

## CANNON–THURSTON MAPS

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### Abstract

We give an overview of the theory of Cannon-Thurston maps which forms one of the links between the complex analytic and hyperbolic geometric study of Kleinian groups. We also briefly sketch connections to hyperbolic subgroups of hyperbolic groups and end with some open questions.

### 1 Kleinian groups and limit sets

The Lie group  $PSL_2(\mathbb{C})$  can be viewed from three different points of view:

1. As a Lie group or a matrix group (**group-theoretic**).
2. As the isometry group of hyperbolic 3-space  $\mathbf{H}^3$ —the upper half space  $\{(x, y, z) : z > 0\}$  equipped with the metric  $ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$ , or equivalently the open ball  $\{(x, y, z) : (x^2 + y^2 + z^2) = r^2 < 1\}$  equipped with the metric  $ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1-r^2)^2}$  (**geometric**).
3. As the group of Möbius transformations of the Riemann sphere  $\hat{\mathbb{C}}$  (**complex dynamic/analytic**).

A finitely generated discrete subgroup  $\Gamma \subset PSL_2(\mathbb{C})$  is called a *Kleinian group*. Depending on how we decide to look at  $PSL_2(\mathbb{C})$ , the group  $\Gamma$  can accordingly be thought of as a discrete subgroup of a Lie group, as the fundamental group of the complete hyperbolic 3-manifold  $\mathbf{H}^3/\Gamma$ , or in terms of its action by holomorphic automorphisms of  $\hat{\mathbb{C}}$ . If  $\Gamma$  is not virtually abelian, it is called *non-elementary*. Henceforth, unless explicitly stated otherwise, we shall assume that *all Kleinian groups in this article are non-elementary*. If  $\Gamma$  can be conjugated by an element of  $PSL_2(\mathbb{C})$  to be contained in  $PSL_2(\mathbb{R})$  it is referred

to as a *Fuchsian group*<sup>1</sup>. If  $\Gamma$  is abstractly isomorphic to  $\pi_1(S)$ , the fundamental group of a closed surface  $S$ , we shall refer to it as a *surface Kleinian group*.

In the 1960's Ahlfors and Bers studied deformations of Fuchsian surface groups in  $PSL_2(\mathbb{C})$ , giving rise to the theory of quasi-Fuchsian groups. Their techniques were largely complex analytic in nature. In the 1970's and 80's, the field was revolutionized by Thurston, who brought in a rich and varied set of techniques from three-dimensional hyperbolic geometry. A conjectural picture of the deep relationships between the analytic and geometric points of view was outlined by Thurston in his visionary paper [Thurston \[1982\]](#). Perhaps the most well-known problem (predating Thurston) in this line of study was the Ahlfors' measure zero conjecture, resolved in the last decade by Brock-Canary-Minsky [Brock, Canary, and Y. N. Minsky \[2012\]](#). Another (more topological) well-known problem that also predates Thurston asks if limit sets of Kleinian groups are locally connected. This is the specific problem that will concern us here. We need to fix some terminology and notation first. Identify the Riemann sphere  $\hat{\mathbb{C}}$  with the sphere at infinity  $S^2$  of  $\mathbf{H}^3$ . Thus,  $S^2$  encodes the 'ideal' boundary of  $\mathbf{H}^3$ , consisting of asymptote classes of geodesics. By adjoining  $S^2$  to  $\mathbf{H}^3$ , we obtain the closed 3-ball  $\mathbf{D}^3$ . The topology on  $S^2$  is the usual one induced by the round metric given by the angle subtended at the origin  $0 \in \mathbf{D}^3$ . The geodesics turn out to be semicircles meeting the boundary  $S^2$  at right angles.

**Definition 1.1.** *The limit set  $\Lambda_\Gamma$  of the Kleinian group  $\Gamma$  is the collection of accumulation points of a  $\Gamma$ -orbit  $\Gamma \cdot z$  for some (any)  $z \in \hat{\mathbb{C}}$ .*

The limit set  $\Lambda_\Gamma$  is independent of  $z$  and may be regarded as the locus of chaotic dynamics of  $\Gamma$  on  $\hat{\mathbb{C}}$ . For non-elementary  $\Gamma$  and any  $z \in \Lambda_\Gamma$ ,  $\Gamma \cdot z$  is dense in  $\Lambda_\Gamma$ . Hence  $\Lambda_\Gamma$  is the smallest closed non-empty  $\Gamma$ -invariant subset of  $\hat{\mathbb{C}}$ . If we take  $z \in \mathbf{H}^3$  instead, then the collection of accumulation points of any  $\Gamma$ -orbit  $\Gamma \cdot z \subset \mathbf{D}^3$  is also  $\Lambda_\Gamma$ .

**Definition 1.2.** *The complement of the limit set  $\hat{\mathbb{C}} \setminus \Lambda_\Gamma$  is called the domain of discontinuity  $\Omega_\Gamma$  of  $\Gamma$ .*

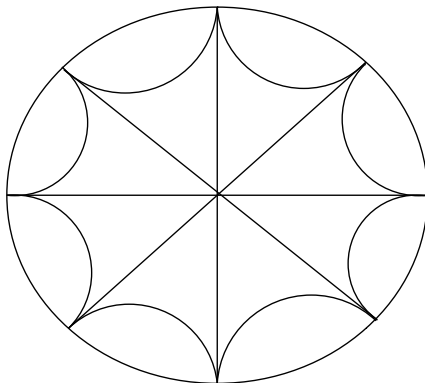
The Kleinian group  $\Gamma$  acts freely and properly discontinuously on  $\Omega_\Gamma$  and  $\Omega_\Gamma/\Gamma$  is a (possibly disconnected) Riemann surface.

**1.1 Fuchsian and Quasi-Fuchsian Groups.** We first give an explicit example of a Fuchsian group.

**An example of a Fuchsian group:** Consider the standard identification space description of the genus two orientable surface  $\Sigma_2$  as an octagon with edge labels  $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}$ . A hyperbolic metric on  $\Sigma_2$  is one where each point has a small neighborhood isometric to a small disk in  $\mathbf{H}^2$ . By the Poincaré polygon theorem, it suffices to find a

<sup>1</sup>Both Fuchsian and Kleinian groups were discovered by Poincaré.

regular hyperbolic octagon (with all sides equal and all angles equal) with each interior angle equal to  $\frac{2\pi}{8}$ . Now, the infinitesimal regular octagon at the tangent space to the origin has interior angles equal to  $\frac{3\pi}{4}$ . Also the ideal regular octagon in  $\mathbb{H}^2$  has all interior angles zero. See figure below.



By the Intermediate Value Theorem, there exists an intermediate regular octagon with all interior angles equal to  $\frac{\pi}{4}$ . The group  $G$  that results from side-pairing transformations corresponds to a Fuchsian group, or equivalently, a discrete faithful representation  $\rho$  of  $\pi_1(\Sigma_2)$  into  $PSL_2(\mathbb{R})$ . Equivalently we may regard  $\rho$  as a representation of  $\pi_1(\Sigma_2)$  into  $PSL_2(\mathbb{C})$  with image that can be conjugated to lie in  $PSL_2(\mathbb{R})$ . Alternately,  $\rho(\pi_1(\Sigma_2))$  preserves a totally geodesic plane in  $\mathbf{H}^3$ . The limit set of  $G = \rho(\pi_1(\Sigma_2))$  is then a round circle.

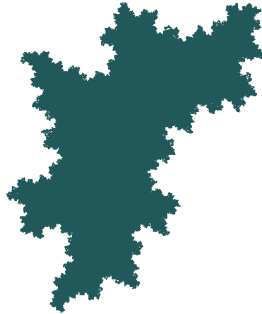
**Quasi-Fuchsian groups:** If we require the limit set to be only topologically a circle, i.e. a Jordan curve, then we obtain a more general class of Kleinian groups:

**Definition 1.3.** *Let  $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$  be a discrete faithful representation such that the limit set of  $G = \rho(\pi_1(S))$  is a Jordan curve in  $S^2$ . Then  $G$  is said to be quasi-Fuchsian. The collection of conjugacy classes of quasi-Fuchsian with the complex analytic structure inherited from  $PSL_2(\mathbb{C})$  is denoted as  $QF(S)$ .*

The domain of discontinuity  $\Omega$  of a quasi-Fuchsian  $G$  consists of two open invariant topological disks  $\Omega_1, \Omega_2$  in  $\hat{\mathbb{C}}$ . Hence the quotient  $\Omega/G$  is the disjoint union  $\Omega_1/G \sqcup \Omega_2/G$  and we have a map  $\tau : QF(S) \rightarrow Teich(S) \times Teich(S)$ , where  $Teich(S)$  denotes the Teichmüller space of  $S$ —the space of marked hyperbolic (or complex) structures on  $S$ . The *Bers simultaneous Uniformization Theorem* asserts:

**Theorem 1.4.**  $\tau : QF(S) \rightarrow Teich(S) \times Teich(S)$  is bi-holomorphic.

Thus, given any two conformal structures  $T_1, T_2$  on a surface, there is a unique discrete quasi-Fuchsian  $G$  (up to conjugacy) whose limit set  $\Lambda_G$  is topologically a circle, and the quotient of whose domain of discontinuity is  $T_1 \sqcup T_2$ . See figure below [Kabaya \[2016\]](#), where the inside and the outside of the Jordan curve correspond to  $\Omega_1, \Omega_2$ .



**1.2 Degenerate groups and the Ending Lamination Theorem.** Quasi-Fuchsian were studied by Ahlfors and Bers analytically as deformations of Fuchsian groups. [Thurston \[1980\]](#) introduced a new set of geometric techniques in the study.

**Definition 1.5.** *The convex hull  $CH_G$  of  $\Lambda_G$  is the smallest non-empty closed convex subset of  $\mathbb{H}^3$  invariant under  $G$ .*

*Let  $M = \mathbf{H}^3/G$ . The quotient of  $CH_G$  by  $G$  is called the convex core  $CC(M)$  of  $M$ .*

The convex hull  $CH_G$  can be constructed by joining all distinct pairs of points on  $\Lambda_G$  by bi-infinite geodesics, iterating this construction and finally taking the closure. It can also be described as the closure of the union of all ideal tetrahedra, whose vertices lie in  $\Lambda_G$ . The convex core  $CC(M)$  is homeomorphic to  $S \times [0, 1]$ .

The distance between the boundary components  $S \times \{0\}$  and  $S \times \{1\}$  in the convex core  $CC(M)$ , measured with respect to the hyperbolic metric, is a crude geometric measure of the complexity of the quasi Fuchsian group  $G$ . We shall call it the *thickness*  $t_G$  of  $CC(M)$ , or of the quasi Fuchsian group  $G$ . (We note here parenthetically that the notions of convex hull  $CH_G$  and convex core  $CC(M)$  go through for any Kleinian group  $G$  and the associated complete hyperbolic manifold  $\mathbf{H}^3/G$ .) For quasi-Fuchsian groups, we ask:

**Question 1.6.** *What happens as thickness tends to infinity?*

To address this question more precisely we need to introduce a topology on the space of representations.

**Definition 1.7.** *A sequence of representations  $\rho_n : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$  is said to converge algebraically to  $\rho_\infty : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$  if for all  $g \in \pi_1(S)$ ,  $\rho_n(g) \rightarrow \rho_\infty(g)$  in  $PSL_2(\mathbb{C})$ . The collection of conjugacy classes of discrete faithful representations of  $\pi_1(S)$  into  $PSL_2(\mathbb{C})$  equipped with the algebraic topology is denoted as  $AH(S)$ .*

It is not even clear a priori that, as  $t_G$  tends to infinity, limits exist in  $AH(S)$ . However, Thurston’s *Double Limit Theorem* Thurston [1986], M. Kapovich [2001], and Otal [2001] guarantees that if we have a sequence  $G_n$  with thickness  $t_{G_n}$  tending to infinity, subsequential limits (in the space of discrete faithful representations with a suitable topology, see Definition 5.7) do in fact exist.

### Geodesic Laminations:

**Definition 1.8.** *A geodesic lamination on a hyperbolic surface is a foliation of a closed subset with geodesics, i.e. it is a closed set given as a disjoint union of geodesics (closed or bi-infinite).*

A geodesic lamination on a surface may further be equipped with a transverse (non-negative) measure to obtain a *measured lamination*. The space of measured (geodesic) laminations on  $S$  is then a positive cone in a vector space and is denoted as  $\mathfrak{ML}(S)$ . It can be projectivized to obtain the space of projectivized measured laminations  $\mathcal{PML}(S)$ . It was shown by Thurston Fathi, Laudenbach, and Poenaru [1979] that  $\mathcal{PML}(S)$  is homeomorphic to a sphere and can be adjoined to  $Teich(S)$  compactifying it to a closed ball.

**Definition 1.9.** *Thurston [1980, Definition 8.8.1] A pleated surface in a hyperbolic three-manifold  $N$  is a complete hyperbolic surface  $S$  of finite area, together with an isometric map  $f : S \rightarrow N$  such that every  $x \in S$  is in the interior of some geodesic segment (in  $S$ ) which is mapped by  $f$  to a geodesic segment (in  $N$ ). Also,  $f$  must take every cusp (corresponding to a maximal parabolic subgroup) of  $S$  to a cusp of  $N$ .*

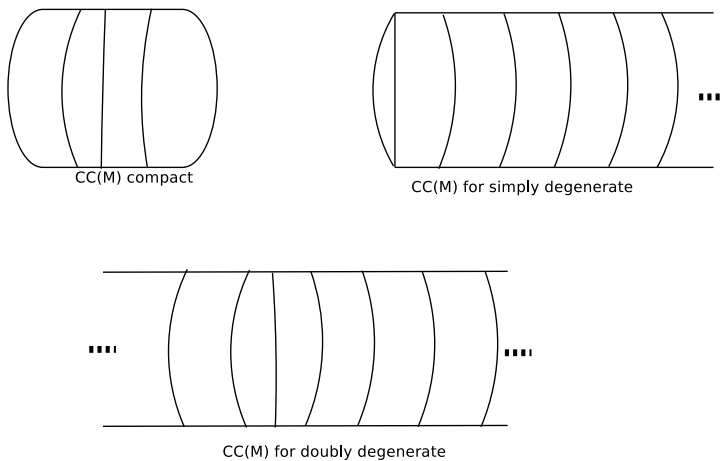
*The pleating locus of the pleated surface  $f : S \rightarrow M$  is the set  $\gamma \subset S$  consisting of those points in the pleated surface which are in the interior of unique geodesic segments mapped to geodesic segments.*

**Proposition 1.10.** *Thurston [ibid., Proposition 8.8.2] The pleating locus  $\gamma$  is a geodesic lamination on  $S$ . The map  $f$  is totally geodesic in the complement of  $\gamma$ .*

Thurston further shows Thurston [ibid., Ch. 8] that the boundary components of the convex core  $CC(M)$  are pleated surfaces.

As thickness (Question 1.6) tends to infinity, the convex core may converge (up to extracting subsequences) to one of two kinds of convex manifolds  $CC(M_\infty)$  (see schematic diagram below):

1.  $CC(M_\infty)$  is homeomorphic to  $S \times [0, \infty)$ . Here  $S \times \{0\}$  is the single boundary component of the convex core  $CC(M_\infty)$ . Such a manifold is called *simply degenerate* and the corresponding Kleinian group a simply degenerate surface Kleinian group. In this case the limit set  $\Lambda_G$  is a dendrite (topologically a tree) and the domain of discontinuity  $\Omega_\Gamma$  is homeomorphic to an open disk with  $\Omega_\Gamma/G$  a Riemann surface homeomorphic to  $S \times \{0\}$ . The end of  $M_\infty$  facing  $\Omega_\Gamma/G$  is called a *geometrically finite end*.
2.  $CC(M_\infty)$  is homeomorphic to  $S \times (-\infty, \infty)$ . Such a manifold is called *doubly degenerate* and the corresponding Kleinian group a doubly degenerate surface Kleinian group. In this case the limit set  $\Lambda_G$  is all of  $S^2$  and the domain of discontinuity  $\Omega_\Gamma$  is empty.



The ends (one for simply degenerate and two for doubly degenerate) of  $CC(M_\infty)$  are called the *degenerate end(s)* of  $M$ . Thurston [1980] and Bonahon [1986] show that any such degenerate end  $E$  is *geometrically tame*, i.e. there is a sequence of simple closed curves  $\{\sigma_n\}$  on  $S$  such that their geodesic realizations in  $CC(M_\infty)$  exit  $E$  as  $n$  tends to  $\infty$ . Further Thurston [1980, Ch. 9], the limit of any such exiting sequence (in  $\mathcal{PM}\mathcal{L}(S)$ ; the topology on  $\mathcal{PM}\mathcal{L}(S)$  is quite close to the Hausdorff topology on  $S$ ) is a unique lamination  $\lambda$  called the *ending lamination* of  $E$ . Thus a doubly degenerate manifold has two ending laminations, one for each degenerate end, while a simply degenerate manifold has a geometrically finite end corresponding to a Riemann surface  $\Omega_\Gamma/G (\in Teich(S))$  at infinity and a degenerate end corresponding to an ending lamination.

These two pieces of information—Riemann surfaces at infinity and ending laminations—give the *end-invariants* of  $M$ . The ending lamination for a geometrically infinite end does

not depend on the reference hyperbolic structure on  $S$ . It may be regarded as a purely topological piece of data associated to such an end.

*We may thus think of the ending lamination as the analog, in the case of geometrically infinite ends, of the conformal structure at infinity for a geometrically infinite end.*

The Ending Lamination Theorem of Brock–Canary–Minsky then takes the place of the Bers’ simultaneous uniformization theorem and asserts:

**Theorem 1.11.** *Y. Minsky [2010] and Brock, Canary, and Y. N. Minsky [2012] Let  $M$  be a simply or doubly degenerate manifold. Then  $M$  is determined up to isometry by its end-invariants.*

Thus, the Ending Lamination Theorem justifies the following and may be considered an analog of Mostow Rigidity for infinite covolume Kleinian groups.

**Slogan 1.12.** *Topology implies Geometry.*

In order to complete the picture of the Ahlfors–Bers theory to study degenerate surface Kleinian groups  $G$  in terms of the dynamics of  $G$  on the Riemann sphere  $\hat{\mathbb{C}}$ , the following issue remains to be addressed:

**Question 1.13.** *Can the data of the ending lamination(s) be extracted from the dynamics of  $G$  on  $\hat{\mathbb{C}}$ ?*

In more informal terms,

**Question 1.14.** *Is the geometric object “at infinity” of the quotient manifold  $\mathbf{H}^3/\Gamma$  (i.e. the ending lamination) determined by the dynamics of  $\Gamma$  at infinity (i.e. the action of  $\Gamma$  on  $S^2$ )?*

We will make these questions precise below. The attempt to make [Question 1.13](#) precise brings us to the following.

## 2 Cannon–Thurston maps

**2.1 The main theorem for closed surface Kleinian groups.** In [Thurston \[1982, Problem 14\]](#), Thurston raised the following question, which is at the heart of the work we discuss here:

**Question 2.1.** *Suppose  $\Gamma$  has the property that  $(\mathbf{H}^3 \cup \Omega_\Gamma)/\Gamma$  is compact. Then is it true that the limit set of any other Kleinian group  $\Gamma'$  isomorphic to  $\Gamma$  is the continuous image of the limit set of  $\Gamma$ , by a continuous map taking the fixed points of an(y) element  $\gamma$  to the fixed points of the corresponding element  $\gamma'$ ?*

A special case of [Question 2.1](#) was answered affirmatively in seminal work of [Cannon and Thurston \[1985, 2007\]](#):

**Theorem 2.2.** *Cannon and Thurston [2007] Let  $M$  be a closed hyperbolic 3-manifold fibering over the circle with fiber  $\Sigma$ . Let  $\widetilde{\Sigma}$  and  $\widetilde{M}$  denote the universal covers of  $F$  and  $M$  respectively. After identifying  $\widetilde{\Sigma}$  (resp.  $\widetilde{M}$ ) with  $\mathbf{H}^2$  (resp.  $\mathbf{H}^3$ ), we obtain the compactification  $\mathbb{D}^2 = \mathbf{H}^2 \cup S^1$  (resp.  $\mathbb{D}^3 = \mathbf{H}^3 \cup S^2$ ) by attaching the circle  $S^1$  (resp. the sphere  $S^2$ ) at infinity. Let  $i : \Sigma \rightarrow M$  denote the inclusion map of the fiber and  $\tilde{i} : \widetilde{\Sigma} \rightarrow \widetilde{M}$  the lift to the universal cover. Then  $\tilde{i}$  extends to a continuous map  $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$ .*

An amazing implication of [Theorem 2.2](#) is that  $\widetilde{\Sigma}$  is an embedded disk in (the ball model) of  $\mathbf{H}^3$  such that its boundary on the sphere  $S^2$  is space-filling! See the following diagram by Thurston [Thurston \[1982, Figure 10\]](#) that illustrates ‘a pattern of identification of a circle, here represented as the equator, whose quotient is topologically a sphere. This defines, topologically a sphere-filling curve.’



A version of [Question 2.1](#) was raised by Cannon and Thurston in the context of closed surface Kleinian groups:

**Question 2.3.** *Cannon and Thurston [2007, Section 6] Suppose that a closed surface group  $\pi_1(S)$  acts freely and properly discontinuously on  $\mathbf{H}^3$  by isometries such that the quotient manifold has no accidental parabolics (here this just means that the image of  $\pi_1(S)$  in  $PSL_2(\mathbb{C})$  has no parabolics). Does the inclusion  $\tilde{i} : \widetilde{S} \rightarrow \mathbf{H}^3$  extend continuously to the boundary?*

Continuous boundary extensions as in [Question 2.3](#), if they exist, are called *Cannon-Thurston maps*. [Question 2.3](#) is intimately related to a much older question c.f. [Abikoff \[1976\]](#) asking if limit sets are locally connected:

**Question 2.4.** *Let  $\Gamma$  be a finitely generated Kleinian group such that the limit set  $\Lambda_\Gamma$  is connected. Is  $\Lambda_\Gamma$  locally connected?*



It is shown in Cannon and Thurston [2007] that for simply degenerate surface Kleinian groups, Questions 2.3 and 2.4 are equivalent, via the Caratheodory extension Theorem.

The following Theorem of Mj [2014a] Mj [2014b] answers questions 2.3 and 2.4 affirmatively:

**Theorem 2.5.** *Let  $\rho(\pi_1(S)) = G \subset PSL_2(\mathbb{C})$  be a simply or doubly degenerate (closed) surface Kleinian group. Let  $M = \mathbb{H}^3/G$  and  $i : S \rightarrow M$  be an embedding inducing a homotopy equivalence. Let  $\tilde{i} : \tilde{S} \rightarrow \mathbb{H}^3$  denote a lift of  $i$  between universal covers. Let  $\mathbb{D}^2, \mathbb{D}^3$  denote the compactifications. Then a Cannon-Thurston map  $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$  exists.*

*Let  $\partial i : S^1 \rightarrow S^2$  denote the restriction of  $\hat{i}$  to the ideal boundaries. Then for  $p \neq q$ ,  $\partial i(p) = \partial i(q)$  if and only if  $p, q$  are the ideal end-points of a leaf of an ending lamination or ideal end-points of a complementary ideal polygon of an ending lamination.*

The second part of Theorem 2.5 shows that the data of the ending lamination can be recovered from the Cannon-Thurston map and so we have an affirmative answer to Question 1.13. In conjunction with the Ending Lamination Theorem 1.11, this establishes the slogan:

**Slogan 2.6.** *Dynamics on the limit set determines geometry in the interior.*

A number of authors have contributed to the resolution of the above questions. Initially it was believed Abikoff [1976] that Question 2.4 had a negative answer for simply degenerate Kleinian groups. Floyd [1980] proved the corresponding theorem for geometrically finite Kleinian groups. Then in the early 80's Cannon and Thurston [1985] proved Theorem 2.2. This was extended by Y. N. Minsky [1994], Klarreich [1999], Alperin, Dicks, and Porti [1999], B. H. Bowditch [2013] and B. H. Bowditch [2007], McMullen [2001], Miyachi [2002] and the author 1998; 2010; 2009-2010; 2016 for various special cases. The general surface group case was accomplished in Mj [2014a] and the general Kleinian group case in Mj [2017a].

**2.2 Geometric Group Theory.** We now turn to a generalization of Question 2.3 to a far more general context. After the introduction of hyperbolic metric spaces by Gromov [1987], Question 2.3 was extended by the author Mitra [1997b], Bestvina [2004], and Mitra [1998c] to the context of a hyperbolic group  $H$  acting freely and properly discontinuously by isometries on a hyperbolic metric space  $X$ . Any hyperbolic  $X$  has a (Gromov) boundary  $\partial X$  given by asymptote-classes of geodesics. Adjoining  $\partial X$  to  $X$  we get the Gromov compactification  $\widehat{X}$ .

There is a natural map  $i : \Gamma_H \rightarrow X$ , sending vertices of  $\Gamma_H$  to the  $H$ -orbit of a point  $x \in X$ , and connecting images of adjacent vertices by geodesic segments in  $X$ . Let  $\widehat{\Gamma}_H, \widehat{X}$  denote the Gromov compactification of  $\Gamma_H, X$  respectively. The analog of Question 2.3 is the following:

**Question 2.7.** *Does  $i : \Gamma_H \rightarrow X$  extend continuously to a map  $\hat{i} : \widehat{\Gamma}_H \rightarrow \widehat{X}$ ?*

Continuous extensions as in [Question 2.7](#) are also referred to as *Cannon-Thurston maps* and make sense when  $\Gamma_H$  is replaced by an arbitrary hyperbolic metric space  $Y$ . A simple and basic criterion for the existence of Cannon-Thurston maps was established in [Mitra \[1998a,b\]](#):

**Lemma 2.8.** *Let  $i : (Y, y) \rightarrow (X, x)$  be a proper map between (based) Gromov-hyperbolic spaces. A continuous extension (also called a Cannon-Thurston map)  $\hat{i} : \widehat{Y} \rightarrow \widehat{X}$  exists if and only if the following holds:*

*There exists a non-negative proper function  $M : \mathbb{N} \rightarrow \mathbb{N}$ , such that if  $\lambda = [a, b]_Y$  is a geodesic lying outside an  $N$ -ball around  $y$ , then any geodesic segment  $[i(a), i(b)]_X$  in  $X$  joining  $i(a), i(b)$  lies outside the  $M(N)$ -ball around  $x = i(y)$ .*

In the generality above [Question 2.7](#) turns out to have a negative answer. An explicit counterexample to [Question 2.7](#) was recently found by [Baker and T. R. Riley \[2013\]](#) in the context of small cancellation theory. The counterexample uses [Lemma 2.8](#) to rule out the existence of Cannon-Thurston maps. Further, [Matsuda and Oguni \[2014\]](#) further developed Baker and Riley’s counterexample to show that given a(ny) non-elementary hyperbolic group  $H$ , there exists hyperbolic group  $G$  such that  $H \subset G$  and there is no Cannon-Thurston map for the inclusion. We shall furnish positive answers to [Question 2.7](#) in a number of special cases in [Section 6](#).

### 3 Closed 3-manifolds

**3.1 3-manifolds fibering over the circle.** We start by giving a sketch of the proof of [Theorem 2.2](#), The proof is coarse-geometric in nature and follows [Mitra \[1998a,b\]](#). We recall a couple of basic Lemmata we shall be needing from [Mitra \[1998b\]](#). The following says that nearest point projection onto a geodesic in a hyperbolic space is coarsely Lipschitz.

**Lemma 3.1.** *Let  $(X, d)$  be a  $\delta$ -hyperbolic metric space. Then there exists a constant  $C \geq 1$  such that the following holds:*

*Let  $\lambda \subset X$  be a geodesic segment and let  $\Pi : X \rightarrow \lambda$  be a nearest point projection. Then  $d(\Pi(x), \Pi(y)) \leq Cd(x, y)$  for all  $x, y \in X$ .*

The next Lemma says that nearest point projections and quasi-isometries almost commute.

**Lemma 3.2.** *Let  $(X, d)$  be a  $\delta$ -hyperbolic metric space. Given  $(K, \epsilon)$ , there exists  $C$  such that the following holds:*

Let  $\lambda = [a, b]$  be a geodesic segment in  $X$ . Let  $p \in X$  be arbitrary and let  $q$  be a nearest point projection of  $p$  onto  $\lambda$ . Let  $\phi$  be a  $(K, \epsilon)$ -quasi-isometry from  $X$  to itself and let  $\Phi(\lambda) = [\phi(a), \phi(b)]$  be a geodesic segment in  $X$  joining  $\phi(a), \phi(b)$ . Let  $r$  be a nearest point projection of  $\phi(p)$  onto  $\Phi(\lambda)$ . Then  $d(r, \phi(q)) \leq C$ .

*Sketch of Proof:* The proof of [Lemma 3.2](#) follows from the fact that a geodesic tripod  $T$  (built from  $[a, b]$  and  $[p, q]$ ) is quasiconvex in hyperbolic space and that a quasi-isometric image  $\phi(T)$  of  $T$  lies close to a geodesic tripod  $T'$  built from  $[\phi(a), \phi(b)]$  and  $[\phi(p), r]$ . Hence the image  $\phi(q)$  of the centroid  $q$  of  $T$  lies close to the centroid  $r$  of  $T'$ .  $\square$

**3.2 The key tool: hyperbolic ladder.** The key idea behind the proof of [Theorem 2.2](#) and its generalizations in [Mitra \[1998a,b\]](#) is the construction of a *hyperbolic ladder*  $\mathfrak{L}_\lambda \subset \widetilde{M}$  for any geodesic in  $\widetilde{\Sigma}$ . The universal cover  $\widetilde{M}$  fibers over  $\mathbb{R}$  with fibers  $\widetilde{\Sigma}$ . Since the context is geometric group theory, we discretize this as follows. Replace  $\widetilde{\Sigma}$  and  $\widetilde{M}$  by quasi-isometric models in the form of Cayley graphs  $\Gamma_{\pi_1(\Sigma)}$  and  $\Gamma_{\pi_1(M)}$  respectively. Let us denote  $\Gamma_{\pi_1(\Sigma)}$  by  $Y$  and  $\Gamma_{\pi_1(M)}$  by  $X$ . The projection of  $\widetilde{M}$  to the base  $\mathbb{R}$  is discretized accordingly giving a model that can be thought of as (and is quasi-isometric to) a *tree of spaces*, where

1.  $T$  is the simplicial tree underlying  $\mathbb{R}$  with vertices at  $\mathbb{Z}$ .
2. All the vertex and edge spaces are (intrinsically) isometric to  $Y$ .
3. The edge space to vertex space inclusions are qi-embeddings.

We summarize this by saying that  $X$  is a tree  $T$  of spaces satisfying the *qi-embedded condition* [Bestvina and Feighn \[1992\]](#).

Given a geodesic segment  $[a, b] = \lambda = \lambda_0 \subset Y$ , we now sketch the promised construction of the *ladder*  $\mathfrak{L}_\lambda \subset X$  containing  $\lambda$ . Index the vertices by  $n \in \mathbb{Z}$ . Since the edge-to-vertex space inclusions are quasi-isometries, this induces a quasi-isometry  $\phi_n$  from the vertex space  $Y_n$  to the vertex space  $Y_{n+1}$  for  $n \geq 0$ . A similar quasi-isometry  $\phi_{-n}$  exists from  $Y_{-n}$  to the vertex space  $Y_{-(n+1)}$ . These quasi-isometries are defined on the vertex sets of  $Y_n$ ,  $n \in \mathbb{Z}$ .  $\phi_n$  induces a map  $\Phi_n$  from geodesic segments in  $Y_n$  to geodesic segments in  $Y_{n+1}$  for  $n \geq 0$  by sending a geodesic in  $Y_n$  joining  $a, b$  to a geodesic in  $Y_{n+1}$  joining  $\phi_n(a), \phi_n(b)$ . Similarly, for  $n \leq 0$ . Inductively define:

- $\lambda_{j+1} = \Phi_j(\lambda_j)$  for  $j \geq 0$ ,
- $\lambda_{-j-1} = \Phi_{-j}(\lambda_{-j})$  for  $j \geq 0$ ,
- $\mathfrak{L}_\lambda = \bigcup_j \lambda_j$ .

$\mathfrak{L}_\lambda$  turns out to be quasiconvex in  $X$ . To prove this, we construct a coarsely Lipschitz retraction  $\Pi_\lambda : \bigcup_j Y_j \rightarrow \mathfrak{L}_\lambda$  as follows.

On  $Y_j$  define  $\pi_j(y)$  to be a nearest-point projection of  $y$  onto  $\lambda_j$  and define

$$\Pi_\lambda(y) = \pi_j(y), \text{ for } y \in Y_j.$$

The following theorem asserts that  $\Pi_\lambda$  is coarsely Lipschitz.

**Theorem 3.3.** *Mitra [1998a,b] and Mj [2010]* There exists  $C \geq 1$  such that for any geodesic  $\lambda \subset Y$ ,

$$d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_X(x, y)$$

for  $x, y \in \bigcup_i Y_i$ .

*Sketch of Proof:*

The proof requires only the hyperbolicity of  $Y$ , but not that of  $X$ . It suffices to show that for  $d_X(x, y) = 1$ ,  $d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C$ . Thus  $x, y$  may be thought of as

1. either lying in the same  $Y_j$ . This case follows directly from [Lemma 3.1](#).
2. or lying vertically one just above the other. Then (up to a bounded amount of error), we can assume without loss of generality, that  $y = \phi_j(x)$ . This case now follows from [Lemma 3.2](#).

Since a coarse Lipschitz retract of a hyperbolic metric space is quasiconvex, we immediately have:

**Corollary 3.4.** *If  $(X, d_X)$  is hyperbolic, there exists  $C \geq 1$  such that for any  $\lambda$ ,  $\mathfrak{L}_\lambda$  is  $C$ -quasiconvex.*

Note here that we have not used any feature of  $Y$  except its hyperbolicity. In particular, we do not need the specific condition that  $Y = \widetilde{\Sigma}$ . We are now in a position to prove a generalization of [Theorem 2.2](#).

**Theorem 3.5.** *Mj [2010]* Let  $(X, d)$  be a hyperbolic tree  $(T)$  of hyperbolic metric spaces satisfying the qi-embedded condition, where  $T$  is  $\mathbb{R}$  or  $[0, \infty)$  with vertex and edge sets  $Y_j$  as above,  $j \in \mathbb{Z}$ . Assume (as above) that the edge-to-vertex inclusions are quasi-isometries. For  $i : Y_0 \rightarrow X$  there is a Cannon-Thurston map  $\hat{i} : \widehat{Y}_0 \rightarrow \widehat{X}$ .

*Proof.* Fix a basepoint  $y_0 \in Y_0$ . By [Lemma 2.8](#) and quasiconvexity of  $\mathfrak{L}_\lambda$  ([Corollary 3.4](#)), it suffices to show that for all  $M \geq 0$  there exists  $N \geq 0$  such that if a geodesic segment  $\lambda$  lies outside the  $N$ -ball about  $y_0 \in Y_0$ , then  $\mathfrak{L}_\lambda$  lies outside the  $M$ -ball around  $y_0 \in X$ . Equivalently, we need a proper function  $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ .

Since  $Y_0$  is properly embedded in  $X$ , there exists a proper function  $g : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\lambda$  lies outside the  $g(N)$ -ball about  $y_0 \in X$ .

Let  $p$  be any point on  $\mathcal{L}_\lambda$ . Then  $p = p_j \in Y_j$  for some  $j$ . Assume without loss of generality that  $j \geq 0$ . It is not hard to see that there exists  $C_0$ , depending only on  $X$ , such that for any such  $p_j$ , there exists  $p_{j-1} \in Y_{j-1}$  with  $d(p_j, p_{j-1}) \leq C_0$ . It follows inductively that there exists  $y \in \lambda = \lambda_0$  such that  $d_X(y, p) \leq C_0 j$ . Hence, by the triangle inequality,  $d_X(y_0, p) \geq g(N) - C_0 j$ .

Next, looking at the ‘vertical direction’,  $d_X(y_0, p) \geq j$  and hence

$$d_X(y_0, p) \geq \max(g(N) - C_0 j, j) \geq \frac{g(N)}{A + 1}.$$

Defining  $M(N) = \frac{g(N)}{A+1}$ , we see that  $M(N)$  is a proper function of  $N$  and we are done.  $\square$

**3.3 Quasiconvexity.** The structure of Cannon-Thurston maps in [Section 3.1](#) can be used to establish quasiconvexity of certain subgroups of  $\pi_1(M)$ . Let  $H \subset \pi_1(\Sigma)$  be a finitely generated infinite index subgroup of the fiber group. Then, due to the LERF property for surface subgroups, a Theorem of Scott [P. Scott \[1978\]](#), there is a finite sheeted cover where  $H$  is geometric, i.e. it is carried by a proper embedded subsurface of (a finite sheeted cover of)  $\Sigma$ . But such a proper subsurface cannot carry a leaf of the stable or unstable foliations  $\mathcal{F}_s$  or  $\mathcal{F}_u$ . This gives us the following Theorem of Scott and Swarup:

**Theorem 3.6.** *G. P. Scott and Swarup [1990] Let  $M$  be a closed hyperbolic 3-manifold fibering over the circle with fiber  $\Sigma$ . Let  $H \subset \pi_1(\Sigma)$  be a finitely generated infinite index subgroup of the fiber group in  $\pi_1(M)$ . Then  $H$  is quasiconvex in  $\pi_1(M)$ .*

[Theorem 3.6](#) has been generalized considerably to the context of convex cocompact subgroups of the mapping class group and  $Out(F_n)$  by a number of authors [Dowdall, Kent, and C. J. Leininger \[2014\]](#), [Dowdall, I. Kapovich, and Taylor \[2016\]](#), [Dowdall and Taylor \[2018, 2017\]](#), and [Mj and Rafi \[2015\]](#).

## 4 Kleinian surface groups: Model Geometries

In this section we shall describe a sequence of models for degenerate ends of 3-manifolds following [Y. N. Minsky \[2001, 1994\]](#) and [Mj \[2010, 2009-2010, 2016\]](#) and [Y. Minsky \[2010\]](#), [Brock, Canary, and Y. N. Minsky \[2012\]](#), and [Mj \[2014a\]](#) and indicate how to generalize the ladder construction of [Section 3.2](#) incorporating electric geometry [Farb \[1998\]](#). Let  $X$  be a hyperbolic metric space, e.g.  $\mathbf{H}^3$ . Let  $\mathcal{H}_X$  be a collection of disjoint convex subsets. Roughly speaking, electrification equips each element of  $\mathcal{H}_X$  with the zero metric,

while preserving the metric on  $X \setminus (\bigcup_{H \in \mathcal{H}_X} H)$ . The resulting electrified space  $\mathcal{E}(X, \mathcal{H}_X)$  is still Gromov hyperbolic under extremely mild conditions and hyperbolic geodesics in  $X$  can be recovered from electric geodesics in the electrified space  $\mathcal{E}(X, \mathcal{H}_X)$ . This will allow us to establish the existence of Cannon-Thurston maps. We shall focus on closed surfaces and follow the summary in [Lecuire and Mj \[2016\]](#) for the exposition.

The topology of each building block is simple: it is homeomorphic to  $S \times [0, 1]$ , where  $S$  is a closed surface of genus greater than one. Geometrically, the top and bottom boundary components in the first three model geometries are uniformly bi-Lipschitz to a fixed hyperbolic structure on  $S$ . Assume therefore that  $S$  is equipped with such a fixed hyperbolic structure. We do so henceforth. The different types of geometries of the blocks make for different model geometries of ends.

**Definition 4.1.** *A model  $E_m$  is said to be built up of blocks of some prescribed geometries glued end to end, if*

1.  $E_m$  is homeomorphic to  $S \times [0, \infty)$
2. There exists  $L \geq 1$  such that  $S \times [i, i + 1]$  is  $L$ -bi-Lipschitz to a block of one of the three prescribed geometries: bounded,  $i$ -bounded or amalgamated (see below).

$S \times [i, i + 1]$  will be called the  $(i + 1)$ th block of the model  $E_m$ .

The thickness of the  $(i + 1)$ th block is the length of the shortest path between  $S \times \{i\}$  and  $S \times \{i + 1\}$  in  $S \times [i, i + 1] (\subset E_m)$ .

**4.1 Bounded geometry.** [Y. N. Minsky \[2001, 1994\]](#) calls an end  $E$  of a hyperbolic 3-manifold to be of bounded geometry if there are no arbitrarily short closed geodesics in  $E$ .

**Definition 4.2.** *Let  $B_0 = S \times [0, 1]$  be given the product metric. If  $B$  is  $L$ -bi-Lipschitz homeomorphic to  $B_0$ , it is called an  $L$ -thick block.*

*An end  $E$  is said to have a model of bounded geometry if there exists  $L$  such that  $E$  is bi-Lipschitz homeomorphic to a model manifold  $E_m$  consisting of gluing  $L$ -thick blocks end-to-end.*

It follows from work of Minsky [Y. N. Minsky \[1993\]](#) that if  $E$  is of bounded geometry, it has a model of bounded geometry. The existence of Cannon-Thurston maps in this setup is then a replica of the proof of [Theorem 3.5](#).

## 4.2 $i$ -bounded Geometry.

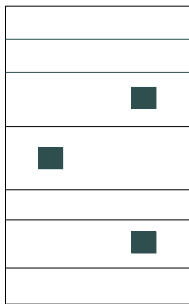
**Definition 4.3.** *[Mj \[2009-2010\]](#) An end  $E$  of a hyperbolic 3-manifold  $M$  has  $i$ -bounded geometry if the boundary torus of every Margulis tube in  $E$  has bounded diameter.*

We give an alternate description. Fix a closed hyperbolic surface  $S$ . Let  $\mathcal{C}$  be a finite collection of (not necessarily disjoint) simple closed geodesics on  $S$ . Let  $N_\epsilon(\sigma_i)$  denote an  $\epsilon$  neighborhood of  $\sigma_i$ ,  $\sigma_i \in \mathcal{C}$ , where  $\epsilon$  is small enough to ensure that lifts of  $N_\epsilon(\sigma_i)$  to  $\tilde{S}$  are disjoint.

**Definition 4.4.** Let  $I = [0, 3]$ . Equip  $S \times I$  with the product metric. Let  $B^c = (S \times I - \cup_j N_\epsilon(\sigma_j) \times [1, 2])$ , equipped with the induced path-metric. Here  $\{\sigma_j\}$  is a subcollection of  $\mathcal{C}$  consisting of disjoint curves. Perform Dehn filling on some  $(1, n)$  curve on each resultant torus component of the boundary of  $B^c$  (the integers  $n$  are quite arbitrary and may vary: we omit subscripts for expository ease). We call  $n$  the twist coefficient. Foliate the relevant torus boundary component of  $B^c$  by translates of  $(1, n)$  curves. Glue in a solid torus  $\Theta$ , which we refer to as a Margulis tube, with a hyperbolic metric foliated by totally geodesic disks bounding the  $(1, n)$  curves.

The resulting copy of  $S \times I$  thus obtained, equipped with the above metric is called a thin block.

**Definition 4.5.** A model manifold  $E_m$  of  $i$ -bounded geometry is built out of gluing  $L$ -thick and thin blocks end-to-end (for some  $L$ ) (see schematic diagram below where the black squares indicate Margulis tubes and the horizontal rectangles indicate the blocks).



It follows from work in Mj [ibid.] that

**Proposition 4.6.** An end  $E$  of a hyperbolic 3-manifold  $M$  has  $i$ -bounded geometry if and only if it is bi-Lipschitz homeomorphic to a model manifold  $E_m$  of  $i$ -bounded geometry.

We give a brief indication of the construction of  $\mathcal{L}_\lambda$  and the proof of the existence of Cannon-Thurston maps in this case. First electrify all the Margulis tubes, i.e. equip them with a zero metric (see Farb [1998] for details on relative hyperbolicity and electric geometry). This ensures that in the resulting electric geometry, each block is of bounded geometry. More precisely, there is a (metric) product structure on  $S \times [0, 3]$  such that each  $\{x\} \times [0, 3]$  has uniformly bounded length in the electric metric.

Further, since the curves in  $\mathcal{C}$  are electrified in a block, Dehn twists are isometries from  $S \times \{1\}$  to  $S \times \{2\}$  in a thin block. This allows the construction of  $\mathcal{L}_\lambda$  to go through as before and ensures that it is quasiconvex in the resulting electric metric.

Finally given an electric geodesic lying outside large balls modulo Margulis tubes one can recover a genuine hyperbolic geodesic tracking it outside Margulis tubes. A relative version of the criterion of [Lemma 2.8](#) can now be used to prove the existence of Cannon-Thurston maps.

**4.3 Amalgamation Geometry.** Again, as in [Definition 4.4](#), start with a fixed closed hyperbolic surface  $S$ , a collection of simple closed curves  $\mathcal{C}$  and set  $I = [0, 3]$ . Perform Dehn surgeries on the Margulis tubes corresponding to  $\mathcal{C}$  as before. Let  $K = S \times [1, 2] \subset S \times [0, 3]$  and let  $K^c = (S \times I - \cup_i N_\epsilon(\sigma_i) \times [1, 2])$ . Instead of fixing the product metric on the complement  $K^c$  of Margulis tubes in  $K$ , allow these complementary components to have *arbitrary geometry* subject only to the restriction that the geometries of  $S \times \{1, 2\}$  are fixed. Equip  $S \times [0, 1]$  and  $S \times [2, 3]$  with the product metrics. The resulting block is said to be a *block of amalgamation geometry*. After lifting to the universal cover, complements of Margulis tubes in the lifts  $\tilde{S} \times [1, 2]$  are termed *amalgamation components*.

**Definition 4.7.** *An end  $E$  of a hyperbolic 3-manifold  $M$  has amalgamated geometry if*

1. *it is bi-Lipschitz homeomorphic to a model manifold  $E_m$  consisting of gluing  $L$ -thick and amalgamation geometry blocks end-to-end (for some  $L$ ).*
2. *Amalgamation components are (uniformly) quasiconvex in  $\tilde{E}_m$ .*

To construct the ladder  $\mathcal{L}_\lambda$  we electrify amalgamation components as well as Margulis tubes. This ensures that in the electric metric,

1. Each amalgamation block has bounded geometry
2. The mapping class element taking  $S \times \{1\}$  to  $S \times \{2\}$  induces an isometry of the electrified metrics.

Quasiconvexity of  $\mathcal{L}_\lambda$  in the electric metric now follows as before. To recover the data of hyperbolic geodesics from quasigeodesics lying close to  $\mathcal{L}_\lambda$ , we use (uniform) quasiconvexity of amalgamation components and existence of Cannon-Thurston maps follows.

**4.4 Split Geometry.** We need now to relax the assumption that the boundary components of model blocks are of (uniformly) bounded geometry. Roughly speaking, split geometry is a generalization of amalgamation geometry where

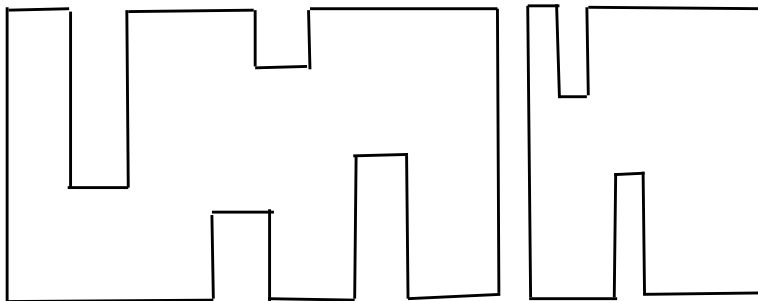
1. A Margulis tube is allowed to travel through a uniformly bounded number of contiguous blocks and split them.



2. The complementary pieces, now called split components, are quasiconvex in a somewhat weaker sense (see [Definition 4.8](#) below).

Each split component is allowed to contain Margulis tubes, called *hanging tubes* that do not go all the way across from the top to the bottom, i.e. they do not split both  $\Sigma_i^s, \Sigma_{i+1}^s$ .

A split component  $B^s \subset B = S \times I$  is topologically a product  $S^s \times I$  for some, necessarily connected  $S^s (\subset S)$ . However, the upper and lower boundaries of  $B^s$  need only be split subsurfaces of  $S^s$  to allow for hanging tubes starting or ending (but not both) within the split block.



*Split Block with hanging tubes*

An end  $E$  of a hyperbolic 3-manifold  $M$  has *weak split geometry* if it is bi-Lipschitz homeomorphic to a model manifold  $E_m$  consisting of gluing  $L$ -thick and split blocks as above end-to-end (for some  $L$ ). Electrifying split components as in [Section 4.3](#), we obtain a new electric metric called the *graph metric*  $d_G$  on  $E$ .

**Definition 4.8.** *A model of weak split geometry is said to be of split geometry if the convex hull of each split component has uniformly bounded  $d_G$ -diameter.*

## 5 Cannon-Thurston Maps for Kleinian groups and Applications

**5.1 Cannon-Thurston maps for degenerate manifolds.** Let  $M$  be a hyperbolic 3-manifold homotopy equivalent to a closed hyperbolic surface  $S$ . Once we establish that  $M$  has split geometry, the proof proceeds as in [Section 4.3](#) by electrifying split components, constructing a hyperbolic ladder  $\mathcal{L}_\lambda$  and finally recovering a hyperbolic geodesic from an electric one. We shall therefore dwell in this subsection on showing that any degenerate end has split geometry. We shall do this under two simplifying assumptions, directing the reader to [Mj \[2014a\]](#) (especially the Introduction) for a more detailed road-map.

We borrow extensively from the hierarchy and model manifold terminology and technology of [Masur and Y. N. Minsky \[2000\]](#) and [Y. Minsky \[2010\]](#). The model manifold

of Y. Minsky [2010] and Brock, Canary, and Y. N. Minsky [2012] furnishes a resolution, or equivalently, a sequence  $\{P_m\}$  of pants decompositions of  $S$  exiting  $E$  and hence a *hierarchy path*. Let  $\tau_m$  denote the simple multicurve on  $S$  constituting  $P_m$ . The  $P_m$  in turn furnish split level surfaces  $\{S_m\}$  exiting  $E$ : a *split level surface*  $S_m$  is a collection of (nearly) totally geodesic embeddings of the pairs of pants comprising  $P_m$ . Next, corresponding to the hierarchy path  $\{\tau_m\}$ , there is a tight geodesic in  $\mathcal{C}(S)$  consisting of the bottom geodesic  $\{\eta_i\}$  of the hierarchy. We proceed to extract a subsequence of the resolution  $\tau_m$  using the bottom geodesic  $\{\eta_i\}$  under two key simplifying assumptions:

1. For all  $i$ , the length of exactly one curve in  $\eta_i$  is sufficiently small, less than the Margulis constant in particular. Call it  $\eta_i$  for convenience.
2. Let  $S_i$  correspond to the first occurrence of the vertex  $\eta_i$  in the resolution  $\tau_m$ . Assume further that the  $S_i$ 's are actually split surfaces and not just split level surfaces, i.e. they all have injectivity radius uniformly bounded below,

It follows that the Margulis tube  $\eta_i$  splits both  $S_i$  and  $S_{i+1}$  and that the tube  $T_i$  is trapped entirely between  $S_i$  and  $S_{i+1}$ . The product region  $B_i$  between  $S_i$  and  $S_{i+1}$  is therefore a split block for all  $i$  and  $T_i$  splits it. The model manifold thus obtained is one of weak split geometry. In a sense, this is a case of ‘pure split geometry’, where all blocks have a split geometry structure (no thick blocks). To prove that the model is indeed of split geometry, it remains to establish the quasiconvexity condition of Definition 4.8.

Let  $K$  be a split component and  $\tilde{K}$  an elevation to  $\tilde{E}$ . Let  $v$  be a boundary short curve for the split component and let  $T_v$  be the Margulis tube corresponding to  $v$  abutting  $K$ . Denote the hyperbolic convex hull by  $CH(\tilde{K})$  and pass back to a quotient in  $M$ . A crucial observation that is needed here is the fact that any pleated surface has bounded  $d_G$ -diameter. This is because thin parts of pleated surfaces lie inside Margulis tubes that get electrified in the graph metric. It therefore suffices to show that any point in  $CH(K)$  lies close to a pleated surface passing near the fixed tube  $T_v$ . This last condition follows from the Brock-Bromberg drilling theorem Brock and Bromberg [2004] and the fact that the convex core of a quasi-Fuchsian group is filled by pleated surfaces Fan [1997]. This completes our sketch of a proof of the following main theorem of Mj [2014a]:

*Theorem 2.5:* Let  $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$  be a simply or doubly degenerate (closed) surface Kleinian group. Then a Cannon-Thurston map exists.

It follows that the limit set of  $\rho(\pi_1(S))$  is a continuous image of  $S^1$  and is therefore locally connected. As a first application of Theorem 2.5, we shall use the following Theorem of Anderson and Maskit [1996] to prove that connected limit sets of Kleinian groups without parabolics are locally connected.

**Theorem 5.1.** *Anderson and Maskit [ibid.]* Let  $\Gamma$  be an analytically finite Kleinian group with connected limit set. Then the limit set  $\Lambda(\Gamma)$  is locally connected if and only if every simply degenerate surface subgroup of  $\Gamma$  without accidental parabolics has locally connected limit set.

Combining the remark after [Theorem 2.5](#) with [Theorem 5.1](#), we immediately have the following affirmative answer to [Question 2.4](#).

**Theorem 5.2.** *Let  $\Gamma$  be a finitely generated Kleinian group without parabolics and with a connected limit set  $\Lambda$ . Then  $\Lambda$  is locally connected.*

[Theorem 2.5](#) can be extended to punctured surfaces [Mj \[2014a\]](#) and this allows [Theorem 5.2](#) to be generalized to arbitrary finitely generated Kleinian groups.

**5.2 Finitely generated Kleinian groups.** In [Mj \[2014b\]](#), we show that the point preimages of the Cannon-Thurston map for a simply or doubly degenerate surface Kleinian group given by [Theorem 2.5](#) corresponds to end-points of leaves of ending laminations. In particular, the ending lamination corresponding to a degenerate end can be recovered from the Cannon-Thurston map. This was extended further in [Das and Mj \[2016\]](#) and [Mj \[2017a\]](#) to obtain the following general version for finitely generated Kleinian groups.

**Theorem 5.3.** *Mj [ibid.]* Let  $G$  be a finitely generated Kleinian group. Let  $i : \Gamma_G \rightarrow \mathbb{H}^3$  be the natural identification of a Cayley graph of  $G$  with the orbit of a point in  $\mathbb{H}^3$ . Further suppose that each degenerate end of  $\mathbb{H}^3/G$  can be equipped with a Minsky model [Y. Minsky \[2010\]](#).<sup>2</sup> Then  $i$  extends continuously to a Cannon-Thurston map  $\hat{i} : \widehat{\Gamma}_G \rightarrow \mathbb{D}^3$ , where  $\widehat{\Gamma}_G$  denotes the (relative) hyperbolic compactification of  $\Gamma_G$ .

Let  $\partial i$  denote the restriction of  $\hat{i}$  to the boundary  $\partial\Gamma_G$  of  $\Gamma_G$ . Let  $E$  be a degenerate end of  $N^h = \mathbb{H}^3/G$  and  $\widetilde{E}$  a lift of  $E$  to  $\widetilde{N}^h$  and let  $M_{g,f}$  be an augmented Scott core of  $N^h$ . Then the ending lamination  $\mathfrak{L}_E$  for the end  $E$  lifts to a lamination on  $\widetilde{M}_{g,f} \cap \widetilde{E}$ . Each such lift  $\mathfrak{L}$  of the ending lamination of a degenerate end defines a relation  $\mathfrak{R}_{\mathfrak{L}}$  on the (Gromov) boundary  $\partial\widetilde{M}_{g,f}$  (or equivalently, the relative hyperbolic boundary  $\partial_r\Gamma_G$  of  $\Gamma_G$ ), given by  $a\mathfrak{R}_{\mathfrak{L}}b$  iff  $a, b$  are end-points of a leaf of  $\mathfrak{L}$ . Let  $\{\mathfrak{R}_i\}$  be the entire collection of relations on  $\partial\widetilde{M}_{g,f}$  obtained this way. Let  $\mathfrak{R}$  be the transitive closure of the union  $\bigcup_i \mathfrak{R}_i$ . Then  $\partial i(a) = \partial i(b)$  iff  $a\mathfrak{R}b$ .

<sup>2</sup>This hypothesis is satisfied for all ends without parabolics as well as for ends incompressible away from cusps - see [Y. Minsky \[2010\]](#) and [Brock, Canary, and Y. N. Minsky \[2012\]](#) and [Mj \[2017a, Appendix\]](#). That the hypothesis is satisfied in general would follow from unpublished work of [B. H. Bowditch \[2016\]](#) and [B. Bowditch \[2005\]](#).

**5.3 Primitive Stable Representations.** In Y. N. Minsky [2013] Minsky introduced an open subset of the  $PSL_2(\mathbb{C})$  character variety for a free group, properly containing the Schottky representations, on which the action of the outer automorphism group is properly discontinuous. He called these *primitive stable representations*. Let  $F_n$  be a free group of rank  $n$ . An element of  $F_n$  is primitive if it is an element of a free generating set. Let  $\mathcal{P} = \dots w w w \dots$  be the set of bi-infinite words with  $w$  cyclically reduced primitive. A representation  $\rho : F_n \rightarrow PSL_2(\mathbb{C})$  is primitive stable if all elements of  $\mathcal{P}$  are mapped to uniform quasigeodesics in  $\mathbb{H}^3$ .

Minsky conjectured that primitive stable representations are characterized by the feature that every component of the ending lamination is blocking.

Using Theorem 5.3, Jeon and Kim [2010], and Jeon, Kim, Ohshika, and Lecuire [2014] resolved this conjecture. We briefly sketch their argument for a degenerate free Kleinian group without parabolics.

Let  $\{D_1, \dots, D_n\} = \mathfrak{D}$  be a finite set of essential disks on a handlebody  $H$  cutting  $H$  into a 3-ball. A free generating set of  $F_n$  is dual to  $\mathfrak{D}$ . For a lamination  $\mathcal{L}$ , the *Whitehead graph*  $Wh(\mathcal{L}, \mathfrak{D})$  is defined as follows. Cut  $\partial H$  along  $\partial \mathfrak{D}$  to obtain a sphere with  $2n$  holes, labeled by  $D_i^\pm$ . The vertices of  $Wh(\mathcal{L}, \mathfrak{D})$  are the boundary circles of  $\partial H$ , with an edge whenever two circles are joined by an arc of  $\mathcal{L} \setminus \mathfrak{D}$ . For the ending lamination  $\mathcal{L}_E$  of a degenerate free group without parabolics,  $Wh(\mathcal{L}_E, \mathfrak{D})$  is connected and has no cutpoints.

Let  $\rho_E$  be the associated representation. If  $\rho$  is not primitive stable, then there exists a sequence of primitive cyclically reduced elements  $w_n$  such that  $\rho(w_n^*)$  is not an  $n$ -quasigeodesic. After passing to a subsequence,  $w_n$  and hence  $w_n^*$  converges to a bi-infinite geodesic  $w_\infty$  in the Cayley graph with two distinct end points  $w_+, w_-$  in the Gromov boundary of  $F_n$ . The Cannon-Thurston map identifies  $w_+, w_-$ . Hence by Theorem 5.3 they are either the end points of a leaf of  $\mathcal{L}_E$  or ideal end-points of a complementary ideal polygon of  $\mathcal{L}_E$ . It follows therefore that  $Wh(w_\infty, \mathfrak{D})$  is connected and has no cutpoints. Since  $w_n$ 's converge to  $w_\infty$ ,  $Wh(w_n, \mathfrak{D})$  is connected and has no cutpoints for large enough  $n$ . A Lemma due to Whitehead says that if  $Wh(w_n, \mathfrak{D})$  is connected and has no cutpoints, then  $w_n$  cannot be primitive, a contradiction.

**5.4 Discreteness of Commensurators.** In C. Leininger, Long, and Reid [2011] and Mj [2011], Theorems 2.5 and 5.3 are used to prove that commensurators of finitely generated, infinite covolume, Zariski dense Kleinian groups are discrete. The basic fact that goes into the proof is that commensurators preserve the structure of point pre-images of Cannon-Thurston maps. The point pre-image structure is known from Theorem 5.3.

**5.5 Radial and horospherical limit sets.**

**Definition 5.4.** A point  $\xi \in S^2$  is a radial or conical limit point of Kleinian group  $\Gamma$ , if for any base-point  $o \in \widetilde{M}$  and any geodesic  $\gamma_\xi$  ending at  $\xi$ , there exist  $C \geq 0$  and infinitely many translates  $g.o \in N_C(\gamma_\xi)$ ,  $g \in \Gamma$ .

A point  $\xi \in S^2$  is a horospherical limit point of Kleinian group  $\Gamma$ , if for any base-point  $o \in \widetilde{M}$  and any horoball  $B_\xi$  based at  $\xi$ , there exist infinitely many translates  $g.o \in B_\xi$ ,  $g \in \Gamma$ .

The collection of radial (resp. horospherical) limit points of  $\Gamma$  is called the radial (resp. horospherical) limit set of  $\Gamma$  and is denoted by  $\Lambda_r$  (resp.  $\Lambda_h$ ).

The multiple limit set  $\Lambda_m$  consists of those point of  $S^2$  that have more than one pre-image under the Cannon-Thurston map.

Several authors [M. Kapovich \[1995\]](#), [Gerasimov \[2012\]](#), and [Jeon, I. Kapovich, C. Leininger, and Ohshika \[2016\]](#) worked on the relationship between  $\Lambda_m$  and  $\Lambda_r$ . They concluded that the conical limit set is strictly contained in the set of injective points of the Cannon-Thurston map, i.e.  $\Lambda_r \subset \Lambda_m^c$ , but the inclusion is proper.

In [Lecuire and Mj \[2016\]](#), we showed:

**Theorem 5.5.**  $\Lambda_m^c = \Lambda_h$ .

**5.6 Motions of limit sets.** We discuss the following question in this section, which paraphrases the second part of [Thurston \[1982, Problem 14\]](#). A detailed survey appears in [Mj \[2017b\]](#).

**Question 5.6.** Let  $G_n$  be a sequence of Kleinian groups converging to a Kleinian group  $G$ . Does the corresponding dynamics of  $G_n$  on the Riemann sphere  $S^2$  converge to the dynamics of  $G$  on  $S^2$ ?

To make [Question 5.6](#) precise, we need to make sense of ‘convergence’ both for Kleinian groups and for their dynamics on  $S^2$ . There are three different notions of convergence for Kleinian groups.

**Definition 5.7.** Let  $\rho_i : H \rightarrow \mathrm{PSL}_2(\mathbb{C})$  be a sequence of Kleinian groups. We say that  $\rho_i$  converges to  $\rho_\infty$  algebraically if for all  $h \in H$ ,  $\rho_i(h) \rightarrow \rho_\infty(h)$ .

Let  $\rho_j : H \rightarrow \mathrm{PSL}_2(\mathbb{C})$  be a sequence of discrete, faithful representations of a finitely generated, torsion-free, nonabelian group  $H$ . If  $\{\rho_j(H)\}$  converges as a sequence of closed subsets of  $\mathrm{PSL}_2(\mathbb{C})$  to a torsion-free, nonabelian Kleinian group  $\Gamma$ ,  $\Gamma$  is called the geometric limit of the sequence.

$G_i (= \rho_i(H))$  converges strongly to  $G (= \rho_\infty(H))$  if the convergence is both geometric and algebraic.

[Question 5.6](#) then splits into the following three questions.

- Question 5.8.** 1. If  $G_n \rightarrow G$  geometrically, then do the corresponding limit sets converge in the Hausdorff topology on  $S^2$ ?
2. If  $G_n \rightarrow G$  strongly then do the corresponding Cannon-Thurston maps converge uniformly?
3. If  $G_n \rightarrow G$  algebraically then do the corresponding Cannon-Thurston maps converge pointwise?

We give the answers straight off and then proceed to elaborate.

- Answers 5.9.** 1. The answer to [Question 5.8 \(1\)](#) is Yes.
2. The answer to [Question 5.8 \(2\)](#) is Yes.
3. The answer to [Question 5.8 \(3\)](#) is No, in general.

The most general answer to [Question 5.8 \(1\)](#) is due to [R. A. Evans \[2000\]](#), [R. Evans \[2004\]](#):

**Theorem 5.10.** [R. A. Evans \[2000\]](#), [R. Evans \[2004\]](#) Let  $\rho_n : H \rightarrow G_n$  be a sequence of weakly type-preserving isomorphisms from a geometrically finite group  $H$  to Kleinian groups  $G_n$  with limit sets  $\Lambda_n$ , such that  $\rho_n$  converges algebraically to  $\rho_\infty : H \rightarrow G_\infty^a$  and geometrically to  $G_\infty^g$ . Let  $\Lambda_a$  and  $\Lambda_g$  denote the limit sets of  $G_\infty^a$  and  $G_\infty^g$ . Then  $\Lambda_n \rightarrow \Lambda_g$  in the Hausdorff metric. Further, the sequence converges strongly if and only if  $\Lambda_n \rightarrow \Lambda_a$  in the Hausdorff metric.

The answer to [Question 5.8 \(2\)](#) is due to the author and [Series Mj and Series \[2017\]](#) in the case that  $H = \pi_1(S)$  for a closed surface  $S$  of genus greater than one. This can be generalized to arbitrary finitely generated Kleinian groups as in [Mj \[2017b\]](#):

**Theorem 5.11.** Let  $H$  be a fixed group and  $\rho_n(H) = \Gamma_n$  be a sequence of geometrically finite Kleinian groups converging strongly to a Kleinian group  $\Gamma$ . Let  $M_n$  and  $M_\infty$  be the corresponding hyperbolic manifolds. Let  $K$  be a fixed complex with fundamental group  $H$ .

Consider embeddings  $\phi_n : K \rightarrow M_n, n = 1, \dots, \infty$  such that the maps  $\phi_n$  are homotopic to each other by uniformly bounded homotopies (in the geometric limit). Then Cannon-Thurston maps for  $\phi_n$  exist and converge uniformly to the Cannon-Thurston map for  $\phi_\infty$ .

Finally we turn to [Question 5.8 \(3\)](#), which turns out to be the subtlest. In [Mj and Series \[2013\]](#) we showed that the answer to [Question 5.8 \(3\)](#) is ‘Yes’ if the geometric limit is geometrically finite. We illustrate this with a concrete example due to [Kerckhoff and Thurston \[1990\]](#)

**Theorem 5.12.** Fix a closed hyperbolic surface  $S$  and a simple closed geodesic  $\sigma$  on it. Let  $tw^i$  denote the automorphism of  $S$  given by an  $i$ -fold Dehn twist along  $\sigma$ . Let  $G_i$  be the quasi-Fuchsian group given by the simultaneous uniformization of  $(S, tw^i(S))$ . Let  $G_\infty$  denote the geometric limit of the  $G_i$ 's. Let  $S_{i-}$  denote the lower boundary component of the convex core of  $G_i$ ,  $i = 1, \dots, \infty$  (including  $\infty$ ). Let  $\phi_i : S \rightarrow S_{i-}$  be such that if  $0 \in \mathbb{H}^2 = \widetilde{S}$  denotes the origin of  $\mathbb{H}^2$  then  $\widetilde{\phi}_i(0)$  lies in a uniformly bounded neighborhood of  $0 \in \mathbb{H}^3 = \widetilde{M}_i$ . We also assume (using the fact that  $M_\infty$  is a geometric limit of  $M_i$ 's) that  $S_{i-}$ 's converge geometrically to  $S_{\infty-}$ . Then the Cannon-Thurston maps for  $\widetilde{\phi}_i$  converge pointwise, but not uniformly, on  $\partial\mathbb{H}^2$  to the Cannon-Thurston map for  $\widetilde{\phi}_\infty$ .

However, if the geometric limit is geometrically infinite, then the answer to [Question 5.8](#) (3) may be negative. We illustrate this with certain examples of geometric limits constructed by Brock in [Brock \[2001\]](#).

**Theorem 5.13.** *Mj and Series [2017]* Fix a closed hyperbolic surface  $S$  and a separating simple closed geodesic  $\sigma$  on it, cutting  $S$  up into two pieces  $S_-$  and  $S_+$ . Let  $\phi$  denote an automorphism of  $S$  such that  $\phi|_{S_-}$  is the identity and  $\phi|_{S_+} = \psi$  is a pseudo-Anosov of  $S_+$  fixing the boundary. Let  $G_i$  be the quasi-Fuchsian group given by the simultaneous uniformization of  $(S, \phi^i(S))$ . Let  $G_\infty$  denote the geometric limit of the  $G_i$ 's. Let  $S_{i0}$  denote the lower boundary component of the convex core of  $G_i$ ,  $i = 1, \dots, \infty$  (including  $\infty$ ). Let  $\phi_i : S \rightarrow S_{i0}$  be such that if  $0 \in \mathbb{H}^2 = \widetilde{S}$  denotes the origin of  $\mathbb{H}^2$  then  $\widetilde{\phi}_i(0)$  lies in a uniformly bounded neighborhood of  $0 \in \mathbb{H}^3 = \widetilde{M}_i$ . We also assume (using the fact that  $M_\infty$  is a geometric limit of  $M_i$ 's) that  $S_{i0}$ 's converge geometrically to  $S_{\infty 0}$ .

Let  $\Sigma$  be a complete hyperbolic structure on  $S_+$  such that  $\sigma$  is homotopic to a cusp on  $\Sigma$ . Let  $\mathcal{L}$  consist of pairs  $(\xi_-, \xi)$  of ideal endpoints (on  $\mathbb{S}_\infty^1$ ) of stable leaves  $\lambda$  of the stable lamination of  $\psi$  acting on  $\widetilde{\Sigma}$ . Also let  $\partial\widetilde{\mathcal{H}}$  denote the collection of ideal basepoints of horodisks given by lifts (contained in  $\widetilde{\Sigma}$ ) of the cusp in  $\Sigma$  corresponding to  $\sigma$ . Let

$$\Xi = \{ \xi : \text{There exists } \xi_- \text{ such that } (\xi_-, \xi) \in \mathcal{L}; \xi_- \in \partial\widetilde{\mathcal{H}} \}.$$

Let  $\partial\phi_i$ ,  $i = 1 \dots, \infty$  denote the Cannon-Thurston maps for  $\widetilde{\phi}_i$ . Then

1.  $\partial\phi_i(\xi)$  does not converge to  $\partial\phi_\infty(\xi)$  for  $\xi \in \Xi$ .
2.  $\partial\phi_i(\xi)$  converges to  $\partial\phi_\infty(\xi)$  for  $\xi \notin \Xi$ .

In [Mj and Ohshika \[2017\]](#), we identify the exact criteria that lead to the discontinuity phenomenon of [Theorem 5.13](#).

## 6 Gromov-Hyperbolic groups

### 6.1 Applications and Generalizations.

**6.1.1 Normal subgroups and trees.** The ladder construction of [Section 3.2](#) has been generalized considerably. We work in the context of an exact sequence  $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ , with  $N$  hyperbolic and  $G$  finitely presented. We observe that for the proof of [Theorem 3.5](#) to go through it suffices to have a qi-section of  $Q$  into  $G$  to provide a ‘coarse transversal’ to flow. Such a qi-section was shown to exist by [Mosher \[1996\]](#). We then obtain the following.

**Theorem 6.1.** *Mitra [1998a]* *Let  $G$  be a hyperbolic group and let  $H$  be a hyperbolic normal subgroup that is normal in  $G$ . Then the inclusion of Cayley graphs  $i : \Gamma_H \rightarrow \Gamma_G$  gives a Cannon-Thurston map  $\hat{i} : \widehat{\Gamma}_H \rightarrow \widehat{\Gamma}_G$ .*

The ladder construction can also be generalized to the general framework of a tree of hyperbolic metric spaces.

**Theorem 6.2.** *Mitra [1998b]* *Let  $(X, d)$  be a tree  $(T)$  of hyperbolic metric spaces satisfying the qi-embedded condition (i.e. edge space to vertex space inclusions are qi-embeddings). Let  $v$  be a vertex of  $T$  and  $(X_v, d_v)$  be the vertex space corresponding to  $v$ . If  $X$  is hyperbolic then the inclusion  $i : X_v \rightarrow X$  gives a Cannon-Thurston map  $\hat{i} : \widehat{X}_v \rightarrow \widehat{X}$ .*

[Theorem 6.1](#) was generalized by the author and Sardar to a purely coarse geometric context, where no group action is present. The relevant notion is that of a metric bundle for which we refer the reader to [Mj and Sardar \[2012\]](#). Roughly speaking, the data of a metric bundle consists of vertex and edge spaces as in the case of a tree of spaces, with two notable changes:

1. The base  $T$  is replaced by an arbitrary graph  $B$ .
2. All edge-space to vertex space maps are quasi-isometries rather than just quasi-isometric embeddings.

With these modifications in place we have the following generalizations of Mosher’s qi-section Lemma [Mosher \[1996\]](#) and [Theorem 6.1](#):

**Theorem 6.3.** *Mj and Sardar [2012]* *Suppose  $p : X \rightarrow B$  is a metric graph bundle satisfying the following:*

1.  $B$  is a Gromov hyperbolic graph.
2. Each fiber  $F_b$ , for  $b$  a vertex of  $B$  is  $\delta$ -hyperbolic (for some  $\delta > 0$ ) with respect to the path metric induced from  $X$ .
3. The barycenter maps  $\partial^3 F_b \rightarrow F_b$ ,  $b \in B$ , sending a triple of distinct points on the boundary  $\partial F_b$  to their centroid, are (uniformly, independent of  $b$ ) coarsely surjective.



4.  $X$  is hyperbolic.

Then there is a qi-section  $B \rightarrow X$ . The inclusion map  $i_b : F_b \rightarrow X$  gives a Cannon-Thurston map  $\hat{i} : \widehat{F}_b \rightarrow \widehat{X}$ .

**6.2 Point pre-images: Laminations.** In Section 3.1, it was pointed out that the Cannon-Thurston map  $\hat{i}$  identifies  $p, q \in S^1$  if and only if  $p, q$  are end-points of a leaf or ideal end-points of a complementary ideal polygon of the stable or unstable lamination.

In Mitra [1997a] an algebraic theory of ending laminations was developed based on Thurston's theory Thurston [1980]. The theory was developed in the context of a normal hyperbolic subgroup of a hyperbolic group  $G$  and used to give an explicit structure for the Cannon-Thurston map in Theorem 6.1.

**Definition 6.4.** *Bestvina, Feighn, and Handel [1997], Coulbois, Hilion, and Lustig [2007, 2008a,b], I. Kapovich and Lustig [2010, 2015], and Mitra [1997a]* An algebraic lamination for a hyperbolic group  $H$  is an  $H$ -invariant, flip invariant, closed subset  $\mathcal{L} \subseteq \partial^2 H = (\partial H \times \partial H \setminus \Delta) / \sim$ , where  $(x, y) \sim (y, x)$  denotes the flip and  $\Delta$  the diagonal in  $\partial H \times \partial H$ .

Let

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

be an exact sequence of finitely presented groups with  $H, G$  hyperbolic. It follows by work of Mosher [1996] that  $Q$  is hyperbolic. In Mitra [1997a], we construct algebraic ending laminations naturally parametrized by points in the boundary  $\partial Q$ . We describe the construction now.

Every element  $g \in G$  gives an automorphism of  $H$  sending  $h$  to  $g^{-1}hg$  for all  $h \in H$ . Let  $\phi_g : \mathcal{U}(\Gamma_H) \rightarrow \mathcal{U}(\Gamma_H)$  be the resulting bijection of the vertex set  $\mathcal{U}(\Gamma_H)$  of  $\Gamma_H$ . This induces a map  $\Phi_g$  sending an edge  $[a, b] \subset \Gamma_H$  to a geodesic segment joining  $\phi_g(a), \phi_g(b)$ .

For some (any)  $z \in \partial Q$  we shall describe an algebraic ending lamination  $\Lambda_z$ . Fix such a  $z$  and let

1.  $[1, z] \subset \Gamma_Q$  be a geodesic ray starting at 1 and converging to  $z \in \partial \Gamma_Q$ .
2.  $\sigma : Q \rightarrow G$  be a qi section.
3.  $z_n$  be the vertex on  $[1, z)$  such that  $d_Q(1, z_n) = n$ .
4.  $g_n = \sigma(z_n)$ .

For  $h \in H$ , let  $S_n^h$  be the  $H$ -invariant collection of all free homotopy representatives (or equivalently, shortest representatives in the same conjugacy class) of  $\phi_{g_n^{-1}}(h)$  in  $\Gamma_H$ .

Identifying equivalent geodesics in  $\mathcal{S}_n^h$  one obtains a subset  $\mathcal{S}_n^h$  of unordered pairs of points in  $\widehat{\Gamma}_H$ . The intersection with  $\partial^2 H$  of the union of all subsequential limits (in the Hausdorff topology) of  $\{\mathcal{S}_n^h\}$  is denoted by  $\Lambda_z^h$ .

**Definition 6.5.** *The set of algebraic ending laminations corresponding to  $z \in \partial\Gamma_Q$  is given by*

$$\Lambda_{EL}(z) = \bigcup_{h \in H} \Lambda_z^h.$$

**Definition 6.6.** *The set  $\Lambda$  of all algebraic ending laminations is defined by*

$$\Lambda_{EL} = \bigcup_{z \in \partial\Gamma_Q} \Lambda_{EL}(z).$$

The following was shown in [Mitra \[1997a\]](#):

**Theorem 6.7.** *The Cannon-Thurston map  $\hat{i}$  of [Theorem 6.1](#) identifies the end-points of leaves of  $\Lambda_{EL}$ . Conversely, if  $\hat{i}(p) = \hat{i}(q)$  for  $p \neq q \in \partial\Gamma_H$ , then some bi-infinite geodesic having  $p, q$  as its end-points is a leaf of  $\Lambda_{EL}$ .*

**6.2.1 Finite-to-one.** The classical Cannon-Thurston map of [Theorem 2.2](#) is finite-to-one. Swarup asked (cf. Bestvina’s Geometric Group Theory problem list [Bestvina \[2004, Prolem 1.20\]](#)) if the Cannon-Thurston maps of [Theorem 6.1](#) are also finite-to-one. Kapovich and Lustig answered this in the affirmative in the following case.

**Theorem 6.8.** *I. Kapovich and Lustig [2015] Let  $\phi \in \text{Out}(F_N)$  be a fully irreducible hyperbolic automorphism. Let  $G_\phi = F_N \rtimes_\phi \mathbb{Z}$  be the associated mapping torus group. Let  $\partial i$  denote the Cannon-Thurston map of [Theorem 6.1](#) in this case. Then for every  $z \in \partial G_\phi$ , the cardinality of  $(\partial i)^{-1}(z)$  is at most  $2N$ .*

[Bestvina, Feighn, and Handel \[1997\]](#) define a closely related set  $\Lambda_{BFH}$  of algebraic laminations in the case covered by [Theorem 6.8](#) using train-track representatives of free group automorphisms. Any algebraic lamination  $\mathcal{L}$  defines a relation  $\mathcal{R}_\mathcal{L}$  on  $\partial F_N$  by  $a\mathcal{R}_\mathcal{L}b$  if  $(a, b) \in \mathcal{L}$ . The transitive closure of  $\mathcal{L}$  will be called its diagonal closure. In [I. Kapovich and Lustig \[2015\]](#), Kapovich and Lustig further show that in the case covered by [Theorem 6.8](#),  $\Lambda_{EL}$  precisely equals the diagonal closure of  $\Lambda_{BFH}$ .

**6.3 Relative hyperbolicity.** The notion of a Cannon-Thurston map can be extended to the context of relative hyperbolicity. This was done in [Mj and Pal \[2011\]](#). Let  $X$  and  $Y$  be relatively hyperbolic spaces, hyperbolic relative to the collections  $\mathcal{H}_X$  and  $\mathcal{H}_Y$  of ‘horosphere-like sets’ respectively. Let us denote the horoballifications of  $X$  and  $Y$  with respect to  $\mathcal{H}_X$  and  $\mathcal{H}_Y$  by  $\mathcal{G}(X, \mathcal{H}_X)$ ,  $\mathcal{G}(Y, \mathcal{H}_Y)$  respectively (see [B. H. Bowditch \[2012\]](#) for details). The horoballification of an  $H$  in  $\mathcal{H}_X$  or  $\mathcal{H}_Y$  is denoted as  $H^h$ . Note that  $\mathcal{G}(X, \mathcal{H}_X)$ ,  $\mathcal{G}(Y, \mathcal{H}_Y)$  are hyperbolic. The electrifications will be denoted as  $\mathcal{E}(X, \mathcal{H}_X)$ ,  $\mathcal{E}(Y, \mathcal{H}_Y)$ .

**Definition 6.9.** A map  $i : Y \rightarrow X$  is strictly type-preserving if

1. for all  $H_Y \in \mathcal{H}_Y$  there exists  $H_X \in \mathcal{H}_X$  such that  $i(H_Y) \subset H_X$ , and
2. images of distinct horospheres-like sets in  $Y$  lie in distinct horosphere-like sets in  $X$ .

Let  $i : Y \rightarrow X$  be a strictly type-preserving proper embedding. Then  $i$  induces a proper embedding  $i_h : \mathfrak{G}(Y, \mathcal{H}_Y) \rightarrow \mathfrak{G}(X, \mathcal{H}_X)$ .

**Definition 6.10.** A Cannon-Thurston map exists for a strictly type-preserving inclusion  $i : Y \rightarrow X$  of relatively hyperbolic spaces if a Cannon-Thurston map exists for the induced map  $i_h : \mathfrak{G}(Y, \mathcal{H}_Y) \rightarrow \mathfrak{G}(X, \mathcal{H}_X)$ .

[Lemma 2.8](#) generalizes to the following.

**Lemma 6.11.** A Cannon-Thurston map for  $i : Y \rightarrow X$  exists if and only if there exists a non-negative proper function  $M : \mathbb{N} \rightarrow \mathbb{N}$  such that the following holds:

Fix a base-point  $y_0 \in Y$ . Let  $\hat{\lambda}$  in  $\mathfrak{E}(Y, \mathcal{H}_Y)$  be an electric geodesic segment starting and ending outside horospheres. If  $\lambda^b = \hat{\lambda} \setminus \bigcup_{K \in \mathcal{H}_Y} K$  lies outside  $B_N(y_0) \subset Y$ , then for any electric quasigeodesic  $\hat{\beta}$  joining the end points of  $\hat{\lambda}$  in  $\mathfrak{E}(X, \mathcal{H}_X)$ ,  $\beta^b = \hat{\beta} \setminus \bigcup_{H \in \mathcal{H}_X} H$  lies outside  $B_{M(N)}(i(y_0)) \subset X$ .

[Theorem 6.2](#) then generalizes to:

**Theorem 6.12.** *Mj and Pal [2011]* Let  $P : X \rightarrow T$  be a tree of relatively hyperbolic spaces satisfying the qi-embedded condition. Assume that

1. the inclusion maps of edge-spaces into vertex spaces are strictly type-preserving
2. the induced tree of electrified (coned-off) spaces continues to satisfy the qi-embedded condition
3.  $X$  is strongly hyperbolic relative to the family  $\mathcal{C}$  of maximal cone-subtrees of horosphere-like sets.

Then a Cannon-Thurston map exists for the inclusion of relatively hyperbolic spaces  $i : X_v \rightarrow X$ , where  $(X_v, d_{X_v})$  is the relatively hyperbolic vertex space corresponding to  $v$ .

**6.4 Problems.** The above survey is conditioned and limited by the author’s bias on the one hand and space considerations on the other. In particular we have omitted the important work on quasigeodesic foliations by [Calegari \[2006, 2000\]](#), [Fenley \[2012\]](#), and [Frankel \[2015\]](#) and the topology of ending lamination spaces by Gabai and others [Gabai \[2014, 2009\]](#), [C. J. Leininger, Mj, and Schleimer \[2011\]](#), and [Hensel and Przytycki \[2011\]](#) as this would be beyond the scope of the present article. A more detailed survey appears in [Papadopoulos \[2007\]](#). We end with some open problems.

**6.4.1 Higher dimensional Kleinian groups.** As a test-case we propose:

**Question 6.13.** *Let  $S$  be a closed surface of genus greater than one and let  $\Gamma = \pi_1(S)$  act freely, properly discontinuously by isometries on  $\mathbf{H}^n$ ,  $n > 3$  (or more generally a rank one symmetric space). Does a Cannon-Thurston map exist in general?*

Can the small cancellation group of Baker-Riley in [Baker and T. R. Riley \[2013\]](#) act geometrically on a rank one symmetric space thus giving a negative answer to [Question 6.13](#) with surface group replaced by free group? Work of [Wise \[2009\]](#) and [Wise \[2004\]](#) guarantees linearity of such small cancellation groups.

The critical problem in trying to answer [Question 6.13](#) is the absence of new examples in higher dimensions. It would be good to find new examples or prove that they do not exist. A version of a question due to [M. Kapovich \[2008\]](#) makes this more precise and indicates our current state of knowledge/ignorance:

**Question 6.14.** *Let  $S$  be a closed surface of genus greater than one and let  $\Gamma$  be a discrete subgroup of  $SO(n, 1)$  (or more generally a rank one Lie group) abstractly isomorphic to  $\pi_1(S)$  acting by isometries on  $\mathbf{H}^n$ ,  $n > 3$  (more generally the associated symmetric space) such that*

1. *orbits are not quasiconvex,*
2. *no element of  $\Gamma$  is a parabolic.*

*Does  $\rho$  factor through a representation to a simply or doubly degenerate (3-dimensional) Kleinian group followed by a deformation of  $SO(3, 1)$  in  $SO(n, 1)$ ?*

A closely related folklore question asks:

**Question 6.15.** *Can a closed higher dimensional  $n > 3$  rank one manifold fiber? In particular over the circle?*

It is known, from the Chern-Gauss-Bonnet theorem that a  $2n$  dimensional rank one manifold cannot fiber over the circle. Unpublished work of [M. Kapovich \[1998\]](#) shows that a complex hyperbolic 4-manifold cannot fiber over a 2-manifold.

**6.4.2 Surface groups in higher rank.** A topic of considerable current interest is higher dimensional Teichmüller theory and Anosov representations of surface groups [Labourie \[2006\]](#), [M. Kapovich, Leeb, and Porti \[2017\]](#), and [Guéritaud, Guichard, Kassel, and Wienhard \[2017\]](#). Kapovich, Leeb and Porti give an equivalent definition of Anosov representations in purely coarse geometric terms as representations that are *asymptotic embeddings*. It will take us too far afield to define these notions precisely here. What we will say however is that if  $\rho : \pi_1(S) \rightarrow \mathfrak{G}$  is a discrete faithful representation into a semi-simple Lie

group  $\mathcal{G}$  and  $\mathcal{G}/\mathcal{P} = \mathcal{B}$  is the Furstenberg boundary, then the Anosov property implies that an orbit  $\rho(\pi_1(S)).o$  ‘extends’ to a  $\rho(\pi_1(S))$ -equivariant embedding of  $\partial\Gamma_{\pi_1(S)} (= S^1)$  into  $\mathcal{B}$ . Thus the boundary map  $\Delta : S^1 \rightarrow \mathcal{B}$  maybe thought of as a *higher rank Cannon-Thurston map*. For the representation to be Anosov,  $\Delta$  is thus an embedding [M. Kapovich, Leeb, and Porti \[2017\]](#).

**Question 6.16.** *What class of representations do we get if we require only that  $\Delta$  is continuous?*

[Question 6.16](#) is basically asking for a ‘nice’ characterization of representations that admit a higher rank Cannon-Thurston map. The core problem in addressing it again boils down to finding some rich class of examples. [Question 6.14](#) has a natural generalization to this context where we replace  $SO(n, 1)$  by  $\mathcal{G}$ .

**6.4.3 Geometric group theory.** As we have seen in [Section 6.1](#), normal hyperbolic subgroups and trees of spaces provide examples where there is a positive answer to [Question 2.7](#). Some sporadic new examples have also been found, e.g. hydra groups [Baker and T. Riley \[2012\]](#). However no systematic theory exists. In the light of the counterexample in [Baker and T. R. Riley \[2013\]](#), the general answer to [Question 2.7](#) is negative. Are there necessary and/or sufficient conditions beyond [Lemma 2.8](#) to guarantee existence of Cannon-Thurston maps?

As illustrated in [Mitra \[1998b\]](#) and [Baker and T. Riley \[2012\]](#), distortion of subgroups [Gromov \[1993\]](#) is irrelevant. Distortion captures the relationship between  $d_H(1, h)$  with  $d_G(1, h)$ . On the other hand Cannon-Thurston maps capture the corresponding relationship between  $d_H(1, [h_1, h_2]_H)$  with  $d_G(1, [h_1, h_2]_G)$ , i.e. existence of Cannon-Thurston maps is equivalent to a proper embedding of ‘pairs of points’ (coding geodesic segments). The function associated with such a proper embedding is closely related to the modulus of continuity of the Cannon-Thurston map [Baker and T. Riley \[2012\]](#).

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