as "a monument, a masterpiece even, that, however impudently, we are expected to revere and bow towards". It also leads to a *sacralisation of syntheses*, that is, of the praxeological organisations elaborated to structure the bodies of knowledge in themes, sectors and domains. The monumentalisation of curriculum goes together with the unquestionability of the lower levels of the scale of didactic codeterminacy.

To avoid assuming this state of things, as researchers—but also as citizens—, the paradigm of visiting works is subsumed into a larger pedagogical paradigm, the *paradigm of questioning the world*, which can also appear as a counter-paradigm because of the important changes it requires in the scale of codeterminacy, from the lowest to the highest levels. The main element to define the paradigm of questioning the world is the notion of *study and research path* (SRP) based on the so-called *Herbartian schema*:

$$[S(X;Y;Q) \curvearrowright M] \hookrightarrow A^{\heartsuit}.$$

In this paradigm, the didactic system $S(X;Y;Q)$ is not formed around a given praxeology $\wp$ to be studied, but around a question Q to which X, with the help of Y, has to provide an answer $A^{\heartsuit}$. The study of Q generates an inquiry process involving a didactic milieu M made up of different types of objects or tools for the inquiry:

$$M = \{A_1^{\diamondsuit}, A_2^{\diamondsuit}, ..., A_m^{\diamondsuit}, W_{m+1}, W_{m+2}, ..., W_n, Q_{n+1}, Q_{n+2}, ..., Q_p, D_{p+1}, D_{p+2}, ..., D_q\}.$$

The heart in superscript in $A^{\heartsuit}$ means both that $A^{\heartsuit}$ is dear to the didactic system's "heart" and will be "at the heart" of the didactic system's activity during the inquiry process: it will be the *official answer* to Q in the class $[X, Y]$.

The $A_i^{\diamond}$ are "ready-made" answers that seem helpful to answer Q (or to answer some questions Q_k derived from Q) that the investigators X, supervised by Y, have discovered in the institutions around them: they are institutional answers to Q, and the lozenge $\diamond$ in superscript indicates that this answer $A^{\diamond}$ is labelled or "hallmarked" by the institution that presents it as the "official" answer to Q. The W_j are works drawn upon to make sense of the $A_i^{\diamond}$, analyse and "deconstruct" them, and to build up $A^{\heartsuit}$. The Q_k are the questions induced by the study of Q, the $A_i^{\diamond}$, and the W_j, as well as the questions raised by the construction of $A^{\heartsuit}$. Finally, the D_l are sets of data of all natures gathered in the course of the inquiry.

In this schema, the "visit of works" does not disappear: in order to find the appropriate labelled answers $A_i^{\diamond}$ that would turn out to be productive for the inquiry, it can be sometimes necessary to explore large domains of knowledge and requiring the help of experts guides. However, the visit in this case is motivated, not by the importance or prestige of $A_i^{\diamond}$, but only by its productivity in the construction of $A^{\heartsuit}$.

The Herbartian schema indicates the main elements of the inquiry process. Its dynamics is captured in terms of some *dialectics* that describe the production, validation and dissemination of $A^{\heartsuit}$. We will consider three of them here. The first one is the *question-answer dialectic*, which will provide a first description of the structure of the process as well as a number of milestones on the paths followed or foreseen during the inquiry. The dialectical character of the questions and answers is related to the notions of study and research: to approach a question Q, one usually searches for available answers $A_i^{\diamond}$ and has to *study* them: that is, to deconstruct and reconstruct to adapt them to Q. This study generates new questions about the validity and limitations of $A_i^{\diamond}$, its adequacy to Q, the adaptations required, etc. The question-answer dialectic is the one that provides visible proof of the progress of the inquiry and contributes to what is called the *chronogenesis* of the process.

Another crucial element of the dynamics of inquiry processes is the *media-milieu dialectic*. *Media* refers to any system emitting messages. A *milieu* in didactics is a system that is supposed to be devoid of intention with respect to the question studied and to which elements of response can be "extorted" Brousseau [1997]. To put the media-milieu dialectic into play, any message from the media has to be confronted with the milieu to test its validity and to collect critical elements providing new information. In a sense, the answers supplied by the media have to be integrated in the milieu—turning into "sure" knowledge—and the elements of the milieu have to be worked out in order to make it

send new messages—to become a media. The evolution of the milieu by the incorporation of new objects and partial answers constitutes the *mesogenesis* of the inquiry (the generation of the milieu).

The third dialectic is the one of *the individual and the collectivity*, which reminds us that the inquirers X act jointly and in cooperation with Y, while the production of the group will also depend on the capacity of each member x and y to contribute to the common project. The way responsibilities are shared in the process and how each member assumes different roles is called the *topogenesis* of the inquiry (the generation of different places or *topos* to teachers and students).

2.1 The herbartian schema as an analytical tool. If we take the chronogenesis, mesogenesis and topogenesis as the main dimensions of the inquiry, we can obtain an outline of the study and research process. Its first description can take the form of a tree or arborescence of derived questions raised and partial answers obtained till the elaboration of the final answer $A^{\heartsuit}$ Bosch and Winsløw [2015], Hansen and Winsløw [2011], Jessen [2014], and Winsløw, Matheron, and Mercier [2013] (Fig. 3). This first model, which shows the progress of the inquiry, including its possible detours and dead-ends, can then be enriched by the description of the evolution of the milieu M. Finally, it is possible to incorporate the didactic sub-systems created and the position and responsibilities of their actors into the model obtained.

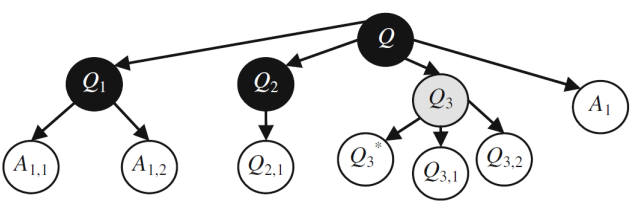

Figure 3. Example of questions-answers map Winsløw, Matheron, and Mercier [2013, p. 271]

With these elements, the Herbartian schema can be used as a tool to describe any kind of study and research process, from the most "transmissive" formats to the most "constructivist" ones; from a traditional course to a PhD research. For instance, the traditional case of a course based on lectures can be depicted with the following schema:

$$[S(X;y;\mathscr{Q}) \curvearrowright \mathscr{M}] \hookrightarrow A^{\heartsuit} = A^{\diamond}_{y} = \wp.$$

In this case, the question Q is usually kept in the shadow (to be discovered later on) and the answer provided by the class $[X, y]$ corresponds to the teacher's answer $A^{\diamond}_{y}$ which is

supposed to reproduce a pre-established praxeology $\wp$. At the other extreme, there may be a purely problem-based activity in which the students are supposed to work out the answer from the sole exploitation of the available milieu, without the possibility to access any external media. In between, one can imagine several different inquiry formats, the most common one being:

$$[S(X; y; Q_\wp) \curvearrowright M] \hookrightarrow A^\heartsuit = \wp.$$

where the teacher poses a question $Q_\wp$ which is supposed to lead to a previously established praxeology $\wp$ and progressively incorporates some derived questions, data and pieces of answers into the milieu for the students to reproduce $\wp$.

The Herbartian schema can thus be exploited as an analytical tool, not only to describe any kind of study and research process, but also to question the choices made in its organisation with respect to many possible alternatives, by asking questions such as:

- What is Q_0? Where does it come from? Who poses it? Why do we need to answer it? What kind of answer is required? To whom is it addressed?
- What derived questions Q_j appear during the process? Which ones fail to do so? Who poses them? How are they addressed? By whom? What kind of answers are searched for? In what types of media? How are they incorporated in the milieu M?
- What are the main initial elements of the milieu M? How are they used? How do they evolve?
- How is the sharing of responsibilities among teacher(s) and students established? Who does what?

Besides these basic methodological elements for the analysis of the pedagogy of the inquiry, the Herbartian schema also appears as a productive tool for the didactic design and explorations of the possibilities open by the new paradigm, as we will see in the next sections.

3 Implementing study and research paths

During the past decade, the Herbartian schema has been used in several research studies to design and implement various types of study and research paths (SRPs) in different educational institutions. These experimentations provide interesting empirical material to study the *economy* of SRPs—the possible ways to implement and manage them—and especially their *ecology*—the institutional conditions and constraints that facilitate or hinder their implementation. They also show possible ways of approaching certain previously

identified didactic phenomena originated in the paradigm of visiting works. A detailed account of these investigations cannot be done here, but partial reviews can be found in Parra and Otero [2017] and Jessen [2017].

3.1 Commonalities in the design and implementation of SRPs. The experimented SRPs were mainly implemented at secondary and tertiary levels—with an interesting exception at pre-school we are not considering here Ruiz-Higueras and García García [2011]. They adopt different instructional formats depending on the school conditions assumed and those that were modified. There are, however, some commonalities that should be emphasised.

3.1.1 The generating question. The starting point of all SRPs was a generating question Q_0 posed by the teacher who was in most cases also a researcher in didactics. When the teacher was not a researcher, there was always a team of researchers closely collaborating with her. In order to give predominance to Q_0—and not to the hypothetical knowledge that was supposed to be activated during the study and research process—a role-playing activity was carried out, in which the class (X, Y) acted as a consultancy team, the teachers Y assuming the role of the leaders and the students X of junior consultants. Q_0 was then introduced as an assignment that came from an instance external to the class—a client—to whom an answer in the form of a report had to be handed in after a given period of time (some weeks or months). During the SRP, some interactions with the client were made possible (for instance requesting more information by e-mail), and some intermediate reports were occasionally required.

3.1.2 Collective work. It is important to point out that there was a unique question addressed to the whole class and, therefore, there was also only one joint report to elaborate at the end of the SRP. However, during the SRP, students could be organised in small teams X_i and different responsibilities could be assigned to each team, according to the derived questions Q_i' generated by the SRP. This organisation differs from many inquiry-based learning formats where students work individually on their own project under the supervision of a teacher. The way to manage the dialectic between the collective and the individual work thus appears as an important condition to take into account, and it soon revealed the lack of pedagogical resources available (at least in the mathematics class) to address it.

In most of the cases, student teams were asked to present weekly reports of the work done, including the questions addressed, the partial results obtained, the difficulties met and the new questions raised. These reports were shared with the large group in different ways, such as oral presentations, peer-reviews or simply saved in a shared web folder.

Sometimes the teacher would name a "secretary" of the class to elaborate a synthesis of the joint work carried out during the session. On some occasions, the students' lack of experience and strategies to work in small teams, for instance to collectively perform a mathematical task, caused difficulties and some simple devices such as what Liljedahl [2016] calls "vertical learning" (letting student teams work in different corners of the room, using the blackboard or blank posters) turn out to be very useful.

3.1.3 Chronogenesis: managing long study processes. With regard to the length of the whole process, all SRPs took place during several sessions that, depending on the school constraints, varied from 50 minutes to 2 hours. This situation was completely new to the students, who are used to being asked to solve several problems within one session, especially in mathematical classrooms. They very rarely have to approach the same problematic question for weeks or months. At times, a feeling of tiredness was observed in some students—"Again the duck populations?"—that can be interpreted in two ways. On the one hand, it attests their poor engagement in the activity and their difficulty in getting rid of the traditional *didactic contract* Brousseau [1997] where students rarely assume the responsibility of "carrying with them" a given problem till they are finally able to solve it... On the other hand, the students' tiredness can also be seen as a consequence of the teacher's failure to keep the chronogenesis of the process alive. An important constraint here is the scarcity of mathematical and didactic resources available—for both the teacher and the students—to describe the inquiry process, identify the results obtained and establish milestones for the work to do. We will come back to this point later on.

3.1.4 Topogenesis: sharing responsibilities between the teacher and the students. The most evident constraint that appeared in almost all SRPs was the difficulty for the teacher to share responsibilities with the students beyond those (few) assigned to them in the traditional didactic contract. At the beginning, students were easily involved in the process, but they were not used to lead the questioning, not even to raise the questions to follow during the inquiry, to select the ways that seem more promising and discard possible dead-ends. At the beginning, even us as researchers had difficulties in assuming the new contract. For instance, in the design of one of the first SRPs, we struggled with the elaboration of a realistic schedule (since we had no elements of contrast) before realising that planning the work was not necessarily the teacher's responsibility, that students could also contribute to it. From then onwards, students were always asked to keep a logbook with a planning that was regularly updated. Again, the lack of words to describe the steps followed and those foreseen appeared as an important constraint for both the teacher and the students.

3.1.5 Mesogenesis: the media-milieu dialectics. The disappearing of the teacher as the main *media* and *milieu of validation* for the students (to tell them what is wrong or correct) appeared as another difficulty—especially for the teacher. Therefore, the inquiry work also required to enrich the traditional media-milieu dialectic to obtain new sources of proof and information. In what concerns the access to the media, in all SRPs, the choice of the initial question was based on a priori analyses that showed the inclusion of some empirical work available to the students and the need of new information to be searched for (in Internet, textbooks, by asking experts, etc.). Again a lack of pedagogical resources appeared: for the teacher to manage new situations requiring knowledge she may not master; for the students to be critical of the information obtained, independently of the source consulted; for both of them to create ad hoc ways to (in)validate answers $A_i^\diamond$ when they came from "expert sources" but were not necessarily appropriate to the specific needs of the inquiry. Very elementary strategies of validation had to be established, besides the classic mathematical ones, like the comparison of different sources, the questions to experts, the rejection of useless answers, etc. On some occasions, however, it was the teacher who finally ended up introducing some key elements to let the inquiry progress.

3.1.6 Openness and assessment. The different levels of openness of the inquiry process are what make it at the same time exciting and disturbing for the study group. The design strategy followed by the research team consists in elaborating an *a priori questions-answers map* (Q-A map) of some expected questions derived from Q_0 and possible available answers, while keeping the end of the story obviously open. This previous analysis of the generating power of Q_0 ensures a minimal viability of the inquiry process and also gives the teacher a first insight of the students' possible proposals. In many SRPs, and in spite of the normal resistance experienced by teachers to let the process advance towards unexpected–and sometimes dead-end–paths, the more freedom was given to the students, the richer the inquiry became. A possible reason is that strong guidance reveals that the initial question is not the real goal of the study, as if the means were more important than the end. This of course contributes to weaken the initial question and, therefore, frustrates the students' efforts made to elaborate the final answer $A^\heartsuit$.

The importance given to the initial question should be made visible at the end of the process with the type of assessment method applied. In the experienced SRPs, the assessment strategy included part of the intermediate oral or written reports required of the students, as well as the final presentation of the answer given to Q_0 or to the derived questions Q_i assigned to each team. A peer-review process among the teams of students was sometimes organised, but on most occasions the feedback was mainly given by the teacher acting as the leader of the consultant group or as the client's representative. Panels with

external teachers and experts were organised at times, and they included oral or poster presentations depending on the number of students to be assessed.

3.2 Questioning the sector and domain levels. Some of the first experiences of the SRPs came as the response to previously identified didactic phenomena that were assumed to be intimately correlated to monumentalism. In the case at hand, this phenomena were also related to some key curricular contents—proportionality, algebra, functions and derivatives—and can thus be located at the level of the sector or domain in the scale of codeterminacy.

In the case of lower secondary school, a first proposal by García [2005] addressed the question of the isolation of proportionality from other functional relationships and the implicit preponderance given to linear growth in school mathematics. In this context, the proposed SRP started with a question that motivated the construction and comparison of different possible relationships:

> How can we save money for the end-of-the-year trip, or any other trip we plan to go on in 6 to 9 months time?

Students were invited to propose different saving strategies, starting with a regular constant instalment (linear growth) and comparing it with other possible plans of increasing and decreasing instalments. It was also part of the SRP to determine the periodicity of the instalments and to make a final decision in accordance with the priorities assumed by the students and the consideration of possible unexpected events (withdrawals, newcomers, etc.). Students were led to carry out an algebraic modelling of the proposed saving plans using Excel simulations as a milieu and study the characteristics of each proposal through the adequate manipulation of equations with parameters (formulas), a work that is very unusual in secondary school mathematics. The design of the SRP was based on a previous reconstruction of the school mathematical domain of proportionality and functions to relate proportionality with other relationships between quantities and present it as a possible model among many others García, Gascón, Ruiz Higueras, and Bosch [2006].

A similar type of SRP was proposed by Ruiz-Munzón [2010] at upper secondary school to facilitate the passage from the algebraic modelling of relationships between quantities to the functional one. This SRP addressed the question of how much money a group of students can make by selling one-print T-shirts, taking into account their unit cost, the selling price and some fixed expenses due to the rent of a stand and/or a store. The data provided to the students were the sales of the previous years and they had to elaborate a plan to reach a given amount of money for an end-of-the-year trip. In this case, students had to search for information about the T-shirts production cost and possible prices, and use functional tools to solve algebraic inequalities with three or four parameters Ruiz-Munzón, Matheron, Bosch, and Gascón [2012]. The important aspect of the work carried

out was to introduce and use functions and functional graphs to answer questions that were not initially formulated in the functional domain. Functions were not to be studied *per se* (as it is usually the case at secondary level) but because they were needed to solve inequalities that could not be solved algebraically.

In continuity with this work, Oliveira Lucas [2015] designed and implemented an SRP aimed at connecting functional modelling with elementary calculus. As in the previous cases, the research was based on an empirical study of the prevailing epistemological model at secondary level about elementary calculus and the elaboration of an alternative praxeological organisation to reconstruct the raison d'être of elementary calculus in a functional modelling context. The SRP started with some data about a Dengue epidemic and requested a forecast for the following weeks. An extensive a priori analysis of the generating question considering two main inquiry paths, depending on the type of data and quantities (discrete/continuous), and the variable used (the original one, its rate of change, its relative rate of change) was performed. Then, continuous models were introduced because of their technical facilities to calculate the variations of the variables considered, reversing in a way the traditional organisation of concepts, where the rate of change appears as a previous step to defining the derivative: here, it is the derivative that is at the service of the rate of change.

In summary, these types of SRPs started by the questioning of the epistemological models about certain themes, sectors or domains that prevail at secondary school and that can be interpreted as a consequence of monumentalism. In these models, mathematical praxeologies like proportionality, algebraic equations, elementary functions and derivatives are organised according to their theoretical components—their logos—, and the types of questions addressed—the praxis—are always presented as applications of already introduced notions or properties. Even in the case where real questions are proposed, their final aim is always to illustrate or construct the previously established praxeologies and, especially, their theoretical components: the concepts of proportionality, function, derivative, etc. In this school organisation of mathematics, modelling activities cannot find their real place, since the resolution of problems is always subordinated to the construction of the notions that structure the curriculum.

These examples illustrate how the problem of the ecology of some mathematical activities taking place at the level of the theme or the domain has to be addressed by questioning the higher levels of the scale of didactic codeterminacy because it is in these higher levels where some raison d'être of the aimed praxeologies can be found. This questioning leads to a redefinition, from the research perspective, of the praxeological organisations that conform the knowledge to be taught and learn. In this context, the role of SRPs is to serve as a study format that breaks some of the main assumptions of monumentalism and offers better conditions of existence for the alternative praxeological organisations.

3.3 Questioning the discipline level at the university. The previous SRPs take as a starting point a previous praxeological analysis of the content to be taught and learnt in order to overcome some identified didactic phenomena, especially related to the isolation and loss of the raison d'être of some curricular praxeologies. In a way, the generating question of the SRP is not the core objective of the inquiry process; the elements produced during the inquiry are. In this case, the SRP is not open, but *finalised*. The previous examples addressed a given theme or domain of school mathematics in order to reconstruct it in a more functional way. It was always foreseen that some or most of the praxeologies that constitute the curriculum would appear as answers to the questions raised during the inquiry. If a given syllabus is defined as a set of mathematical praxeologies $\{A_1^\diamond, A_2^\diamond, ..., A_r^\diamond\}$, the design of a *finalised SRP* consists in finding a sequence of questions $\{Q_1, Q_2, ..., Q_k\}$ that could be derived from an initial question Q_0, the study of which is highly likely to activate a subset of the targeted praxeologies $A_i^\diamond$.

In the case of university education, the constraints imposed by the curricula are usually weaker than in secondary education, and lecturers have a greater degree of freedom to select and rearrange the subject matter content. The research carried out by Barquero [2009] proposes a finalised SRP that covers almost all the mathematical content of a first year course of Natural Sciences degrees. In this case, the level of the scale of didactic codeterminacy addressed is the discipline one. A unique question about the study of the dynamics of populations was proposed to a group of first year students of a degree in Chemical Engineering during four consecutive academic years. The SRP was proposed as a "mathematical modelling workshop" running parallel to the normal course during the whole year. Its main aim was to establish appropriate conditions for mathematics to be learnt as a modelling tool, starting from a generating question in the domain of natural science and using some of the main praxeologies included in the course syllabus. The initial question was formulated as follows:

> Given the size of a population over previous periods of time, how can we predict the long-term behaviour of its size? What sort of assumptions about the population, its growth and its surroundings should be made? What kind of forecasts can be made and how to test them?

This question was specified with different populations: pheasants, fish and yeast. The first ones were modelled with discrete models and considered two cases: independent and mixed generations; with the third population a similar path was reproduced for the continuous case. The proposed SRP was divided into four branches (Fig. 4).

The design of the SRP also included the elaboration of a productive enough milieu to produce the emergence of the derived questions and the deconstruction and reconstruction of the new praxeological organisations that were required to help the inquiry progress. Some of these praxeological organisations were introduced by the lecturer in the "normal course"

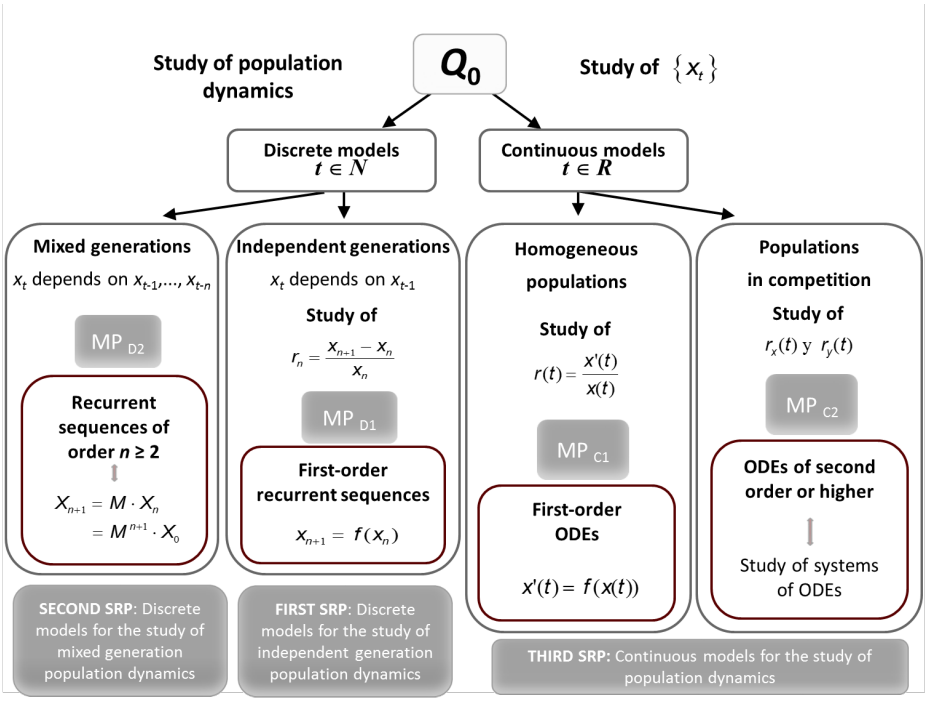

Figure 4. Structure of an SRP on population dynamics Barquero and Bosch [2015, p. 265]

(sequences, functions, derivatives, matrices, ordinary differential equations, etc.) and the students were asked to search for the more specific ones (Malthus and Verhulst models, transition and Leslie matrices, etc.) in the media available. During the four implementations of the SRP, the relationship between the course and the workshop evolved towards greater integration. It seemed as if the logic of the questions and answers derived from the SRP was gradually modifying the traditional organisation of the course contents since the lecturer agreed to introduce them when needed at the workshop, sometimes modifying the traditional organisation based on the theoretical coherence of the *syntheses*. Therefore, a workshop that was initially implemented as a complement to the course—to illustrate the main applications of the mathematical content introduced—started to acquire a prominent role, letting the course run as a nourisher of the inquiry process. This is in fact the ideal situation of the paradigm of questioning the world: a curriculum structured around a set of crucial questions to be studied and some flexible modules "on demand", aimed at supplying the inquiry processes, when required, with some of the more basic needed praxeologies.

Unfortunately, this was not the destiny of the SRP on population dynamics, even if its evolution towards a more traditional teaching format can be easily explained by some constraints originated at the higher levels of didactic codeterminacy (see Barquero, Bosch, and Gascón [2013].

3.4 Weakening the school and pedagogical constraints. The previous examples of SRPs were designed to ensure their viability by introducing new conditions and didactic resources given the common constraints encountered at the levels of schools, pedagogies and disciplines. What happens when we try to weaken these constraints and implement *more open* SRPs?

Quintana [2005] implemented an SRP at upper secondary level generated by the question of how to determine which of the three mobile companies operating at this moment in Spain was giving the best rate plans depending on the customer profile. This question was chosen by the research team to propose a situation where students had to build their own functional models and use the graph sketching as a tool to solve a problem (inequalities) that was too complex to be solved algebraically Rodríguez, Bosch, and Gascón [2008]. The SRP lasted for 18 two-hour sessions and was implemented as an after-class activity for two consecutive academic years with groups of 10-12 volunteer students. The mathematics teacher presented it to the students as a mathematical workshop that would help them with the subject.

It is worth mentioning that this SRP, one of the first to be experienced, revealed important features about the traditional didactic contract that were implicitly assumed by the research team. Maybe the fact that the SRP was proposed as an optional after-class activity facilitated the participation of motivated students and weakened the school, the pedagogical and even the didactic constraints. Many unexpected outcomes appeared thanks to the students' initiative. For instance, during the very first session, once the class had raised some initial derived questions to approach the problem, and because there was no empirical information available (no Internet connection in the class), the students decided to invent some basic cases to start with. They were spontaneously creating their own exercises for a functional purpose! Another interesting anecdote to report is the teacher asking the students to stop comparing rates, once she saw that many of the most interesting functions had already been used and that the comparison was taking a lot of time. "There is no point in making comparisons if we do not compare everything with everything!", the students answered, thus reminding the teacher that the SRP was about providing an answer to the initial question, and not only about using functions to solve inequalities...

Another interesting and unexpected outcome was the fact that the students proposed to present the final report as an open interactive internet site where people could enter

their regular consumption and get advice in return. Three options were established depending on the kind of information required from the consumer: "normal people", "lazy people" and "very lazy people" —the last one only asking for the minutes of conversation per day and the number of messages sent. Also, in one of the editions, the students decided to use their own invoices as a validation of their final answers, thus enriching the expected work with functions with some basic statistics description. Finally, surprised at the complexity of the work involved, they decided to write a letter to the Minister for Economy and Finance to complain about the consumers' situation and the opacity of the information provided by companies. This illustrates how the inquiry into a question soon penetrates activity domains of different sorts, breaking the limits established at schools between disciplines and between domains within a discipline.

4 The evolution of SRPs: integrating didactic tools into the inquiry process

It seems clear that the paradigm of questioning the world represents important changes in the organisation of study processes at the different levels of the scale of didactic codeterminacy. Those at the level of civilisations are possibly the most hidden ones, since they correspond to beliefs or assumptions that are difficult to identify, unless we move to another civilisation, through the space or the time. The last case presented shows that the act of questioning, of posing queries about any aspect of our surrounding reality, has not always been assumed with normality by all civilisations and is still not clearly approved in all domains. It is not clear either that the access to any kind of labelled answer $A^\diamond$ is seen as possible or appropriate for everybody at any time and in all the domains: each civilisation has its own forbidden spaces and implicit regulations. The hierarchy established among different types of knowledge—some being noble, others plebeian—is another variable to take into account. Societies also establish certain compartments in the organisation of knowledge and not everybody can easily move from one part to another. They also promote a given model of teacher that makes erudition prevail over inquiry, and do not succeed in making education evolve beyond the paradigm of visiting works: curricula formulated as lists of works, individual conception of learning, final school examinations based on tightly identified content, etc.

However, changes performed at the higher levels of the scale will remain limited if they do not come with the corresponding modifications at the lower levels. The inclusion of "competencies" as a key tool to impulse university teaching renovation shows the limitations of proposals that do not easily surpass the pedagogical level... But the lower levels of the scale introduced in figure 2 correspond to the paradigm of visiting works, where didactic systems $S(X, Y, Q)$ are established around previously determined pieces of knowledge

located in relation to a given discipline. This cannot be assumed in the paradigm of questioning the world, where didactic systems are not formed around some selected works but a set of selected questions. In this case, the scale of levels of codeterminacy should end at the level of the didactic system, since what appears below will not only depend on the type of question addressed, but also on the decisions made by the inquirers about the possible works that are candidates to provide partial answers to the derived questions. Moreover, the choice of disciplines, domains or sectors where some existent labelled answers $A_i^\diamond$ could be found is not a simple issue and, in any case, should be included as a question to deal with in the inquiry process. And it is not always a simple issue: let us just remember that, in World War 2, it took the British secret intelligence services a certain amount of time to associate the problem of deciphering the Nazi codes with the discipline of mathematics, a question that was traditionally associated with linguistics...

Therefore, in the paradigm of questioning the world, questions do not belong to any pre-established field of knowledge. Moreover, it is part of the inquiry process to investigate the possible sources of useful answers and, in particular, to mix praxeologies of a different nature, size and degree of "honourability". Thus, the specific levels corresponding to the given disciplines have to be located below—or after—the didactic system (Fig. 5).

Humanity
↑↓
Civilisations
↑↓
Societies
↑↓
Schools
↑↓
Pedagogies
↑↓
Didactic system $S(X, Y, Q)$
↑↓
Disciplines ⇄ Sectors ⇄ Domains ⇄ Themes ⇄ Questions

Figure 5. Scale of codeterminacy in the paradigm of questioning the world

An important constraint the experimented SRPs have revealed is precisely related to this intermediate level of the didactic system in the scale. It refers to a certain lack of knowledge resources to guide the inquiry that are not easy to locate in any of the official

descriptions of the disciplines. We have already mentioned this issue in the description of the SRPs' chronogenesis, concerning the difficulties of proposing explicit milestones to structure the inquiry process: teachers, together with students, need to discern where they are, what has been done so far and what seems to remain to be done at some crucial moments of the inquiry. When the inquiry is not a pre-established path—which it rarely is—the words and concepts to describe its main steps are not always available. In the paradigm of visiting works, the evolution of teaching processes can be formulated in terms of the praxeologies that have been visited, by naming their main components, the practical ones as well as the theoretical ones. The logos of the praxeologies provide descriptions of their main elements and present already-made assemblages of types of tasks and techniques, of techniques and properties or theorems, etc. As the study programme is predetermined, there are words, expressions and labels to designate the paths followed during the visits carried out in the study process: "We have covered limits of functions, we can now start with derivatives".

The situation is very different when one is in the middle of a long inquiry process. There is no "official" discourse to *name* the elements of the inquiry process: the provisional results obtained, the questions derived, the paths selected and those that have been ruled out, etc. The work carried out in an SRP is always in need of new words, concepts and discourses. It is also in need of didactic—or epistemological—resources, for instance to manage the media-milieu dialectic, now that the school can no longer ignore that the access to external and unfiltered information is absolutely unavoidable.

The currently available school, pedagogical and didactic resources are unable to support the teachers' and the students' work in the same way textbooks, treatises, encyclopaedias, documentaries, etc. do in the paradigm of visiting works. In our first explorations of the ecology of the paradigm of questioning the world, it was up to teachers and students to elaborate their own narrative of the inquiry process, and to establish their own milestones to mark the path and set the pace. In the last experimented SRPs Florensa, Bosch, and Gascón [2016], some of the elements of the Herbartian schema, such as the questions-answers maps, are starting being used as explicit tools for teachers and students to manage the inquiry process. It is not impossible than others, like the media-milieu dialectics for instance, will also turn out to be productive at this respect. The lack of epistemological resources to manage inquiry processes seems to be one of the main open problems raised by the research on SRPs. Its solution will certainly require the contribution of scholars of different fields—and also mathematicians—to create new knowledge means to fill the gap. Our current efforts go in this direction.

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