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CRITICAL ORBITS AND ARITHMETIC EQUIDISTRIBUTION

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Abstract

These notes present recent progress on a conjecture about the dynamics of rational maps on $\mathbb{P}^1(\mathbb{C})$, connecting critical orbit relations and the structure of the bifurcation locus to the geometry and arithmetic of postcritically finite maps within the moduli space M_d. The conjecture first appeared in a 2013 publication by Baker and DeMarco. Also presented are some related results and open questions.

1 The critical orbit conjecture

These lecture notes are devoted to a conjecture presented in Baker and DeMarco [2013] and the progress made over the past five years. The setting for this problem is the dynamics of rational maps

$$f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$

of degree $d > 1$. Such a map has exactly $2d - 2$ critical points, when counted with multiplicity, and it is well known in the study of complex dynamical systems that the critical orbits of f play a fundamental role in understanding its general dynamical features. For example, hyperbolicity on the Julia set, linearizability near a neutral fixed point, and stability in families can all be characterized in terms of critical orbit behavior. The postcritically finite maps – those for which each of the critical points has a finite forward orbit – play a special role within the family of all maps of a given degree d.

The critical orbit conjecture, in its most basic form, is the following:

Conjecture 1.1. *Let $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ be a nontrivial algebraic family of rational maps of degree $d > 1$, parametrized by t in a quasiprojective complex algebraic curve X. There are infinitely many $t \in X$ for which f_t is postcritically finite if and only if the family f_t has at most one independent critical orbit.*

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Remark 1.2. Favre and Gauthier have recently announced a proof of Conjecture 1.1 for all families f_t of polynomials, building on the series of works Baker and DeMarco [2011], Ghioca, Hsia, and Tucker [2013], Baker and DeMarco [2013], Ghioca and Ye [2016], Favre and Gauthier [2016]. Other forms of this conjecture appear as Baker and DeMarco [2013, Conjecture 1.10], Ghioca, Hsia, and Tucker [2015, Conjecture 2.3], DeMarco [2016a, Conjecture 6.1], DeMarco [2016b, Conjecture 4.8], treating also the higher-dimensional parameter spaces X, where much less is known.

An algebraic family is, by definition, one for which the coefficients of f_t are meromorphic functions of t on a compactification $\overline{X}$. We also assume that f_t is a holomorphic family on the Riemann surface X, in the sense that it determines a holomorphic map $f : X \times \mathbb{P}^1 \to \mathbb{P}^1$. The family is said to be trivial if all f_t, for $t \in X$, are Möbius conjugate rational maps.

The notion of having "at most one independent critical orbit" is a bit subtle to define. I will give two candidate definitions of this notion in Section 2, so Conjecture 1.1 is actually two distinct conjectures. But, roughly speaking, if $c_i : X \to \mathbb{P}^1$, $i = 1, \ldots, 2d-2$, parametrize the critical points of f_t, then "at most one independent critical orbit" should mean that, for every pair $i \neq j$, either (1) at least one of c_i or c_j is persistently preperiodic, so that $f_t^n(c_i(t)) = f_t^m(c_i(t))$ or $f_t^n(c_j(t)) = f_t^m(c_j(t))$ for some $n > m$ and all $t \in X$; or (2) there is an orbit relation of the form $f_t^n(c_i(t)) = f_t^m(c_j(t))$ holding for all t. (Assuming condition (1) or (2) for every pair $\{i, j\}$ easily implies that there are infinitely many postcritically finite maps in the family f_t, but this assumption is too strong for a characterization: these conditions do not capture the possible symmetries in the family f_t.)

Let us put this conjecture into context. From a complex-dynamical point of view, the independent critical orbits in a holomorphic family f_t induce bifurcations. Indeed, a holomorphic family f_t with holomorphically-parametrized critical points $c_i(t)$ (for t in a disk $D \subset \mathbb{C}$) is structurally stable on its Julia set if and only if each of the critical orbits determines a normal family of holomorphic functions $\{t \mapsto f_t^n(c(t))\}_{n \geq 0}$ from D to $\mathbb{P}^1$ Mañé, Sad, and Sullivan [1983], Lyubich [1983]. For nontrivial algebraic families as in Conjecture 1.1, McMullen proved that the family is stable on all of X if and only if all of the critical points are persistently preperiodic McMullen [1987]; in other words, the family will be postcritically finite for *all* $t \in X$. Thurston's Rigidity Theorem states that the only nontrivial families of postcritically finite maps are the flexible Lattès maps, meaning that they are quotients of holomorphic maps on a family of elliptic curves Douady and Hubbard [1993]; thus, we obtain a complete characterization of stable algebraic families. From this perspective, then, Conjecture 1.1 is an attempt to characterize a slight weakening of stability, where the number of independent critical orbits is allowed to be equal to the dimension of the parameter space. One then expects interesting geometric consequences:

for example, the postcritically finite maps should be uniformly distributed with respect to the bifurcation measure (defined in DeMarco [2001] when $\dim X = 1$ and Bassanelli and Berteloot [2007] in general) on any such parameter space X.

But Conjecture 1.1 was in fact motivated more from the perspective of arithmetic geometry and the principle of unlikely intersections, as exposited in Zannier [2012]. The moduli space M_d of rational maps on $\mathbb{P}^1$ of degree $d > 1$ is naturally an affine scheme defined over $\mathbb{Q}$ Silverman [1998]. From Thurston's Rigidity Theorem, we may deduce that the postcritically finite maps lie in $\mathrm{M}_d(\overline{\mathbb{Q}})$, except for the 1-parameter families of flexible Lattès examples. Furthermore, the postcritically finite maps form

(a) a Zariski dense subset of M_d DeMarco [2016b, Theorem A], and

(b) a set of bounded Weil height in $\mathrm{M}_d(\overline{\mathbb{Q}})$, after excluding the flexible Lattès families Benedetto, Ingram, Jones, and Levy [2014, Theorem 1.1].

It is then natural to ask which algebraic subvarieties V of M_d also contain a Zariski-dense subset of postcritically finite maps. The general form of the conjecture states that this is a very special property of the variety V: it should hold if and only if V is itself defined by critical orbit relations.

This type of question is reminiscent of some famous questions and conjectures in algebraic and arithmetic geometry. To name a few, we may consider the Manin-Mumford Conjecture about abelian varieties (now theorems of Raynaud [1983a,b]) or the multiplicative version due to Lang [1960] – where a subvariety contains "too many" torsion points if and only if it is itself a subgroup (or closely related to one) – and the André–Oort conjecture which is a moduli-space analogue Klingler and Yafaev [2014], Pila [2011], and Tsimerman [2018]. In fact, our conjecture has been called the "Dynamical André–Oort Conjecture" in the literature; however, unlike for the "Dynamical Manin–Mumford Conjecture" of Zhang (S.-W. Zhang [2006] and Ghioca, Tucker, and S. Zhang [2011]), there is no overlap between the original conjecture and its dynamical analogue, at least not in the setting presented here for critical orbits and the moduli space M_d. On the other hand, there are generalizations of each – of our critical orbit conjecture and of these geometric conjectures – which *do* have overlap, and some of this is discussed in Section 5. Here I state a sample result, from my recent joint work with N. M. Mavraki, extending the work of Masser and Zannier [2010, 2014] and closely related to that of Ullmo [1998] and S.-W. Zhang [1998a].

Theorem 1.3. *DeMarco and Mavraki [2017] Let B be a quasiprojective algebraic curve defined over $\overline{\mathbb{Q}}$. Suppose $A \to B$ is a family of abelian varieties defined over $\overline{\mathbb{Q}}$ which is isogenous to a fibered product of $m \geq 2$ elliptic surfaces over B. Let $\mathfrak{L}$ be a line bundle on A which restricts to an ample and symmetric line bundle on each fiber A_t, and let $\hat{h}_t$ be the induced Néron-Tate canonical height on A_t, for each $t \in B(\overline{\mathbb{Q}})$. Finally, let $P : B \to A$*

be a section defined over $\overline{\mathbb{Q}}$. Then there exists an infinite sequence of points $t_n \in B$ for which

$$\hat{h}_{t_n}(P_{t_n}) \to 0$$

if and only if P is special.

Of course, I have not given any of the definitions of the words in this statement, so it is perhaps meaningless at first glance. My goal is merely to illustrate the breadth of concepts that connect back to the dynamical statement of Conjecture 1.1 and the existing proofs of various special cases. To make the analogy explicit: fixing a section P in $A = E_1 \times_B \cdots \times_B E_m$ would correspond, in a dynamical setting, to marking m critical points of a family of rational functions; the parameters $t \in B(\overline{\mathbb{Q}})$ where $\hat{h}_t(P_t) = 0$ correspond to the postcritically finite maps in the family; and the "specialness" of P corresponds to the family f_t having at most one independent critical orbit. In Theorem 1.3, however, the novelty is the treatment of parameters t with small (positive) height and not only height 0.

Outline. I begin by defining critical orbit relations in Section 2. Section 3 contains the sketch of a proof of a theorem from Baker and DeMarco [2011] that inspired our formulation of Conjecture 1.1 and many of the proofs that appeared afterwards, especially the cases of treated in Baker and DeMarco [2013]. Section 4 brings us up to date with what is now known about Conjecture 1.1. Finally, in Section 5, I discuss a generalization of the conjecture which motivated Theorem 1.3 and related results.

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2 Critical orbit relations

In this section we formalize the notion of dependent critical orbits to make Conjecture 1.1 precise.

Let f_t be a nontrivial algebraic family of rational maps of degree $d > 1$, parameterized by t in a quasiprojective, complex algebraic curve X. By passing to a branched cover of X, we may assume that each of the critical points of f_t can be holomorphically parameterized by $c_i : X \to \mathbb{P}^1$, $i = 1, \ldots, 2d - 2$. A critical point c_i is *persistently preperiodic* if it satisfies a relation of the form $f_t^n(c_i(t)) = f_t^m(c_i(t))$, with $n \neq m$, for all t. A pair of non-persistently-preperiodic critical points (c_i, c_j) is said to be *coincident* if we have

$$c_i(t) \text{ is preperiodic for } f_t \iff c_j(t) \text{ is preperiodic for } f_t \tag{2-1}$$

for all but finitely many $t \in X$; see DeMarco [2016a, Section 6]. If the relation (2-1) holds for every pair of non-persistently-preperiodic critical points, then the bifurcation locus of the family f_t – in the sense of Mañé, Sad, and Sullivan [1983], Lyubich [1983] – is determined by the orbit of a single critical point. That is, choosing any $i \in \{1, \ldots, 2d-2\}$ for which c_i is not persistently preperiodic, the sequence of holomorphic maps

$$\{t \mapsto f_t^n(c_i(t)) : n \geq 1\}$$

forms a normal family on an open set $U \subset X$ if and only if the family $\{f_t\}$ is J-stable on U. (See McMullen [1994, Chapter 4], Dujardin and Favre [2008a, Lemma 2.3].)

Definition 2.1 (One independent critical orbit: weak notion)**.** We say that an algebraic family f_t has at most one independent critical orbit if every pair of non-persistently-preperiodic critical points is coincident.

The relation (2-1) is implied by a more traditional notion of critical orbit relation, namely that there exist integers $n, m \geq 0$, so that

$$f_t^n(c_i(t)) = f_t^m(c_j(t)) \tag{2-2}$$

for all t. Because of the possibility of symmetries of f_t, we cannot expect (2-1) to be equivalent to (2-2). Examples are given in Baker and DeMarco [2013]. In that article, we formulated a more general notion of orbit relation that accounts for these symmetries and still implies coincidence. To define this, we let $\overline{X}$ be a smooth compactification of X, and consider the family f_t as one rational map defined over the function field $k = \mathbb{C}(\overline{X})$; it acts on $\mathbb{P}^1_{\bar{k}}$. A pair $a, b \in \mathbb{P}^1(k)$ is *dynamically related* if the point $(a, b) \in \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ lies on an algebraic curve

$$V \subset \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}} \tag{2-3}$$

which is forward invariant for the product map

$$(f, f) : (\mathbb{P}^1_{\bar{k}})^2 \to (\mathbb{P}^1_{\bar{k}})^2.$$

For example, if the point a is persistently preperiodic, then it is dynamically related to any other point b, taking $V = \{(x, y) : f^n(x) = f^m(x)\}$ to depend only on one coordinate. The relation (2-2) implies that (c_i, c_j) are dynamically related, taking $V = \{(x, y) : f^n(x) = f^m(y)\}$. But also, if f commutes with a rational function A of degree ≥ 1, then points a and $b = A(a)$ are dynamically related by the invariant curve $V = \{(x, y) : y = A(x)\}$.

Definition 2.2 (One independent critical orbit: strong notion)**.** We say that an algebraic family f_t has at most one independent critical orbit if every pair of critical points is dynamically related.

I expect the two notions of "one independent critical orbit" to be equivalent DeMarco [2016a, Conjecture 6.1], but we can easily show:

Lemma 2.3. *The strong notion implies the weak notion.*

Proof. Let f_t be a nontrivial algebraic family of rational maps, for $t \in X$, and assume that it has at most one independent critical orbit, in the strong sense. Let $F_t = (f_t, f_t)$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ for all $t \in X$. Assume that neither c_i nor c_j is persistently preperiodic. Then there exists an algebraic curve $V \subset \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ (defined over $k = \mathbb{C}(\overline{X})$, or perhaps a finite extension) so that the specializations satisfy

$$F_t^n(c_i(t), c_j(t)) \in V_t$$

for all n and all but finitely many t. Note that V_t cannot contain the vertical component $\{c_i(t)\} \times \mathbb{P}^1$ for infinitely many t: indeed, the bi-degree of the specialization V_t within $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is equal to the bi-degree (k, ℓ) of V in $\mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ for all but finitely many t, the curve V_t is invariant under (f_t, f_t) so a vertical component through $c_i(t)$ implies a vertical component through $f_t^n(c_i(t))$ for all n, and there are only finitely many t where the orbit of c_i has length $\leq \ell$. Thus, for all but finitely many t, if $c_i(t)$ is preperiodic, then the orbit of $c_j(t)$ is confined to lie in a finite subset, and so $c_j(t)$ will also be preperiodic. By symmetry, the same holds when $c_j(t)$ is preperiodic, and the proof is complete. □

One might ask why we only consider dynamical relations between *pairs* of critical points and not arbitrary tuples of critical points (as was first formulated in Baker and DeMarco [2013] and DeMarco [2016a]). In fact, the model-theoretic approach of Medvedev [2007] and Medvedev and Scanlon [2014] implies that it is sufficient to consider only the relations between two points.

Theorem 2.4. *Medvedev [2007, Theorem 10], Medvedev and Scanlon [2014, Fact 2.25] Suppose that f is a rational map of degree > 1, defined over a field k of characteristic 0, and assume that it is not conjugate to a monomial map, $\pm$ a Chebyshev polynomial, or a Lattès map. Let*

$$V \subset \mathbb{P}^1_{\bar{k}} \times \cdots \times \mathbb{P}^1_{\bar{k}}$$

be forward-invariant by the action of $(f, \dots, f)$. Then each component of V is a component of the intersection

$$\bigcap_{1 \leq i \leq j \leq n} \pi_{i,j}^{-1} \pi_{i,j}(V),$$

where $\pi_{i,j}$ is the projection to the product of the i-th and j-th factors in $(\mathbb{P}^1_{\bar{k}})^n$.

In the setting of non-trivial algebraic families f_t, as in Conjecture 1.1, the monomials and Chebyshev polynomials do not arise because they would be trivial, and the flexible Lattès maps have all their critical points persistently preperiodic. Thus, we may apply Theorem 2.4 and restrict our attention to dynamical relations among critical points that depend only on two of the critical points at a time.

Even having narrowed our concept of dynamical relation to (2-3), depending on only two points, we still do not have an explicit description of all possible relations. The article Medvedev and Scanlon [ibid.] provides a careful and complete treatment of the case of polynomials f, building on the work of Ritt [1922]. Their classification of invariant curves for polynomial maps of the form (f, f) appears as a key step in the proof of the main theorem of my work with Baker and DeMarco [2013], where we prove special cases of Conjecture 1.1.

The classifcation of invariant curves in $\mathbb{P}^1 \times \mathbb{P}^1$ for a product of rational maps is still an open problem. Works by Pakovich and Zieve (see, e.g., Pakovich [2016] and Zieve [2007]) take steps towards such a classification. I posed the following question in DeMarco [2016b, Conjecture 4.8], as this represents the form of all relations I know (including for pairs of points that are not necessarily critical):

Question 2.5. *Let f_t be a non-trivial algebraic family of rational maps of degree >1, parameterized by $t \in X$, and suppose that $a, b \in \mathbb{P}^1(k)$ are two non-persistently-preperiodic points, for $k = \mathbb{C}(\overline{X})$. If a and b are dynamically related in the sense of* (2-3)*, then do there exist rational functions A, B of degrees ≥ 1 defined over $\bar{k}$ and an integer $\ell \geq 1$ so that*

$$f^\ell \circ A = A \circ f^\ell, \quad f^\ell \circ B = B \circ f^\ell, \quad \textit{and} \quad A(a) = B(b)?$$

Note that A and B might themselves be iterates of f. It is known that if two rational maps of degree > 1 commute, and if they aren't monomial, Chebyshev, or Lattès, then they must share an iterate Ritt [1923].

3 Proof strategy: heights and equidistribution

In this section, I present the sketch of a proof of a closely related result from Baker and DeMarco [2011], one which initially inspired the formulation of Conjecture 1.1 and its generalizations. The ideas in the proof given here have gone into the proofs of all of the successive results related to Conjecture 1.1, though of course distinct technical issues and complications arise in each new setting.

Before getting started, I need to introduce one important tool, the canonical height of a rational function Call and Silverman [1993]. If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map of degree

$d > 1$, defined over a number field, then its canonical height function

$$\hat{h}_f : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$$

is defined by

$$\hat{h}_f(\alpha) = \lim_{n\to\infty} \frac{1}{d^n}\, h(f^n(\alpha))$$

where h is the usual logarithmic Weil height on $\mathbb{P}^1(\overline{\mathbb{Q}})$. It is characterized by the following two important properties: (1) there exists a constant $C = C(f)$ so that $|h - \hat{h}_f| < C$ and (2) $\hat{h}_f(f(\alpha)) = d\,\hat{h}_f(\alpha)$ for all $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$. As a consequence, we have $\hat{h}_f(\alpha) = 0$ if and only if α has finite forward orbit for f Call and Silverman [1993, Corollary 1.1.1].

Theorem 3.1. *Baker and DeMarco [2011] Fix points $a_1, a_2 \in \mathbb{C}$. Let $f_t(z) = z^2 + t$ be the family of quadratic polynomials, for $t \in \mathbb{C}$, and define*

$$S(a_i) := \{t \in \mathbb{C} : a_i \textit{ is preperiodic for } f_t\}.$$

Then the intersection $S(a_1) \cap S(a_2)$ is infinite if and only if $a_1 = \pm a_2$.

Sketch of Proof. Step 1 is to treat the easy implication: assume that $a_1 = \pm a_2$ and deduce that $S(a_1) \cap S(a_2)$ is an infinite set. This uses a standard argument from complex dynamics. For any given point a, we first observe that the family of functions $\{t \mapsto f_t^n(a)\}$ cannot be normal on all of $\mathbb{C}$. Indeed, for all t large, we find that $f_t^n(a) \to \infty$ as $n \to \infty$, while for $t = a - a^2$, the point a is fixed by f_t. In fact, via Montel's Theorem on normal families, there must be infinitely many values of t for which a is preperiodic, and therefore $S(a)$ is infinite. If $a_1 = \pm a_2$, then $f_t(a_1) = f_t(a_2)$ for all t, and therefore $S(a_1) = S(a_2)$.

The goal of Step 2 is to show that $S(a_1) \cap S(a_2)$ being infinite implies that a_1 and a_2 are coincident: we will see that $S(a_1) = S(a_2)$. First assume that a_1 and a_2 are algebraic numbers, and suppose K is a number field containing both a_1 and a_2. We define a height function on $\mathbb{P}^1(\overline{K})$ associated to each a_i. Indeed, for each $t \in \overline{K}$, we set

$$h_i(t) := \hat{h}_{f_t}(a_i)$$

where $\hat{h}_{f_t}$ is the canonical height function of f_t. In particular, we see that

$$h_i(t) = 0 \iff t \in S(a_i).$$

It turns out that h_i is the height associated to a continuous adelic metric of non-negative curvature on the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ (in the sense of S. Zhang [1995]) and an adelic measure (in the sense of Baker and Rumely [2006] and Favre and Rivera-Letelier [2006]).

Therefore, we may apply the equidistribution theorems of Baker and Rumely [2010], Favre and Rivera-Letelier [2006], and Chambert-Loir [2006] to see that the elements of $S(a_i)$ are uniformly distributed with respect to a natural measure μ_i on $\mathbb{P}^1(\mathbb{C})$. More precisely, given any sequence of finite subsets $S_n \subset S(a_i)$ which are $\mathrm{Gal}(\overline{K}/K)$-invariant and with $|S_n| \to \infty$, the discrete probablity measures

$$\mu_{S_n} = \frac{1}{|S_n|} \sum_{s \in S_n} \delta_s$$

on $\mathbb{P}^1(\mathbb{C})$ will converge weakly to μ_i. In particular, when $S(a_1) \cap S(a_2)$ is infinite, this set – because it is $\mathrm{Gal}(\overline{K}/K)$ invariant – will be uniformly distributed with respect to *both* μ_1 and μ_2, allowing us to deduce that $\mu_1 = \mu_2$. Even more, by the nondegeneracy of a pairing between heights of this form, we can also conclude that $h_1 = h_2$. Therefore $S(a_1) = S(a_2)$.

Step 2 will be complete if we can also treat the case where at least one of the a_i is not algebraic. In this setting, the smallest field K containing the a_i and $\mathbb{Q}$ will have a nonzero transcendence degree over $\mathbb{Q}$, so we treat K as a function field. The arithmetic equidistribution theorems (Baker and Rumely [2010], Favre and Rivera-Letelier [2006], and Chambert-Loir [2006]) work just as well in this setting. However, the equidistribution in question – of Galois-invariant subsets of $S(a_i)$ becoming uniformly distributed with respect to a natural measure μ_i – is no longer taking place on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. Instead, we obtain a geometric convergence statement on a family of Berkovich projective lines $\mathbf{P}^1_v$, one for each place v of the function field K. Nevertheless, one concludes from equidistribution that if $S(a_1) \cap S(a_2)$ is infinite, then the heights h_1 and h_2 on $\mathbb{P}^1(\overline{K})$ must coincide, and therefore

$$a_1(t) \text{ is preperiodic for } f_t \iff a_2(t) \text{ is preperiodic for } f_t$$

for all $t \in \overline{K}$. But if there is some element t of $\mathbb{C}$ where $a_1(t)$ is preperiodic, then that t can be identified with an element of $\overline{K}$, so in fact

$$a_1(t) \text{ is preperiodic for } f_t \iff a_2(t) \text{ is preperiodic for } f_t$$

for all $t \in \mathbb{C}$.

Step 3 is to show that coincidence implies an explicit relation between the two points a_1 and a_2. When a_1 and a_2 are both algebraic, a closer look at the definitions of the heights h_i reveals that the measure μ_i on $\mathbb{P}^1(\mathbb{C})$ is the equilibrium measure on the boundary of a "Mandelbrot-like" set associated to the point a_i. That is, we consider the sets

$$M_i = \{t \in \mathbb{C} : \sup_n |f_t^n(a_i)| < \infty\},$$

and μ_i is the harmonic measure for the domain $\hat{\mathbb{C}} \setminus M_i$ centered at ∞. (Note that if $a_i = 0$, then M_i is the usual Mandelbrot set M, and μ_i is the bifurcation measure for the family f_t.) But even for non-algebraic points a_i, having concluded from Step 2 that $S(a_1) = S(a_2)$, we see that $M_1 = M_2$; this is because the set M_i is obtained from the closure $\overline{S(a_i)}$ by filling in the bounded complementary components. Now, just as in the original proof that the Mandelbrot set M is connected, which uses a dynamical construction of the Riemann map to $\hat{\mathbb{C}} \setminus M$, we investigate the uniformizing map near ∞ for the sets $M_1 = M_2$. The injectivity of that map – built out of the Böttcher coordinates near ∞ for the maps f_t with t large – allows us to deduce that $f_t(a_1) = f_t(a_2)$ for all t large. Therefore, we have $a_1 = \pm a_2$. □

It is worth observing at this point, as was observed in Baker and DeMarco [2011], that the proof of Theorem 3.1 gives a stronger statement. The arithmetic equidistribution theorems allow us to treat intersections of points of small height and not only those of height 0 (for the heights h_i introduced in the proof). For example, the proof provides:

Theorem 3.2. *Baker and DeMarco [ibid.] Fix points $a_1, a_2 \in \overline{\mathbb{Q}}$. Let $f_t(z) = z^2 + t$ be the family of quadratic polynomials, for $t \in \mathbb{C}$, and define*

$$S(a_i) := \{t \in \mathbb{C} : a_i \text{ is preperiodic for } f_t\}.$$

Then $S(a_i) \subset \overline{\mathbb{Q}}$, and the following are equivalent:

1. *there exists an infinite sequence $t_n \in \overline{\mathbb{Q}}$ for which $h_1(t_n) \to 0$ and $h_2(t_n) \to 0$;*
2. *the intersection $S(a_1) \cap S(a_2)$ is infinite;*
3. *$S(a_1) = S(a_2)$;*
4. *$\mu_1 = \mu_2$; and*
5. *$a_1 = \pm a_2$.*

The original motivation for statements like Theorem 3.2, and specifically the inclusion of condition (1), includes the Bogomolov Conjecture, proved by Ullmo and Zhang Ullmo [1998] and S.-W. Zhang [1998a], building on the equidistribution theorem of Szpiro, Ullmo, and S. Zhang [1997]. I will return to this theme in Section 5.

4 What is known

A good deal of work has gone into proving Conjecture 1.1 and its generalizations in various settings. Here, I mention some of the key recent developments. Most progress has been made in the context of polynomial dynamics. One important advantage of working with polynomials is that the conjecture itself is easier to state in a more precise form: the critical orbit relations, in the sense of (2-3), have been classified, as discussed in Section 2. But also, we have the advantage of extra tools: the uniformizing Böttcher coordinates of a complex polynomial near ∞ have proved immensely useful (as in Step 3 of the proof of Theorem 3.1), and the height functions (as defined in Step 2 of Theorem 3.1) are easier to work with. For example, the main theorem of Favre and Gauthier [2017] addresses an important property of the dynamically-defined local height functions, used in their proof of Conjecture 1.1 for families of polynomials; the analogous result fails for general families of rational maps DeMarco and Okuyama [2017].

Conjecture 1.1 is trivially satisfied for polynomials in degree 2, where the moduli space has dimension 1 and can be parameterized by the family $f_t(z) = z^2 + t$ with $t \in \mathbb{C}$ with exactly one independent critical point at $z = 0$. There are infinitely many postcritically finite polynomials in this family. Furthermore, it is known that the postcritically finite maps are uniformly distributed with respect to the bifurcation measure in this family (which is equal to the equilibrium measure μ_M on the boundary of Mandelbrot set) Levin [1989]. In addition, Baker and Hsia proved an arithmetic version of the equidistribution theorem, deducing that any sequence of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subsets of the postcritically finite parameters is also uniformly distributed with respect to μ_M Baker and Hsia [2005].

In Baker and DeMarco [2013], we proved Conjecture 1.1 for families f_t of polynomials of arbitrary degree, parametrized by $t \in \mathbb{C}$ with coefficients that were polynomial in t. In degree 3, but for arbitrary algebraic families, the following result was obtained a few years ago, independently by Favre-Gauthier and Ghioca-Ye.

Theorem 4.1. *Favre and Gauthier [2016] and Ghioca and Ye [2016] Let X be an irreducible complex algebraic curve in the space $\mathcal{P}_3 \simeq \mathbb{C}^2$ of polynomials of the form*

$$f_{a,b}(z) = z^3 - 3a^2 z + b,$$

with critical points at $\pm a$. There are infinitely many $t \in X$ for which f_t is postcritically finite if and only if one of the following holds:

1. *either a or $-a$ is persistently preperiodic on X;*

2. *there is a symmetry of the form*

$$f_t(-z) = -f_t(z)$$

for all $t \in X$ and all $z \in \mathbb{C}$, and $X = \{(a, b) \in \mathbb{C}^2 : b = 0\}$; or

3. *there exist non-negative integers n and m so that*

$$f_t^n(a(t)) = f_t^m(-a(t))$$

for all $t \in X$.

Furthermore, in each of these cases, the postcritically finite maps will be (arithmetically) equidistributed with respect to the bifurcation measure on X.

Remark 4.2. Favre and Gauthier have recently announced a proof of Conjecture 1.1 for families f_t of polynomials in arbitrary degree, extending Theorem 4.1.

The proof of Theorem 4.1 has the same outline as the proof of Theorem 3.1. The idea is to consider the two points a_t and $-a_t$ for $t \in X$ and follow the same three steps. Step 1, the "easy" implication, follows as before, with the additional input that any non-persistently-preperiodic critical point along the curve X must be undergoing bifurcations Dujardin and Favre [2008a, Theorem 2.5].

For Step 2, we assume that neither critical point is persistently preperiodic and that there are infinitely many postcritically finite maps on X, and we aim to show coincidence of the two points. There is one simplifying condition in the setting of Theorem 4.1: the postcritically finite maps are algebraic points in $\mathcal{P}_3$, so X must itself be defined over $\overline{\mathbb{Q}}$. Thus we can avoid the arguments needed for the transcendental case of Theorem 3.1. Nevertheless, there are new difficulties that arise; for example, it is not obvious that the height functions

$$h_{\pm a}(t) := \hat{h}_{f_t}(\pm a_t)$$

defined on $X(\overline{\mathbb{Q}})$ will satisfy the hypotheses of the existing arithmetic equidistribution theorems. This is checked with some careful estimates near the cusps of X. Then one can apply the equidistribution theorems of Yuan [2008], Thuillier [2005], and Chambert-Loir [2006] and conclude that $h_a = h_{-a}$ on X. In particular, we then have that $a(t)$ is preperiodic if and only if $-a(t)$ is preperiodic, for all $t \in X$.

Finally, one needs to deduce the explicit algebraic relations on the critical points, as in Step 3 of Theorem 3.1. One strategy is provided in my work with Baker and DeMarco [2013], to first produce an *analytic* relation between the critical points a and $-a$ via the Böttcher coordinates at infinity (similar to what was done for Theorem 3.1 but in a more general setting). From there, we used iteration to promote the analytic relation to an algebraic relation which is invariant under the dynamics. This strategy is followed in Ghioca and Ye [2016]; an alternative approach is given in Favre and Gauthier [2016]. To obtain the explicit form of the relation, Baker and I used results of Medvedev and Scanlon [2014], classifying the invariant curves for (f, f) acting on $\mathbb{P}^1 \times \mathbb{P}^1$ (over the function field $\mathbb{C}(X)$). The results of Favre-Gauthier and Ghioca-Ye for cubic polynomials simplify the relations further to give the possibilities appearing in Theorem 4.1.

One can also formulate a version of Conjecture 1.1 for tuples of polynomials or rational maps, rather than a single family of rational maps. The following result answered a question posed by Patrick Ingram, inspired by the result of André about complex-multiplication pairs in the moduli space $\mathfrak{M}_1 \times \mathfrak{M}_1 \simeq \mathbb{C}^2$ of pairs of elliptic curves André [1998].

Theorem 4.3. *Ghioca, Krieger, Nguyen, and Ye [2017] Let X be an irreducible complex algebraic curve in the space $\mathcal{P}_2 \times \mathcal{P}_2 \simeq \mathbb{C}^2$ of pairs of quadratic polynomials of the form $f_t(z) = z^2 + t$. If X contains infinitely many pairs (t_1, t_2) for which both f_{t_1} and f_{t_2} are postcritically finite, then X is*

1. *a vertical line $\{t_1\} \times \mathbb{C}$ where f_{t_1} is postcritically finite;*
2. *a horizontal line $\mathbb{C} \times \{t_2\}$ where f_{t_2} is postcritically finite; or*
3. *the diagonal $\{(t,t) : c \in \mathbb{C}\}$.*

By contrast, in the case of pairs of elliptic curves, there is an infinite collection of modular curves in $\mathfrak{M}_1 \times \mathfrak{M}_1$, all of which contain infinitely many CM pairs. Thus, Theorem 4.3 tells us that there is no analogue of these modular curves in the quadratic family. To see this, the authors prove an important rigidity property of the Mandelbrot set M: it is not invariant under nontrivial algebraic correspondences. This rigidity was recently extended to a local, analytic rigidity statement in Luo [2017]: Luo proved that any conformal isomorphism between domains $U, V \subset \mathbb{C}$ intersecting the boundary ∂M, sending $U \cap \partial M$ to $V \cap \partial M$, must be the identity.

For non-polynomial rational maps, Conjecture 1.1 is only known for some particular families. For example, in the moduli space of quadratic rational maps $\mathrm{M}_2 \simeq \mathbb{C}^2$, for each $\lambda \in \mathbb{C}$, one may consider the dynamically-defined subvariety

$$\mathrm{Per}_1(\lambda) = \{f \text{ in } \mathrm{M}_2 \text{ with a fixed point of multiplier } \lambda\},$$

where the multiplier of a fixed point is simply the derivative of f at that point Milnor [1993]. (Similarly, one can define the algebraic curves $\mathrm{Per}_n(\lambda)$ for maps with a cycle of period n and multiplier λ, but one should take care in the definition when $\lambda = 1$.) Observe that the curve $\mathrm{Per}_1(0)$ is defined by a critical orbit relation, the existence of a fixed critical point. Thus, the curve $\mathrm{Per}_1(0)$ coincides with the family of polynomials within M_2; in particular, it contains infinitely many postcritically finite maps.

Theorem 4.4. *DeMarco, Wang, and Ye [2015] Fix $\lambda \in \mathbb{C}$. The curve $\mathrm{Per}_1(\lambda)$ in the moduli space of quadratic rational maps contains infinitely many postcritically finite maps if and only if $\lambda = 0$.*

Remark 4.5. The analogous result for $\mathrm{Per}_1(\lambda)$ in the space of cubic polynomials was obtained in Baker and DeMarco [2013]. The theorem is proved for curves $\mathrm{Per}_n(\lambda)$, for every n, in the space of cubic polynomials in Favre and Gauthier [2016].

All of these theorems are closely related to questions about the geometry of the bifurcation locus the family f_t of rational maps, as seen in Step 3 in the proof of Theorem 3.1, or the statement about the rigidity of the Mandelbrot set used to prove Theorem 4.3. In the case of $\mathrm{Per}_1(\lambda)$, we should first pass to a double cover $\widehat{\mathrm{Per}}_1(\lambda)$ where the two critical points can be holomorphically and independently parameterized by c_1 and c_2. The bifurcation measure μ_i^λ of the critical point c_i reflects the failure of the family $\{t \mapsto f_t^n(c_i(t))\}$ to be normal on $\widehat{\mathrm{Per}}_1(\lambda)$. Let $S(c_i)$ denote the set of parameters t where the critical point c_i is preperiodic. It is known that $S(c_i)$ is uniformly distributed with respect to this measure μ_i^λ for all $\lambda \in \mathbb{C}$ Dujardin and Favre [2008a,b]. The proof of Theorem 4.4 uses the following two strengthenings of this equidistribution statement:

Theorem 4.6. *DeMarco, Wang, and Ye [2015] For each $\lambda \in \mathbb{C} \setminus \{0\}$, we have $\mu_1^\lambda \neq \mu_2^\lambda$ on $\widehat{\mathrm{Per}}_1(\lambda)$.*

Theorem 4.7. *DeMarco, Wang, and Ye [2015] and Mavraki and Ye [2015] For each $\lambda \in \overline{\mathbb{Q}} \setminus \{0\}$, we have arithmetic equidistribution of $S(c_1)$ and $S(c_2)$. That is, for any infinite sequence t_n in $S(c_i)$, the discrete measures*

$$\frac{1}{|\operatorname{Gal}(\overline{\mathbb{Q}(\lambda)}/\mathbb{Q}(\lambda)) \cdot t_n|} \sum_{t \in \operatorname{Gal}(\overline{\mathbb{Q}(\lambda)}/\mathbb{Q}(\lambda)) \cdot t_n} \delta_t$$

converge weakly to the bifurcation measure μ_i^λ on $\widehat{\mathrm{Per}}_1(\lambda)$.

Remark 4.8. The height functions $h_i(t) := \hat{h}_{f_t}(c_i(t))$ on $\widehat{\mathrm{Per}}_1(\lambda)$ provided the first examples of this type that are *not* adelic – in the sense of S. Zhang [1995], Baker and Rumely [2010], and Favre and Rivera-Letelier [2006] – and therefore did *not* satisfy the hypotheses of the existing equidistribution theorems. The article Mavraki and Ye [2015] extends the equidistribution theorems of Baker and Rumely [2010], Favre and Rivera-Letelier [2006], and Chambert-Loir [2006] for heights on $\mathbb{P}^1(\overline{\mathbb{Q}})$ to the setting of quasi-adelic heights.

Remark 4.9. Despite Theorem 4.6, it is not yet known if $\operatorname{supp} \mu_1^\lambda \neq \operatorname{supp} \mu_2^\lambda$ for all $\lambda \in \mathbb{C}$; see DeMarco, Wang, and Ye [2015, Question 2.4]. One can ask, much more generally, about the bifurcation loci associated to independent critical points in algebraic families f_t in every degree and if they can ever coincide; see DeMarco [2016b, Question 2.5].

An assortment of results is known for other families of rational functions. For example, in Ghioca, Hsia, and Tucker [2015], the authors treat maps of the form

$$f_t(z) = g(z) + t$$

for $t \in \mathbb{C}$, where $g \in \overline{\mathbb{Q}}(z)$ is a rational function of degree > 2 with a superattracting fixed point at ∞. They show the weaker form of the conjecture, deducing coincidence of the critical orbits if there are infinitely many postcritically finite maps; this follows from an equidistribution result associated to the dynamically-defined height functions on $\mathbb{P}^1(\overline{\mathbb{Q}})$ (similar to Step 2 in the proofs of Theorems 3.1 and 4.1).

Almost nothing is known about Conjecture 1.1 in the context of higher-dimensional parameter spaces X, apart from the "easy" implication of the conjecture, as in Step 1 of Theorem 3.1; see DeMarco [2016a]. A general form of the arithmetic equidistribution theorem exists for higher-dimensional arithmetic varieties Yuan [2008], but the challenge lies in understanding when a dynamically-defined height will satisfy the stated hypotheses; see, e.g., Favre and Gauthier [2015] and Remark 4.8 above.

5 Arbitrary points

In this final section, I present some results about a generalization of Conjecture 1.1 that connects with interesting results and questions about elliptic curves (or more general families of abelian varieties). As in Theorem 3.1, one can study the orbits of arbitrary points, not only the critical orbits. This may seem less motivated in the context of studying complex dynamical systems, as the critical points are the ones that induce bifurcations (in the traditional dynamical sense), but the problem is quite natural from another point of view.

As an example, the following result was proved by Masser and Zannier, motivated by a conjecture of Pink [2005]:

Theorem 5.1. *Masser and Zannier [2010, 2012] Let E_t be a non-isotrivial family of elliptic curves over a quasiprojective curve B, so defining an elliptic curve E over the function field $k = \mathbb{C}(\overline{B})$. Suppose that P and Q are non-torsion elements of $E(k)$. There are infinitely many $t \in B(\mathbb{C})$ for which P_t and Q_t are both torsion on E_t if and only if there exist nonzero integers n and m so that $nP + mQ = 0$ on E.*

Because the multiplication-by-m maps on an elliptic curve descend to rational maps on $\mathbb{P}^1$, and because the torsion points on the elliptic curve project to the preperiodic points on $\mathbb{P}^1$, Theorem 5.1 has a direct translation into a dynamical statement:

Theorem 5.2. *Let f_t be a family of flexible Lattès maps on $\mathbb{P}^1$ parameterized by $t \in B$, induced from an endomorphism of a non-isotrivial elliptic curve E over $k = \mathbb{C}(\overline{B})$. Fix non-persistently-preperiodic points $P, Q \in \mathbb{P}^1(k)$. Then there are infinitely many $t \in B$ for which both P_t and Q_t are preperiodic for f_t if and only if there exist Lattès maps g_t and h_t (also induced by endomorphisms of E) for which*

$$g_t(P_t) = h_t(Q_t)$$

for all t.

In fact, it was Theorem 5.1 that inspired Theorem 3.1 in the first place: Baker and I were answering a question posed by Umberto Zannier. And compare the conclusion of Theorem 5.2 to that of Question 2.5: note that the Lattès maps g_t and h_t will commute with f_t. Conjecture 1.1 is just a special case of conjectures presented in Ghioca, Hsia, and Tucker [2015] or DeMarco [2016a], addressing algebraic families f_t and arbitrary (non-critical) pairs of marked points, and for which Theorems 5.1 and 5.2 are also a special case:

Conjecture 5.3. *Let $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ be a nontrivial algebraic family of rational maps of degree $d > 1$, parametrized by t in a quasiprojective, complex algebraic curve X. Suppose that $a, b : X \to \mathbb{P}^1$ are meromorphic functions on a compactification $\overline{X}$. There are infinitely many $t \in X$ for which both $a(t)$ and $b(t)$ are preperiodic for f_t if and only if a and b are dynamically related.*

When E is the Legendre family of elliptic curves, and the points P and Q lie in $\mathbb{P}^1(\mathbb{C})$, X. Wang, H. Ye, and I gave a dynamical proof of Theorem 5.2, building on the same ideas that went into the proof of Theorem 3.1. As in Theorem 3.2, we obtain the stronger statement about parameters of small height, which does not follow from the proofs given in Masser and Zannier [2010, 2012]. For algebraic points, our theorem can be stated as:

Theorem 5.4. *DeMarco, Wang, and Ye [2016] Let $E_t = \{(x, y) : y^2 = x(x-1)(x-t)\}$ be the Legendre family of elliptic curves, with $t \in \mathbb{C} \setminus \{0, 1\}$. Fix $a, b \in \overline{\mathbb{Q}} \setminus \{0, 1\}$. The following are equivalent:*

1. $|\mathrm{Tor}(a) \cap \mathrm{Tor}(b)| = \infty$*;*
2. $\mathrm{Tor}(a) = \mathrm{Tor}(b)$*;*
3. *there is an infinite sequence $\{t_n\} \subset \overline{\mathbb{Q}}$ so that $\hat{h}_a(t_n) \to 0$ and $\hat{h}_b(t_n) \to 0$;*
4. *$\mu_a = \mu_b$ on $\mathbb{P}^1(\mathbb{C})$; and*
5. *$a = b$.*

Here, $\mathrm{Tor}(a) = \{t \in \mathbb{C} : (a, \sqrt{a(a-1)(a-t)})$ is torsion on $E_t\}$. The height $\hat{h}_a(t)$ is the Néron-Tate canonical height of the point $(a, \sqrt{a(a-1)(a-t)})$ in E_t for $t \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, so that $\hat{h}_a(t) = 0$ if and only if $t \in \mathrm{Tor}(a)$. As in the theorems of the previous section, the geometry of the "bifurcation locus" associated to the marked points a and b plays a key role; the $\mathrm{Gal}(\overline{\mathbb{Q}(a)}/\mathbb{Q}(a))$-invariant subsets of $\mathrm{Tor}(a)$ are uniformly distributed with respect to a natural measure μ_a. As in the dynamical examples, Theorems 3.2 or 4.1, the measures μ_a at the *archimedean places* (the limiting distributions on the underlying complex curve) are sufficient to characterize the existence of a dynamical relation. It is not known if this will be the case for all families of rational maps in Conjecture 5.3.

And this returns us, finally, to the statement of Theorem 1.3 from the Introduction. That result proves a special case of a conjecture of S.-W. Zhang [1998b] from his ICM lecture notes from exactly 20 years ago, which was posed as an extension of the Bogomolov Conjecture to non-trivial families of abelian varieties. The notion of a "special" section is carefully defined in citeDM:variation, building on the work of Masser and Zannier [2012, 2014]. Our proof was inspired by the combination of ideas presented here, connecting dynamical orbit relations and equidistribution theorems with the geometry of abelian varieties. These ideas are, in turn, closely related to the original proofs of Ullmo and Zhang of the Bogomolov Conjecture Ullmo [1998] and S.-W. Zhang [1998a], relying on the (arithmetic) equidistribution of the torsion points within an abelian variety defined over $\overline{\mathbb{Q}}$ Szpiro, Ullmo, and S. Zhang [1997]. We have not yet been able to give a purely dynamical proof of Theorem 1.3, in the flavor of Theorems 3.1 and 5.4. Instead, we used the work of Silverman [1992, 1994a,b] to provide the technical statements needed to show that our height functions on $B(\overline{\mathbb{Q}})$ satisfy all the hypotheses needed to apply the arithmetic equidistribution theorems of Thuillier and Yuan Thuillier [2005] and Yuan [2008]. Via the equidistribution theorem, we were able to reduce the statement of Theorem 1.3 to a more general form of Theorem 5.1 proved by Masser and Zannier [2014].

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