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FROM CONTINUOUS RATIONAL TO REGULOUS FUNCTIONS

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Abstract

Let X be an algebraic set in $\mathbb{R}^n$. Real-valued functions, defined on subsets of X, that are continuous and admit a rational representation have some remarkable properties and applications. We discuss recently obtained results on such functions, against the backdrop of previously developed theories of arc-symmetric sets, arc-analytic functions, approximation by regular maps, and algebraic vector bundles.

1 Introduction

Our purpose is to report on some new developments in real algebraic geometry, focusing on functions that have a rational representation. Let us initially consider the simplest case. A function $f : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^k$, where k is a nonnegative integer, is said to have a *rational representation* if there exist two polynomial functions p, q on $\mathbb{R}^n$ such that q is not identically 0 and $f = p/q$ on $\{q \neq 0\}$. A typical example is

(1.1) $f_k : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f_k(x, y) = \frac{x^{3+k}}{x^2 + y^2} \text{ for } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0.$$

To the best of our knowledge, Kucharz [2009] was the first paper devoted to the systematic study of such functions. This line of research was continued by several mathematicians Bilski, Kucharz, A. Valette, and G. Valette [2013], Fichou, Huisman, Mangolte, and Monnier [2016], Fichou, Monnier, and Quarez [2017], Kollár, Kucharz, and

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Kurdyka [2018], Kollár and Nowak [2015], Kucharz [2013b, 2014a,b, 2015b, 2016a,b], Kucharz and Kurdyka [2016a,b, 2017, 2015], Kucharz and Zieliński [2018], Monnier [2018], Zieliński [2016], frequently with $k = 0$, where the functions admitting a rational representation are only continuous. Let us note that the complex case is quite different. Any continuous function from $\mathbb{C}^n$ into $\mathbb{C}$ that has a rational representation is a polynomial function.

Henceforth we work with real algebraic sets, which is equivalent to the approach adopted in Bochnak, Coste, and Roy [1998]. By a *real algebraic set* we mean an algebraic subset of $\mathbb{R}^n$ for some n. One can realize real projective d-space $\mathbb{P}^d(\mathbb{R})$ as a real algebraic set using the embedding

$$\mathbb{P}^d(\mathbb{R}) \ni (x_0 : \cdots : x_d) \mapsto \left(\frac{x_i x_j}{x_0^2 + \cdots + x_d^2} \right) \in \mathbb{R}^{(d+1)^2}. \tag{1.2}$$

Thus any algebraic subset of $\mathbb{P}^d(\mathbb{R})$ is an algebraic subset of $\mathbb{R}^{(d+1)^2}$. Consequently, many useful constructions can be performed within the class of real algebraic sets, blowing-up being an important example. One can also view any real algebraic set as the set of real points $V(\mathbb{R})$ of a quasiprojective variety V defined over $\mathbb{R}$.

Unless explicitly stated otherwise, we always assume that real algebraic sets and their subsets are endowed with the Euclidean topology, which is induced by the usual metric on $\mathbb{R}$. For a real algebraic set X, its singular locus $\mathrm{Sing}(X)$ is an algebraic, Zariski nowhere dense subset of X. We say that X is *smooth* if $\mathrm{Sing}(X)$ is empty. The following examples illustrate some phenomena that do not occur in the complex setting.

(1.3) The algebraic curve

$$C := (x^4 - 2x^2y - y^3 = 0) \subset \mathbb{R}^2$$

is irreducible and $\mathrm{Sing}(C) = \{(0,0)\}$. Actually, C is an analytic submanifold of $\mathbb{R}^2$.

(1.4) The algebraic curve

$$C := (x^3 - x^2 - y^2 = 0) \subset \mathbb{R}^2$$

is irreducible and $\mathrm{Sing}(C) = \{(0,0)\}$. It has two connected components, the singleton $\{(0,0)\}$ and the unbounded branch $C \setminus \{(0,0)\}$.

(1.5) The algebraic curve

$$C := (x^2(x^2-1)(x^2-4) + y^2 = 0) \subset \mathbb{R}^2$$

is irreducible and $\mathrm{Sing}(C) = \{(0,0)\}$. It has three connected components, the singleton $\{(0,0)\}$ and two ovals.

(1.6) The Cartan umbrella

$$S := (x^3 - z(x^2 + y^2) = 0) \subset \mathbb{R}^3$$

is an irreducible algebraic surface with $\operatorname{Sing}(S) = (z\text{-axis})$. The surface S is connected and $S \setminus \operatorname{Sing}(S)$ is not dense in S. Furthermore, S is not coherent when regarded as an analytic subset of $\mathbb{R}^3$.

It will be convenient to consider regular functions in a more general setting than usual. Let $X \subset \mathbb{R}^n$ be an algebraic set and let $f: W \to \mathbb{R}$ be a function defined on some subset W of X. We say that f is *regular at a point* $x \in W$ if there exist two polynomial functions p, q on $\mathbb{R}^n$ such that $q(x) \neq 0$ and $f = p/q$ on $W \cap \{q \neq 0\}$. We say that f is a *regular function* if it is regular at each point of W. For any algebraic set $Y \subset \mathbb{R}^p$, a map $\varphi = (\varphi_1, \dots, \varphi_p): W \to Y$ is *regular* if all the components $\varphi_i: W \to \mathbb{R}$ are regular functions. These notions are independent of the algebraic embeddings $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^p$.

Any rational function R on X determines a regular function $R: X \setminus \operatorname{Pole}(R) \to \mathbb{R}$, where $\operatorname{Pole}(R)$ stands for the polar set of R.

Contents. In Section 2 we recall briefly main facts about arc-symmetric sets and arc-analytic functions. These notions, introduced 30 years ago by the second-named author, describe some rigidity phenomena (of an analytic type) of real algebraic sets. They form a background for the subsequent sections in which we present recent developments in the context of rational functions.

Section 3 contains presentation of new results on the geometry defined by regulous functions, that is, continuous functions which admit a strong version of rational representation.

In Section 4 we recall some theorems on approximation of continuous maps with values in spheres by regular maps and give new results in which approximating maps are allowed to be regulous.

In Section 5 we discuss topological, algebraic and regulous vector bundles. Regulous vector bundles have many desirable properties of algebraic vector bundles but are more flexible.

2 Arc-symmetric sets and arc-analytic functions

Arc-symmetric sets and arc-analytic functions were introduced in Kurdyka [1988]. They were further investigated and applied in Adamus and Seyedinejad [2017], Bierstone and Milman [1990], Bierstone, Milman, and Parusiński [1991], Fichou [2005], Koike and

Parusiński [2003], Kucharz [2005], Kurdyka [1991, 1994, 1999], Kurdyka and Parusiński [2007, 2012], Kurdyka and Rusek [1988], McCrory and Parusiński [2003], Parusiński [1994, 2001, 2004], Parusiński and Păunescu [2017].

2.1 Arc-symmetric sets. We say that a subset $E \subset \mathbb{R}^n$ is *arc-symmetric* if for every analytic arc $\gamma\colon (-1,1) \to \mathbb{R}^n$ with $\gamma((-1,0)) \subset E$, we have $\gamma((-1,1)) \subset E$. We are mostly interested in semialgebraic arc-symmetric sets.

Recall that a topological space is called *Noetherian* if every descending chain of its closed subsets is stationary. In particular, $\mathbb{R}^n$ with the Zariski topology is a Noetherian topological space. In Kurdyka [1988], the following is proved.

Theorem 2.1. *The semialgebraic arc-symmetric subsets of $\mathbb{R}^n$ are precisely the closed sets of a certain Noetherian topology on $\mathbb{R}^n$.*

Following Kurdyka [ibid.], we call this topology on $\mathbb{R}^n$ the $\mathfrak{AR}$ *topology*. Thus a subset of $\mathbb{R}^n$ is $\mathfrak{AR}$-closed if and only if it is semialgebraic and arc-symmetric. It follows from the curve selection lemma that each $\mathfrak{AR}$-closed subset of $\mathbb{R}^n$ is closed (in the Euclidean topology), cf. Kurdyka [ibid.]. Clearly, any connected component of an $\mathfrak{AR}$-closed subset of $\mathbb{R}^n$ is also $\mathfrak{AR}$-closed. Furthermore, any irreducible analytic component of an algebraic subset of $\mathbb{R}^n$ is $\mathfrak{AR}$-closed. However, an $\mathfrak{AR}$-closed set need not be analytic at every point.

Example 2.2. The set

$$E = \{(x,y,z) \in \mathbb{R}^3\colon x^3 - z(x^2+y^2) = 0,\ x^2+y^2 \neq 0\} \cup \{(0,0,0)\}$$

(the "cloth" of the Cartan umbrella (1.6)) is $\mathfrak{AR}$-closed, but it is not analytic at the origin of $\mathbb{R}^3$.

Given a semialgebraic subset $E \subset \mathbb{R}^n$, we say that a point $x \in E$ is *regular in dimension d* if for some open neighborhood $U_x \subset \mathbb{R}^n$ of x, the intersection $E \cap U_x$ is a d-dimensional analytic submanifold of U_x. We let $\operatorname{Reg}_d(E)$ denote the locus of regular points of E in dimension d. The dimension of E, written $\dim E$, is the maximum d with $\operatorname{Reg}_d(E)$ nonempty. If V is the Zariski closure of E in $\mathbb{R}^n$, then $\dim E = \dim V$, cf. Bochnak, Coste, and Roy [1998].

By a *resolution of singularities* of a real algebraic set X we mean a proper regular map $\pi : \tilde{X} \to X$ where $\tilde{X}$ is a smooth real algebraic set and π is birational.

The following is the key result of Kurdyka [1988].

Theorem 2.3. *Let $X \subset \mathbb{R}^n$ be a d-dimensional real algebraic set and let $E \subset \mathbb{R}^n$ be an $\mathfrak{AR}$-closed irreducible subset with $E \subset X$ and* $\dim E = d$. *If $\pi : \tilde{X} \to X$ is a resolution of singularities of X, then there exists a unique connected component $\tilde{E}$ of $\tilde{X}$ such that $\pi(\tilde{E})$ is the closure (in the Euclidean topology) of* $\operatorname{Reg}_d(E)$.

This is illustrated by an example below.

Example 2.4. The real cubic $C := (x^3 - x - y^2 = 0) \subset \mathbb{R}^2$ is smooth and irreducible. It has two connected components, C_1 which is compact and C_2 which is noncompact. Consider the cone $X := (x^3 - xz^2 - y^2 z = 0) \subset \mathbb{R}^3$ over C. Note that X is irreducible and $\operatorname{Sing}(X) = \{(0,0,0)\}$. Clearly,

$$\pi : \tilde{X} := C \times \mathbb{R} \to X, \quad (x, y, z) \mapsto (xz, yz, z)$$

is a resolution of singularities of X. The connected components $C_1 \times \mathbb{R}$ and $C_2 \times \mathbb{R}$ of $\tilde{X}$ correspond via π to the $\mathfrak{AR}$-irreducible components of X.

The notion of arc-symmetric set turns out to be related to a notion introduced by Nash in his celebrated paper Nash [1952]. We adapt his definition to the case of $\mathfrak{AR}$-closed sets.

Definition 2.5. Let E be an $\mathfrak{AR}$-closed subset of $\mathbb{R}^n$. We say that a subset $S \subset E$ is a (Nash) *sheet* of E if the following conditions are satisfied:

(i) for any two points x_0, x_1 in S there exists an analytic arc $\gamma : [0,1] \to \mathbb{R}^n$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$, and $\gamma([0,1]) \subset S$;

(ii) S is maximal in the class of subsets satisfying the condition (i);

(iii) the interior of S in X is nonempty.

The following result of Kurdyka [1988] gives a positive and precise answer to Nash's conjecture on sheets of an algebraic set Nash [1952].

Theorem 2.6. *Let X be an algebraic subset or more generally an $\mathfrak{AR}$-closed subset of $\mathbb{R}^n$. Then:*

(i) *There are finitely many sheets in X.*

(ii) *Each sheet in X is semialgebraic and closed (in the Euclidean topology).*

(iii) *X is the union of its sheets.*

The proof of this theorem is based on Theorem 2.3 and the notion of *immersed component* of an $\mathfrak{AR}$-closed set. An immersed component of X is an $\mathfrak{AR}$-irreducible subset of X with nonempty interior in X. In general, X may have more immersed components than $\mathfrak{AR}$-irreducible components. For instance, the Whitney umbrella $(xy^2 - z^2 = 0) \subset \mathbb{R}^3$ is $\mathfrak{AR}$-irreducible, but it has two immersed components.

Compact $\mathfrak{AR}$-closed sets share with compact real algebraic sets all known local and global topological properties. In particular, each compact $\mathfrak{AR}$-closed set carries the mod 2 fundamental class. It is conjectured that any compact $\mathfrak{AR}$-closed set is semialgebraically homeomorphic to a real algebraic set.

Recall that a Nash manifold $X \subset \mathbb{R}^n$ is an analytic submanifold which is a semialgebraic set. Building on Thom's representability theorem Thom [1954] and Theorem 2.3, the following was established in Kucharz [2005].

Theorem 2.7. *Let $X \subset \mathbb{R}^n$ be a compact Nash manifold, and d an integer satisfying $0 \leq d \leq \dim X$. Then each homology class in $H_d(X; \mathbb{Z}/2)$ can be represented by an $\mathfrak{AR}$-closed subset of $\mathbb{R}^n$, contained in X.*

Now we recall a result of Kurdyka and Rusek [1988] which was motivated by the problem of surjectivity of injective selfmaps.

Theorem 2.8. *Let $X \subset \mathbb{R}^n$ be an $\mathfrak{AR}$-closed subset of dimension d, with $0 \leq d \leq n-1$. Then the homotopy group $\pi_{n-d-1}(\mathbb{R}^n \setminus X)$ is nontrivial.*

As demonstrated in Kurdyka and Rusek [ibid.], Theorem 2.8 implies the following result of Białynicki-Birula and Rosenlicht [1962].

Theorem 2.9. *Any injective polynomial map from $\mathbb{R}^n$ into itself is surjective.*

One should mention that Theorem 2.9, with $n = 2$, was established earlier by Newman [1960]. Ax [1969] proved that any injective regular map of a complex algebraic variety into itself is surjective. Ax's proof is based on the Lefschetz principle and a reduction to the finite field case. By extending the idea of Białynicki-Birula and Rosenlicht [1962], Borel [1969] gave a topological proof of Ax's theorem that works also for injective regular maps of a smooth real algebraic set into itself. Finally, combining Borel's argument with the geometry of $\mathfrak{AR}$-closed sets, the second-named author proved in Kurdyka [1999] the following.

Theorem 2.10. *Let X be a real algebraic set (possibly singular) and let $f : X \to X$ be an injective regular map. Then f is surjective.*

In fact, there is a more general version of Theorem 2.10 due to Parusiński [2004], cf. also Kurdyka and Parusiński [2007].

2.2 Arc-analytic maps. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^p$ be some subsets. A map $f : X \to Y$ is said to be *arc-analytic* if for every analytic arc $\gamma : (-1, 1) \to \mathbb{R}^n$ with $\gamma((-1, 1)) \subset X$, the composite $f \circ \gamma : (-1, 1) \to \mathbb{R}^p$ is an analytic map. We are mostly interested in the case where X and Y are $\mathfrak{AR}$-closed, and f is semialgebraic.

The function $f_k : \mathbb{R}^2 \to \mathbb{R}$ in (1.1) is arc-analytic and of class $\mathcal{C}^k$, but it is not of class $\mathcal{C}^{k+1}$. The following fact is recorded in Kurdyka [1988].

Proposition 2.11. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^p$ be $\mathcal{AR}$-closed subsets, and let $f : X \to Y$ be a semialgebraic arc-analytic map. Then:*

(i) *The graph of f is an $\mathcal{AR}$-closed subset of $\mathbb{R}^n \times \mathbb{R}^p$.*

(ii) *If $Z \subset Y$ is an $\mathcal{AR}$-closed set, then so is $f^{-1}(Z)$.*

(iii) *f is continuous (in the Euclidean topology).*

Arc-analytic functions do not have nice properties without some additional assumptions. For instance, an arc-analytic function on $\mathbb{R}^n$ need not be subanalytic Kurdyka [1991] or continuous Bierstone, Milman, and Parusiński [1991], and even for $n = 2$ it may have a nondiscrete singular set Kurdyka [1994].

In complex algebraic geometry, the image of an algebraic set by a proper regular map is again an algebraic set. This is trivially false in the real case; consider $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. There is also a more interesting example. Let

$$X := (x - y^2 - 1 = 0) \subset \mathbb{R}^2 \text{ and } Y := (x^3 - x^2 - y^2 = 0) \subset \mathbb{R}^2.$$

Then $f : X \to Y$, $(x, y) \mapsto (x, xy)$ is an injective, proper regular map. However, $f(X)$ is not an algebraic set. Therefore the following embedding theorem of Kurdyka [1988] is of interest.

Theorem 2.12. *Let $X \subset \mathbb{R}^n$ be an $\mathcal{AR}$-closed subset and let $f : X \to \mathbb{R}^p$ be a semialgebraic arc-analytic map that is injective and proper. Then $f(X) \subset \mathbb{R}^p$ is an $\mathcal{AR}$-closed subset.*

Given an $\mathcal{AR}$-closed subset $X \subset \mathbb{R}^n$, we denote by $\mathcal{A}_a(X)$ the ring of semialgebraic arc-analytic functions on X. According to Kurdyka [ibid.], the ring $\mathcal{A}_a(X)$ is not Noetherian if $\dim X \geq 2$. However, any ascending chain of prime ideals of $\mathcal{A}_a(X)$ is stationary. Furthermore, by Kurdyka [ibid.], there exists a function $f \in \mathcal{A}_a(\mathbb{R}^n)$ such that $X \subset f^{-1}(0)$ and $\dim(f^{-1}(0) \setminus X) < \dim X$. This latter result has been recently strengthened by Adamus and Seyedinejad [2017], who proved that actually $X = f^{-1}(0)$ for some $f \in \mathcal{A}_a(\mathbb{R}^n)$. This enabled them to obtain the Nullstellensatz for the ring $\mathcal{A}_a(X)$, generalizing thereby the weak Nullstellensatz of Kurdyka [1988].

2.3 Blow-Nash and blow-analytic functions.

Let X be a smooth real algebraic set. A *Nash function* on X is an analytic function which is semialgebraic. A function on X is said to be *blow-Nash* if it becomes a Nash function after composing with a finite sequence of blowups with smooth nowhere dense centers. It was conjectured by K. Kurdyka (1987) that a function is blow-Nash if and only if it is arc-analytic and semialgebraic. The first proof of this conjecture was published by Bierstone and Milman [1990]. They developed techniques which later turned out to be useful in their approach to the resolution of singularities Bierstone and Milman [1997]. There is also a second proof due to Parusiński [1994]. It is based on the rectilinearization theorem for subanalytic functions Parusiński [ibid.], which is a prototype of the preparation theorem for subanalytic functions, cf. for example Parusiński [2001].

Less is known on arc-analytic functions which are subanalytic. Any such function is continuous and can be made analytic after composing with finitely many local blowups with smooth centers, cf. Bierstone and Milman [1990] and Parusiński [1994]. It is not known whether one can use global blowups, that is, whether arc-analytic subanalytic functions coincide with *blow-analytic* functions of T. C. Kuo [1985], cf. also Fukui, Koike, and T.-C. Kuo [1998]. In Kurdyka and Parusiński [2012] it is proved that the locus of nonanalyticity of an arc-analytic subanalytic function is arc-symmetric and subanalytic. Another result of Kurdyka and Parusiński [ibid.] asserts that in the blow-Nash case, the centers of blowups can be chosen in the locus of nonanalyticity.

2.4 Some applications. Recently arc-symmetric sets were used in the construction of new invariants in the singularity theory. These invariants include the virtual Betti numbers of real algebraic sets McCrory and Parusiński [2003] and arc-symmetric sets Fichou [2005]. Other invariants, analogous to the zeta function of Denef and Loeser, proved to be useful in the classification of germs of functions with respect to blow-analytic and blow-Nash equivalence, cf. Koike and Parusiński [2003] and Fukui, Koike, and T.-C. Kuo [1998]. Arc-analytic homeomorphisms were recently used in Parusiński and Păunescu [2017] to construct nice trivializations in the stratification theory.

3 Regulous functions

3.1 Functions regular on smooth algebraic arcs. All results presented in this subsection come from our joint paper Kollár, Kucharz, and Kurdyka [2018].

Let X be a real algebraic set. A subset $A \subset X$ is called a *smooth algebraic arc* if its Zariski closure C is an irreducible algebraic curve, $A \subset C \setminus \operatorname{Sing}(C)$, and A is homeomorphic to $\mathbb{R}$.

An open subset $U \subset X$ is said to be *smooth* if it is contained in $X \setminus \operatorname{Sing}(X)$.

Theorem 3.1. *Let X be a real algebraic set and let $f : U \to \mathbb{R}$ be a function defined on a connected smooth open subset $U \subset X$. Assume that the restriction of f is regular on* *each smooth algebraic arc contained in* U. *Then there exists a rational function* R *on* X *such that* $P := U \cap \mathrm{Pole}(R)$ *has codimension at least* 2 *and* $f|_{U\setminus P} = R|_{U\setminus P}$.

There are two main steps in the proof of Theorem 3.1. Assuming that f is a semialgebraic function (so U is a semialgebraic set), one first obtains a local variant of the assertion by means of Bertini's theorem, and then extends it along smooth algebraic arcs. The general case is reduced to the semialgebraic one via some subtle Hartogs-like results on analytic functions due to Błocki [1992] and Siciak [1990].

A function regular on smooth algebraic arcs need not be continuous.

Example 3.2. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \frac{x^8 + y(x^2 - y^3)^2}{x^{10} + (x^2 - y^3)^2} \text{ for } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0$$

is regular on each smooth algebraic arc in $\mathbb{R}^2$, but it is not locally bounded on the curve $x^2 - y^3 = 0$.

Let $X = X_1 \times \cdots \times X_n$ be the product of real algebraic sets and let $\pi_i : X \to X_i$ be the projection on the ith factor. A subset $K \subset X$ is said to be *parallel to the* ith *factor of* X if $\pi_j(K)$ consists of one point for each $j \neq i$.

Theorem 3.3. *Let* $X = X_1 \times \cdots \times X_n$ *be the product of real algebraic sets and let* $f : U \to \mathbb{R}$ *be a function defined on a connected smooth open subset* $U \subset X$. *Assume that the restriction of* f *is regular on each smooth algebraic arc contained in* U *and parallel to one of the factors of* X. *Then there exists a rational function* R *on* X *such that* $P := U \cap \mathrm{Pole}(R)$ *has codimension at least* 2 *and* $f|_{U\setminus P} = R|_{U\setminus P}$.

Theorem 3.3, for $n = 1$, coincides with Theorem 3.1. The general case is proved by induction on n, but a detailed argument is fairly long.

As a direct consequence, we get the following.

Corollary 3.4. *Let* $f : U \to \mathbb{R}$ *be a function defined on a connected open subset* $U \subset \mathbb{R}^n$. *Assume that the restriction of* f *is regular on each open interval contained in* U *and parallel to one of the coordinate axes. Then there exists a rational function* R *on* $\mathbb{R}^n$ *such that* $P := U \cap \mathrm{Pole}(R)$ *has codimension at least* 2 *and* $f|_{U\setminus P} = R|_{U\setminus P}$.

Results similar to Corollary 3.4 have been known earlier, but they were obtained under the restrictive assumption that f is an analytic function on U, cf. Bochner and Martin [1948].

3.2 Introducing regulous functions. Let X be a real algebraic set, $f : W \to \mathbb{R}$ a function defined on some subset $W \subset X$, and Y the Zariski closure of W in X.

Definition 3.5. A rational function R on Y is said to be a *rational representation* of f if there exists a Zariski open dense subset $Y^0 \subset Y \setminus \mathrm{Pole}(R)$ such that $f|_{W\cap Y^0} = R|_{W\cap Y^0}$.

While the definition makes sense for an arbitrary subset W, it is sensible only if W contains a sufficiently large portion of Y. The key examples of interest are open subsets and semialgebraic subsets, with $W = X$ being the most important case.

One readily checks that the following conditions are equivalent:

(3.6) For every algebraic subset $Z \subset X$ the restriction $f|_{W\cap Z}$ has a rational representation.

(3.7) There exists a sequence of algebraic subsets

$$X = X_0 \supset X_1 \supset \cdots \supset X_{m+1} = \varnothing$$

such that the restriction of f is regular on $W \cap (X_i \setminus X_{i+1})$ for $i = 0, \dots, m$.

(3.8) There exists a finite stratification $\mathcal{S}$ of X, with Zariski locally closed strata, such that the restriction of f is regular on $W \cap S$ for every $S \in \mathcal{S}$.

Definition 3.9. We say that f is a *regulous function* if it is continuous and the equivalent conditions (3.6), (3.7), (3.8) are satisfied.

In some papers, regulous functions are called *hereditarily rational* Kollár, Kucharz, and Kurdyka [2018] and Kollár and Nowak [2015] or *stratified-regular* Kucharz [2015b], Kucharz and Kurdyka [2016b, 2015], and Zieliński [2016]. The short name "regulous", derived from "regular" and "continuous", was introduced in Fichou, Huisman, Mangolte, and Monnier [2016]. A continuous function that has a rational representation is often called simply a *continuous rational function* Kollár, Kucharz, and Kurdyka [2018], Kollár and Nowak [2015], Kucharz [2009, 2013b, 2014b,a, 2016a], and Kucharz and Kurdyka [2016a, 2017].

Evidently, any regulous function is continuous and has a rational representation. The converse holds in an important special case.

Proposition 3.10. *Let X be a real algebraic set and let $W \subset X$ be a smooth open subset. For a function $f : W \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is regulous.*

(b) *f is continuous and has a rational representation.*

The nontrivial implication (b)⇒(a) is proved in Kollár and Nowak [2015]. Suppose that (b) holds, and let R be a rational representation of f. Since f is continuous and W

is smooth, one gets $f|_{W\setminus P} = R|_{W\setminus P}$, where $P = W \cap \operatorname{Pole}(R)$. Furthermore, it is not hard to see that P has codimension at least 2. Finally, condition (3.6) can be verified by induction on codim Z.

As demonstrated in Kollár and Nowak [ibid.] and recalled below, the smoothness assumption in Proposition 3.10 cannot be dropped.

Example 3.11. Consider the algebraic surface

$$S := (x^3 - (1+z^2)y^3 = 0) \subset \mathbb{R}^3$$

and the function $f : S \to \mathbb{R}$ defined by $f(x,y,z) = (1+z^2)^{1/3}$. Note that $\operatorname{Sing}(S) =$ z-axis and $f(x,y,z) = x/y$ on $S \setminus (z\text{-axis})$. In particular, f is continuous and has a rational representation. However, f is not regulous since $f|_{z\text{-axis}}$ does not have a rational representation. It is also interesting that S is an analytic submanifold of $\mathbb{R}^3$.

The main result of Kollár and Nowak [ibid.] can be stated as follows.

Theorem 3.12. *Let X be a smooth real algebraic set and let $f : W \to \mathbb{R}$ be a function defined on an algebraic subset $W \subset X$. Then the following conditions are equivalent:*

(a) *f is regulous.*

(b) *$f = \tilde{f}|_W$, where $\tilde{f} : X \to \mathbb{R}$ is a continuous function that has a rational representation.*

The proof of (a)⇒(b) by induction on dim W is tricky. Roughly speaking, one first finds a function on X that extends f and has a rational representation. However, such an extension may not be continuous and has to be corrected. This is achieved by analyzing liftings of functions to the blowup of X at a suitably chosen ideal. The argument relies on a version of the Łojasiewicz inequality given in Bochnak, Coste, and Roy [1998, Theorem 2.6.6].

The implication (b)⇒(a) follows from Proposition 3.10.

As it was noted on various occasions (see for example Kucharz [2009, p. 528] or Fichou, Huisman, Mangolte, and Monnier [2016, Théorème 3.11]), Hironaka's theorem on resolution of indeterminacy points Hironaka [1964] implies immediately the following.

Proposition 3.13. *Let X be a smooth real algebraic set. For a function $f : X \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is continuous and has a rational representation.*

(b) *There exists a regular map $\pi : X' \to X$, which is the composite of a finite sequence of blowups with smooth Zariski nowhere dense centers, such that the function $f \circ \pi : X' \to \mathbb{R}$ is regular.*

Fefferman and Kollár [2013] study the following problem. Consider a linear equation

$$f_1 y_1 + \cdots + f_r y_r = g,$$

where g and the f_i are regular (or polynomial) functions on $\mathbb{R}^n$. Assume that it admits a solution where the y_i are continuous functions on $\mathbb{R}^n$. Then, according to Fefferman and Kollár [ibid., Section 2], it has also a continuous semialgebraic solution. One could hope to prove that it has a regulous solution. This is indeed the case for $n = 2$ Kucharz and Kurdyka [2017, Corollary 1.7], but fails for any $n \geq 3$ Kollár and Nowak [2015, Example 6]. It would be interesting to decide which linear equations have regulous solutions. Of course, the problem can be considered in a more general setting, replacing $\mathbb{R}^n$ by a real algebraic set.

3.3 Curve-regulous and arc-regulous functions. All results discussed in this subsection come from our joint paper Kollár, Kucharz, and Kurdyka [2018], where it is proved that regulous functions can be characterized by restrictions to algebraic curves or algebraic arcs.

Definition 3.14. Let X be a real algebraic set and let $f : W \to \mathbb{R}$ be a function defined on some subset $W \subset X$.

We say that f is *regulous on algebraic curves* or *curve-regulous* for short if for every irreducible algebraic curve $C \subset X$ the restriction $f|_{W \cap C}$ is regulous (equivalently, $f|_{W \cap C}$ is continuous and has a rational representation).

Furthermore, we say that f is *regulous on algebraic arcs* or *arc-regulous* for short if for every irreducible algebraic curve $C \subset X$ and every point $x \in W \cap C$ there exists an open neighborhood $U_x \subset W$ of x such that the restriction $f|_{U_x \cap C}$ is regulous (equivalently, $f|_{U_x \cap C}$ is continuous and has a rational representation).

Obviously, any curve-regulous function is arc-regulous. The converse does not hold for a rather obvious reason. For instance, consider the hyperbola $H \subset \mathbb{R}^2$ defined by $xy - 1 = 0$. Any function on H that is constant on each connected component of H is arc-regulous, but it must be constant to be regulous.

In Kollár, Kucharz, and Kurdyka [ibid.], curve-regulous (resp. arc-regulous) functions are called *curve-rational* (resp. *arc-rational*).

Our main result on curve-regulous functions is the following.

Theorem 3.15. *Let X be a real algebraic set and let $W \subset X$ be a subset that is either open or semialgebraic. For a function $f : W \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is regulous.*

(b) *f is curve-regulous.*

The corresponding result for arc-regulous functions takes the following form.

Theorem 3.16. *Let X be a real algebraic set and let $W \subset X$ be a connected smooth open subset. For a function $f : W \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is regulous.*

(b) *f is arc-regulous.*

The crucial ingredient in the proofs of Theorems 3.15 and 3.16 is Theorem 3.1. In both cases, only the implication (b)⇒(a) is not obvious. It is essential that testing curves and arcs are allowed to have singularities.

The main properties of arc-regulous functions on semialgebraic sets can be summarized as follows.

Theorem 3.17. *Let X be a real algebraic set and let $f : W \to \mathbb{R}$ be an arc-regulous function defined on a semialgebraic subset $W \subset X$. Then f is continuous and there exists a sequence of semialgebraic sets*

$$W = W_0 \supset W_1 \supset \cdots \supset W_{m+1} = \varnothing,$$

which are closed in W, such that the restriction of f is a regular function on each connected component of $W_i \setminus W_{i+1}$ for $i = 0, \ldots, m$. In particular, f is a semialgebraic function.

We also establish a connection between arc-regulous functions and, discussed in Section 2, arc-analytic functions.

Theorem 3.18. *Let X be a real algebraic set and let $f : W \to \mathbb{R}$ be an arc-regulous function defined on an open subset $W \subset X$. Then f is continuous and arc-analytic.*

In Kollár, Kucharz, and Kurdyka [ibid.] there are several other related results.

3.4 Constructible topology and k-regulous functions.

We consider regulous functions of class $\mathcal{C}^k$.

Definition 3.19. Let X be a smooth real algebraic set and let $f : U \to \mathbb{R}$ be a function defined on an open subset $U \subset X$.

We say that f is a *k-regulous* function, where k is a nonnegative integer, if it is of class $\mathcal{C}^k$ and regulous; or equivalently, by Proposition 3.10, if it is of class $\mathcal{C}^k$ and has a rational representation.

The set $\mathcal{R}^k(U)$ of all k-regulous functions on U forms a ring. An example of a k-regulous function on $\mathbb{R}^2$ is provided by (1.1).

A function on U which is of class $\mathcal{C}^\infty$ and regulous is actually regular, cf. Kucharz [2009]. Therefore one gains no new insight by considering such functions.

All results discussed in the remainder of this subsection come from the paper of Fichou, Huisman, Mangolte, and Monnier [2016], where they are stated for functions defined on $\mathbb{R}^n$. The ring $\mathcal{R}^k(\mathbb{R}^n)$ is not Noetherian if $n \geq 2$. Nevertheless it has some remarkable properties.

Given a collection F of real-valued functions on $\mathbb{R}^n$, we set

$$Z(F) := \{x \in \mathbb{R}^n : f(x) = 0 \text{ for all } f \in F\}$$

and write $Z(f)$ for $Z(F)$ if $F = \{f\}$.

The following is a variant of the classical Nullstellensatz for the ring $\mathcal{R}^k(\mathbb{R}^n)$.

Theorem 3.20. *Let I be an ideal of the ring $\mathcal{R}^k(\mathbb{R}^n)$. If a function f in $\mathcal{R}^k(\mathbb{R}^n)$ vanishes on $Z(I)$, then f^m belongs to I for some positive integer m.*

Recall that the Nullstellensatz for the ring of polynomial or regular functions on $\mathbb{R}^n$ requires an entirely different formulation, cf. Bochnak, Coste, and Roy [1998].

The subsets of $\mathbb{R}^n$ of the form $Z(I)$ for some ideal I of $\mathcal{R}^k(\mathbb{R}^n)$ can be characterized in terms of constructible sets. A subset of $\mathbb{R}^n$ is said to be *constructible* if it belongs to the Boolean algebra generated by the algebraic subsets of $\mathbb{R}^n$; or equivalently if it is a finite union of Zariski locally closed subsets of $\mathbb{R}^n$.

Theorem 3.21. *For a subset $E \subset \mathbb{R}^n$, the following conditions are equivalent:*

(a) *$E = Z(I)$ for some ideal I of $\mathcal{R}^k(\mathbb{R}^n)$.*

(b) *$E = Z(f)$ for some function f in $\mathcal{R}^k(\mathbb{R}^n)$.*

(c) *E is closed and constructible.*

Theorem 3.21 can be illustrated as follows.

Example 3.22. Consider the Cartan umbrella $S \subset \mathbb{R}^3$ defined in (1.6), and let E be the closure of $S \setminus (z\text{-axis})$. It is clear that E is a closed constructible set. Moreover, $E = Z(f)$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is the regulous function defined by

$$f(x, y, z) = z - \frac{x^3}{x^2 + y^2} \text{ on } \mathbb{R}^3 \setminus (z\text{-axis}) \text{ and } f(x, y, z) = z \text{ on the } z\text{-axis}.$$

The collection of all closed constructible subsets of $\mathbb{R}^n$ forms the family of closed subsets for a Noetherian topology on $\mathbb{R}^n$, called the *constructible topology*. Any subset of $\mathbb{R}^n$ which is constructible-closed is actually $\mathcal{QR}$-closed. The converse does not hold if $n \geq 2$.

In what follows we consider $\mathbb{R}^n$ endowed with the constructible topology. The assignment $\mathcal{R}^k : U \mapsto \mathcal{R}^k(U)$, where U runs through the open subsets of $\mathbb{R}^n$, is a sheaf of rings on $\mathbb{R}^n$, and $(\mathbb{R}^n, \mathcal{R}^k)$ is a locally ringed space. Sheaves of $\mathcal{R}^k$-modules on $\mathbb{R}^n$ are called *k-regulous sheaves*.

It follows from Theorem 3.20 that the ringed space $(\mathbb{R}^n, \mathcal{R}^k)$ carries essentially the same information as the affine scheme $\operatorname{Spec}(\mathcal{R}^k(\mathbb{R}^n))$. In particular, Cartan's theorems A and B are available for k-regulous sheaves.

Theorem 3.23. *For any quasi-coherent k-regulous sheaf $\mathcal{F}$ on $\mathbb{R}^n$, the following hold:*

(A) *$\mathcal{F}$ is generated by global sections.*

(B) *$H^i(\mathbb{R}^n, \mathcal{F}) = 0$ for $i \geq 1$.*

As is well-known, Cartan's theorems A and B fail for coherent algebraic sheaves on $\mathbb{R}^n$.

Let $V \subset \mathbb{R}^n$ be a constructible-closed subset. The sheaf $\mathcal{J}_V \subset \mathcal{R}^k$ of ideals of k-regulous functions vanishing on V is a quasi-coherent k-regulous sheaf on $\mathbb{R}^n$, and the quotient sheaf $\mathcal{R}^k/\mathcal{J}_V$ has support V. Endow V with the induced (constructible) topology, and let $\mathcal{R}^k_V$ be the restriction of the sheaf $\mathcal{R}^k_V/\mathcal{J}_V$ to V. The locally ringed space $(V, \mathcal{R}^k_V)$ is called a *closed k-regulous subvariety of* $(\mathbb{R}^n, \mathcal{R}^k)$. One can consider k-regulous sheaves on V, which are just sheaves of $\mathcal{R}^k_V$-modules. By a standard argument, Theorem 3.23 implies that Cartan's theorems A and B hold also for quasi-coherent k-regulous sheaves on V.

An *affine k-regulous variety* is a locally ringed space isomorphic to a closed k-regulous subvariety of $\mathbb{R}^n$ for some n. An *abstract k-regulous variety* can be defined in the standard way. The geometry of k-regulous varieties is to be developed.

4 Homotopy and approximation

In this section we discuss some homotopy and approximation results in the framework of real algebraic geometry, focusing on maps with values in the unit p-sphere

$$\mathbb{S}^p = (u_0^2 + \cdots + u_p^p - 1 = 0) \subset \mathbb{R}^{p+1}.$$

Approximation of continuous maps means approximation in the compact-open topology.

4.1 Approximation by regular maps. The theory of regular maps between real algebraic sets was developed by J. Bochnak and the first-named author Bochnak and Kucharz

[1987a,b, 1988, 1989b, 1991b, 1993, 1999, 2007] who joined forces with R. Silhol working on Bochnak, Kucharz, and Silhol [1997]. Regular maps are studied also in Ghiloni [2006, 2007], Joglar-Prieto and Mangolte [2004], Kucharz [2010, 2013a, 2015a], Loday [1973], Ozan [1995], Peng and Tang [1999], Turiel [2007], and Wood [1968]. We make no attempt to survey this theory, but give instead a sample of results that motivated later work described in the next subsection.

Problem 4.1. Let X be a compact real algebraic set. For a continuous map

$$f : X \to \mathbb{S}^p,$$

consider the following questions:

(i) Is f homotopic to a regular map?

(ii) Can f be approximated by regular maps?

It is expected that questions (i) and (ii) are equivalent, however, the proof is available only for special values of p, cf. Bochnak and Kucharz [1987a].

Theorem 4.2. *Let X be a compact real algebraic set. For a continuous map*

$$f : X \to \mathbb{S}^p,$$

where $p \in \{1, 2, 4\}$, the following conditions are equivalent:

(a) *f is homotopic to a regular map.*

(b) *f can be approximated by regular maps.*

Basic topological properties of regular maps between unit spheres still remain mysterious.

Conjecture 4.3. *For any pair (n, p) of positive integers, the following assertions hold:*

(i) *Each continuous map from $\mathbb{S}^n$ into $\mathbb{S}^p$ is homotopic to a regular map.*

(ii) *Each continuous map from $\mathbb{S}^n$ into $\mathbb{S}^p$ can be approximated by regular maps.*

Conjecture 4.3 (i) is known to be true in several cases Bochnak, Coste, and Roy [1998], Bochnak and Kucharz [1987a,b], Peng and Tang [1999], Turiel [2007], and Wood [1968]; for example if $n = p$ or $(n, p) = (2q + 14, 2q + 1)$ with $q \geq 7$.

Conjecture 4.3 (ii) holds if either $n < p$ (trivial) or $p \in \{1, 2, 4\}$ Bochnak and Kucharz [1987a]. Nothing is known for other pairs (n, p).

However, a complete solution to Problem 4.1 is known in several cases. The simplest one, noted in Bochnak and Kucharz [ibid.], is the following.

Proposition 4.4. *Let X be a compact smooth real algebraic curve. Then each continuous map from X into $\mathbb{S}^1$ can be approximated by regular maps.*

Going beyond curves is a lot harder. Nevertheless, it can happen also in higher dimension that the behavior of regular maps is determined entirely by the topology of the real algebraic sets involved.

Consider a compact $\mathcal{C}^\infty$ manifold M. A smooth real algebraic set diffeomorphic to M is called an *algebraic model* of M. By the Nash–Tognoli theorem Nash [1952] and Tognoli [1973], M has algebraic models. Actually, according to Bochnak and Kucharz [1991a], there exists an uncountable family of pairwise birationally nonequivalent algebraic models of M, provided that $\dim M \geq 1$. If $\dim M \leq 2$, the existence of algebraic models of M follows easily from the well-known classification of such manifolds. As X runs through the class of all algebraic models of M, the topological properties of regular maps from X into $\mathbb{S}^p$ may vary; this phenomenon is extensively investigated in Bochnak and Kucharz [1987b, 1988, 1989b, 1990, 1993] and Kucharz [2010].

A detailed study of regular maps into $\mathbb{S}^1$ is contained in Bochnak and Kucharz [1989b] where in particular the following result is proved.

Theorem 4.5. *Let M be a compact $\mathcal{C}^\infty$ manifold. Then there exists an algebraic model X of M such that each continuous map from X into $\mathbb{S}^1$ can be approximated by regular maps.*

For simplicity, we state the next result of Bochnak and Kucharz [ibid.] only for surfaces.

Theorem 4.6. *Let M be a connected, compact $\mathcal{C}^\infty$ surface. Then the following conditions are equivalent:*

(a) *For any algebraic model X of M, each continuous map from X into $\mathbb{S}^1$ can be approximated by regular maps.*

(b) *M is homeomorphic to the unit 2-sphere or the real projective plane or the Klein bottle.*

In Bochnak and Kucharz [1987b], one finds the following.

Theorem 4.7. *Let M be a compact $\mathcal{C}^\infty$ manifold of dimension p. Then there exists an algebraic model X of M such that each continuous map from X into $\mathbb{S}^p$ is homotopic to a regular map.*

Theorem 4.7, for $p = 1$, is of course weaker than Proposition 4.4. The cases $p = 2$ and $p = 4$ are of particular interest in view of Theorem 4.2.

Numerous results on algebraic models and regular maps into even-dimensional spheres are included in Bochnak and Kucharz [1988, 1990, 1993] and Kucharz [2010]. The following comes from Bochnak and Kucharz [1988].

Theorem 4.8. *Let M be a connected, compact $\mathcal{C}^\infty$ surface. Then the following conditions are equivalent:*

(a) *For any algebraic model X of M, each continuous map from X into $\mathbb{S}^2$ can be approximated by regular maps.*

(b) *M is nonorientable of odd genus.*

The true complexity of Problem 4.1 becomes apparent for surfaces of other types.

Consider smooth cubic curves in $\mathbb{P}^2(\mathbb{R})$. Each such cubic is either connected or has two connected components, and its Zariski closure in $\mathbb{P}^2(\mathbb{C})$ is also smooth. If C_1 and C_2 are smooth cubic curves in $\mathbb{P}^2(\mathbb{R})$, then $C_1 \times C_2$ can be oriented in such a way that for each regular map $\varphi\colon C_1 \times C_2 \to \mathbb{S}^2$, the topological degree $\deg(\varphi|_A)$ of the restriction of φ to a connected component A of $C_1 \times C_2$ does not depend on the choice of A. Moreover, the set

$$\mathrm{Deg}_{\mathcal{R}}(C_1, C_2) := \{m \in \mathbb{Z} : m = \deg(\psi|_A) \text{ for some regular map } \psi\colon C_1 \times C_2 \to \mathbb{S}^2\}$$

is a subgroup of $\mathbb{Z}$. These assertions are proved in Bochnak and Kucharz [1993, Theorem 3.1]. Define $b(C_1, C_2)$ to be the unique nonnegative integer satisfying

$$\mathrm{Deg}_{\mathcal{R}}(C_1, C_2) = b(C_1, C_2)\mathbb{Z}.$$

According to Hopf's theorem and Theorem 4.2, a continuous map $f : C_1 \times C_2 \to \mathbb{S}^2$ is homotopic to a regular map (or equivalently can be approximated by regular maps) if and only if for every connected component A of $C_1 \times C_2$, one has $\deg(f|_A) = b(C_1, C_2)r$ for some integer r independent of A. Thus, in this context, Problem 4.1 is reduced to the computation of the numerical invariant $b(C_1, C_2)$.

For any real number α in $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, set

$$\tau_\alpha = \frac{1}{2}(1 + \alpha\sqrt{-1}) \text{ if } \alpha > 0 \text{ and } \tau_\alpha = \alpha\sqrt{-1} \text{ if } \alpha < 0.$$

The lattice $\Lambda_\alpha := \mathbb{Z} + \mathbb{Z}\tau_\alpha$ in $\mathbb{C}$ is stable under complex conjugation. Hence the numbers

$$g_2(\tau_\alpha) = 60 \sum_\omega \omega^{-4}, \quad g_3(\tau_\alpha) = 140 \sum_\omega \omega^{-6}$$

(summation over $\omega \in \Lambda_\alpha \setminus \{0\}$) are real and

$$D_\alpha := \{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) \colon y^2 z = 4x^3 - g_2(\tau_\alpha)xz^2 - g_3(\tau_\alpha)z\}$$

is a smooth cubic curve in $\mathbb{P}^2(\mathbb{R})$. Each smooth cubic curve in $\mathbb{P}^2(\mathbb{R})$ is biregularly isomorphic to exactly one cubic D_α. Thus $\mathbb{R}^*$ can be regarded as a moduli space of smooth cubic curves in $\mathbb{P}^2(\mathbb{R})$. For $\alpha > 0$ (resp. $\alpha < 0$) the cubic D_α is connected (resp. has two connected components).

The invariant $b(D_{\alpha_1}, D_{\alpha_2})$ is explicitly computed in Bochnak and Kucharz [ibid.] for all pairs (α_1, α_2). In particular, it can take any nonnegative integer value. We recall only two cases.

First we deal with generic pairs (α_1, α_2).

Theorem 4.9. *For (α_1, α_2) in $\mathbb{R}^* \times \mathbb{R}^*$, the following conditions are equivalent:*

(a) *Each regular map from $D_{\alpha_1} \times D_{\alpha_2}$ into $\mathbb{S}^2$ is null homotopic.*

(b) $b(D_{\alpha_1}, D_{\alpha_2}) = 0$.

(c) *The product $\alpha_1\alpha_2$ is an irrational number.*

From the viewpoint of approximation, the following case is of greatest interest.

Theorem 4.10. *For (α_1, α_2) in $\mathbb{R}^* \times \mathbb{R}^*$, the following conditions are equivalent:*

(a) *Each continuous map from $D_{\alpha_1} \times D_{\alpha_2}$ into $\mathbb{S}^2$ can be approximated by regular maps.*

(b) $\alpha_1 > 0$, $\alpha_2 > 0$, *and* $b(D_{\alpha_1}, D_{\alpha_2}) = 1$.

(c) $\alpha_i = (p_i/q_i)\sqrt{d}$ *for $i = 1, 2$, where p_i, q_i, d are positive integers, p_i and q_i are relatively prime, d is square free, $d \equiv 3 \pmod 4$, $p_1 p_2 q_1 q_2 \equiv 1 \pmod 2$, and $p_1 p_2 d$ is divisible by $q_1 q_2$.*

Theorems 4.9 and 4.10 show that a small perturbation of (α_1, α_2) can drastically change topological properties of regular maps from $D_{\alpha_1} \times D_{\alpha_2}$ into $\mathbb{S}^2$. Thus, in general, one cannot hope to find a comprehensive solution to Problem 4.1, even for X smooth with $\dim X = p$. It is therefore desirable to introduce maps which have good features of regular maps but are more flexible.

4.2 Approximation by regulous maps. Let X and $Y \subset \mathbb{R}^q$ be smooth real algebraic sets, and let k be a nonnegative integer. A map $f = (f_1, \dots, f_q)\colon X \to Y$ is said to be *k-regulous* if its components $f_i : X \to \mathbb{R}$ are k-regulous functions; 0-regulous maps are called *regulous*.

If f is a regulous map, we denote by $P(f)$ the smallest algebraic subset of X such that $f|_{X \setminus P(f)}\colon X \setminus P(f) \to Y$ is a regular map. Obviously, $P(f)$ is Zariski nowhere dense in X. We say that f is *nice* if $f(P(f)) \neq Y$.

We state the next result for $\mathcal{C}^\infty$ maps. This is convenient since such maps have regular values by Sard's theorem.

Theorem 4.11. *Let X be a compact smooth real algebraic set, $f : X \to \mathbb{S}^p$ a $\mathcal{C}^\infty$ map with $p \geq 1$, and $y \in \mathbb{S}^p$ a regular value of f. Assume that the $\mathcal{C}^\infty$ submanifold $f^{-1}(y)$ of X is isotopic to a smooth Zariski locally closed subset of X. Then:*

(i) *f is homotopic to a nice k-regulous map, where k is an arbitrary nonnegative integer.*

(ii) *f can be approximated by nice regulous maps.*

Theorem 4.11 is due to the first-named author. Part (i) is a simplified version of Kucharz [2009, Theorem 2.4]. The proof is based on the Pontryagin construction (framed cobordism), Łojasiewicz inequality, and Hironaka's resolution of singularities. In turn (ii) follows from Kucharz [2014a, Theorem 1.2] since, by Kucharz [1985, Theorem 2.1], $f^{-1}(y)$ can be approximated by smooth Zariski locally closed subsets of X.

Theorem 4.11 provides information also on continuous maps since they can be approximated by $\mathcal{C}^\infty$ maps.

According to Akbulut and King [1992, Theorem A], any compact $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^n$ (resp. $\mathbb{S}^n$) is isotopic to a smooth Zariski locally closed subset of $\mathbb{R}^n$ (resp. $\mathbb{S}^n$). In particular, Theorem 4.11 yields the following.

Corollary 4.12. *Let (n, p) be a pair of positive integers. Then:*

(i) *Each continuous map from $\mathbb{S}^n$ into $\mathbb{S}^p$ is homotopic to a nice k-regulous map, where k is an arbitrary nonnegative integer.*

(ii) *Each continuous map from $\mathbb{S}^n$ into $\mathbb{S}^p$ can be approximated by nice regulous maps.*

With notation as in Theorem 4.11 we have $\dim f^{-1}(y) = \dim X - p$. Hence we get immediately the following.

Corollary 4.13. *Let X be a compact smooth real algebraic set of dimension p. Then:*

(i) *Each continuous map from X into $\mathbb{S}^p$ is homotopic to a nice k-regulous map, where k is an arbitrary nonnegative integer.*

(ii) *Each continuous map from X into $\mathbb{S}^p$ can be approximated by nice regulous maps.*

Comparing Theorems 4.8, 4.9 and 4.10 with Corollary 4.13 we see that k-regulous maps are indeed more flexible than regular ones. However, for each integer $p \geq 1$ there exist a compact smooth real algebraic set Y and a continuous map $g : Y \to \mathbb{S}^p$ such that $\dim Y = p + 1$ and g is not homotopic to any regulous map, cf. Kucharz and Kurdyka [2016b, Theorem 7.9]. In particular, Theorem 4.11 does not hold without some assumption on the $\mathcal{C}^\infty$ submanifold $f^{-1}(y) \subset X$. It would be very useful to formulate an appropriate assumption in terms of bordism. This is related to a certain conjecture, which has nothing to do with regulous maps and originates from the celebrated paper of Nash [1952] and the subsequent developments due to Tognoli [1973], Akbulut and King [1992], and other mathematicians.

For a real algebraic set X, a bordism class in the unoriented bordism group $\mathfrak{N}_*(X)$ is said to be *algebraic* if it can be represented by a regular map from a compact smooth real algebraic set into X.

Conjecture 4.14. *For any smooth real algebraic set X, the following holds: If M is a compact $\mathcal{C}^\infty$ submanifold of X and the unoriented bordism class of the inclusion map $M \hookrightarrow X$ is algebraic, then M is ε-isotopic to a smooth Zariski locally closed subset of X.*

Here "ε-isotopic" means isotopic via a $\mathcal{C}^\infty$ isotopy that can be chosen arbitrarily close, in the $\mathcal{C}^\infty$ topology, to the inclusion map. A slightly weaker assertion than the one in Conjecture 4.14 is known to be true: If the unoriented bordism class of the inclusion map $M \hookrightarrow X$ is algebraic, then the $\mathcal{C}^\infty$ submanifold $M \times \{0\}$ of $X \times \mathbb{R}$ is ε-isotopic to a smooth Zariski locally closed subset of $X \times \mathbb{R}$, cf. Akbulut and King [ibid., Theorem F].

Remark 4.15. Let X be a compact smooth real algebraic set and let $f : X \to \mathbb{S}^p$ be a continuous map. According to Kucharz and Kurdyka [2016a, Proposition 1.4], if Conjecture 4.14 holds, then the following conditions are equivalent:

(a) f is homotopic to a nice regulous map.

(b) f can be approximated by nice regulous maps.

Using a method independent of Conjecture 4.14, the first-named author proved in Kucharz [2016a] the following weaker result.

Theorem 4.16. *Let X be a compact smooth real algebraic set and let p be an integer satisfying* $\dim X + 3 \leq 2p$. *For a continuous map $f : X \to \mathbb{S}^p$, the following conditions are equivalent:*

(a) *f is homotopic to a nice regulous map.*

(b) *f can be approximated by nice regulous maps.*

Other results on topological properties of regulous maps can be found in Kucharz [2013b, 2014b, 2016a], Kucharz and Kurdyka [2016b], and Zieliński [2016].

5 Vector bundles

Let $\mathbb{F}$ stand for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). We consider only left $\mathbb{F}$-vector spaces. When convenient, $\mathbb{F}$ will be identified with $\mathbb{R}^{d(\mathbb{F})}$, where $d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}$.

5.1 Algebraic versus topological vector bundles. Let X be a real algebraic set. For any nonnegative integer n, let

$$\varepsilon_X^n(\mathbb{F}) = (X \times \mathbb{F}^n, p, X)$$

denote the product $\mathbb{F}$-vector bundle of rank n on X, where $X \times \mathbb{F}^n$ is regarded as a real algebraic set and $p\colon X \times \mathbb{F}^n \to X$ is the canonical projection.

An *algebraic* $\mathbb{F}$*-vector bundle on* X is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon_X^n(\mathbb{F})$ for some n (cf. Bochnak, Coste, and Roy [1998] for various characterizations of algebraic $\mathbb{F}$-vector bundles). The category of algebraic $\mathbb{F}$-vector bundles on X is equivalent to the category of finitely generated projective left $\mathcal{R}(X, \mathbb{F})$-modules, where $\mathcal{R}(X, \mathbb{F})$ is the ring of $\mathbb{F}$-valued regular functions on X.

Any algebraic $\mathbb{F}$-vector bundle on X can be regarded also as a topological $\mathbb{F}$-vector bundle. A topological $\mathbb{F}$-vector bundle is said to *admit an algebraic structure* if it is topologically isomorphic to an algebraic $\mathbb{F}$-vector bundle.

Problem 5.1. Which topological $\mathbb{F}$-vector bundles on X admit an algebraic structure?

It is convenient to bring into play the reduced Grothendieck group $\tilde{K}_{\mathbb{F}}(X)$ of topological $\mathbb{F}$-vector bundles on X. Since X has the homotopy type of a compact polyhedron Bochnak, Coste, and Roy [ibid.], the Abelian group $\tilde{K}_{\mathbb{F}}(X)$ is finitely generated Atiyah and Hirzebruch [1961] and Dyer [1969]. We let $\tilde{K}_{\mathbb{F}\text{-alg}}(X)$ denote the subgroup of $\tilde{K}_{\mathbb{F}}(X)$ generated by the classes of all topological $\mathbb{F}$-vector bundles on X that admit an algebraic structure.

If X is compact, then Problem 5.1 is equivalent to providing a description of $\tilde{K}_{\mathbb{F}\text{-alg}}(X)$. More precisely, the following holds.

Theorem 5.2. *Let X be a compact real algebraic set. Then:*

(i) *Two algebraic $\mathbb{F}$-vector bundles on X are algebraically isomorphic if and only if they are topologically isomorphic.*

(ii) *A topological $\mathbb{F}$-vector bundle on X admits an algebraic structure if and only if its class in $\tilde{K}_{\mathbb{F}}(X)$ belongs to $\tilde{K}_{\mathbb{F}\text{-alg}}(X)$.*

Theorem 5.2 follows from Swan [1977, Theorem 2.2], and a geometric proof is given in Bochnak, Coste, and Roy [1998]. Note that $\tilde{K}_{\mathbb{F}\text{-alg}}(X) = 0$ if and only if each algebraic $\mathbb{F}$-vector bundle on X is algebraically stably trivial. In turn, $\tilde{K}_{\mathbb{F}\text{-alg}}(X) = \tilde{K}_{\mathbb{F}}(X)$ if and only if each topological $\mathbb{F}$-vector bundle on X admits an algebraic structure.

According to Fossum [1969] and Swan [1977], we have the following.

Theorem 5.3. *For the unit n-sphere $\mathbb{S}^n$, the equality $\tilde{K}_{\mathbb{F}\text{-alg}}(\mathbb{S}^n) = \tilde{K}_{\mathbb{F}}(\mathbb{S}^n)$ holds.*

Benedetti and Tognoli [1980] proved that algebraization of topological vector bundles on a compact $\mathcal{C}^\infty$ manifold is always possible.

Theorem 5.4. *Let M be a compact $\mathcal{C}^\infty$ manifold. Then there exists an algebraic model X of M such that $\tilde{K}_{\mathbb{F}\text{-alg}}(X) = \tilde{K}_{\mathbb{F}}(X)$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$.*

The groups $\tilde{K}_{\mathbb{F}\text{-alg}}(-)$ have been extensively investigated by Bochnak and Kucharz [1989a, 1990, 1992] and Bochnak, Buchner, and Kucharz [1989]. In many cases, $\tilde{K}_{\mathbb{F}\text{-alg}}(-)$ are "small" subgroups of $\tilde{K}_{\mathbb{F}}(-)$. The following is a simplified version of Bochnak, Buchner, and Kucharz [ibid., Theorem 7.1].

Theorem 5.5. *Let M be a compact $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^{n+1}$, with* $\dim M = n \geq 1$. *Then M is ε-isotopic to a smooth algebraic subset X of $\mathbb{R}^{n+1}$ such that the group $\tilde{K}_{\mathbb{F}\text{-alg}}(X)$ is finite for $\mathbb{F} = \mathbb{R}$, $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$.*

The conclusion of Theorem 5.5 can be strengthened in some cases.

Example 5.6. Recall that $\tilde{K}_{\mathbb{F}}(\mathbb{S}^{4d}) = \mathbb{Z}$ for every positive integer d, cf. Husemoller [1975]. Hence, by Theorem 5.5, $\mathbb{S}^{4d}$ is ε-isotopic in $\mathbb{R}^{4d+1}$ to a smooth algebraic subset Σ^{4d} such that

$$\tilde{K}_{\mathbb{F}\text{-alg}}(\Sigma^{4d}) = 0 \text{ and } \tilde{K}_{\mathbb{F}}(\Sigma^{4d}) = \mathbb{Z}$$

for $\mathbb{F} = \mathbb{R}$, $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$.

One readily sees that each topological $\mathbb{R}$-vector bundle on a smooth real algebraic curve admits an algebraic structure. Next we discuss some results on vector bundles on a product of real algebraic curves.

Example 5.7. Let $X = C_1 \times \cdots \times C_n$, where $C_1, \ldots, C_n$ are connected, compact smooth real algebraic curves. Then $\tilde{K}_{\mathbb{R}\text{-alg}}(X) = \tilde{K}_{\mathbb{R}}(X)$ if $n = 2$ or $n = 3$. This assertion is a special case of Bochnak and Kucharz [1989a, Theorem 1.6].

In Example 5.7, one cannot take $n \geq 4$.

Example 5.8. Let $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ be the n-fold product of $\mathbb{S}^1$. According to Bochnak and Kucharz [1987b], $\tilde{K}_{\mathbb{C}\text{-alg}}(\mathbb{T}^n) = 0$. Furthermore, by Kucharz and Kurdyka [2016b, Example 1.11], we have $\tilde{K}_{\mathbb{R}\text{-alg}}(\mathbb{T}^n) \neq \tilde{K}_{\mathbb{R}}(\mathbb{T}^n)$ and $\tilde{K}_{\mathbb{H}\text{-alg}}(\mathbb{T}^n) \neq \tilde{K}_{\mathbb{H}}(\mathbb{T}^n)$ for $n \geq 4$.

Let $\mathbb{K}$ be a subfield of $\mathbb{F}$, where $\mathbb{K}$ (as $\mathbb{F}$) stands for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Any $\mathbb{F}$-vector bundle ξ can be regarded as a $\mathbb{K}$-vector bundle, which is indicated by $\xi_{\mathbb{K}}$.

Example 5.9. Let λ be a nontrivial topological $\mathbb{C}$-line bundle on $\mathbb{T}^2$. By Example 5.7, the $\mathbb{R}$-vector bundle $\lambda_{\mathbb{R}}$ on $\mathbb{T}^2$ admits an algebraic structure. However, in view of Example 5.8, λ does not admit an algebraic structure. The rth tensor power $\lambda^{\otimes r}$ is a nontrivial $\mathbb{C}$-line bundle, hence it does not admit an algebraic structure.

The next two theorems come from Bochnak and Kucharz [1992]. We use smooth real cubic curves $D_\alpha \subset \mathbb{P}^2(\mathbb{R})$, with $\alpha \in \mathbb{R}^*$, introduced in Section 4.1.

Theorem 5.10. *Let $X = D_\alpha \times \cdots \times D_\alpha$ be the n-fold product of D_α, where α is in $\mathbb{R}^*$ and $n \geq 2$. Then the following conditions are equivalent:*

(a) $\tilde{K}_{\mathbb{C}\text{-alg}}(X) = 0$.

(b) *The number α^2 is irrational.*

In this context, the equality $\tilde{K}_{\mathbb{C}\text{-alg}}(-) = \tilde{K}_{\mathbb{C}}(-)$ is characterized as follows.

Theorem 5.11. *Let $X = D_{\alpha_1} \times \cdots \times D_{\alpha_n}$, where $\alpha_1, \dots, \alpha_n$ are in $\mathbb{R}^*$ and $n \geq 2$. Then the following conditions are equivalent:*

(a) $\tilde{K}_{\mathbb{C}\text{-alg}}(X) = \tilde{K}_{\mathbb{C}}(X)$.

(b) $\alpha_i > 0$ *for all i, and $b(D_{\alpha_i}, D_{\alpha_j}) = 1$ for $i \neq j$.*

The pairs (α_i, α_j) with $b(D_{\alpha_i}, D_{\alpha_j}) = 1$ are explicitly described in Theorem 4.10.

In the next subsection we deal with vector bundles of a new type, which occupy an intermediate position between algebraic and topological vector bundles.

5.2 Regulous versus topological vector bundles. Let X be a real algebraic set. As in (3.8), by a stratification of X we mean a finite collection $\mathcal{S}$ of pairwise disjoint Zariski locally closed subsets whose union is X. An $\mathcal{S}$-*algebraic* $\mathbb{F}$-*vector bundle on* X is a topological $\mathbb{F}$-vector subbundle of $\varepsilon^n_X(\mathbb{F})$, for some n, such that the restriction $\xi|_S$ of ξ to each stratum $S \in \mathcal{S}$ is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon^n_S(\mathbb{F})$. If ξ and η are $\mathcal{S}$-algebraic $\mathbb{F}$-vector bundles on X, then an $\mathcal{S}$-*algebraic morphism* $\varphi\colon \xi \to \eta$ is a morphism of topological $\mathbb{F}$-vector bundles which induces a morphism of algebraic $\mathbb{F}$-vector bundles $\varphi_S\colon \xi|_S \to \eta|_S$ for each stratum $S \in \mathcal{S}$.

Definition 5.12. A *regulous* $\mathbb{F}$-*vector bundle on* X is an $\mathcal{S}$-algebraic $\mathbb{F}$-vector bundle for some stratification $\mathcal{S}$ of X. If ξ and η are regulous $\mathbb{F}$-vector bundles on X, then a *regulous morphism* $\varphi\colon \xi \to \eta$ is an $\mathcal{S}$-algebraic morphism for some stratification $\mathcal{S}$ of X such that both ξ and η are $\mathcal{S}$-algebraic $\mathbb{F}$-vector bundles.

In our joint paper Kucharz and Kurdyka [2016b], we introduced and investigated regulous (= stratified-algebraic) vector bundles. The main focus of Kucharz and Kurdyka [ibid.] and the subsequent papers Kucharz [2015b, 2016b], Kucharz and Kurdyka [2015], Kucharz and Zieliński [2018] is on comparison of algebraic, regulous, and topological vector bundles.

Regulous $\mathbb{F}$-vector bundles on X (together with regulous morphisms) form a category, which is equivalent to the category of finitely generated projective left $\mathcal{R}^0(X,\mathbb{F})$-modules, where $\mathcal{R}^0(X,\mathbb{F})$ is the ring of $\mathbb{F}$-valued regulous functions on X, cf. Kucharz and Kurdyka [2016b, Theorem 3.9].

A topological $\mathbb{F}$-vector bundle on X is said to *admit a regulous structure* if it is topologically isomorphic to a regulous $\mathbb{F}$-vector bundle. We let $\tilde{K}_{\mathbb{F}\text{-reg}}(X)$ denote the subgroup of $\tilde{K}_{\mathbb{F}}(X)$ generated by the classes of all topological $\mathbb{F}$-vector bundles on X that admit a regulous structure.

We have the following counterpart of Theorem 5.2, cf. Kucharz and Kurdyka [ibid.].

Theorem 5.13. *Let X be a compact real algebraic set. Then:*

(i) *Two regulous $\mathbb{F}$-vector bundles on X are regulously isomorphic if and only if they are topologically isomorphic.*

(ii) *A topological $\mathbb{F}$-vector bundle on X admits a regulous structure if and only if its class in $\tilde{K}_{\mathbb{F}}(X)$ belongs to $\tilde{K}_{\mathbb{F}\text{-reg}}(X)$.*

Hence $\tilde{K}_{\mathbb{F}\text{-reg}}(X) = \tilde{K}_{\mathbb{F}}(X)$ if and only if each topological $\mathbb{F}$-vector bundle on X admits a regulous structure.

The following result of Kucharz and Kurdyka [ibid.] should be compared with Theorem 5.3 and Example 5.6.

Theorem 5.14. *Let X be a compact real algebraic set that is homotopically equivalent to the unit n-sphere $\mathbb{S}^n$. Then $\tilde{K}_{\mathbb{F}\text{-reg}}(X) = \tilde{K}_{\mathbb{F}}(X)$.*

In contrast to Example 5.8 and Theorems 5.10 and 5.11, we proved in Kucharz and Kurdyka [ibid.] the following.

Theorem 5.15. *Let $X = X_1 \times \cdots \times X_n$, where X_i is a compact real algebraic set that is homotopically equivalent to the unit d_i-sphere $\mathbb{S}^{d_i}$ for $1 \le i \le n$. Then*

$$2\tilde{K}_{\mathbb{R}}(X) \subset \tilde{K}_{\mathbb{R}\text{-reg}}(X),\ \tilde{K}_{\mathbb{C}\text{-reg}}(X) = \tilde{K}_{\mathbb{C}}(X) \text{ and } \tilde{K}_{\mathbb{H}\text{-reg}}(X) = \tilde{K}_{\mathbb{H}}(X).$$

It is possible that $\tilde{K}_{\mathbb{R}\text{-reg}}(X) = \tilde{K}_{\mathbb{R}}(X)$ in Theorem 5.15, but no proof is available even for $X = \mathbb{T}^n$ with $n \ge 4$.

Our next result comes from Kucharz and Kurdyka [2015].

Theorem 5.16. *Let X be a compact real algebraic set that is homotopically equivalent to $\mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_n}$. Then the quotient group $\tilde{K}_{\mathbb{F}}(X)/\tilde{K}_{\mathbb{F}\text{-reg}}(X)$ is finite.*

If $n \ge 5$, then there exists an algebraic model X of $\mathbb{T}^n$ with $\tilde{K}_{\mathbb{F}}(X)/\tilde{K}_{\mathbb{F}\text{-reg}}(X) \ne 0$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$, cf. Kucharz and Kurdyka [2016b, Example 7.10].

For any real algebraic set X, we let $\tilde{K}_{\mathbb{F}}^{(\mathrm{crk})}(X)$ denote the subgroup of $\tilde{K}_{\mathbb{F}}(X)$ generated by the classes of all topological $\mathbb{F}$-vector bundles of constant rank. Define $\Gamma_{\mathbb{F}}(X)$ to be the quotient group

$$\Gamma_{\mathbb{F}}(X) := \tilde{K}_{\mathbb{F}}^{(\mathrm{crk})}(X)/(\tilde{K}_{\mathbb{F}\text{-reg}}(X) \cap \tilde{K}_{\mathbb{F}}^{(\mathrm{crk})}(X))$$

(cf. Kucharz and Kurdyka [2015] for an equivalent description). Evidently, $\Gamma_{\mathbb{F}}(X) = \tilde{K}_{\mathbb{F}}(X)/\tilde{K}_{\mathbb{F}\text{-reg}}(X)$ if X is connected. Note, however, that for the real algebraic curve C of (1.5), one has $\Gamma_{\mathbb{R}}(C) = 0$, while the group $\tilde{K}_{\mathbb{R}}(C)/\tilde{K}_{\mathbb{R}\text{-reg}}(C)$ is infinite.

Conjecture 5.17. *For any compact real algebraic set X, the group $\Gamma_{\mathbb{F}}(X)$ is finite.*

In Kucharz and Kurdyka [ibid.], we proved that Conjecture 5.17 holds in low dimensions.

Theorem 5.18. *If X is a compact real algebraic set of dimension at most 8, then the group $\Gamma_{\mathbb{F}}(X)$ is finite.*

For $\mathbb{C}$-line bundles, the following is expected.

Conjecture 5.19. *For any compact real algebraic set X and any topological $\mathbb{C}$-line bundle λ on X, the $\mathbb{C}$-line bundle $\lambda^{\otimes 2}$ admits a regulous structure.*

According to Kucharz [2015b], Conjecture 4.14 implies Conjecture 5.19. If $\dim X \leq 8$, then for some positive integer r, the $\mathbb{C}$-line bundle $\lambda^{\otimes r}$ admits a regulous structure, cf. Kucharz and Kurdyka [2015]. This should be compared with Example 5.9.

The key role in the proofs of the results presented in this subsection plays the following theorem of Kucharz and Kurdyka [2016b].

Theorem 5.20. *Let X be a compact real algebraic set. A topological $\mathbb{F}$-vector bundle ξ on X admits a regulous structure if and only if the $\mathbb{R}$-vector bundle $\xi_{\mathbb{R}}$ admits a regulous structure.*

By Example 5.9, one cannot substitute "algebraic" for "regulous" in Theorem 5.20.

References

Janusz Adamus and Hadi Seyedinejad (2017). "A proof of Kurdyka's conjecture on arc-analytic functions". *Math. Ann.* 369.1-2, pp. 387–395. MR: 3694650 (cit. on pp. 739, 743).

Selman Akbulut and Henry King (1992). "On approximating submanifolds by algebraic sets and a solution to the Nash conjecture". *Invent. Math.* 107.1, pp. 87–98. MR: 1135465 (cit. on pp. 756, 757).

Michael F. Atiyah and Friedrich Hirzebruch (1961). “Vector bundles and homogeneous spaces”. In: *Proc. Sympos. Pure Math., Vol. III*. American Mathematical Society, Providence, R.I., pp. 7–38. MR: 0139181 (cit. on p. 758).

James Ax (1969). “Injective endomorphisms of varieties and schemes”. *Pacific J. Math.* 31, pp. 1–7. MR: 0251036 (cit. on p. 742).

Riccardo Benedetti and Alberto Tognoli (1980). “On real algebraic vector bundles”. *Bull. Sci. Math. (2)* 104.1, pp. 89–112. MR: 560747 (cit. on p. 759).

Andrzej Białynicki-Birula and Maxwell Rosenlicht (1962). “Injective morphisms of real algebraic varieties”. *Proc. Amer. Math. Soc.* 13, pp. 200–203. MR: 0140516 (cit. on p. 742).

Edward Bierstone and Pierre D. Milman (1990). “Arc-analytic functions”. *Invent. Math.* 101.2, pp. 411–424. MR: 1062969 (cit. on pp. 739, 744).

– (1997). “Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant”. *Invent. Math.* 128.2, pp. 207–302. MR: 1440306 (cit. on p. 744).

Edward Bierstone, Pierre D. Milman, and Adam Parusiński (1991). “A function which is arc-analytic but not continuous”. *Proc. Amer. Math. Soc.* 113.2, pp. 419–423. MR: 1072083 (cit. on pp. 739, 743).

Marcin Bilski, Wojciech Kucharz, Anna Valette, and Guillaume Valette (2013). “Vector bundles and regulous maps”. *Math. Z.* 275.1-2, pp. 403–418. MR: 3101814 (cit. on p. 737).

Zbigniew Błocki (1992). “Singular sets of separately analytic functions”. *Ann. Polon. Math.* 56.2, pp. 219–225. MR: 1159993 (cit. on p. 745).

Jacek Bochnak, Michael Buchner, and Wojciech Kucharz (1989). “Vector bundles over real algebraic varieties”. *K-Theory* 3.3, pp. 271–298. MR: 1040403 (cit. on p. 759).

Jacek Bochnak, Michel Coste, and Marie-Françoise Roy (1998). *Real Algebraic Geometry*. Vol. 36. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, pp. x+430. MR: 1659509 (cit. on pp. 738, 740, 747, 750, 752, 758).

Jacek Bochnak and Wojciech Kucharz (1987a). “Algebraic approximation of mappings into spheres”. *Michigan Math. J.* 34.1, pp. 119–125. MR: 873026 (cit. on pp. 751, 752).

– (1987b). “Realization of homotopy classes by algebraic mappings”. *J. Reine Angew. Math.* 377, pp. 159–169. MR: 887406 (cit. on pp. 751–753, 759).

– (1988). “On real algebraic morphisms into even-dimensional spheres”. *Ann. of Math. (2)* 128.2, pp. 415–433. MR: 960952 (cit. on pp. 751–753).

– (1989a). “K-theory of real algebraic surfaces and threefolds”. *Math. Proc. Cambridge Philos. Soc.* 106.3, pp. 471–480. MR: 1010372 (cit. on p. 759).

Jacek Bochnak and Wojciech Kucharz (1989b). “Algebraic models of smooth manifolds”. *Invent. Math.* 97.3, pp. 585–611. MR: 1005007 (cit. on pp. 751–753).
– (1990). “On vector bundles and real algebraic morphisms”. In: *Real analytic and algebraic geometry (Trento, 1988)*. Vol. 1420. Lecture Notes in Math. Springer, Berlin, pp. 65–71. MR: 1051204 (cit. on pp. 753, 759).
– (1991a). “Nonisomorphic algebraic models of a smooth manifold”. *Math. Ann.* 290.1, pp. 1–2. MR: 1107659 (cit. on p. 753).
– (1991b). “Polynomial mappings from products of algebraic sets into spheres”. *J. Reine Angew. Math.* 417, pp. 135–139. MR: 1103909 (cit. on pp. 751, 752).
– (1992). “Vector bundles on a product of real cubic curves”. *K-Theory* 6.6, pp. 487–497. MR: 1204824 (cit. on pp. 759, 760).
– (1993). “Elliptic curves and real algebraic morphisms”. *J. Algebraic Geom.* 2.4, pp. 635–666. MR: 1227471 (cit. on pp. 751–755).
– (1999). “The Weierstrass approximation theorem for maps between real algebraic varieties”. *Math. Ann.* 314.4, pp. 601–612. MR: 1709103 (cit. on pp. 751, 752).
– (2007). “Real algebraic morphisms represent few homotopy classes”. *Math. Ann.* 337.4, pp. 909–921. MR: 2285743 (cit. on pp. 751, 752).
Jacek Bochnak, Wojciech Kucharz, and Robert Silhol (1997). “Morphisms, line bundles and moduli spaces in real algebraic geometry”. *Inst. Hautes Études Sci. Publ. Math.* 86, 5–65 (1998). MR: 1608561 (cit. on p. 752).
Salomon Bochner and William T. Martin (1948). *Several Complex Variables*. Princeton Mathematical Series, vol. 10. Princeton University Press, Princeton, N. J., pp. ix+216. MR: 0027863 (cit. on p. 745).
Armand Borel (1969). “Injective endomorphisms of algebraic varieties”. *Arch. Math. (Basel)* 20, pp. 531–537. MR: 0255545 (cit. on p. 742).
Eldon Dyer (1969). *Cohomology theories*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York-Amsterdam, pp. xiii+183. MR: 0268883 (cit. on p. 758).
Charles Fefferman and János Kollár (2013). “Continuous solutions of linear equations”. In: *From Fourier analysis and number theory to Radon transforms and geometry*. Vol. 28. Dev. Math. Springer, New York, pp. 233–282. MR: 2986959 (cit. on p. 748).
Goulwen Fichou (2005). “Motivic invariants of arc-symmetric sets and blow-Nash equivalence”. *Compos. Math.* 141.3, pp. 655–688. MR: 2135282 (cit. on pp. 739, 744).
Goulwen Fichou, Johannes Huisman, Frédéric Mangolte, and Jean-Philippe Monnier (2016). “Fonctions régulues”. *J. Reine Angew. Math.* 718, pp. 103–151. MR: 3545880 (cit. on pp. 737, 746, 747, 750).
Goulwen Fichou, Jean-Philippe Monnier, and Ronan Quarez (2017). “Continuous functions on the plane regular after one blowing-up”. *Math. Z.* 285.1-2, pp. 287–323. MR: 3598813 (cit. on p. 737).

Robert M. Fossum (1969). "Vector bundles over spheres are algebraic". *Invent. Math.* 8, pp. 222–225. MR: 0250298 (cit. on p. 758).

Toshizumi Fukui, Satoshi Koike, and Tzee-Char Kuo (1998). "Blow-analytic equisingularities, properties, problems and progress". In: *Real analytic and algebraic singularities (Nagoya/Sapporo/Hachioji, 1996)*. Vol. 381. Pitman Res. Notes Math. Ser. Longman, Harlow, pp. 8–29. MR: 1607662 (cit. on p. 744).

Riccardo Ghiloni (2006). "On the space of morphisms into generic real algebraic varieties". *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 5.3, pp. 419–438. MR: 2274786 (cit. on p. 752).

– (2007). "Second order homological obstructions on real algebraic manifolds". *Topology Appl.* 154.17, pp. 3090–3094. MR: 2355513 (cit. on p. 752).

Heisuke Hironaka (1964). "Resolution of singularities of an algebraic variety over a field of characteristic zero. I". *Ann. of Math. (2)* 79, pp. 109–203. MR: 0199184 (cit. on p. 747).

Dale Husemoller (1975). *Fibre bundles*. Second Edition. Graduate Texts in Mathematics, No. 20. Springer-Verlag, New York-Heidelberg, pp. xv+327. MR: 0370578 (cit. on p. 759).

Nuria Joglar-Prieto and Frédéric Mangolte (2004). "Real algebraic morphisms and del Pezzo surfaces of degree 2". *J. Algebraic Geom.* 13.2, pp. 269–285. MR: 2047699 (cit. on p. 752).

Satoshi Koike and Adam Parusiński (2003). "Motivic-type invariants of blow-analytic equivalence". *Ann. Inst. Fourier (Grenoble)* 53.7, pp. 2061–2104. MR: 2044168 (cit. on pp. 739, 744).

János Kollár, Wojciech Kucharz, and Krzysztof Kurdyka (2018). "Curve-rational functions". *Math. Ann.* 370.1-2, pp. 39–69. MR: 3747483 (cit. on pp. 737, 744, 746, 748, 749).

János Kollár and Krzysztof Nowak (2015). "Continuous rational functions on real and p-adic varieties". *Math. Z.* 279.1-2, pp. 85–97. MR: 3299844 (cit. on pp. 738, 746–748).

Wojciech Kucharz (1985). "On homology of real algebraic sets". *Invent. Math.* 82.1, pp. 19–25. MR: 808106 (cit. on p. 756).

– (2005). "Homology classes represented by semialgebraic arc-symmetric sets". *Bull. London Math. Soc.* 37.4, pp. 514–524. MR: 2143731 (cit. on pp. 740, 742).

– (2009). "Rational maps in real algebraic geometry". *Adv. Geom.* 9.4, pp. 517–539. MR: 2574137 (cit. on pp. 737, 746, 747, 750, 756).

– (2010). "Complex cycles on algebraic models of smooth manifolds". *Math. Ann.* 346.4, pp. 829–856. MR: 2587094 (cit. on pp. 752, 753).

– (2013a). "Regular maps into real Fermat varieties". *Bull. Lond. Math. Soc.* 45.5, pp. 1086–1090. MR: 3105000 (cit. on p. 752).

Wojciech Kucharz (2013b). "Regular versus continuous rational maps". *Topology Appl.* 160.12, pp. 1375–1378. MR: 3072698 (cit. on pp. 738, 746, 757).

– (2014a). "Approximation by continuous rational maps into spheres". *J. Eur. Math. Soc. (JEMS)* 16.8, pp. 1555–1569. MR: 3262451 (cit. on pp. 738, 746, 756).

– (2014b). "Continuous rational maps into the unit 2-sphere". *Arch. Math. (Basel)* 102.3, pp. 257–261. MR: 3181715 (cit. on pp. 738, 746, 757).

– (2015a). "Complex cycles as obstructions on real algebraic varieties". *Glasg. Math. J.* 57.2, pp. 343–347. MR: 3333945 (cit. on p. 752).

– (2015b). "Some conjectures on stratified-algebraic vector bundles". *J. Singul.* 12, pp. 92–104. MR: 3317142 (cit. on pp. 738, 746, 760, 762).

– (2016a). "Continuous rational maps into spheres". *Math. Z.* 283.3-4, pp. 1201–1215. MR: 3520001 (cit. on pp. 738, 746, 757).

– (2016b). "Stratified-algebraic vector bundles of small rank". *Arch. Math. (Basel)* 107.3, pp. 239–249. MR: 3538519 (cit. on pp. 738, 760).

Wojciech Kucharz and Krzysztof Kurdyka (2015). "Comparison of stratified-algebraic and topological K-theory". arXiv: 1511.04238 (cit. on pp. 738, 746, 760–762).

– (2016a). "Some conjectures on continuous rational maps into spheres". *Topology Appl.* 208, pp. 17–29. MR: 3506967 (cit. on pp. 738, 746, 757).

– (2016b). "Stratified-algebraic vector bundles". *J. Reine Angew. Math.* (Cit. on pp. 738, 746, 756, 757, 759–762).

– (2017). "Linear equations on real algebraic surfaces". *Manuscripta Math.* 154.3-4, pp. 285–296. MR: 3713914 (cit. on pp. 738, 746, 748).

Wojciech Kucharz and Maciej Zieliński (2018). "Regulous vector bundles". *Math. Nachr.* (Cit. on pp. 738, 760).

Tzee Char Kuo (1985). "On classification of real singularities". *Invent. Math.* 82.2, pp. 257–262. MR: 809714 (cit. on p. 744).

Krzysztof Kurdyka (1988). "Ensembles semi-algébriques symétriques par arcs". *Math. Ann.* 282.3, pp. 445–462. MR: 967023 (cit. on pp. 739–741, 743).

– (1991). "A counterexample to subanalyticity of an arc-analytic function". In: *Proceedings of the Tenth Conference on Analytic Functions (Szczyrk, 1990)*. Vol. 55, pp. 241–243. MR: 1141439 (cit. on pp. 740, 743).

– (1994). "An arc-analytic function with nondiscrete singular set". *Ann. Polon. Math.* 59.3, pp. 251–254. MR: 1282776 (cit. on pp. 740, 743).

– (1999). "Injective endomorphisms of real algebraic sets are surjective". *Math. Ann.* 313.1, pp. 69–82. MR: 1666793 (cit. on pp. 740, 742).

Krzysztof Kurdyka and Adam Parusiński (2007). "Arc-symmetric sets and arc-analytic mappings". In: *Arc spaces and additive invariants in real algebraic and analytic geometry*. Vol. 24. Panor. Synthèses. Soc. Math. France, Paris, pp. 33–67. MR: 2409688 (cit. on pp. 740, 742).

– (2012). "On the non-analyticity locus of an arc-analytic function". *J. Algebraic Geom.* 21.1, pp. 61–75. MR: 2846679 (cit. on pp. 740, 744).

Krzysztof Kurdyka and Kamil Rusek (1988). "Surjectivity of certain injective semialgebraic transformations of $\mathbf{R}^n$". *Math. Z.* 200.1, pp. 141–148. MR: 972400 (cit. on pp. 740, 742).

Jean-Louis Loday (1973). "Applications algébriques du tore dans la sphère et de $S^p \times S^q$ dans S^{p+q}", 79–91. Lecture Notes in Mathematics, Vol. 342. MR: 0368034 (cit. on p. 752).

Clint McCrory and Adam Parusiński (2003). "Virtual Betti numbers of real algebraic varieties". *C. R. Math. Acad. Sci. Paris* 336.9, pp. 763–768. MR: 1989277 (cit. on pp. 740, 744).

Jean-Philippe Monnier (2018). "Semi-algebraic geometry with rational continuous functions". *Math. Ann.* (Cit. on p. 738).

John Nash (1952). "Real algebraic manifolds". *Ann. of Math. (2)* 56, pp. 405–421. MR: 0050928 (cit. on pp. 741, 753, 757).

Donald J. Newman (1960). "One-one polynomial maps". *Proc. Amer. Math. Soc.* 11, pp. 867–870. MR: 0122817 (cit. on p. 742).

Yildiray Ozan (1995). "On entire rational maps in real algebraic geometry". *Michigan Math. J.* 42.1, pp. 141–145. MR: 1322195 (cit. on p. 752).

Adam Parusiński (1994). "Subanalytic functions". *Trans. Amer. Math. Soc.* 344.2, pp. 583–595. MR: 1160156 (cit. on pp. 740, 744).

– (2001). "On the preparation theorem for subanalytic functions". In: *New developments in singularity theory*. Vol. 21. NATO Science Series II: Mathematics, Physics and Chemistry. Kluwer Academic Publishers, Dordrecht, pp. viii+472. MR: 1849301 (cit. on pp. 740, 744).

– (2004). "Topology of injective endomorphisms of real algebraic sets". *Math. Ann.* 328.1-2, pp. 353–372. MR: 2030380 (cit. on pp. 740, 742).

Adam Parusiński and Laurenţiu Păunescu (2017). "Arc-wise analytic stratification, Whitney fibering conjecture and Zariski equisingularity". *Adv. Math.* 309, pp. 254–305. MR: 3607278 (cit. on pp. 740, 744).

Jiagui Peng and Zizhou Tang (1999). "Algebraic maps from spheres to spheres". *Sci. China Ser. A* 42.11, pp. 1147–1154. MR: 1749862 (cit. on p. 752).

Józef Siciak (1990). "Singular sets of separately analytic functions". *Colloq. Math.* 60/61.1, pp. 281–290. MR: 1096378 (cit. on p. 745).

Richard G. Swan (1977). "Topological examples of projective modules". *Trans. Amer. Math. Soc.* 230, pp. 201–234. MR: 0448350 (cit. on p. 758).

René Thom (1954). "Quelques propriétés globales des variétés différentiables". *Comment. Math. Helv.* 28, pp. 17–86. MR: 0061823 (cit. on p. 742).

Alberto Tognoli (1973). "Su una congettura di Nash". *Ann. Scuola Norm. Sup. Pisa (3)* 27, pp. 167–185. MR: 0396571 (cit. on pp. 753, 757).

Francisco-Javier Turiel (2007). "Polynomial maps and even dimensional spheres". *Proc. Amer. Math. Soc.* 135.8, pp. 2665–2667. MR: 2302590 (cit. on p. 752).

Reginald Wood (1968). "Polynomial maps from spheres to spheres". *Invent. Math.* 5, pp. 163–168. MR: 0227999 (cit. on p. 752).

Maciej Zieliński (2016). "Homotopy properties of some real algebraic maps". *Homology Homotopy Appl.* 18.1, pp. 287–294. MR: 3498647 (cit. on pp. 738, 746, 757).

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