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RENORMALIZATION AND RIGIDITY

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Abstract

The ideas of renormalization was introduced into dynamics around 40 years ago. By now renormalization is one of the most powerful tools in the asymptotic analysis of dynamical systems. In this article we discuss the main conceptual features of the renormalization approach, and present a selection of recent results. We also discuss open problems and formulate related conjectures.

1 Renormalization Group in Statistical Mechanics and Critical Phenomena

The ideas of renormalization and universality emerged in statistical mechanics in the 1960s in the works of L.Kadanoff, M.Fisher, A.Patashinski, V.Pokrovski, B.Widom, and K.Wilson in connection with a problem of phase transitions and critical phenomena. As parameters of physical system change the system may go through a dramatic change of its behaviour. This phenomenon is called phase transition. The simplest examples are provided by lattice spin systems. When temperature T decreases the system changes from a state of almost independent (statistically) spins to a highly correlated state. This transition happens at a particular value of temperature $T = T_{cr}$, called the critical temperature. It was discovered by physicists that at the critical temperature one can observe highly non-trivial scaling invariant structures. Moreover, this scaling invariant structures have strong universality properties. It means that their statistical properties and scalings exponents do not depend on the details of the interaction, but only on global characteristics, such as dimension, symmetries etc. Physicists also developed technical tools to calculate critical exponents and other parameters of the asymptotic statistical objects. These tools were based on the concept of renormalization group, or renormalization. The main idea is to look at large and increasing chunks of a system, but also rescale them in order to deal with objects of order 1. The transformation describing transition from one block size to

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the next one, typically constant time larger, is called renormalization group transformation. The core of the renormalization group theory is a concept of the renormalization fixed point. It was realized that under the above procedure the system will converge to a statistical state which is a fixed point of the renormalization transformation. If the temperature $T > T_{cr}$ the renormalization converges to a trivial, or Gaussian, fixed point, which corresponds to the Central Limit Theorem (CLT) for statistically weakly depended spins. The system should be renormalized by a factor $N^{1/2}$ where N is a number of spins in a block. Thus the scaling exponent in this case is a trivial CLT exponent $\gamma = 2 \times 1/2 = 1$. Interesting behaviour can be observed only at a critical temperature $T = T_{cr}$ and at the temperatures infinitesimally close to it. It turns out that at the critical point the system under renormalization converges to a non-trivial statistical state which is also a fixed point for renormalization group transformation but with non-trivial critical scaling exponent γ. In the case of 2D Ising model $\gamma = 7/4$ which corresponds to a scaling $N^{7/8}$ instead of Gaussian scaling $N^{1/2}$.

Next step of a conceptual picture is a discussion of a stability of the above fixed points. One can linearize renormalization transformation near a fixed point and ask about the number of unstable directions. It turns out that there exists a single non-trivial unstable direction. According to the hyperbolic theory, it means that in the infinite-dimensional space of systems there exists a co-dimension 1 stable manifold of systems converging to a fixed point under renormalization. A one-parameter family of systems, parametrized by the temperature, will typically intersect this stable manifold. Such an intersection point corresponds to a critical temperature $T = T_{cr}$ at which the system converges to the critical renormalization fixed point. For temperatures $T > T_{cr}$ the system will converge to a trivial Gaussian fixed point which can be shown to be stable. For temperatures $T > T_{cr}$ and close to T_{cr} the system first comes close to a critical fixed point, and then goes away following a one-dimensional unstable manifold corresponding to the unique unstable direction.

This is a very brief sketch of the physical theory. I started with it to provide a historical background, and also to emphasize similarities with dynamical renormalization which I discuss below. I finish this physical detour with a few remarks. Firstly, the wording "renormalization group" is used less frequently in our days. In fact, one can speak of a semi-group of iterates of the renormalization transformation. In what follows I use only the word renormalization. Secondly, when I write above "it was discovered", "it was realized", "it turns out" etc, I mean statements established on the physical level. Mathematically rigorous results in this direction are very few, and they are incredibly difficult to prove. However, the conceptual picture described above is extremely simple and very attractive. It provided a completely different point of view, and played a crucial role in shaping of modern understanding of critical phenomena. Whenever physicists see a universal behaviour and universal numbers they immediately think about renormalization.

Thirdly, physical renormalization theory is much more sophisticated and elaborated than the sketch above. It also provides tools to calculate things. That is why the 1982 Nobel Prize in Physics was awarded to K.Wilson for his work on critical phenomena.

2 Dynamical Renormalization

In the context of the theory of dynamical systems renormalization was introduced by M.Feigenbaum and P.Coullet, Ch.Tresser in the middle of the 70s (Feigenbaum [1978] and Tresser and Coullet [1978]). Feigenbaum was playing around with two families of unimodal maps, the logistic family and the sine family. He looked at the sequences of the period-doubling bifurcations trying to find numerically the bifurcations parameter values. Feigenbaum observed that parameter values seems to converge exponentially fast to a limiting value. To help with numerics he estimated the rate of the exponential convergence. Astonishingly the rate looked the same for both families, $\mu_{n+1} - \mu_n \sim C\delta^{-n}$, where $\delta = 4.669\ldots$ is the famous Feigenbaum constant. As I pointed out above, physicists would look for a renormalization explanation when they come across universal numbers. Feigenbaum developed renormalization theory which explained the universality phenomenon. He defined a renormalization transformation and showed numerically that it has a non-trivial hyperbolic fixed point with essentially unique unstable direction. This unique unstable eigenvalue is exactly equal to δ. A large block of spins is replaced by an exponentially increasing sequence of iterates of a map. In the period-doubling case the n-th step of the renormalization procedure, or, in other words, n-th iterate of a renormalization transformation, corresponds to 2^n iterate of an original map. It turns out that interesting dynamics for 2^n iterate happens near the critical point, in fact, exponentially close to it. Thus it makes sense to rescale a space variable so that the effective dynamics is described in terms of order 1 maps. Next $(n+1)$-step requires iterating twice the rescaled n-th step map, as well as an additional rescaling. What is described in words is exactly period doubling transformation. Namely, consider the space of unimodal maps $f(x)$ with non-degenerate point of maximum at $x = 0$, and normalized in such a way that $f(0) = 1$. Then the renormalization transformation can be defined in the following way:

(1)
$$Rf(x) = -\alpha f(f(-\alpha^{-1}x)), \ \alpha^{-1} = -f(1).$$

The concept of metrical universality was so revolutionary and novel for mathematicians at the time that it was initially even difficult to make them believe in it. Shortly, however, mathematicians realized the importance of the discovery. O.Lanford (Lanford [1982]) gave the first rigorous computer-assisted proof of the existence of the Feigenbaum fixed point with one unstable direction. Later E.Vul, Ya.Sinai and K.Khanin developed the thermodynamic formalism for the Feigenbaum attractor (Vul, Sinai, and Khanin [1984]), important Epstein classes were introduced by H.Epstein(Epstein [1989]). However, a real mathematical theory, so-called Sullivan-McMullen-Lyubich theory was developed only in the 90s (Sullivan [1992], McMullen [1994], and Lyubich [1999]). In fact, the theory covers a much more general case. The accumulation points of period-doubling bifurcations corresponds to a particular combinatorics of the so-called infinitely renormalizable maps. For other combinatorial types the renormalization transformation can also be defined. On the other hand, in this more general situation it does not make sense to speak about fixed points of renormalization since the renormalization transformation itself changes at every step. However, it still make sense to speak about convergence of renormalization. Namely, consider two different analytic infinitely renormalizable unimodal maps T_1 and T_2 with the same order of critical points, and the same combinatorics. It means that at every step the same renormalization transformation is applied to both of them. Denote by $f_n^{(1)}$ and $f_n^{(2)}$ the n-th step renormalization for T_1 and T_2 respectively. It follows from the Sullivan-McMullen-Lyubich theory that $\| f_n^{(1)} - f_n^{(2)} \| \to \infty$ as $n \to \infty$ exponentially fast. At the same time, changing of the combinatorial type results in an exponential instability. One can show that there are no other unstable directions. In other words, the hyperbolicity of renormalization with one unstable direction is valid in full generality. The set of maps with a given combinatorial type form a stable manifold of co-dimension one. The fact that this set of maps has a smooth manifold structure follows from general fact of the hyperbolicity theory. This is a kind of dream result for the renormalization ideology. Indeed, it provides a full justification of the renormalization picture. The only drawback is the requirement that the order of critical points is given by even integer numbers. At the same time all the experts agree that similar result must hold for all orders greater than 1. While the analyticity assumption can be significantly relaxed, currently there are almost no results on convergence of renormalization for maps with critical points of non-integer order. In our opinion this is one of the central open problems in the theory of renormalization. We shall discuss this problem in more details later.

Our brief discussion of renormalization for unimodal maps would be incomplete without mentioning the paper by S. Davie on period-doubling for $C^{2+\epsilon}$ maps (Davie [1996]), another paper by M. Martens providing a construction of a large class of periodic point for renormalization transformation (Martens [1998]), and a recent important paper by A. Avila and M. Lyubich where a new approach to convergence of renormalization for unimodal maps is developed (Avila and Lyubich [2011]).

Many aspects of the renormalization theory can be presented in a cleaner and easier way in the case of circle homeomorphisms. This will be the main object in what we discuss below. The problem for interval maps is very similar. However circle maps have several advantages. Firstly, the combinatorial types are completely determined by the rotation numbers. Infinitely renormalizable are simply maps with irrational rotation number. But most importantly, exponential instability in changing of a combinatorial type is obvious in the circle case and is highly non-trivial fact for interval maps.

3 Renormalization and Rigidity for Circle Homeomorphisms

Let T be a homeomorphism of the unit circle $\mathbb{S}^1$ with irrational rotation number ρ. Although renormalization can be defined for any homeomorphism, a meaningful theory requires certain regularity. We shall either consider the case when T is a smooth diffeomorphism, or assume that T is smooth outside of a finite set of singular points. In this section we discuss the case of one singularity. It can be either a critical inflection point, for so-called critical circle maps, or a break point, that is a point where the first derivative has a jump discontinuity.

To implement a general renormalization scheme one has to determine a sequence of times such that an iterate of a initial point comes close to itself, and then rescale the space coordinate. Let the continued fraction expansion for ρ be given by $\rho = [k_1, k_2, \dots, k_n, \dots]$, where $k_i \in \mathbb{N}$, $i \in \mathbb{N}$ is a sequence of partial quotients. It is well known that denominators q_n of convergents $p_n/q_n = [k_1, k_2, \dots, k_n]$ correspond to a sequence of times of closest returns for a linear rotation $T_\rho : x \mapsto x + \rho \pmod 1$. Moreover, $T_\rho^{q_{2n}} x_0$ converges to x_0 from the right, and $T_\rho^{q_{2n-1}} x_0$ from the left. To define a sequence of renormalization one has to fix a point x_0, called a marked point, about which the renormalization will be defined, then for n-level renormalization consider a closed interval I_n containing x_0 with the end points $x_{q_{n-1}} = T^{q_{n-1}} x_0$ and $x_{q_n} = T^{q_n} x_0$. For simplicity we assume that n is even, so $I_n = [x_{q_{n-1}}, x_{q_n}]$. The first return map from I_n into itself has two branches. The first one is given by $T^{q_n} : [x_{q_{n-1}}, x_0] \to [x_{q_n+q_{n-1}}, x_{q_n}]$, the second branch corresponds to $T^{q_{n-1}} : [x_0, x_{q_n}] \to [x_{q_{n-1}}, x_{q_n+q_{n-1}}]$. Here and below the trajectory of x_0 is denoted by $x_i = T^i x_0$, $i \in \mathbb{Z}$. Next we choose the n-th level renormalized coordinate, denoted by z, such that the length of I_n in the coordinate z will be order one. Usually z is defined by an affine change of variables such that $z(x_{q_{n-1}}) = -1, z(x_0) = 0$. Hence, $z(x) = (x - x_0)/(x_0 - x_{q_{n-1}})$. To simplify notations we do not indicate dependence of the coordinate z on the renormalization level n. Now we can define $R^n(T)$ as a pair of first return maps $T^{q_{n-1}}$, T^{q_n} expressed in the renormalized coordinate z. Denote $a_n = z(x_{q_n}), -b_n = z(x_{q_n+q_{n-1}})$. Then $R^n(T) = (f_n(z), g_n(z))$, $f_n(z) : [-1, 0] \to [-b_n, a_n]$, $g_n(z) : [0, a_n] \to [-1, -b_n]$, where

(2)
$$f_n(z) = \frac{T^{q_n}(x_0 + z(x_0 - x_{q_{n-1}})) - x_0}{x_0 - x_{q_{n-1}}}, \quad g_n(z) = \frac{T^{q_{n-1}}(x_0 + z(x_0 - x_{q_{n-1}})) - x_0}{x_0 - x_{q_{n-1}}}.$$

A simple but important observation is that in order to find maps (f_{n+1}, g_{n+1}) one does not need to know the original map T. Indeed, maps (f_{n+1}, g_{n+1}) are completely determined by the maps (f_n, g_n). Since $q_{n+1} = k_{n+1}q_n + q_{n-1}$, we have $f_{n+1} = A_{n+1}^{-1} \circ f_n^{k_{n+1}} \circ g_n \circ A_{n+1}$, $g_{n+1} = A_{n+1}^{-1} \circ f_n \circ A_{n+1}$, where A_{n+1} is a linear rescaling $A_{n+1}(z) = -a_n z$, and $a_n = f_n(0)$. An integer number k_{n+1} is also determined by f_n. It is just a number of iterates of f_n such that a trajectory of point (-1) reaches positive semi-axis. Namely, $f_n^i(-1) < 0, i \leq k_{n+1}$ and $f_n^{k_{n+1}+1}(-1) > 0$. The transformation from a pair (f_n, g_n) to a pair (f_{n+1}, g_{n+1}) is called the *renormalization transformation*, and is denoted by R. Note that maps g_{n+1} are obtained from f_n by linearly rescaling the coordinates. Hence, in most cases, it is enough to keep track only of a sequence of maps f_n.

The main goal of the renormalization theory is to show that the renormalization transformation is hyperbolic, and circle maps with the same irrational rotations numbers and with the same local structure of their singular points belong to the same stable manifold for R. In other words, renormalizations $f_n^{(1)}$, $f_n^{(2)}$ constructed from two such maps T_1, T_2 converge to each other with exponential rate. Moreover, the rate of convergence is universal. It means that it does not depend on the rotation number and the maps T_1, T_2, but only on local characteristics of singular points. Circle maps which we consider below satisfy the Denjoy property. Namely, they all topologically conjugate to the linear rotation with the same rotation number. This allows us to formulate renormalization conjecture in a more precise way. We shall consider two type of singularities: critical points of finite order x_{cr} and break points x_{br}. At a critical point the derivative $T'(x_{cr}) = 0$, and a map T locally behaves like $T(x) - T(x_{cr}) \sim A(x - x_{cr})|x - x_{cr}|^{\alpha-1}$, where $A > 0$ and $\alpha > 1$ is the order of the critical point. At a break point x_{br} the first derivative of T has a jump discontinuity. Namely, both one sided derivatives exist and both are positive, but $T'(x_{br}-) \neq T'(x_{br}+)$. Parameter $c = \sqrt{T'(x_{br}-)/T'(x_{br}+)}$ is called the size of a break. We take square root in the formula for c to simplify some formulas below. When we say above “local characteristics of singular points” we mean precisely parameters α and c. Notice that both α and c are smooth invariants. In other words, they are preserved by smooth changes of variables.

We shall say that two circle maps T_1 and T_2 with the same irrational rotation number are *singularity equivalent* if there exists a topological conjugacy ϕ which is also a bijection between the singular points of T_1 and T_2. Moreover, conjugated singular points are of the same type (critical or break), and with the same value of parameters α or c.

Renormalization Conjecture. Suppose circle homeomorphisms T_1 and T_2 have a finite number of singular points and are singularity equivalent. Assume also that T_1 and T_2 are $C^{2+\alpha}$-smooth outside of a set of singularity points. Then the renormalizations $f_n^{(1)}$, $f_n^{(2)}$

constructed from the conjugated marked points converge to each other with a universal exponential rate.

Although convergence of renormalizations is interesting in its own right, it is also directly related to the rigidity theory. In many cases convergence of renormalization implies that two maps which a priori are only topologically conjugate are, in fact, smoothly conjugate to each other. Such an upgrade from topological equivalence to a smooth one is called the *rigidity*. Below we discuss rigidity results in parallel with results on renormalization convergence. In this section we consider only the unimodal setting, that is maps with a single singularity which is either a critical point or a break point. In this case it is natural to take the singularity point as a marked point for the renormalization construction. However, we start with the case of $C^{2+\alpha}$-smooth diffeomorphisms where the renormalization picture is much simpler.

Linearization of circle diffeomorphisms. Since in the smooth setting linear rotations form a distinguished class, rigidity in this case is usually discussed in terms of the linearization problem. The main question here is when a smooth circle diffeomorhism with irrational rotations number ρ is smoothly conjugate to a linear rotation T_ρ. This problem was first addressed in a classical paper by V.Arnold (Arnold [1961]). Arnold proved that smooth (in fact, analytic) linearization holds for analytic circle diffeomorphisms close to linear rotations, provided their rotation numbers are typical, i.e. satisfy certain conditions of a Diophantine type. He also showed that smooth linearization cannot be extended to all irrational rotation numbers, and conjectured that local condition, that is closeness to linear rotations, can be removed. Global result was proved by M.Herman (Herman [1979]) and extended to a larger class of rotation numbers by J-C.Yoccoz (Yoccoz [1984a]). Although renormalization was not explicitly used by Herman and Yoccoz, the linearization problem is closely related to the Renormalization Conjecture above. It is easy to see that for linear rotations T_ρ the renormalization $f_n(z) = a_n + z$, where $a_n = [k_{n+1}, k_{n+2}, \ldots]$. Hence, convergence of renormalization reduces to showing that in a general case $\max_{z\in[-1,0]} |f_n'(z) - 1| \to 0$ as $n \to \infty$, and the convergence is fast enough. Since $f_n'(z) = (T^{q_n})'(x)$, the task is to prove that $\epsilon_n = \max_{x\in\mathbb{S}^1} |\log (T^{q_n})'(x)| \to 0$ as $n \to \infty$. This case is simpler than the case of maps with singularities since renormalization converge to a "trivial fixed-point" given by a family of linear maps $f_a(z) = z + a$. What is much harder is to get sharp estimates for ϵ_n which are needed for rigidity results. Indeed, even if we know that f_n is ϵ_n-close to a linear family, after many iterates one can lose control, and iterated maps may be not close to linear ones anymore. This may and will happen when k_{n+1} are very large. Convergence rate for ϵ_n is related to a growth rate of the denominators q_n. The following estimate was proved in Khanin and Teplinsky [2009].

Lemma 3.1. *Let $T \in C^{2+\alpha}(\mathbb{S}^1)$. Denote by $l_n = \max_{x\in\mathbb{S}^1} |T^{q_n}x - x|$. Then there exists a constant $C > 0$ such the following estimate holds:*

$$\epsilon_n \leq C\left[l_{n-1}^{\alpha} + \frac{l_n}{l_{n-1}} l_{n-2}^{\alpha} + \frac{l_n}{l_{n-2}} l_{n-3}^{\alpha} + \cdots + \frac{l_n}{l_0}\right]$$

It is easy to show that the sequence l_n decays exponentially fast, and, uniformly in n, the ratio l_n/l_{n-k} is exponentially small in k. It follows immediately that $\epsilon_n \to 0$ exponentially fast as $n \to \infty$. But, in fact, the lemma above gives much more. Its proof is quite elementary. It is based on renormalization ideology and cross-ratio distortion relations. Cross-ratio distortion methods were first used by Yoccoz (Yoccoz [1984b]) in his study of analytical critical circle maps. As we will see below it is a powerful tool in renormalization theory. The lemma above allows to prove the following sharp rigidity result. Denote by $D_\delta = \{\rho : |q\rho - p| \geq C(\rho)q^{-1-\delta},\ \forall p \in \mathbb{Z}, q \in \mathbb{N}\}$. It is well known that for any $\delta > 0$ the set D_δ has full Lebesgue measure.

Theorem 3.2. *(Sinai and Khanin [1989] and Khanin and Teplinsky [2009])Let $T \in C^{2+\alpha}(\mathbb{S}^1)$, $\rho(T) \in D_\delta$, and $\alpha > \delta$. Then T is $C^{1+\alpha-\delta}$-smoothly conjugate to T_ρ.*

It is well known that C^1-rigidity cannot hold for typical rotation numbers when the map T is only C^2-smooth. Hence $C^{2+\alpha}$ is a natural setting here. Note also that the smoothness result above is sharp. For maps T of higher smoothness the best result is due to Y. Katznelson and D.Ornstein (Katznelson and Ornstein [1989]).

Theorem 3.3. *Let $T \in C^{k+\alpha}(\mathbb{S}^1, k \geq 2,\ \rho(T) \in D_\delta$, and $k - 1 + \alpha - \delta > 1$. Then, for any arbitrary small ϵ, the map T is $C^{k-1+\alpha-\delta-\epsilon}$-smoothly conjugate to T_ρ.*

Note extra ϵ which has to be subtracted from the smoothness exponent which is not necessary in the case $k = 2$. I was recently informed by D.Ornstein that he can now improve the last result and remove ϵ from the estimate above (Ornstein [2017]). Finally, notice that C^1 linearization implies regularity of invariant measure for T. It is well known that any circle homeomorphism with irrational rotation number is uniquely ergodic. In other words, it has a unique invariant probability measure. This measure is absolutely continuous with respect to the Lebesgue measure with a positive continuous density if and only if the map T is C^1 linearizable.

I should also mention a closely related problem of simultaneous linearization of commuting circle diffeomorphisms. In this case one can speak about a higher rank action by the group $\mathbb{Z}^d$ where d is the number of diffeomorphisms. A conjecture by J. Moser stated that Herman theory can be extended to this setting in the following sense. Suppose a vector $\vec{\rho} = (\rho_1, \dots \rho_d)$, where $\rho_i, 1 \le i \le d$ are the rotation numbers of commuting C^∞ circle diffeomorphisms $T_1, \dots T_d$, is a Diophantine vector in the sense of simultaneous rational approximations. Then the diffeomorphisms $T_1, \dots T_d$ can be simultaneously linearized, and the corresponding conjugacy is C^∞ smooth. Moser proved a local version of the above conjecture. The global result was proved in Fayad and Khanin [2009].

Summarizing, we see that in the case of smooth circle diffeomorphisms the convergence of renormalization holds for all irrational rotation numbers. At the same time, the rigidity results require conditions of a Diophantine type.

Critical circle maps. Convergence of renormalization for critical circle maps was studied by E. de Faria, W. de Melo (de Faria and de Melo [1999, 2000] and M. Yampolsky (Yampolsky [2002] and Yampolsky [2001]). Although they used different approach, in both cases analysis was based on a combination of real-analytic methods and methods from the holomorphic dynamics. This required an assumption that the order of critical points is given by odd integer numbers: $\alpha = 3, 5, 7, \dots$. In de Faria and de Melo [1999] convergence of renormalization was proved in the C^∞ setting for rotation numbers of bounded type. Yampolsky developed a new approach based on parabolic renormalization which allowed him to prove exponential convergence of renormalizations in the analytic case C^ω for all irrational rotation numbers. Below I formulate the result in the analytic setting. Note that in the critical case we always choose a critical point as a marked point.

Consider a double-infinite sequence of natural numbers $\mathbf{k} = \{k_i,\, k_i \in \mathbb{N},\, i \in \mathbb{Z}\}$, and form two irrational numbers $\rho_+ = [k_1, k_2, \dots, k_n \dots]$, $\rho_- = [k_0, k_{-1}, \dots, k_{-n} \dots]$. Denote by $\hat{G}$ a natural extension of a Gauss map $G : \rho \mapsto \rho^{-1} \pmod 1$, $\rho \in [0, 1]$. The transformation $\hat{G}$ acts on pairs (ρ_-, ρ_+) : $\hat{G}(\rho_-, \rho_+) = (([{\rho_+}^{-1}] + \rho_-)^{-1}, G\rho_+)$, where $[\cdot]$ denotes the integer part of a number. It is easy to see that $\hat{G}$ corresponds to the unit shift of a sequence $\mathbf{k}$. Denote by $\mathfrak{F}_\alpha$ the space of pairs of commuting analytic functions $(f(z), g(z)$ with a critical point of the order α at the origin. The following statements describing a hyperbolic horseshoe attractor for the renormalization transformation R follow from the results in Yampolsky [2002].

• Fix $\alpha = 2k + 1, k \in \mathbb{N}$. Then for every irrational $\rho_+ \in \mathbb{S}^1$ there exists a smooth co-dimension 1 manifold $\Gamma_s(\rho_+)$ which consists of pairs $(f(z), g(z)) \in \mathfrak{F}_\alpha$ with a "forward" rotation number ρ_+. Also, for every irrational $\rho_- \in \mathbb{S}^1$ there exists a smooth one-dimensional manifold $\Gamma_u(\rho_-)$ which consists of pairs $(f(z), g(z)) \in \mathfrak{F}_\alpha$ with a "backward" rotation number ρ_-. For every pair $(\rho_+, \rho_-) \in \mathbb{S}^1 \times \mathbb{S}^1$ the manifolds $\Gamma_s(\rho_+)$ and $\Gamma_u(\rho_-)$ intersect transversally at a unique point $(f_{\rho_-,\rho_+}(z), g_{\rho_-,\rho_+}(z))$.

• A set $A \subset \mathfrak{F}_\alpha$ which consists of all intersection points $(f_{\rho_-,\rho_+}(z), g_{\rho_-,\rho_+}(z))$ for different pairs (ρ_-, ρ_+) is a compact subset of $\mathfrak{F}_\alpha$ invariant for the renormalization transformation R :

$$R(f_{\rho_-,\rho_+}(z), g_{\rho_-,\rho_+}(z)) = (f_{\hat{G}(\rho_-,\rho_+)}(z), g_{\hat{G}(\rho_-,\rho_+)}(z)).$$

• The set A is a Cantor-type hyperbolic attractor for the renormalization transformation R. At every point $(f_{\rho_-,\rho_+}, g_{\rho_-,\rho_+}) \in A$ the global stable and unstable manifolds are precisely $\Gamma_s(\rho_+)$ and $\Gamma_u(\rho_-)$.

• Let T be an analytic critical circle map with a critical point of the order α and with an irrational rotation number ρ. Denote $\rho_+^{(n)} = G^n \rho = [k_{n+1}, k_{n+2}, \dots]$, $\rho_-^{(n)} = [k_n, k_{n-1}, \dots, k_1, 1, 1, \dots]$. Then $\|(f_n, g_n) - (f_{\rho_-^{(n)}, {\rho^{(n)}}_+}, g_{{\rho^{(n)}}_-, {\rho^{(n)}}_+})\| \to 0$ exponentially fast as $n \to \infty$.

Note that filling in the tail of $\rho_-^{(n)}$ with 1s is not essential since the family $(f_{\rho_-,\rho_+}, g_{\rho_-,\rho_+})$ depends exponentially weakly on the tails of a sequence $\mathbf{k}$. One can say that all critical circle maps with the same irrational rotation number and the same order α belong to the same stable manifold for renormalization transformation R. The analyticity assumptions can and was substantially weakened. In Guarino and de Melo [2017] convergence of renormalizations has been proved in the C^4 setting. However the condition of odd integer orders of critical points is used in all existing results (apart from some perturbative results for α close to odd integers). At the same time there are no doubts that the renormalization conjecture must be true for all orders $\alpha > 1$. This is still an open problem of central importance.

We shall now discuss rigidity results for critical circle maps. We always consider a particular conjugacy which maps a critical point into another critical point. It turns out that critical maps are more rigid than diffeomorphisms. So-called *robust rigidity* result was proved in Khanin and Teplinsky [2007].

Theorem 3.4. *Let T_1 and T_2 be two analytical critical circle maps with the same order of critical points, and with the same irrational rotation number $\rho = \rho(T_1) = \rho(T_2)$. Then T_1 and T_2 are C^1-smoothly conjugate to each other.*

Note that C^1 rigidity holds for all irrational rotation numbers. Also note that the above result is sharp. A.Avila (Avila [2013]) proved that it cannot be extended even on a level of a modulus of continuity of the conjugacy between T_1 and T_2. In fact, the result in Khanin and Teplinsky [2007] is more general. It says that as long as convergence of renormalization is established, robust rigidity holds. In particular, in view of Guarino and de Melo [2017], it can be applied to C^4 critical circle maps with the same odd order of critical points.

To explain the mechanism of the robust rigidity we introduce a sequence of dynamical partitions which are closely related to renormalization, and discuss geometry of these partitions. Let x_0 be a marked point for renormalization. A partition $\xi_n(x_0)$ of the n-th level is a partition on $q_{n-1}+q_n$ intervals whose endpoints are given by a finite trajectory of a point $x_0 : \{x_i = T^i x_0, 0 \le i < q_{n-1}+q_n\}$. Denote $\Delta_0^{(n)}$ a closed interval with endpoints x_0 and x_{q_n}. It is easy to see that all the elements of the partition $\xi_n(x_0)$ belong to two sequences of closed intervals: $\Delta_i^{(n-1)} = T^i \Delta_0^{(n-1)}, 0 \le i < q_n$ and $\Delta_j^{(n)} = T^j \Delta_0^{(n)}, 0 \le j < q_{n-1}$. These two sequences do not intersect except at their endpoints, and cover the whole unit circle. It turn out that in the case of critical circle maps the geometry of partitions is uniformly bounded. Swiatek (Swiatek [1988]) showed that for any such T there exists a constant $C > 1$ such that for any two neighbouring intervals Δ_1, Δ_2 of a partition $\xi_n(x_0)$ the ratio of their lengths is bounded by C: $C^{-1} \le |\Delta_1|/|\Delta_2| \le C$. Moreover, a constant C is asymptotically universal. Namely, there exists a constant $C > 1$ which depends only on the order of a critical point such that the above estimate holds for all T if n is large enough. Note that in the diffeomorphism case the geometry is unbounded. Obviously $|\Delta_0^{(n)}|/|\Delta_0^{(n-1)}| \sim k_{n+1}^{-1}$ and can be arbitrary small for large k_{n+1}. Due to the bounded geometry, one can show that in the critical case when k_{n+1} is large renormalization $f_n(z)$ has a unique point of almost parabolic tangency with the diagonal. Iterations near such points of almost tangency are very regular. This fact together with convergence of renormalization is responsible for the robust rigidity.

For typical rotation numbers the conjugacy between two maps is $C^{1+\beta}$, $0 < \beta < 1$ smooth. It was first proved in de Faria and de Melo [2000] for rotation numbers of bounded type and later extended to the Lebesgue typical case. Although I don't know rigorous results in this direction, it is expected that in general the smoothness of conjugacy cannot be improved to C^2 or above even for rotation numbers of bounded type. This feature makes the critical case substantially different from the case of smooth diffeomorphism.

Concluding, one can prove the renormalization conjecture in full generality under a rather annoying condition on the order of critical points. The "fixed point family" given by the attractor $A = \{(f_{\rho_-,\rho_+}(z), g_{\rho_-,\rho_+}(z))\}$ is highly non-trivial. Smooth C^1 rigidity holds for all irrational rotation numbers.

Circle maps with breaks. Circle maps with breaks were introduced in Khanin and Vul [1991]. One possible motivation for their study can be explained if we adopt a point of view of generalized interval exchange transformations. It is well known that linear circle rotations can be viewed as interval exchange transformations of two intervals. Imagine now that the maps transforming two intervals are still smooth and monotone but are non-linear. The endpoints will be still matched. However a condition of matching of the derivatives of two branches of non-linear maps at the endpoints is rather unnatural. If the derivatives are not matched then we get two break points with break sizes c_1 and c_2. In fact, since both break points belong to a single trajectory, the renormalization for such maps is equivalent to renormalization for maps with one break point with a size of a break $c = c_1 c_2$.

The renormalization theory for maps with a single break point can be considered as a one-parameter extension of the Herman theory where parameter is a size of a break c. The value $c = 1$ corresponds to the diffeomorphisms case where, as we have seen above, renormalization converge to a trivial fixed-family consisting of linear maps with slope 1. In the break case renormalization converge to a space of Möbius transformations. The renormalization transformation has a non-trivial Cantor-type hyperbolic attractor similar to the critical case. Appearance of Möbius transformations is a conceptual fact. In the holomorphic dynamics this fact is related to the Köbe principle. Here we give a real-analytic explanation. Suppose we have a sequence of C^3-smooth one-dimensional maps $T_i, 0 \le i \le n-1$ acting on intervals $\Delta_i = [a_i, b_i], 0 \le i \le n$, $T_i : \Delta_i \to \Delta_{i+1}$. For each interval Δ_i define a relative coordinate $z_i(x) = (x - a_i)/(b_i - a_i)$. Assume that all intervals Δ_i are smaller than ϵ and $\sum_{i=0}^{n-1} |\Delta_i| = O(1)$. Assume also that all derivatives of T_i are uniformly bounded by a constant $C > 1$ and $T_i'(x) \ge C^{-1}$. Then the function $z_n(z_0)$ is order ϵ close to a fractional-linear function. The proof is based on a simple application of cross-ratio distortion estimates. If the maps T_i are only $C^{2+\alpha}$-smooth than closeness above will be of the order ϵ^α. In terms of renormalization this property implies that the renormalization $(f_n(z), g_n(z))$ are getting exponentially close to the space of pairs of fractional-linear functions as $n \to \infty$. It is also easy to see that asymptotically as $n \to \infty$ the limiting pairs $(f(z), g(z))$ must satisfy the commutativity relation $f \circ g(c^2 x) = g \circ f(x)$. Here there is a little twist. Since on every step of renormalization the orientation changes, c is also changing to c^{-1}, and back to c on the next step. Thus for odd n the commutativity relation is replaced by $f \circ g(c^{-2} x) = g \circ f(x)$. It is convenient to define $c_n = c^{(-1)^n}$. Taking into account the commutativity relations one can show that $(f_n(z), g_n(z))$ converges to invariant two-parameter family of pairs of fractional-linear functions which can be written explicitly. It is possible to write explicit formulas using geometrically define parameters a_n and b_n. However it is more convenient to replace b_n by another parameter $v_n = (c - a_n)/b_n - 1$ which characterizes the nonlinearity of the map $f_n(z)$. Let us define a family

(3) $$F_{a,v,c}(z) = \frac{a + cz}{1 - vz}, \; G_{a,v,c}(z) = \frac{a(z-c)}{ac + z(1 + v - c)}$$

Then $(f_n(z), g_n(z))$ gets exponentially close to $F_{a_n,v_n,c_n}(z) = G_{a_n,v_n,c_n}(z)$ as $n \to \infty$. The limiting family (3) is invariant for the renormalization transformation R. The action of R on the parameter c is trivial: $Rc = 1/c$. The hyperbolic properties of R acting on the two-dimensional plane of parameters (a, v) were studied in Khanin and Khmelev [2003] and Khanin and Teplinsky [2013]. An important role is played by the special time-reversible symmetry. Define an involution $I(a, v, c) = ((c - 1 - v)/av, -v/c, 1/c)$.

Then, $R^{-1} = I \circ R \circ I$. Since R is essentially a two-dimensional transformation, the time-reversible symmetry provides duality between stable and unstable directions. The results proved in Khanin and Teplinsky [2013] are very similar to the convergence of renormalization results for critical circle maps. Again, there exist smooth stable and unstable manifolds $\Gamma_s(\rho_+)$ and $\Gamma_u(\rho_-)$, in this case they both are one-dimensional, parametrized by forward and backward rotation numbers ρ_+ and ρ_- respectively, and a hyperbolic attractor A which consists of all points of intersection between $\Gamma_s(\rho_+)$ and $\Gamma_u(\rho_-)$. As in the critical case, for any pair (ρ_+, ρ_-) the manifolds $\Gamma_s(\rho_+)$ and $\Gamma_u(\rho_)$ intersect transversally in a single point. All Lyapunov exponents are uniformly bounded away from zero by a positive constant which depends only on c. The main difference with the critical case is that the whole picture is now two-dimensional. Another, more conceptual, proof of the above statement can be found in Khanin and Yampolsky [2015]. It is based on the following idea. There is always one unstable direction for the renormalization transformation. This direction is related to a change of a rotation number. The other direction must be stable by the time-reversible symmetry. Together these two statements imply the hyperbolicity.

Although renormalization converge exponentially to the two-dimensional invariant family, and renormalization dynamics restricted to this family is hyperbolic and well understood, it is highly non-trivial to combine this two facts into the statement of global convergence of renormalization. The main difficulty is a strongly unbounded geometry when k_{n+1} are getting large. Assume that $c > 1$. Then for even n renormalization $f_n(z)$ will be convex, and for odd n it will concave. Thus, when k_{n+1} is large and n is even the function $f_n(z)$ has an almost parabolic tangency with the diagonal, like in the case of critical circle maps. This is a good case of bounded geometry. However, when n is odd, function $f_n(z)$ come close to the diagonal at a point $z = 0$ corresponding to the break point. One can show that in this case $f'(-1)$ is close to c, and $f'(0)$ is close to $1/c$. As a result, geometry is strongly unbounded. Namely, $|\Delta_0^{(n)}|/|\Delta_0^{(n-1)}| \sim c^{-k_{n+1}/2}$ which decays extremely fast as $k_{n+1} \to \infty$. Recall, that in the diffeomorphisms case $|\Delta_0^{(n)}|/|\Delta_0^{(n-1)}| \sim k_{n+1}^{-1}$. In the case $c < 1$ the situation is dual: the geometry is strongly unbounded for even n, and bounded for odd n. Despite the above difficulty the following global convergence result was proved in Khanin and Kocić [2014]. Consider two circle maps T_1 and T_2 with a break of size c. Assume that they are $C^{2+\alpha}$ smooth outside of break points. Denote $f_n^{(1)}(z)$, $f_n^{(2)}(z)$ renormalization for T_1 and T_2 respectively.

Theorem 3.5. *For any c there exist a constant $\lambda(c) \in (0,1)$ such that for all T_1 and T_2 as above and all n large enough we have $\| f_n^{(1)}(z) - f_n^{(2)}(z)\|_{C^2[-1,0]} \le \lambda(c)^n$, provided $\rho(T_1) = \rho(T_2)$.*

Theorem 3.5 allows to prove the rigidity result for maps with breaks. Due to unbounded geometry robust rigidity does not hold in this case (Khanin and Kocić [2013]). One has to impose certain restrictions for a growth rate of k_{n+1} for odd n in case $c > 1$, and even n in the case $c < 1$. Assume that in both cases the corresponding sequence of k_{n+1} is bounded by λ_1^{-n} for large enough n of the right parity. Here $\lambda_1 < 1$ is an arbitrary constant greater than $\lambda(c)$. Then the following result holds (Khanin, Kocić, and Mazzeo [2017]).

Theorem 3.6. *Suppose $\rho = \rho(T_1) = \rho(T_2)$ satisfies the above condition. Then T_1 and T_2 are C^1-smoothly conjugate to each other.*

In this theorem we again assumed that $c(T_1) = c(T_2)$, and both T_1 and T_2 are $C^{2+\alpha}$ smooth outside of their break points. We have seen above that in the case of critical circle maps $C^{1+\beta}$ rigidity holds for Lebesgue almost all rotation numbers. S.Kocic showed that this is not the case for circle maps with breaks (Kocić [2016]). Of course $C^{1+\beta}$ rigidity still holds for rotation numbers of bounded type, or if k_{n+1} grow slowly enough.

4 Critical behaviour and parameter dependence

The renormalization for maps with singularities is very different from the diffeomorphism case. Using the statistical mechanics analogy we can say that it demonstrate the critical behaviour. Non-trivial scalings and fractal, or, more precisely, multifractal, structure of the renormalization attractor are just two of many manifestation of criticality.

Another feature is singularity of invariant measure. The singularity follows from Graczyk and Swiatek [1993] in the case of critical circle maps and from Dzhalilov and Khanin [1998] in the case of maps with a break point. Properties of invariant measure for maps with several breaks were studied in Dzhalilov, Liousse, and Mayer [2009].

It was known that if breaks belong to the same trajectory and the product of their sizes is equal to 1, then they can effectively compensate each other, and the invariant measure can be absolutely continuous. For a while it was conjectured that in all other cases the invariant measure is singular. Recently A. Teplinsky (Teplinsky [2018]) constructed interesting example of a piecewise linear circle maps with four breaks where the invariant measure is absolutely continuous, although the breaks do not compensate each other completely.

Another interesting property of critical behaviour is related to a parameter dependence. For any circle homeomorphism it is natural to include it in a one-parameter family which allows to change a rotation number. In can be done in different ways. Below we discuss the simplest construction when a one parameter family is given by $T_\omega(x) = T(x) + \omega \pmod 1$, $\omega \in [0, 1]$. An object of interest here is a function $\rho(\omega)$. Obviously, $\rho(\omega) = \omega$ in the linear case $T(x) = x$. In a nonlinear case $\rho(\omega)$ is a monotone, but not a strictly monotone, function. It has flat pieces $\omega \in I(p/q) = [a(p/q), b(p/q)]$ for all rational $0 \le \rho = p/q < 1$. These closed intervals $I(p/q)$ sometimes are called mode-locking intervals. On the contrary for any irrational $0 < \rho < 1$ there exists a unique value $\omega = \omega(\rho)$ corresponding to it. Denote by I_{ir} a Cantor-type of "irrational" parameter values, that is parameter values corresponding to irrational rotation numbers. It follows from the KAM theory that the set I_{ir} has a positive Lebesgue measure provided T is smooth enough. On the contrary, in the case of critical circle maps (Swiatek [1988]) and in the case of circle maps with a break point (Khanin and Vul [1991]) $\mathfrak{L}(I_{ir}) = 0$, where $\mathfrak{L}$ is the Lebesgue measure.

We next discuss how to define the notion of typical irrational rotation number in a one-parameter family. Since the parameter space is equipped with the Lebesgue measure it is natural to consider the conditional distribution on $\omega \in [0,1]$ under condition that the rotation number is irrational. It is easy to define such a conditional distribution in the diffeomorphism case when $\mathfrak{L}(I_{ir}) > 0$. In the other two cases it is much less straightforward. A natural approach here is to consider a decreasing sequence of sets $I_n, n \in \mathbb{N}$ such that for $\omega \in I_n$ the continued fraction expansion for the rotation number $\rho(\omega)$ is longer than n. Obviously $\cap_n I_n = I_{ir}$. Since $\mathfrak{L}(I_n) > 0$ one can define the conditional distribution μ_n under the condition $\omega \in I_n$. Then the conditional distribution under the condition $\omega \in I_{ir}$ can be defined as $\mu(T) = \lim_{n\to\infty} \mu_n$. Using the hyperbolicity of renormalization one can proof that such a limit exists (Dolgopyat, Fayad, Khanin, and Kocic [2018]). Next one can consider the measure $\nu(T)$ which is a pushforward of $\mu(T)$ by the map $\omega \mapsto \rho(\omega)$. This measure provides a natural probability distribution on the irrational rotation numbers in a one-parameter family T_ω. Although measure $\nu(T)$ depends on a map T its asymptotic properties are universal. One way to see it is to consider pushforward of $\nu(T)$ by iterates of the Gauss map G. It turns out that in the case of critical circle maps $G^m\nu(T) \to \nu_\alpha$ as $m \to \infty$, where the limiting measure ν_α is already universal and depends only on the order of a critical point α (Dolgopyat, Fayad, Khanin, and Kocic [ibid.]). In the symbolic representation corresponding to the continued fraction expansion the measure ν_α has a Gibbs structure with a nice Hölder continuous potential. Note that in the diffeomorphism case the corresponding measure ν is absolutely continuous with the density $\frac{1}{\ln 2}\frac{1}{1+\rho}$. The statistical properties of measure ν are very different from ν_α. One can show that a probability with respect to ν_α of ρ such that an entry k_n in a continued fraction expansion for ρ takes a large value $k_n = k$ decays as k^{-3} while in the diffeomorphism case it decays as k^{-2}.

Similar statements can be proved for circle maps with a break point. The only difference is that in this case the sequence $G^m\nu(T)$ converges to a periodic orbit of the period two: $G^{2n}\nu(T) \to \nu_c^e$, $G^{2n+1}\nu(T) \to \nu_c^o$ as $n \to \infty$. This periodic orbit $G\nu_c^e = \nu_c^o$, $G\nu_c^o = \nu_c^e$ is also universal and depends only on a break size c. Obviously, $\nu_c^e = \nu_{1/c}^o$, $\nu_c^o = \nu_{1/c}^e$. In the case $c > 1$ a probability of large values of $k_n = k$ decays again as k^{-3} for odd n. For even n the probability decays exponentially with k. In the case $c < 1$ the parity is opposite. It looks plausible that for two families $T_{1,\omega}$, $T_{2,\omega}$ for typical "irrational" parameter values

ω_1, ω_2 the conjugacy between T_{1,ω_1} and T_{2,ω_2} will be $C^{1+\beta}$ smooth provided $\rho(T_{1,\omega_1}) = \rho(T_{2,\omega_2})$.

5 Beyond dimension one

All the results which we discussed above were one-dimensional. The combinatorics of trajectories which is present in the one-dimensional setting either in terms of rotation numbers, or in terms of kneading sequences is a crucially important feature in the universality phenomenon. This is especially true in the case of critical behaviour.

The multidimensional results are either related to the linear KAM regime (Mackay [1982], Koch [1999], and Khanin, Lopes Dias, and Marklof [2007, 2006]), or to the cases where one-dimensional structure is effectively present. By this I mean situations where dynamics is essentially one-dimensional with dissipation in other directions. This approach goes back to the paper by Collet, Eckmann, and Koch [1981]. Important results related to the geometry of embeddings of quasi one-dimensional attractors into two-dimensional space were studied by De Carvalho, Lyubich, and Martens [2005] and by Gaidashev and Yampolsky [2016].

Very important renormalization problems were studied in connection with critical invariant curves for area-preserving twist maps of a cylinder. An interesting and rich critical behaviour was discovered there by R.MacKay who also developed a renormalization scheme for such maps (Mackay [1982]). Unfortunately most of the results in this direction are still not rigorous. Recently Koch, using computer-assisted methods, was able to construct a corresponding fixed point for critical invariant curves with the golden mean rotation number $\rho = \frac{\sqrt{5}-1}{2}$ (Koch [2008]). Although dynamics in this case is really two-dimensional without dissipation, a footprint of the 1D case is still present through the invariant curve.

In the KAM regime renormalization schemes are based on the multidimensional continued fraction algorithm (Khanin, Lopes Dias, and Marklof [2007, 2006]). One can consider either a problem of constructing smooth invariant tori, or study the linearization problem for smooth diffeomorphisms of the torus $\mathbb{T}^d$. In both cases, in addition to the coordinate rescaling, on every step of the renormalization procedure one also have to implement certain nonlinear coordinate changes. These coordinate changes are eliminating some non-essential unstable direction. There are not essential precisely because they are related to coordinate changes. In fact, in the KAM regime all the eigen-directions, even stable ones, can be removed by means of proper coordinate changes. This fact can be seen as a renormalization explanation of a fast convergence of the linearization approximations in the KAM setting.

It is well known that KAM type of results on linearization in the multidimensional case require closeness of the corresponding maps to the linear ones. At the same time linearization results in the one-dimensional case are global. One of the main difficulties is related to Denjoy theory. In the 1D case we know when a diffeomorphism is topologically conjugated to a linear one, while in the multidimensional situation it is not the case. In certain sense this is the only obstacle to global results. The following conjecture by R.Krikorian states that the global rigidity results hold in any dimension.

Conjecture (R.Krikorian). Let T be a C^∞ diffeomorphism of $\mathbb{T}^d$. Assume that T is topologically conjugate to a linear translation $T_{\vec{\omega}} : \vec{x} \mapsto \vec{x} + \vec{\omega} \pmod 1)\, \vec{x} \in \mathbb{T}^d$ with a Diophantine rotation vector $\vec{\omega} = (\omega_1, \dots, \omega_d)$. Then the conjugacy between T and $T_{\vec{\omega}}$ is C^∞ smooth.

It is obvious that in the presence of periodic orbits smooth linearization is virtually impossible. In the above conjecture there are no periodic orbits for T since it is conjugate to an irrational translation. Diophantine condition guarantees that there are also no orbits which are too close to periodic ones. The conjecture basically says that there are no other obstructions to smooth linearization.

It looks natural to propose the following generalization of the above conjecture. Let M be a compact Riemannian manifold. Let T_1 and T_2 be C^∞ diffeomorphisms of M. Assume also that T_1 and T_2 satisfy the Diophantine property. Namely, there exists $\tau, C > 0$ such that for $x \in M$ and all $n \in N$ we have: $dist(T_1^n x, x) \geq Cn^{-\tau}$, $dist(T_2^n x, x) \geq Cn^{-\tau}$.

Rigidity Conjecture. Suppose T_1 and T_2 are topologically conjugate. Then the conjugacy is C^∞ smooth.

Probably it is enough to require the Diophantine condition for only one of the maps T_1 and T_2.

6 Concluding remarks

We have seen above how the renormalization ideology can be implemented in several dynamical problems. Although the conceptual picture is very simple and appealing the proofs are quite difficult. A variety of different techniques is required in various settings. At the same time the phenomenon seems to be extremely general. In all the cases studied we see the hyperbolicity of renormalization, and convergence of renormalization for maps which are equivalent topologically and have the same structure of critical points.

The renormalization behaviour is extremely rigid. The geometrical properties of trajectories are completely determined by the backward and the forward rotation numbers and parameters (like α and c) characterizing the local behaviour near singular points.

Above we formulated a number of conjectures and discussed several open problem. As we have emphasized repeatedly, the central open problem in renormalization theory is the problem of convergence of renormalization for critical maps with an arbitrary order of critical points. Below we formulate few more open problems.

Interesting problems on convergence of renormalization arise in the multimodal setting when the map has several singular points. Consider for example the case of a map T_1 which has several break points u_i with break sizes c_i, $1 \le i \le k$. We shall present below an heuristic argument behind the hyperbolicity of renormalization. According to the Renormalization Conjecture, in order to have convergence of renormalization the second map T_2 should have the same rotation number and a matching structure of its break points. Denote v_i, $1 \le i \le k$ the break points for T_2. Matching means that their break sizes are also c_i, $1 \le i \le k$, and $\mu_1([u_i, u_{i+1}] = \mu_2([v_i, v_{i+1}]$, $1 \le i < k$, where μ_1, μ_2 are probability invariant measures for T_1 and T_2 respectively. Changing the values of ρ and $m_i = \mu_1([u_i, u_{i+1}]$, $1 \le i < k$ correspond to a k-dimensional unstable manifold. The task is to show that all other directions are stable. If there are no other unstable directions then by fixing the values of ρ and m_i, $1 \le i < k$ we put T_1 and T_2 on the same stable manifold for renormalization transformation. In the case of k break points one has $(k+1)$ smooth branches of renormalization. Like in the case of one break point, all branches after rescaling will converge to the space of fractional-linear maps. Using k commutativity conditions one can reduce the number of parameters to $2k$. Renormalization transformation corresponds to some inducing scheme. It is possible to use the Rauzy induction. Indeed, at every step of renormalization the map can be viewed as a nonlinear interval exchange transformation of $(k+1)$ intervals. However one can develop a different inducing scheme which is more suitable for our purposes (Khanin and Teplinsky [2018]). This scheme corresponds to interchanges between k disjoint intervals, moreover, each of them is subdivided onto two subintervals. More precisely, each break point is surrounded by two intervals, one to the right and one to the left of it. These intervals are mapped by proper iterates of T. The union of their images covers the initial collection of intervals. On the next step of renormalization intervals get smaller and number of iterates get larger. An inductive scheme of the above type can be constructed in such a way that the renormalization transformation R will again have an explicit time-reversible symmetry provided by an involution I (Khanin and Teplinsky [ibid.]). We have seen above that such an involution exists in the case of one break. Of course the involution I is more complicated in the multiple breaks case. It acts on the subset of $\mathbb{R}^{2k}$ corresponding to parameters of fractional-linear functions. It also acts on a finite set of combinatorial types. In the case $k = 1$ it was just two combinatorial types related to a change of orientation.

Interchanges of k subdivided intervals have a more complicated combinatorial structure. Because of the time-reversible symmetry the number of stable and unstable directions are the same. This means that as long as the existence of k unstable directions is established, all other directions must be stable.

A simpler case of maps with k breaks under additional condition $\prod_{i=1}^{k} c_i = 1$ was considered in Cunha and Smania [2014, 2013]. In this case the renormalization converge to the space of linear maps with different slopes. Cunha and Smania used the Rauzy induction and proved convergence of renormalization and smooth rigidity. One can say that inductive schemes of the Rauzy type have three different representations: the usual one in the space of linear maps with slop 1, another one in the space of linear maps with different slopes studied in Cunha and Smania [2014, 2013], and the third one in the space of fractional-linear functions. Moreover, all three representations are hyperbolic.

Another interesting open problem is related to critical circle maps with asymmetric critical points. Namely, the orders α_1 to the right of the critical point and α_2 to the left of it are different. Since the maps in this case are not quasisymmetric the results of Yoccoz (Yoccoz [1984b]) cannot be applied. We even do not know whether such maps are topologically conjugate to linear rotations, although this should be expected. We believe that the renomalization for such maps will behave in the following way. The geometry of the dynamical partition ξ_n will be strongly unbounded. The intervals $\Delta_0^{(n)}$ and $\Delta_0^{(n-1)}$ will be exponentially small with different exponents. However, their images $\Delta_1^{(n)} = T(\Delta_0^{(n)})$ and $\Delta_1^{(n-1)} = T(\Delta_0^{(n-1)})$ will be already of the same order. In other words, a meaningful renormalization theory can be developed near the critical value, rather than near the critical point. One can expect that renormalization will still converge within the universality class.

This paper provides a relatively brief introduction into the theory of dynamical renormalization. It was certainly impossible to comment on all important contributions made in this area in the last 40 years. Many important papers were not discussed above. The reason for their omission is a lack of space rather than lack of respect.

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