Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 2 (1105–1132)

CONSTRUCTING GROUP ACTIONS ON QUASI-TREES

Koji Fujiwara (藤原　耕二)

Abstract

A quasi-tree is a geodesic metric space quasi-isometric to a tree. We give a general construction of many actions of groups on quasi-trees. The groups we can handle include non-elementary hyperbolic groups, CAT(0) groups with rank 1 elements, mapping class groups and the outer automorphism groups of free groups. As an application, we show that mapping class groups act on finite products of Gromov-hyperbolic spaces so that orbit maps are quasi-isometric embeddings. It implies that mapping class groups have finite asymptotic dimension.

1 Introduction

1.1 Overview. Group actions are useful in the study of infinite (discrete) groups. One example is the theory of groups acting on simplicial trees by automorphisms, called *Bass-Serre theory*, Serre [1980]. Serre observed that $SL(2,\mathbb{Z})$ properly and co-compactly acts on an infinite simplicial tree, which is embedded in the upper half plane. On the other hand he proved that if $SL(3,\mathbb{Z})$ acts on any simplicial tree by automorphisms then there is a fixed point. Using the theory, he obtained a geometric proof of the theorem by Ihara saying that every torsion-free discrete subgroup of $SL(2,\mathbb{Q}_p)$ is free.

A central idea in Geometric group theory is to use *hyperbolicity* (in the sense of Gromov) of a space to prove algebraic properties of a group that acts on it. A tree is a most elementary example of a hyperbolic space. This method created the theory of *hyperbolic groups*, Gromov [1987]. Another example is an approach by Masur-Minsky to the mapping class group of a surface using the hyperbolicity of the curve complex of the surface. We will rely on their theory for our application.

This note is a survey of the work by Bestvina, Bromberg, and Fujiwara [2015], which uses *quasi-trees* to study groups. A quasi-tree is a geodesic metric space that is *quasi-isometric* to (ie, "looks like", see the precise definition later) a simplicial tree. A quasi-tree

I would like to thank my parents. This work is supported by KAKENHI (23244005, 15H05739).
MSC2010: 20F65.
Keywords: quasi-trees, projection complex, mapping class groups, hyperbolic groups, asymptotic dimension.

is always hyperbolic in the sense of Gromov. There are many advantages in this approach. One is that quasi-trees are more flexible than trees, so that there are in fact more groups that act on quasi-trees than on trees. Since quasi-trees are hyperbolic, many techniques and results that are obtained for groups acting on hyperbolic spaces apply, but moreover, we sometimes obtain stronger conclusions since quasi-trees are special and easier to handle.

Also, we introduce new methods to produce quasi-trees, called *projection complex*, equipped with an isometric group action by a given group. For that we only need to check a small set of axioms, which are satisfied by many examples (see Examples 4.3), including *hyperbolic groups*, mapping class groups and the outer automorphism group $Out(F_n)$ of a free group F_n of rank n, and are able to produce many actions that are sometimes hidden at a first glance. We also construct closely related space called *quasi-tree of metric spaces*. Using those constructions we prove new theorems and also recover some known ones.

1.2 Intuitive description of the main construction. To explain the idea by an example, consider a discrete group Γ of isometries of hyperbolic n-space $\mathbb{H}^n$ and let $\gamma \in \Gamma$ be an element with an axis (ie, a γ-invariant geodesic that γ acts on by a translation) $\ell \subset \mathbb{H}^n$. Denote by $\mathbf{Y}$ the set of all Γ-translates of ℓ, i.e. the set of axes of conjugates of γ. Now we will construct a quasi-tree Q with a G-action from the disjoint union of the translates of ℓ by joining pairs of translates by edges following a certain rule. We want: the resulting space Q is connected; Q looks like a tree, so Q should not contain larger and larger embedded circles (then Q is not a quasi-tree); so that we put edges as few as possible just enough to make Q to be connected; the connecting rule is G-equivariant to have a G action on Q. Intuitively, the rule we will use is simple. For elements (ie, lines) $A, B \in \mathbf{Y}$ we do not want to join them when there is another element C that is "between A and B" (then we would rather join A and C, and also C and B). So, we join A and B by an edge only when there is no element C between A and B.

But how do we define that "C is between A and B"? When $A, C \in \mathbf{Y}$, $A \neq C$, denote by $\pi_C(A) \subset C$ the image of A under the nearest point projection $\pi_C : \mathbb{H}^n \to C$. We call this set the *projection* of A to C and we observe:

(P0) The diameter $\operatorname{diam} \pi_C(A)$ is uniformly bounded by some constant $\theta \geq 0$, independently of $A, C \in \mathbf{Y}$.

This is a consequence of discreteness of G, because a line in $\mathbb{H}^n$ will have a big projection to another line only if the two lines have long segments with small Hausdorff distance between them, so that there is a uniform upper bound unless they coincide, since G is discrete. When $B \neq A \neq C$ we define a "distance" by

$$d_C^{\pi}(A, B) = \operatorname{diam}(\pi_C(A) \cup \pi_C(B)).$$

Now fix a constant $K >> \theta$. We say C is between A and B if $d^\pi_C(A,B) \geq K$. (In fact we slightly perturb the distance functions d^π_C at once in advance.) Now the rule is that if there is no such $C \in \mathbf{Y}$ then we join A and B by an edge connecting $\pi_A(B)$ and $\pi_B(A)$, which are small (imagine θ is small) sets by (P0). The resulting space, $\mathcal{C}(\mathbf{Y})$, turns out to be a quasi-tree.

One may think this example is special since each A is a line. Of course, $\mathcal{C}(\mathbf{Y})$ would not be a quasi-tree if $A \in \mathbf{Y}$ were not a quasi-tree since A is embedded in $\mathcal{C}(\mathbf{Y})$. By collapsing each A to a point in $\mathcal{C}(\mathbf{Y})$ we obtain a new space $\mathcal{P}(\mathbf{Y})$ (then the geometry of A becomes irrelevant), which is again a quasi-tree. Our discovery is that if we start with a given abstract data: a set $\mathbf{Y}$ and maps(projections) between any two elements in $\mathbf{Y}$ satisfying a small set of "Axioms", and construct a space $\mathcal{P}(\mathbf{Y})$ by the rule we explained, $\mathcal{P}(\mathbf{Y})$ is always a quasi-tree. It does not matter how we obtain the data as long as it satisfies the axioms.

The technical difficulty in the construction is to perturb the distance functions d^π_A by a bounded amount as we said (see Section 2.1 for details). Without this perturbation the resulting space may contain larger and larger loops and is not a quasi-tree even if the initial data satisfies the axioms (see Bestvina, Bromberg, and Fujiwara [2015] for a counter example).

1.3 Axioms for quasi-tree of metric spaces and projection complex. We continue with the example, and explain the Axioms behind the construction. First, the distance function d^π_Y is always symmetric and satisfies the triangle inequality (nothing to do with discreteness of G):

(P3) $d^\pi_Y(X,Z) = d^\pi_Y(Z,X)$;

(P4) $d^\pi_Y(X,Z) + d^\pi_Y(Z,W) \geq d^\pi_Y(X,W)$,

but in general we have $d^\pi_A(B,B) > 0$, so this is a pseudo-distance function (we frequently drop "pseudo") . We observe further, again since Γ is discrete, for a perhaps larger constant θ:

(P1) For any triple $A, B, C \in \mathbf{Y}$ of distinct elements, at most one of the following three numbers is greater than θ:

$$d^\pi_A(B,C), d^\pi_B(A,C), d^\pi_C(A,B).$$

(P2) For any $A, B \in \mathbf{Y}$ the following set is finite:

$$\{C \in \mathbf{Y} \mid d^\pi_C(A,B) > \theta\}.$$

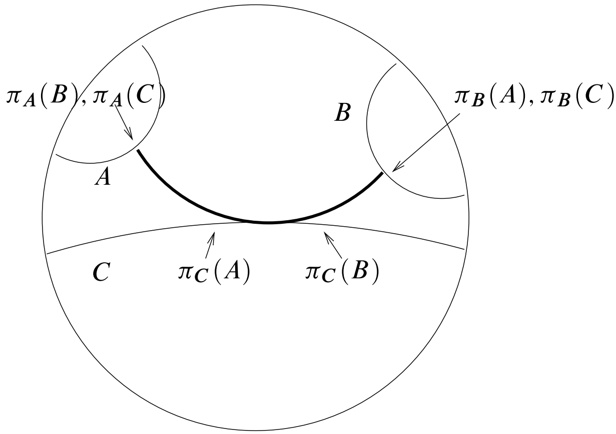

Figure 1: Axiom (P1). The bold line is the shortest segment between A and B in $\mathbb{H}^n$. Note that C and this segment stay close for a long time, therefore $d^\pi_C(A, B)$ is large, while $d^\pi_A(B, C)$ and $d^\pi_B(A, C)$ are small. (Figure from Bestvina, Bromberg, and Fujiwara [2015])

For an even more basic example where (P0)-(P2) hold with $\theta = 0$, consider the Cayley tree (Cayley graph) of the free group $F_2 = \langle a, b\rangle$ and for $\mathbf{Y}$ take the F_2-orbit of the axis of a.

We can show that (P0), (P1) and (P2) are sufficient *Axioms* for the space we construct to be a quasi-tree. We start with a collection of metric spaces $\mathbf{Y}$ and a collection of subsets $\pi_A(B) \subset A$ for $A \neq B$ with d^π_A satisfying (P0)-(P2). Note that (P3) and (P4) always hold for d^π_A. Then we construct a space by putting edges between (disjoint union of) the metric spaces in $\mathbf{Y}$ following our rule. Here is a summary theorem. (1) applies to the $\mathbb{H}^n$ example. In a way we "reconstruct" the ambient space with the group action from $\mathbf{Y}$. But the punch line is that we do not have to start with an ambient space and its subspaces in the theorem.

Theorem 1.1. *Bestvina, Bromberg, and Fujiwara [2015] Suppose $\mathbf{Y}$ is a collection of geodesic metric spaces and for every $A, B \in \mathbf{Y}$ with $A \neq B$ we are given a subset $\pi_A(B) \subset A$ such that (P0)-(P2) hold for the distance functions d^π_A for a constant θ.*

Then there is a geodesic metric space $\mathcal{C}(\mathbf{Y})$ that contains isometrically embedded, totally geodesic, pairwise disjoint copies of each $A \in \mathbf{Y}$ such that for all $A \neq B$ the nearest point projection of B to A in $\mathcal{C}(\mathbf{Y})$ is a uniformly bounded set uniformly close to $\pi_A(B)$ such that

(1) *The construction is equivariant, namely, if a group G acts isometrically on the disjoint union of the spaces in $\mathbf{Y}$ preserving projections, i.e., $g(\pi_A(B)) = \pi_{gA}(gB)$ for any $A, B \in \mathbf{Y}$ and $g \in G$, then the group action extends to $\mathcal{C}(\mathbf{Y})$.*

(2) *The quotient $\mathcal{C}(\mathbf{Y})/\mathbf{Y}$ obtained by collapsing the embedded copies of each $Y \in \mathbf{Y}$ to a point is a quasi-tree.*

Some explanations are in order. A subset $\mathcal{Y} \subset \mathcal{X}$ is *totally geodesic* if any geodesic in $\mathcal{X}$ joining two points in $\mathcal{Y}$ is contained in $\mathcal{Y}$. The space $\mathcal{C}(\mathbf{Y})$ is called a *quasi-tree of metric spaces*. Its construction will depend on the choice of a sufficiently large parameter $K >> \theta$, and it would be more precise to denote the space by $\mathcal{C}_K(\mathbf{Y})$. If $K < K'$ there is a natural Lipschitz map $\mathcal{C}_K(\mathbf{Y}) \to \mathcal{C}_{K'}(\mathbf{Y})$ which is in general not a quasi-isometry, and in fact unbounded sets may map to bounded sets.

The space $\mathcal{C}(\mathbf{Y})/\mathbf{Y}$ is called the *projection complex*, denoted by $\mathcal{P}(\mathbf{Y}) = \mathcal{P}_K(\mathbf{Y})$, which depends on K. We can think of the quasi-tree of metric spaces $\mathcal{C}(\mathbf{Y})$ as being obtained from $\mathcal{P}(\mathbf{Y})$, which is a quasi-tree by (2), by blowing up vertices to corresponding metric spaces. This explains the terminology.

Many properties that hold uniformly for the spaces in $\mathbf{Y}$ carry over to $\mathcal{C}(\mathbf{Y})$. For example (there will be the item (iii) later):

Theorem 1.2. *Bestvina, Bromberg, and Fujiwara [ibid.]. Let $\mathcal{C}(\mathbf{Y})$ be the quasi-tree of metric spaces $\mathbf{Y}$ constructed in Theorem 1.1.*

(i) *If each $X \in \mathbf{Y}$ is isometric to $\mathbb{R}$ then $\mathcal{C}(\mathbf{Y})$ is a quasi-tree; more generally, if all $X \in \mathbf{Y}$ are quasi-trees with a uniform bottleneck constant then $\mathcal{C}(\mathbf{Y})$ is a quasi-tree.*

(ii) *If each $X \in \mathbf{Y}$ is δ-hyperbolic with the same δ, then $\mathcal{C}(\mathbf{Y})$ is hyperbolic.*

Here, a geodesic metric space X satisfies the *bottleneck property* if there exists $\Delta \geq 0$ such that for any two points $x, y \in X$ the midpoint z of a geodesic between x and y satisfies the property such that any path from x to y intersects the Δ-ball centered at z. The constant Δ is called the *bottleneck constant*. Manning [2005] showed that X satisfying the bottleneck property is equivalent to X being a quasi-tree. Note that (i) in particular says that the space $\mathcal{C}(\mathbf{Y})$ obtained from an orbit of axes in $\mathbb{H}^n$ as in the example is a quasi-tree and not (quasi-isometric to) $\mathbb{H}^n$.

1.4 Asymptotic dimension. We give another example of a property that descends from spaces in $\mathbf{Y}$ to $\mathcal{C}(\mathbf{Y})$. The notion of *asymptotic dimension* was introduced by Gromov Gromov [1993] as a large-scale analog of the covering dimension.

Definition 1.3 (Asymptotic dimension). A metric space X has *asymptotic dimension* $\operatorname{asdim}(X) \leq n$ if for every $R > 0$ there is a covering of X by uniformly bounded sets such that every metric R-ball intersects at most $n+1$ of the sets in the cover. More generally, a collection of metric spaces has asdim at most n *uniformly* if for every R there are covers of each space as above whose elements are uniformly bounded over the whole collection.

In **Theorem 1.2** we also have:

(iii) *If the collection* $\mathbf{Y}$ *has* asdim $\le n$ *uniformly, then* $\operatorname{asdim}(\mathcal{C}(\mathbf{Y})) \le n+1$.

As an application we have:

Theorem 1.4. *Bestvina, Bromberg, and Fujiwara [2015]. Let Σ be a closed orientable surface, possibly with punctures, and $MCG(\Sigma)$ its mapping class group. Then* $\operatorname{asdim}(MCG(\Sigma)) < \infty$.

The *Coarse Baum-Connes conjecture* (for torsion free subgroups of finite index) and therefore the *Novikov conjecture* for $MCG(\Sigma)$ follows from Theorem 1.4 by a work of Yu [1998], cf. Roe [2003]. Various other statements that imply the Novikov conjecture for $MCG(\Sigma)$ were known earlier (see Kida [2008], Hamenstädt [2009], and J. A. Behrstock and Minsky [2008]).

1.5 Some basic notions. We collect some standard definitions we use.

Let X, Y be two metric spaces. We often denote the distance between x, y by $|x-y|$. A map $f : X \to Y$ is a (K, L)-*quasi-isometric embedding* if for all points $x, y \in X$,

$$\frac{|x-y|}{K} - L \le |f(x) - f(y)| \le K|x-y| + L.$$

If it additionally satisfies that for all point $y \in Y$ there exists $x \in X$ such that $|y - f(x)| \le L$, then we say f is a *quasi-isometry*, and X and Y are *quasi-isometric*. In those definitions, only the existence of constants K, L is important, and we sometimes omit them. Quasi-isometry is an equivalence relation among metric spaces.

Suppose a group G acts on a metric space X by isometries. We say the action is *co-compact/co-bounded* if the quotient is compact/bounded. We say the action is *proper* (or, *properly discontinuous*) if for any $R > 0$ and $x \in X$ the number of elements $g \in G$ with $|x - gx| \le R$ is finitely many.

Let G be a finitely generated group and S a finite set of generators. We assume that if $s \in S$ then $s^{-1} \in S$. We form a graph as follows: there is a vertex for each element $g \in G$. We join two vertices $g, h \in G$ if there is $s \in S$ with $h = gs$. The graph is called a *Cayley graph* of G, denoted by $Cay(G, S)$. Since S generates G, the graph $Cay(G, S)$ is connected. G acts on the graph by automorphisms from the left: An element $g \in G$ sends a vertex $h \in G$ to a vertex $gh \in G$. By declaring each edge has length 1, $Cay(G, S)$ becomes a geodesic space. The action by G is proper and co-compact. The distance between the identity and $g \in G$ is denoted by $|g|$ and called the *word norm* of g.

Let $\Delta(a,b,c)$ be a geodesic triangle in the hyperbolic plane $\mathbb{H}^2$, where the three sides a,b,c are geodesics. Gauss-Bonnet theorem says that the area of Δ is at most π. It then follows that each side is contained in the 2-neighborhood of the union of the other two sides. Gromov turned this uniform thinness of geodesic triangles into a definition. A geodesic metric space X is *δ-hyperbolic* for a constant δ if all geodesic triangle in X is *δ-thin*, namely, each side is contained in the δ-neighborhood of the union of the other two. We often suppress δ and say X is *hyperbolic*. The hyperbolic spaces $\mathbb{H}^n$ are hyperbolic, trees are hyperbolic, but the Euclidean plane is not hyperbolic. A finitely generated group is a *(word) hyperbolic group* if it acts on a hyperbolic space properly and co-compactly by isometries, equivalently, if its Cayley graph is hyperbolic. If G contains $\mathbb{Z}^2$ then it is not hyperbolic. Another important class of spaces is of *CAT(0) spaces* (or *Hadamard spaces*), which are, roughly speaking, complete, simply connected, and "non-positively curved" geodesic spaces. See for example Ballmann [1995]. This class is a good source of examples.

The *translation length* $\tau(g)$ of an isometry $g : X \to X$ of a metric space X is

$$\tau(g) := \lim_{k\to\infty} \frac{d_X(x, g^k(x))}{k}.$$

The limit exists and is independent of $x \in X$. We say the isometry is *hyperbolic* if $\tau(g) > 0$.

We organize the rest of this note as follows: in Section 2 we define projection complex and quasi-tree of metric spaces from the beginning, which is independent from Section 1 (so that there is an overlap). In Section 3 we discuss application to mapping class groups. In Section 4 we give many examples that satisfy the axioms, and also discuss other applications.

Acknowledgments. I would like to thank Mladen Bestvina and Kenneth Bromberg. Most of the work presented here is from a long collaboration with them. I am grateful to Bestvina for reading a draft and giving useful suggestions.

2 Definition of projection complex and quasi-tree of metric spaces

We start over and will give a precise setting and conditions for our construction, Bestvina, Bromberg, and Fujiwara [2015]. To define the projection complex, we do not really need the projections $\pi_A(B)$ as in the example on $\mathbb{H}^n$; we only need the pseudo-distances $d^\pi_C(A,B)$.

2.1 Projection complex axioms. Let $\mathbf{Y}$ be a set, and assume that for each $Y \in \mathbf{Y}$ we have a function

$$d^{\pi}_{Y} : (\mathbf{Y} \setminus \{Y\}) \times (\mathbf{Y} \setminus \{Y\}) \longrightarrow [0, \infty).$$

Let $\theta \geq 0$ be a constant. Assume the following (PC1) - (PC4) are satisfied (they are same as (P1)-(P4) except for the order). We call them *projection complex axioms*.

(PC 1) $d^{\pi}_{Y}(X, Z) = d^{\pi}_{Y}(Z, X)$ for all distinct X, Y, Z;

(PC 2) $d^{\pi}_{Y}(X, Z) + d^{\pi}_{Y}(Z, W) \geq d^{\pi}_{Y}(X, W)$ for all distinct X, Y, Z, W (triangle inequality);

(PC 3) $\min\{d^{\pi}_{Y}(X, Z), d^{\pi}_{Z}(X, Y)\} \leq \theta$ for all distinct X, Y, Z;

(PC 4) for all $X, Z \in \mathbf{Y}$, $\#\{Y | d^{\pi}_{Y}(X, Z) > \theta\}$ is finite.

As an analog of (P0), uniform boundedness of the projections $\pi_Y(Z)$, we could require (PC0), but this will not be used to define a projection complex.

(PC 0) $d^{\pi}_{Y}(Z, Z) \leq \theta$ for all distinct Y, Z.

Before we define the projection complex, there is one technical difficulty we have to deal with. Given distance functions that satisfy (PC1) - (PC4), we modify them by a bounded amount for our purpose. This modification is a key to define an order on a set $\mathbf{Y}_K(X, Z)$ we define later.

For $X, Z \in \mathbf{Y}$ with $X \neq Z$ let $\mathcal{H}(X, Z)$ be the set of pairs $(X', Z') \in \mathbf{Y} \times \mathbf{Y}$ with $X' \neq Z'$ such that one of the following four holds:

- both $d^{\pi}_{X}(X', Z'), d^{\pi}_{Z}(X', Z') > 2\theta$;
- $X = X'$ and $d^{\pi}_{Z}(X, Z') > 2\theta$;
- $Z = Z'$ and $d^{\pi}_{X}(X', Z) > 2\theta$;
- $(X', Z') = (X, Z)$.

We then define the modified distance functions

$$d_Y : (\mathbf{Y} \setminus \{Y\}) \times (\mathbf{Y} \setminus \{Y\}) \to [0, \infty)$$

by $d_Y(X, Z) = 0$ if Y is contained in a pair in $\mathcal{H}(X, Z)$, and otherwise,

$$d_Y(X, Z) = \inf_{(X', Z') \in \mathcal{H}(X, Z)} d^{\pi}_{Y}(X', Z').$$

For example, if $d^\pi_Y(W,Z) > 2\theta$, then $(W,Z) \in \mathcal{H}(Y,Z)$ and $d_W(Y,Z) = 0$. Note that it is clear from the definition that $d_Y(X,Z) \le d^\pi_Y(X,Z)$ and therefore (PC3) and (PC4) still hold for d_Y with the same constant. (PC1) is trivial, but we have to modify (PC2), the triangle inequality.

One can prove that the modification is bounded, namely, for distinct X, Y, Z,

$$d^\pi_Y(X,Z) - 2\theta \le d_Y(X,Z) \le d^\pi_Y(X,Z).$$

It then follows from (PC2) that

(PC 2') $d_Y(X,Z) + d_Y(Z,W) + 4\theta \ge d_Y(X,W)$ (modified triangle inequality)

For a constant $K \ge \theta$ and distinct Y, Z we define a set, which is finite by (PC 4), as follows:

$$\mathbf{Y}_K(X,Z) = \{Y \in \mathbf{Y} | d_Y(X,Z) > K\}.$$

We say an element Y in $\mathbf{Y}_K(X,Z)$ is *between* X, Z. We are ready to define the projection complex.

Definition 2.1 (Projection complex). For a constant $K > 0$, the *projection complex* $\mathcal{P}_K(\mathbf{Y})$ is the following graph. The vertex set of $\mathcal{P}_K(\mathbf{Y})$ is $\mathbf{Y}$. Two distinct vertices X and Z are connected with an edge (of length 1) if $\mathbf{Y}_K(X,Z)$ is empty. Denote the distance function for this graph by $d(\ ,\ )$.

Note that for different values of K the spaces $\mathcal{P}_K(\mathbf{Y})$ are not necessarily quasi-isometric to each other (the vertex sets are the same, but for larger K there are more edges).

Here is the first main theorem.

Theorem 2.2. *Bestvina, Bromberg, and Fujiwara [2015, Theorem 3.16] Suppose functions $d^\pi_Y, Y \in \mathbf{Y}$ satisfy (PC1)- (PC4). Modify them to d_Y. If K is sufficiently large compared to θ, then the projection complex $\mathcal{P}_K(\mathbf{Y})$ is a quasi-tree.*

We make some comments on the proof. Suppose K is large. We first show that for X, Z, the subset $\mathbf{Y}_K(X,Z) \subset \mathbf{Y}$ gives a path between them in $\mathcal{P}_K(\mathbf{Y})$. In particular it implies

$$d(X,Z) \le |\mathbf{Y}_K(X,Z)| + 1.$$

In fact this path is a *quasi-geodesic* (ie, a quasi-isometric embedding of a segment) between the two points. This is the part where the modification to d_Y plays a role (without a modification the theorem is not true). Using the distances d_Y, we put a total order on $\mathbf{Y}_K(X,Z) \cup \{X,Z\}$ with X least and Z greatest: $Y < W$ if $d_W(X,Y) < \theta$. It takes some work to show this is well defined and gives a total order on the set $\mathbf{Y}_K(X,Z) \cup \{X,Z\}$. We then show this set form a path in $\mathcal{P}_K(\mathbf{Y})$ in this order. Recall the $\mathbb{H}^n$ example from the introduction. For axes X, Z, let σ be the shortest geodesic between them in $\mathbb{H}^n$. Then the set $\mathbf{Y}_K(X, Z)$ is roughly the collections of axes in $\mathbf{Y}$ that stay close to σ at least for distance K (cf. Figure 1). Because of (P 0), there is a bounded overlap between any two of them, so that there is an obvious order on $\mathbf{Y}_K(X, Z)$ in this picture.

There is also a lower bound of $d(X, Z)$. For a sufficiently large K' compared to K (roughly $5K$ is enough), if $Y \in \mathbf{Y}_{K'}(X, Z)$ then every geodesic from X to Z in $\mathcal{P}_K(\mathbf{Y})$ contains Y. This implies

$$d(X, Z) \geq |\mathbf{Y}_{K'}(X, Z)| + 1.$$

One can say that if X and Z has a "large projection" (ie, $\geq K'$) to Y, then every geodesic from X to Z has to pass Y.

2.2 Quasi-tree of metric spaces. To define a quasi-tree of metric spaces we need that each $Y \in \mathbf{Y}$ is a metric space. Here is the precise setup. Let $\mathbf{Y} = \{(Y, \rho_Y)\}$ be a collection of metric spaces and for each distinct $Y, Z \in \mathbf{Y}$ assume that we have sets $\pi_Y(Z) \subseteq Y$ and $\pi_Z(Y) \subseteq Z$. The π_Y are called *projection maps*. Fix a constant $\theta > 0$. Assume that for any $X \neq Y$,

(P 0) $\mathrm{diam}(\pi_X(Y)) \leq \theta$.

For any $X \neq Y \neq Z$, set

$$d_Y^\pi(X, Z) = \mathrm{diam}(\pi_Y(X) \cup \pi_Y(Z)),$$

where diam is ρ_Y-diameter. Then d_Y^π satisfy (PC1) and (PC2) trivially. Notice that (P0) implies (PC0). We assume that they also satisfy (PC3) and (PC4). Families of metric spaces with projection maps satisfying (PC0), (PC3) and (PC4) occur naturally in many contexts, see Examples 4.3.

Definition 2.3 (Quasi-tree of metric space). For $K > 0$, we build the *quasi-tree of metric spaces*, $\mathcal{C}_K(\mathbf{Y})$, by taking the union of the metric spaces in $\mathbf{Y}$ with an edge of length $L > 0$ (L depends on K) connecting every pair of points in $\pi_Y(X)$ and $\pi_X(Y)$ if $d_{\mathcal{P}_K(\mathbf{Y})}(X, Y) = 1$.

In other words, if $\mathbf{Y}_K(X, Y)$ is empty, we put edges between the sets $\pi_X(Y) \subset X$ and $\pi_Y(X) \subset Y$. $\mathcal{C}_K(\mathbf{Y})$ is a metric space. For example if each Y is a path metric space (which is the case in our applications), then $\mathcal{C}_K(\mathbf{Y})$ is path connected and there is an obvious metric on it. We denote the metric by $d_{\mathcal{C}_K(\mathbf{Y})}$.

To equip a group action on $\mathcal{C}_K(\mathbf{Y})$, consider a metric ρ (that is possibly infinite) on the disjoint union of elements of $\mathbf{Y}$ by setting $\rho(x_0, x_1) = \rho_X(x_0, x_1)$ if $x_0, x_1 \in X$, for some $X \in \mathbf{Y}$; and $\rho(x_0, x_1) = \infty$ if x_0 and x_1 are in different spaces in $\mathbf{Y}$. Assume that the group G acts isometrically on $\mathbf{Y}$ with this metric and that the projections π_X are G-invariant, i.e. $\pi_{gX}(gY) = g(\pi_X(Y))$. Then G naturally acts on $\mathcal{C}_K(\mathbf{Y})$ by isometries. This occurs in most examples.

2.3 Distance formula. In $\mathcal{P}_K(\mathbf{Y})$ we gave upper and lower bounds of $d(X, Z)$ using $\mathbf{Y}_K(X, Z)$ and $\mathbf{Y}_{K'}(X, Z)$. There are similar formula in $\mathcal{C}_K(\mathbf{Y})$ as follows. The projection $\pi_Y(X)$ is defined for $X \in \mathbf{Y}\backslash\{Y\}$, but since Y is a metric space in the current setting we extend the projection to each point $x \in X$ by $\pi_Y(x) = \pi_Y(X)$. Also, for a point $y \in Y$, we set $\pi_Y(y) = \{y\}$. The projection π_Y is now defined on $\mathcal{C}_K(\mathbf{Y})$, except for the edges.

Remark 2.4. Since $\mathcal{C}_K(\mathbf{Y})$ is a metric space we can also define $\pi_Y(x)$ using the nearest point projection. Then the Hausdorff distance between the sets we obtain in Y by the two different definitions is bounded. It is a part of Theorem 1.1.

As before, we then set $d_Y(x, z) = \operatorname{diam}(\pi_Y(x) \cup \pi_Y(z))$ for $x, z \in \mathcal{C}_K(\mathbf{Y})$. We also define

$$\mathbf{Y}_J(x, z) = \{Y \in \mathbf{Y} | d_Y(x, z) > J\}.$$

It is possible for X or Z, where $x \in X$ and $z \in Z$, to be in $\mathbf{Y}_J(x, z)$. Here is a *distance formula* in $\mathcal{C}_K(\mathbf{Y})$.

Theorem 2.5. *Bestvina, Bromberg, and Fujiwara [2015, Theorem 4.13] Let $K' > K$ be sufficiently large. Then for $x \in X, z \in Z$ we have*

$$\frac{1}{2} \sum_{W \in \mathbf{Y}_{K'}(x,z)} d_W(x, z) \leq d_{\mathcal{C}_K(\mathbf{Y})}(x, z) \leq 6K + 4 \sum_{W \in \mathbf{Y}_K(x,z)} d_W(x, z).$$

Some explanation is in order. In a way the lower bound is harder to get. Recall that if $W \in \mathbf{Y}_{K'}(X, Z)$ then every geodesic γ from X to Z in $\mathcal{P}_K(\mathbf{Y})$ has to visit the vertex W. In $\mathcal{C}_K(\mathbf{Y})$ the vertex W is replaced by the metric space W, so one changes γ to a path γ' in $\mathcal{C}_K(\mathbf{Y})$ by replacing the vertex W by a geodesic in W joining the subsets $\pi_W(x), \pi_W(z)$. Since the distance between those two sets is roughly $d_W(x, z)$, after those replacement the length of the path γ' is roughly bounded below by $\sum_{W \in \mathbf{Y}_{K'}(x,z)} d_W(x, z)$, which appears in the lower bound. The above formula is an analogy of the *Masur-Minsky distance formula* for a mapping class group (see Theorem 3.6).

3 Application to mapping class groups of surfaces

The family of mapping class groups of surfaces is an interesting object to study in Geometric group theory. We discuss applications of our construction to mapping class groups.

We recommend a book Farb and Margalit [2012] for a general reference of the theory of mapping class groups and a survey Minsky [2006] for the Masur-Minsky theory.

3.1 Definitions. Let $\Sigma = \Sigma_{g,b}$ be an orientable compact surface with genus g and b boundary components. The group of orientation preserving homeomorphisms of an oriented surface Σ to itself, taken modulo isotopy, is called the *mapping class group* of Σ, denoted by $MCG(\Sigma)$. $MCG(\Sigma)$ is a finitely presented group. An *essential* simple closed curve in Σ is an embedded circle in Σ that is homotopically non-trivial and not homotopic into the boundary (non-peripheral). We may just say simple closed curves. A mapping class, ie, an element in $MCG(\Sigma)$, that preserves a system of disjoint essential simple closed curves on Σ is called *reducible*. For example Dehn twists are reducible. Thurston classified the nontrivial conjugacy classes in $MCG(\Sigma)$ as reducible, finite-order, and *pseudo-Anosov*. A pseudo-Anosov mapping class does not preserve any finite set of closed curves, but instead preserves a pair of *measured geodesic laminations*.

For example, $MCG(\Sigma_{0,0})$ is trivial. For $\Sigma_{1,0}, \Sigma_{1,1}$, $MCG(\Sigma)$ is isomorphic to $SL(2,\mathbb{Z})$, and $MCG(\Sigma_{0,4})$ maps to $PSL(2,\mathbb{Z})$ with the kernel $(\mathbb{Z}/2\mathbb{Z})^2$. In particular they are hyperbolic groups. But $MCG(\Sigma)$ is not word-hyperbolic if $g \geq 2$ since it contains $\mathbb{Z}^2$ generated by commuting Dehn twists. One natural metric space $MCG(\Sigma)$ acts on by isometries is *the Teichmüller space*. The Teichmüller space for $\Sigma_{g,0}, g > 0$ is diffeomorphic to the Euclidean space of dimension $6g - 6$, with the Teichmüller metric. The action by $MCG(\Sigma)$ is proper but not co-compact. There are a lot of negative curvature aspects on a Teichmuller space (cf. Minsky [1996]), but it is not δ-hyperbolic.

3.1.1 Curve graph. We recall another object which $MCG(\Sigma)$ acts on, and is useful for our approach. Let $\mathcal{C}_0(\Sigma)$ be the set of homotopy classes of essential simple closed curves and properly embedded simple arcs on Σ (when $\partial\Sigma$ is not empty) that are essential (not homotopic into $\partial\Sigma$). We then define the *curve graph*, $\mathcal{C}(\Sigma)$, to be the 1-complex obtained by attaching an edge to a pair of disjoint closed curves or arcs in $\mathcal{C}_0(\Sigma)$.

Remark 3.1. The graph we have constructed is often called the *curve and arc graph*, Masur and Minsky [2000]. The usual curve graph, whose vertices are only curves, is quasi-isometric to the curve and arc graph and so we will use the less cumbersome name of curve graph.

$\mathcal{C}(\Sigma)$ is connected, and $MCG(\Sigma)$ naturally acts on $\mathcal{C}(\Sigma)$ by automorphisms since homeomorphisms preserve disjointness, and the quotient is finite. The action is far from proper, but the homomorphism $MCG(\Sigma) \to Aut(\mathcal{C}(\Sigma))$ has at most finite kernel, and the index of the image is finite in $Aut(\mathcal{C}(\Sigma))$, (Ivanov, Luo, Korkmaz).

Masur and Minsky [1999], Masur and Minsky [2000] studies the geometry of a curve complex, and their work has a significant impact on the study of hyperbolic 3-manifolds and mapping class groups. The following result is the first important theorem (cf. Bestvina and Fujiwara [2007] for a non-orientable Σ). More recently, it is proved that δ is uniform for all Σ, Hensel, Przytycki, and Webb [2015].

Theorem 3.2. *$\mathcal{C}(\Sigma)$ is a δ-hyperbolic space, and $g \in MCG(\Sigma)$ is a hyperbolic isometry on $\mathcal{C}(\Sigma)$ if and only if g is a pseudo-Anosov element.*

Moreover, for a given surface Σ, there is a uniform positive lower bound on the translation length of a pseudo-Anosov element.

3.2 Applications.

We explain a setting for us to apply projection complex to the study of $MCG(\Sigma)$. A set $\mathbf{Y}$ we take is very different from the $\mathbb{H}^n$ example in the introduction. $\mathbf{Y}$ is not a collection of subsets in some hyperbolic space, say, the curve graph of Σ, but it will be a certain collection of (isotopy classes of) *essential* subsurfaces $Y \subset \Sigma$. A subsurface is *essential* if it is π_1-injective and non-peripheral. To define a quasi-tree of metric spaces, we also need a metric space for each Y (here, we distinguish Y and the metric space associated to it). For that we take the curve complex $\mathcal{C}(Y)$ for each Y. We use *subsurface projection* to define projections between two subsurfaces in $\mathbf{Y}$, then apply our method after checking the axioms.

3.2.1 Subsurface projection.

We say two essential subsurfaces *overlap* if $\partial Y \cap \partial Z \neq \emptyset$ (this means that the intersection is nonempty even after any isotopy). Following Masur and Minsky [2000], if Y and Z overlap, we define the *subsurface projection* $\pi_Y(Z) \subset \mathcal{C}(Y)$ by taking the intersection of ∂Z with Y and identifying homotopic curves and arcs. Also, when β is a simple closed curve that cannot be isotoped to be disjoint from Y, we similarly define a projection $\pi_Y(\beta) \subset \mathcal{C}(Y)$.

Also, we will need the curve graph for an essential simple closed curve γ. The definition has a somewhat different flavor and we do not recall a definition here. $\mathcal{C}(\gamma)$ is quasi-isometric to $\mathbb{Z}$, and the Dehn twist along γ, which leaves γ invariant, acts by a hyperbolic isometry. We will call γ a subsurface too. When a curve γ and the boundary of a subsurface Y intersect, we already defined $\pi_Y(\gamma)$ but we will also need $\pi_\gamma(Y) \subset \mathcal{C}(\gamma)$. More generally Y can be a curve. See for example Bestvina, Bromberg, and Fujiwara [2015, §5.1] for the precise definition of $\mathcal{C}(\gamma)$ and the projection.

Now we want to check that the projection complex axioms are satisfied.

Theorem 3.3. *Let $\mathbf{Y}$ be a collection of essential subsurfaces in Σ such that any two distinct subsurfaces intersect. For distinct $X, Y, Z \in \mathbf{Y}$ define $d^\pi_Y(X,Z) = \operatorname{diam}(\pi_Y(X) \cup \pi_Y(Z))$, where the diameter is measured in $\mathcal{C}(Y)$. Then $\{d^\pi_Y\}$ satisfy (P0)-(P2) for some constant $\theta(\Sigma)$, which depends only on Σ.*

Thus for every such family $\mathbf{Y}$ we obtain the projection complex $\mathcal{P}_K(\mathbf{Y})$ for a large K, and the quasi-tree of curve complexes $\mathcal{C}_K(\mathbf{Y})$ for $\{\mathcal{C}(Y), Y \in \mathbf{Y}\}$. We make comments on verifying the axioms (P0)-(P2) in this setting. Axiom (P0) follows easily from definitions. Axiom (P1) was established by J. A. Behrstock [2006]. We sometimes refer to Axiom (P1) in general as *Behrstock's inequality*. Axiom (P2) is by Masur-Minsky (a consequence of the Theorem 4.6 and Lemma 4.2 in Masur and Minsky [2000]). A central idea in Masur and Minsky [ibid.] is the notion of a *hierarchy* and this is used in the original verification of (P1) and (P2). This is a powerful tool but it is complicated to define and difficult to use. Leininger gave a very simple, hierarchy free proof of (P1) (see Mangahas [2010, 2013]) and also (P2) has a direct, hierarchy free proof (Bestvina, Bromberg, and Fujiwara [2015]).

3.2.2 Embedding MCG. Having Theorem 3.3, we now use Theorem 1.1 to embed the mapping class group in a *finite* product of *quasi-trees of curve complexes*. To use Theorem 3.3, we group the essential subsurfaces in Σ into finitely many subcollections $\mathbf{Y}^1, \mathbf{Y}^2, \cdots, \mathbf{Y}^k$, such that any X, Y in each family overlap, hence the projection $\pi_X(Y)$ is defined. The subcollections are the orbits of a certain subgroup $\mathcal{S}$ in $MCG(\Sigma)$, given in the following lemma.

Lemma 3.4 (Color preserving subgroup)**.** *Bestvina, Bromberg, and Fujiwara [ibid.]. There is a coloring $\phi : \mathcal{C}_0(\Sigma) \to F$ of the set of simple closed curves on Σ with a finite set F of colors so that if a, b span an edge then $\phi(a) \neq \phi(b)$. Moreover, there is a finite index subgroup $\mathcal{S}$ of the mapping class group $MCG(\Sigma)$ (where Σ is closed) such that every element of $\mathcal{S}$ preserves the colors.*

We call this subgroup *the color preserving subgroup*. Note that there are only finitely many $\mathcal{S}$-orbits of subsurfaces of Σ and any two subsurfaces in each $\mathcal{S}$-oribit overlap. Having Theorem 3.3, for each orbit $\mathbf{Y}^i$ we apply our construction to $\{\mathcal{C}(Y) | Y \in \mathbf{Y}^i\}$ and obtain $\mathcal{C}_K(\mathbf{Y}^i)$ for a large enough constant K. Everything (for example projection $\pi_X(Y)$) is done equivariantly in the construction, so that we have an equivariant orbit map

$$\Phi : MCG(\Sigma) \to \mathcal{C}_K(\mathbf{Y}^1) \times \mathcal{C}_K(\mathbf{Y}^2) \times \cdots \times \mathcal{C}_K(\mathbf{Y}^k),$$

sending $g \in MCG(\Sigma)$ to $g(o)$, where o is an arbitrary base point in the product. Note that an element in $\mathcal{S}$ preserves each factor, while other elements permute the factors. The choice of a base point will not be important for our purpose. We put the ℓ^1-metric on the product.

Since $MCG(\Sigma)$ is finitely generated, the map Φ is a Lipschitz map. Moreover, by some compactness argument regarding the set of curves on Σ, one can show Φ is a coarse embedding. A map between two metric spaces $f : X \to Y$ is a *coarse embedding* if there are constants A, B and a function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(t) \to \infty$ as $t \to \infty$ such that

$$\Phi(|x - x'|) \leq |f(x) - f(x')| \leq A\,|x - x'| + B.$$

Moreover, it turns out that Φ is a quasi-isometric (QI) embedding. As each factor is hyperbolic by Theorem 1.2 (ii), we have the following theorem:

Theorem 3.5. *Bestvina, Bromberg, and Fujiwara [ibid.]. $MCG(\Sigma)$ equivariantly quasi-isometrically embeds in a finite product of hyperbolic spaces.*

The argument to show that Φ is a QI-embedding is by reinterpreting the remarkable Masur-Minsky distance formula (Theorem 6.12 in Masur and Minsky [2000]). To state it, let α be a finite *binding* collection of simple closed curves in Σ, ie, every essential curve in Σ intersects at least one curve in α. For x, M, the number $[x]_M$ is defined as x if $x > M$ and as 0 if $x \leq M$. Fix a finite generating set of $MCG(\Sigma)$, and let $|g|$ be the word norm.

Theorem 3.6 (Masur-Minsky distance formula)**.** *Suppose M is sufficiently large. Then there exist K, L such that for any $g \in MCG(\Sigma)$,*

$$\frac{1}{K}|g| - L \leq \sum_Y [d_{\mathcal{C}(Y)}(\pi_Y(\alpha), \pi_Y(g(\alpha)))]_M \leq K|g| + L,$$

where the sum is over all essential subsurfaces Y in Σ.

After arranging $K' = M$, notice that the sum in the middle of the above theorem appears in the left hand side (to be precise, we add them over all $\mathbf{Y}^i$'s) of the distance formula in a quasi-tree of metric spaces (Theorem 2.5) with $x = \pi_Y(\alpha), z = \pi_Y(g(\alpha))$. Combing those two estimates we obtain a desired estimate to show that Φ is a QI-embedding.

The following result follows easily from the definition of *asymptotic cones* (see J. Behrstock, Druţu, and Sapir [2011b,a]) and Theorem 3.5 since the asymptotic cone of a hyperbolic space is an $\mathbb{R}$-tree.

Theorem 3.7 (Behrstock-Druţu-Sapir)**.** *Every asymptotic cone of $MCG(\Sigma)$ embeds by a bi-Lipschitz map in a finite product of $\mathbb{R}$-trees.*

In fact they prove more including some information on the geometry of the image of the embedding, but their theorem does not imply Theorem 3.5. They use the notion of *tree-graded space* introduced in Druţu and Sapir [2005].

3.2.3 Asymptotic dimension. We discuss a further application to asymptotic dimension, which we stated as Theorem 1.4. We recall a few basic properties. Let $\mathcal{X}, \mathcal{Y}$ be metric spaces. If $\mathcal{X} \subset \mathcal{Y}$, with a metric on $\mathcal{Y}$ restricted to $\mathcal{X}$, then $\mathrm{asdim}(\mathcal{X}) \leq \mathrm{asdim}(\mathcal{Y})$.

We have Product Formula: $\operatorname{asdim}(\mathcal{X} \times \mathcal{Y}) \le \operatorname{asdim}(\mathcal{X}) + \operatorname{asdim}(\mathcal{Y})$. It is straightforward from the definition that the asymptotic dimension is not only a quasi-isometric invariant but is also a *coarse invariant*. In particular $\operatorname{asdim}(\mathcal{X}) \le \operatorname{asdim}(\mathcal{Y})$ if there exists a coarse embedding $f : \mathcal{X} \to \mathcal{Y}$ (Roe [2003]).

It is a theorem of Bell and Fujiwara [2008] that each curve complex has finite asymptotic dimension. Thus from Theorems 1.2 (iii) and 3.5 we obtain the following theorem, which motivated the work Bestvina, Bromberg, and Fujiwara [2015]:

Theorem 3.8. *(Theorem 1.4) Let Σ be a closed orientable surface, possibly with punctures. Then* $\operatorname{asdim}(MCG(\Sigma)) < \infty$.

The exact value of $\operatorname{asdim}(MCG(\Sigma))$ is unknown. Webb Webb [2015] found explicit bounds on the asymptotic dimension of curve complexes, which was improved to a linear bound by Bestvina and Bromberg [n.d.] by a different method. As a consequence, $\operatorname{asdim}(MCG(\Sigma))$ is bounded by an exponential function in the *complexity* of the surface, $\kappa(\Sigma_{g,b}) = 3g + b$.

We can also prove:

Theorem 3.9. *Bestvina, Bromberg, and Fujiwara [2015]. The Teichmüller space of Σ, with either the Teichmüller metric or the Weil-Petersson metric, has finite asymptotic dimension.*

3.2.4 Dehn twists as hyperbolic isometry. When X is a quasi-tree, an isometry with unbounded orbits is always hyperbolic, Manning [2006]. The following theorem uses the observation that the $MCG(\Sigma)$-orbit of a curve α in a surface Σ of even genus that separates Σ into subsurfaces of equal genus consists of pairwise intersecting curves. We then form a quasi-tree of metric spaces for the collection of those curves, viewed as subsurfaces as we explained (and their curve complexes), where the Dehn twist along α will be hyperbolic with an axis $\mathcal{C}(\alpha)$.

Theorem 3.10. *Bestvina, Bromberg, and Fujiwara [2015]. The mapping class groups in even genus can act on quasi-trees with at least one Dehn twist having unbounded orbits.*

In the case of odd genus one has to pass to the color preserving subgroup $\mathcal{S}$. In any case, it follows that for each Dehn twist g, the word norm $|g^n|$ has linear growth on n in $MCG(\Sigma)$ (not only in $\mathcal{S}$). Thus we recover the theorem by Farb, Lubotzky, and Minsky [2001] (the other types of elements are easy):

Theorem 3.11. *For each element g of infinite order in $MCG(\Sigma)$, the word norm $|g^n|$ has linear growth.*

3.2.5 Promoting actions to CAT(0) spaces or trees. Theorem 3.10 provides a sharp contrast to a result of Bridson [2010], who showed that in semi-simple actions of mapping class groups (of genus > 2) on complete CAT(0) spaces Dehn twists are always elliptic. A group action is *semi-simple* if each element has either a bounded orbit or positive translation length.

By a *thickening* of a metric space X we mean a quasi-isometric embedding $X \to Y$. When X is a graph with edges of length 1 and $d \geq 1$, there is a particular thickening $X \to P_d(X)$ called the *Rips complex* of X. The space $P_d(X)$ is a simplicial complex with the same vertex set as X and with simplices consisting of finite collections of vertices with pairwise distance at most d. The Dehn twists that are hyperbolic in Theorem 3.10 stay hyperbolic in any thickning of the quasi-tree. Now by the theorem of Bridson, a thickning is never CAT(0). This give the following theorem:

Theorem 3.12. *Bestvina, Bromberg, and Fujiwara [2015]. There is an isometric action of a group on a graph X which is a quasi-tree such that no equivariant thickening admits an equivariant $CAT(0)$ metric. In particular, for no $d \geq 1$, the Rips complex $P_d(X)$ admits an equivariant $CAT(0)$ metric.*

We make some comments on the background. It is a long-standing open question whether every hyperbolic group acts co-compactly and properly by isometries on a CAT(0) space. One approach is to consider the Rips complex $P_d(X)$ for the Cayley graph X of the group and large d. Theorem 3.12 is not a counterexample to this approach since our X is not locally finite, but it does point out difficulties. Note that in light of Mosher, Sageev, and Whyte [2003] the quasi-trees that arise in our construction are necessarily locally infinite, since otherwise we would be able to promote our group actions on quasi-trees to group actions on simplicial trees without fixed points, which is not possible for certain groups, for example, non-elementary hyperbolic groups that have property (T), eg, uniform lattices in $\mathrm{Sp}(n, 1)$.

3.2.6 Uniform uniform exponential growth of $MCG(\Sigma)$. We discuss applications to the exponential growth of groups. Let Γ be a group and S a finite set in Γ. Assume that $1 \in S$ and $S = S^{-1}$. Set

$$h(S) := \lim_{n\to\infty} \frac{1}{n} \log |S^n|.$$

Let Γ be a finitely generated group. Set $h(\Gamma) = \inf_S\{h(S) | \langle S\rangle = \Gamma\}$, where S runs over all finite generating subsets. If $h(\Gamma) > 0$ we say Γ has *uniform exponential growth*, of *growth rate* $h(\Gamma)$.

Using quasi-trees of metric spaces, Breuillard and Fujiwara [n.d.] recovers the following theorem by Mangahas, Mangahas [2010]:

Theorem 3.13 (UUEG of MCG). *Let Σ be a compact oriented surface possibly with punctures. Then there exists a constant $N(\Sigma)$ such that for any finite set $S \subset MCG(\Sigma)$ with $S = S^{-1}$, either $\langle S \rangle$ is virtually abelian or S^N contains g, h that produces a free semi-group. In particular: $h(S) \geq \frac{1}{N}\log 2$.*

It follows that for each surface Σ there is a constant $c(\Sigma) > 0$ such that for any finite set S in $MCG(\Sigma)$, $h(S) \geq c(\Sigma)$ unless $\langle S \rangle$ is virtually abelian. We say $MCG(\Sigma)$ has "uniform uniform exponential growth" (UUEG). The proof of Theorem 3.13 in Breuillard and Fujiwara [n.d.] applies a standard "Ping-Pong" argument to the actions of the color preserving subgroup $\mathcal{S}$ on the $\mathcal{C}_K(\mathbf{Y}^i)$'s. A key property of those actions is that any nontrivial element of infinite order in $\mathcal{S}$ is hyperbolic for at least one of the actions. Mangahas' proof is different and does not use our complexes. Also she proves more, that g, h produces a free group of rank-two, maybe for a larger constant $N(\Sigma)$.

3.2.7 Stable commutator length. Let G be a group, and $[G, G]$ its commutator subgroup. For an element $g \in [G, G]$, let $cl(g) = cl_G(g)$ denote the *commutator length* of g, the least number of commutators whose product is equal to g. We define $cl(g) = \infty$ for an element g not in $[G, G]$. For $g \in G$, the *stable commutator length*, $\mathrm{scl}(g) = \mathrm{scl}_G(g)$, is defined by

$$\mathrm{scl}(g) = \liminf_{n\to\infty} \frac{cl(g^n)}{n} \leq \infty.$$

It is clear that $\mathrm{scl}(g^n) = n\,\mathrm{scl}(g)$ and $\mathrm{scl}(hgh^{-1}) = \mathrm{scl}(g)$. We recommend a monograph Calegari [2009] as a reference on scl.

One theme in the subject is to classify elements g in a given group for which $scl(g) > 0$. To verify $scl(g) > 0$ for an element g, the notion of quasi-morphisms is useful. A function $H : G \to \mathbb{R}$ is a *quasi-morphism* if

$$\Delta(H) := \sup_{x,y\in G} |H(xy) - H(x) - H(y)| < \infty.$$

$\Delta(H)$ is called the *defect* of H. It is a simple exercise to show that if there exists a quasi-morphism $f : G \to \mathbb{R}$ which is unbounded on the powers of g, then $scl(g) > 0$. Also, the converse holds by so called *Bavard duality*. For example, Brooks [1981] showed that in free groups G, $scl(g) > 0$ for every nontrivial element g by constructing a quasi-morphism $f : G \to \mathbb{R}$ which is unbounded on the powers of g.

On the other hand, it is straightforward from the definition that in the following situations $cl(g^n)$ is bounded and therefore $scl(g) = 0$:

(a) g has finite order,

(b) more generally, g is *achiral*, i.e. g^k is conjugate to g^{-k} for some $k \neq 0$.

For example, Epstein and Fujiwara [1997], generalizing the Brooks construction of quasi-morphisms, proved that in hyperbolic groups G the above obstruction (b) is the only one, namely, if g is *chiral* (i.e. not achiral) then $scl(g) > 0$. They use the hyperbolicity of the Cayley graph of G to construct a suitable quasi-morphism for g.

There are three more conditions one can directly check from the definition for $cl(g^n)$ being bounded and therefore $scl(g) = 0$, cf. Bestvina, Bromberg, and Fujiwara [2016b]:

(c) $g = g_1 g_2^{-1}$ such that $g_1 g_2 = g_2 g_1$, and g_1 is conjugate to g_2,

(d) more generally, g is expressed as a commuting product $g = g_1 \cdots g_p$ such that $g_i^{n_i}$ are all conjugate for some $n_i \neq 0$ and that $\sum_i \frac{1}{n_i} = 0$,

(e) $g = g_1 \cdots g_p$ is a commuting product and $cl(g_i^n)$ are bounded for all i.

Now, we are interested in the question to decide when $scl(g) = 0$ for $g \in MCG(\Sigma)$. Some partial answers were known. Using 4-manifold invariants, Endo and Kotschick [2001] and Korkmaz [2004] prove that $scl(g) > 0$ if g is a Dehn twist. Endo and Kotschick [2007] also note the obstruction (c): in $MCG(\Sigma)$ for example this occurs if g_1, g_2 are Dehn twists in disjoint curves in the same $MCG(\Sigma)$-orbit. By contrast, Calegari and Fujiwara [2010] prove that if g is pseudo-Anosov and chiral then $scl(g) > 0$, ie, (b) is the only obstruction among pseudo-Anosov elements. They use Theorem 3.2 to construct a suitable quasi-morphism for g.

We state our result in the following vague form. See the precise statement in Bestvina, Bromberg, and Fujiwara [2016b]. It covers all cases in a unified way and in particular we recover the result on Dehn-twists.

Theorem 3.14. *Let $G < MCG(\Sigma)$ be a subgroup of finite index and $g \in G$. Then there is a characterization of elements g with $scl_G(g) > 0$ in terms of the "Nielsen-Thurston form" of g.*

The new and more complicated case is on a reducible element g. By Nielsen-Thurston form, a power of such g is written as a commuting product of powers of Dehn twists and pseudo-Anosov maps on disjoint subsurfaces on Σ, after removing a system of g-invariant curves. We argue that either $cl(g^n)$ is bounded applying (a)-(e) to the Nielsen-Thurston form of g, or else we can construct a suitable quasi-morphism on $\mathcal{S} \cap G$ to show $scl(g) > 0$. A key point is g is hyperbolic on one of the $\mathcal{C}_K(\mathbf{Y}^i)$'s. Also we prove the following theorem. The *Torelli subgroup* is the kernel of the action of $MCG(\Sigma)$ on $H_1(\Sigma, \mathbb{Z})$. It has infinite index as a subgroup.

Theorem 3.15. *If $G = \mathcal{T}$ is the Torelli subgroup in $MCG(\Sigma)$ and $1 \neq g \in G$ then $scl_G(g) > 0$.*

4 Other applications

4.1 Contracting geodesics and WPD. In the introduction we explained how we build a quasi-tree from a discrete group G of isometries of $\mathbb{H}^n$, and extracted the axioms we need. The axioms are satisfied since $\mathbb{H}^n$ is hyperbolic and G is discrete. We can relax the assumptions keeping the axioms satisfied.

Let X be a geodesic space, and $Y \subset X$ a subset. For a constant $B > 0$ we say that Y is *B-contracting* if the nearest point projection (this is a *coarse map*, ie, the image of a point may contain more than one point) to Y of any metric ball disjoint from Y has diameter bounded by B. See Bestvina and Fujiwara [2009]. For example if X is δ-hyperbolic and Y is a geodesic, then Y is 10δ-contracting. Let γ be a hyperbolic isometry of X, and let O be the γ-orbit of a point x in X. We say γ is *rank 1* if O is B-contracting for some B. This notion does not depend on the choice of x, and also, one can take an axis of γ, if it exists, instead of an orbit O for an equivalent definition, Bestvina, Bromberg, and Fujiwara [n.d.]. Any hyperbolic isometry on a hyperbolic space is rank 1.

Assume that a group Γ acts by isometries on a geodesic metric space X, and $\gamma \in \Gamma$ acts hyperbolically. We say γ is a *WPD element* if for all $D > 0$ and $x \in X$ there exists $M > 0$ such that the set

$$\{g \in \Gamma \mid d(x, g(x)) \le D, d(f^M(x), gf^M(x)) \le D\}$$

is finite. Bestvina and the author Bestvina and Fujiwara [2002] introduced this notion and proved that every pseudo-Anosov element in $MCG(\Sigma)$ is WPD on $\mathcal{C}(\Sigma)$, in view of an application to computing the *second bounded cohomology* of a subgroup A, $H_b^2(A, \mathbb{R})$, in $MCG(\Sigma)$. Note that if the action of G on X is proper, then any hyperbolic isometry is WPD. WPD stands for *weak proper discontinuous*. There is even a weaker notion, called *WWPD*, introduced in Bestvina, Bromberg, and Fujiwara [2015]. This notion is needed to prove Theorem 3.14. Delzant [2016] found an application of WWPD and the projection complex to Kähler groups. Handel and Mosher [n.d.] found an application of WWPD to computing the second bounded cohomology of subgroups in $Out(F_n)$.

The $\mathbb{H}^n$ example in the introduction is a special case of the following theorem.

Theorem 4.1. *Bestvina, Bromberg, and Fujiwara [2015], Bestvina, Bromberg, and Fujiwara [n.d.]. Let Γ act on a geodesic metric space X such that some $\gamma \in \Gamma$ is a hyperbolic WPD element. Assume the γ-orbit of some point, O, is B-contracting for some B (ie, γ is rank 1). Then the collection of parallel classes of Γ-translates of the orbit O with nearest point projections satisfies (P0)-(P2) and thus Γ acts on a quasi-tree, Q. In addition, in this action γ is a hyperbolic WPD element.*

Some explanation is in order. We say that two orbits are *parallel* if their Hausdorff distance is finite. The quasi-tree Q in the theorem is a quasi-tree of metric spaces for the

Γ-translates of O, each of which is quasi-isometric to a line. We do not assume that X is hyperbolic nor CAT(0). The main part of the proof consists of verifying (P0)-(P2) and applying Theorem 1.1 in this situation.

Remark 4.2. If an infinite geodesic γ bounds a Euclidean half plane, then clearly γ is not B-contracting for any B. This condition, *the flat half plane condition*, was important in the study of $CAT(0)$-space X, see Ballmann [1995]. If X is a locally compact $CAT(0)$-space and γ is an axis of a hyperbolic isometry g then γ being B-contracting for some B is equivalent to that γ does not bound a flat half plane, Bestvina and Fujiwara [2009]. See also Charney and Sultan [2015] for a recent development. The flat half plane conditions and the notion of rank 1 are first defined in the study of Riemannian manifolds of nonpositive sectional curvature.

Examples 4.3. The following examples all satisfy Theorem 4.1. One considers the translates of an axis, or more generally the orbit of a point, of a hyperbolic WPD element, γ.

(1) As a generalization of discrete subgroups in $\mathbb{H}^n$, Γ is a group of isometries of a δ-hyperbolic space X that contains a hyperbolic, WPD element, γ. For example, in the action of a hyperbolic group on its Cayley graph, any element of infinite order is hyperbolic and WPD. The class of hyperbolic groups contains many groups with Kazhdan's property (T) and therefore every isometric action on a simplicial tree has a fixed point (cf. de la Harpe and Valette [1989]). For the action of $MCG(\Sigma)$ on the curve complex, every pseudo-Anosov element is hyperbolic and WPD Bestvina and Fujiwara [2002].

(2) Let G be the fundamental group of a *rank-1 manifold* M, ie, M is a complete Riemannian manifold of non-positive curvature of finite volume such that the universal cover X of M is not a Riemannian product nor a symmetric space of non-compact type of rank at least two. Then G properly acts on X and by the *Rank rigidity theorem* Ballmann [1985], G contains a hyperbolic isometry γ which is rank-1.

(3) Γ is a discrete group of isometries (i.e. the group action is proper) of a $CAT(0)$-space that contains a hyperbolic rank-1 element γ. Also, pseudo-Anosov mapping classes are rank 1 elements in the action of $MCG(\Sigma)$ on the Weil-Petersson completion of Teichmüller space, which is CAT(0). Those elements are WPD although the action of $MCG(\Sigma)$ is not properly discontinuous. See Bestvina and Fujiwara [2009]. There are classifications of rank 1 elements in Coxeter groups Caprace and Fujiwara [2010], right angled Artin groups J. Behrstock and Charney [2012] and cube complexes Caprace and Sageev [2011].

(4) Γ is $MCG(\Sigma)$ acting on Teichmüller space with Teichmüller metric, and γ is a pseudo-Anosov mapping class. By Minsky [1996] the axis of γ is B-contracting. It is WPD since the action is properly discontinuous.

(5) $\Gamma = Out(F_n)$ acting on Culler-Vogtmann's Outer space CV_n Culler and Vogtmann [1986], equipped with the Lipschitz metric (not symmetric, see Algom-Kfir and Bestvina [2012]). The action is properly discontinuous. CV_n is not δ-hyperbolic. See for example Vogtmann [2006] for more information on $Out(F_n)$ and Outer space. As an analogue of a pseudo-Anosov element in a mapping class group, an element f of $Out(F_n)$ is *fully irreducible* if there are no conjugacy classes of proper free factors of F_n which are f-periodic. Such elements are hyperbolic with axes in CV_n, see Bestvina [2011], which are B-contracting (Algom-Kfir [2011]) .

(6) *The Cremona group*, G, of all birational transformations of the projective plane $\mathbb{P}^2_k$, where k is an algebraically closed field. G acts on a hyperbolic space, and it contains a hyperbolic WPD element, which was shown by Cantat and Lamy [2013]. It then follows that Cremona groups are not simple. We explain this implication in the next section.

4.2 Acylindrically hyperbolic groups and hyperbolically embedded subgroups. Dahmani, Guirardel, and Osin [2017] introduced the notion of *hyperbolically embedded subgroups*, a generalization of the concept of a *relatively hyperbolic group* (see their paper for the precise definition). They proved

Theorem 4.4. *If G is not virtually cyclic and acts on a hyperbolic space X such that G contains γ that is hyperbolic and WPD, then G contains a proper infinite hyperbolically embedded subgroup H.*

Here, we can take H to be virtually cyclic containing $\langle \gamma \rangle$. They use projection complex as a key tool in the argument. They further proved that for a sufficiently large N, γ^N normally generates a free subgroup (of maybe infinite rank) whose non-trivial elements are all hyperbolic on X. In particular G is not simple. This is the implication we mentioned in Examples 4.3 (6), and it applies to groups G in Examples 4.3 by Theorem 4.1. Also, by this method, they produce a free normal subgroup in $MCG(\Sigma)$, unless it is virtually cyclic, whose non-trivial elements are all pseudo-Anosov.

An isometric group action is *acylindrical* if for every $D > 0$ there exist $R, N > 0$ such that $d(x, y) \geq R$ implies that the set

$$\{g \in G \mid d(x, g(x)) \leq D, d(y, g(y)) \leq D\}$$

has cardinality at most N. Notice that this property implies WPD for any hyperbolic element. If a group action is proper and co-compact then it is acylindrical. Sela [1997] introduced the acylindricity of a group action on simplicial trees, then Bowditch [2008] formulated this definition for hyperbolic spaces and proved that the action of $MCG(\Sigma)$ on $\mathcal{C}(\Sigma)$ is acylindrical. Based on this definition, Osin [2016] develops a theory of *acylindrically hyperbolic groups*: these are groups that admit a non-elementary acylindrical isometric action on a hyperbolic space. Here, an action of G on a hyperbolic spce X is *non-elementary* if the limit set of the G-orbit of a point in X contains at least three points. He proved the following theorem.

Theorem 4.5. *Osin [ibid.] Let a group Γ, which is not virtually cyclic, act on a δ-hyperbolic metric space X such that $\gamma \in \Gamma$ is a hyperbolic WPD element. Then Γ is an acylindrically hyperbolic group. Thus all groups in Examples 4.3 are acylindrically hyperbolic.*

We make comments on his argument. By Theorem 4.4, G contains a hyperbolically embedded subgroup H. To construct a hyperbolic space for G to act on acylindrically, he uses an idea similar to projection complex with $\mathbf{Y}$ to be translates of an orbit of H. It has been improved in Balasubramanya [n.d.] so that the hyperbolic space in Theorem 4.5 can be taken to be a quasi-tree. In Bestvina, Bromberg, Fujiwara, and Sisto [n.d.] we recover this improvement by a different axiomatic construction: we start with $\mathbf{Y}$, the translates of an orbit of γ in X, which satisfies (PC0) -(PC4), then slightly change the definition of the projection $\pi_X(Y)$. For this new projection, the resulting projection complex by the usual definition is a quasi-tree, acted by G acylindrically.

4.3 Actions on CAT(0) square complex. Recall that Burger and Mozes [2000] constructed an example of a simple group, which acts freely and co-compactly on the product of two trees. Thus the quotient is a finite non-positively curved square complex with finitely-presented, infinite simple fundamental group. A square complex Z, built from unit Euclidean squares, is *non-positively curved* if the universal cover is CAT(0). Caprace and Delzant pointed out the following curious corollary of Theorem 4.1, which can be thought of a converse of the Burger-Mozes theorem.

Corollary 4.6 (see Bestvina, Bromberg, and Fujiwara [2015]). *Suppose Z is a finite non-positively curved square complex with no free edges whose fundamental group is simple. Then the universal cover $\tilde{Z}$ is isometric to the product of two trees.*

They argue that by the Ballmann-Brin Rank Rigidity Theorem Ballmann and Brin [1995, Th C] (see also Caprace and Sageev [2011]) the universal cover $\tilde{Z}$ is either the product of two trees or the deck group contains a rank 1 element. But in the latter case, by Theorem 4.1, $\pi_1(Z)$ acts on a quasi-tree and contains a hyperbolic WPD element γ. Also, $\pi_1(Z)$ is non virtually cyclic since it is simple and torsion-free. Now as we explained $\pi_1(Z)$ is not simple, impossible.

4.4 Bounded cohomology and QFA. Manning [2005] gave a construction of an action of a group G on a quasi-tree starting with a quasi-morphism $G \to \mathbb{R}$ but it is not clear when such actions are *non-elementary* (i.e. have unbounded orbits and do not fix an end nor a pair of ends). Groups G in Examples 4.3 have isometric actions on quasi-trees, and if G is non-elementary (ie, not virtually cyclic), then the action is non elementary. Conversely, if one has actions of a group G on a quasi-tree (with a hyperbolic WPD element), one can use such actions to give unified constructions of quasi-morphisms on G (cf. Epstein and Fujiwara [1997], Fujiwara [2000], Fujiwara [1998], Bestvina and Fujiwara [2002]), and even *quasi-cocycles* with coefficients in unitary representations in "uniformly convex" Banach spaces, for example, the regular representation on $\ell^2(G)$, which is of particular importance (see Monod [2006]). As a consequence, we prove

Theorem 4.7. *Bestvina, Bromberg, and Fujiwara [2016a] Let G be an acylindrically hyperbolic group with no non-trivial finite normal subgroup, and ρ a unitary representation of G in a uniformly convex Banach space, then the second bounded cohomology $H_b^2(G;\rho)$ is infinite dimensional.*

By contrast, there are many groups that do not admit nontrivial (namely, orbits are unbounded) actions on a quasi-tree. A group G satisfies *QFA* if every action on a quasi-tree has bounded orbits. For example, $SL_n(\mathbb{Z})$, $n \geq 3$ satisfies QFA, Manning [2006]. More recently, Haettel [n.d.] proves that if G is a lattice in (a product of) a higher rank semi-simple Lie group with finite center, then G satisfies QFA. He even proved that an action on any hyperbolic space X by such G has either a bounded orbit in X or has a fixed point in the ideal boundary of X.

4.5 $Out(F_n)$. A version of Theorem 3.5 for $Out(F_n)$ is known. There are several analogs of the curve graph $\mathcal{C}(\Sigma)$, for example the complex of free factors and the complex of free splittings. Both have been shown to be hyperbolic, the former in Bestvina and Feighn [2014a] and the latter in Handel and Mosher [2013]. The analog of subsurface projections was defined by Bestvina-Feighn in Bestvina and Feighn [2014b] and they show using the projection complex technique:

Theorem 4.8. *Bestvina and Feighn [ibid.] $Out(F_n)$ acts isometrically on a finite product of hyperbolic spaces so that every element of "exponential growth" acts with positive translation length.*

It is unknown if $Out(F_n)$ acts on a finite product of hyperbolic spaces that gives a QI-embedding. While a finite product of hyperbolic spaces satisfies a quadratic *isoperimetric inequality*, it is known that the isoperimetric inequality of $Out(F_n)$ is exponential (cf. Bridson and Vogtmann [1995]), but that is not an obstruction for QI-embeddings because we do not require that an embedding is (quasi-)convex. Theorem 1.4 (finiteness of asymptotic dimension) is unknown for $Out(F_n)$, but recently Bestvina-Guirardel-Horbez proved that $Out(F_n)$ is *boundary amenable*, therefore satisfies the Novikov conjecture on higher signatures Bestvina, Guirardel, and Horbez [n.d.].

4.6 Farrell-Jones conjecture for MCG. Bartels and Bestvina [n.d.] prove the *Farrell-Jones Conjecture* for mapping class groups:

Theorem 4.9 (Bestvina-Bartels)**.** *The mapping class group $Mod(\Sigma)$ of any oriented surface Σ of finite type satisfies the Farrell-Jones Conjecture.*

The main step of the proof is the verification of a regularity condition, called *finite $\mathfrak{F}$-amenability* (see Bartels and Bestvina [ibid.] for a precise definition). Using subsurface projections by Masur-Minsky, combined with the projection complex technique, they prove the action of $MCG(\Sigma)$ on the space $\mathcal{PMF}$ of projective measured foliations on Σ is finitely $\mathfrak{F}$-amenable, for a certain family $\mathfrak{F}$ of subgroups in $MCG(\Sigma)$. Theorem 4.9 is then a consequence of the axiomatic results of Lück, Reich and Bartels for the Farrell-Jones Conjecture (cf. Bartels and Lück [2012]) and an induction on the complexity of the surface.

$MCG(\Sigma)$ has finite asymptotic dimension (Theorem 1.4). As we said this implies the integral Novikov conjecture, i.e., the integral injectivity of the assembly maps in algebraic K-theory and L-theory relative to the family of finite subgroups. This is related to the Farrell-Jones conjecture (see Bartels and Bestvina [n.d.]).

References

Yael Algom-Kfir (2011). “Strongly Contracting Geodesics in Outer Space”. *Geometry and Topology* 15.4, pp. 2181–2234 (cit. on p. 1126).

Yael Algom-Kfir and Mladen Bestvina (2012). “Asymmetry of outer space”. *Geom. Dedicata* 156, pp. 81–92. MR: 2863547 (cit. on p. 1126).

Sahana Balasubramanya (n.d.). “Acylindrical group actions on quasi-trees”. arXiv: 1602.03941 (cit. on p. 1127).

Werner Ballmann (1985). “Nonpositively curved manifolds of higher rank”. *Ann. of Math. (2)* 122.3, pp. 597–609. MR: 819559 (cit. on p. 1125).

– (1995). *Lectures on spaces of nonpositive curvature*. Vol. 25. DMV Seminar. With an appendix by Misha Brin. Birkhäuser Verlag, Basel, pp. viii+112. MR: 1377265 (cit. on pp. 1111, 1125).

Werner Ballmann and Michael Brin (1995). “Orbihedra of nonpositive curvature”. *Inst. Hautes Études Sci. Publ. Math.* 82, 169–209 (1996). MR: 1383216 (cit. on p. 1127).

Arthur Bartels and Mladen Bestvina (n.d.). “The Farrell-Jones Conjecture for mapping class groups”. arXiv: 1606.02844 (cit. on p. 1129).

Arthur Bartels and Wolfgang Lück (2012). "The Borel conjecture for hyperbolic and CAT(0)-groups". *Ann. of Math. (2)* 175.2, pp. 631–689. MR: 2993750 (cit. on p. 1129).

Jason A. Behrstock (2006). "Asymptotic geometry of the mapping class group and Teichmüller space". *Geom. Topol.* 10, pp. 1523–1578. MR: 2255505 (cit. on p. 1118).

Jason A. Behrstock and Yair N. Minsky (2008). "Dimension and rank for mapping class groups". *Ann. of Math. (2)* 167.3, pp. 1055–1077. MR: 2415393 (cit. on p. 1110).

Jason Behrstock and Ruth Charney (2012). "Divergence and quasimorphisms of right-angled Artin groups". *Math. Ann.* 352.2, pp. 339–356. MR: 2874959 (cit. on p. 1125).

Jason Behrstock, Cornelia Druţu, and Mark Sapir (2011a). "Addendum: Median structures on asymptotic cones and homomorphisms into mapping class groups". *Proc. Lond. Math. Soc. (3)* 102.3, pp. 555–562. MR: 2783136 (cit. on p. 1119).

– (2011b). "Median structures on asymptotic cones and homomorphisms into mapping class groups". *Proc. Lond. Math. Soc. (3)* 102.3, pp. 503–554. MR: 2783135 (cit. on p. 1119).

Gregory C. Bell and Koji Fujiwara (2008). "The asymptotic dimension of a curve graph is finite". *J. Lond. Math. Soc. (2)* 77.1, pp. 33–50. MR: 2389915 (cit. on p. 1120).

Mladen Bestvina (2011). "A Bers-like proof of the existence of train tracks for free group automorphisms". *Fund. Math.* 214.1, pp. 1–12. MR: 2845630 (cit. on p. 1126).

Mladen Bestvina, Ken Bromberg, and Koji Fujiwara (2015). "Constructing group actions on quasi-trees and applications to mapping class groups". *Publ. Math. Inst. Hautes Études Sci.* 122, pp. 1–64. MR: 3415065 (cit. on pp. 1105, 1107–1111, 1113, 1115, 1117–1121, 1124, 1127).

– (2016a). "Bounded cohomology with coefficients in uniformly convex Banach spaces". *Comment. Math. Helv.* 91.2, pp. 203–218. MR: 3493369 (cit. on p. 1128).

– (2016b). "Stable commutator length on mapping class groups". *Ann. Inst. Fourier (Grenoble)* 66.3, pp. 871–898. MR: 3494163 (cit. on p. 1123).

Mladen Bestvina and Kenneth Bromberg (n.d.). "On the asymptotic dimension of a curve complex". arXiv: 1508.04832 (cit. on p. 1120).

Mladen Bestvina, Kenneth Bromberg, and Koji Fujiwara (n.d.). "Bounded cohomology via quasi-trees". arXiv: 1306.1542 (cit. on p. 1124).

Mladen Bestvina, Kenneth Bromberg, Koji Fujiwara, and Alessandro Sisto (n.d.). "Acylindrical actions on projection complexes". in preparation (cit. on p. 1127).

Mladen Bestvina and Mark Feighn (2014a). "Hyperbolicity of the complex of free factors". *Advances in Mathematics* 256, pp. 104–155 (cit. on p. 1128).

– (2014b). "Subfactor projections". *Journal of Topology*. 34 pages (cit. on p. 1128).

Mladen Bestvina and Koji Fujiwara (2002). "Bounded cohomology of subgroups of mapping class groups". *Geom. Topol.* 6, 69–89 (electronic). MR: 1914565 (cit. on pp. 1124, 1125, 1128).

– (2007). "Quasi-homomorphisms on mapping class groups". *Glas. Mat. Ser. III* 42(62).1, pp. 213–236. MR: 2332668 (cit. on p. 1117).

– (2009). "A characterization of higher rank symmetric spaces via bounded cohomology". *Geom. Funct. Anal.* 19.1, pp. 11–40. MR: 2507218 (cit. on pp. 1124, 1125).

Mladen Bestvina, Vincent Guirardel, and Camille Horbez (n.d.). "Boundary amenability of $Out(F_n)$". arXiv: 1705.07017 (cit. on p. 1129).

Brian H. Bowditch (2008). "Tight geodesics in the curve complex". *Invent. Math.* 171.2, pp. 281–300. MR: 2367021 (cit. on p. 1127).

Emmanuel Breuillard and Koji Fujiwara (n.d.). "On the joint spectral radius for isometries of non-positively curved spaces and uniform growth". arXiv: 1804.00748 (cit. on pp. 1121, 1122).

Martin R. Bridson (2010). "Semisimple actions of mapping class groups on CAT(0) spaces". In: *Geometry of Riemann surfaces*. Vol. 368. London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, pp. 1–14. MR: 2665003 (cit. on p. 1121).

Martin R. Bridson and Karen Vogtmann (1995). "On the geometry of the automorphism group of a free group". *Bull. London Math. Soc.* 27.6, pp. 544–552. MR: 1348708 (cit. on p. 1128).

Robert Brooks (1981). "Some remarks on bounded cohomology". In: *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*. Vol. 97. Ann. of Math. Stud. Princeton, N.J.: Princeton Univ. Press, pp. 53–63. MR: 624804 (cit. on p. 1122).

Marc Burger and Shahar Mozes (2000). "Lattices in product of trees". *Inst. Hautes Études Sci. Publ. Math.* 92, 151–194 (2001). MR: 1839489 (cit. on p. 1127).

Danny Calegari (2009). *scl*. Vol. 20. MSJ Memoirs. Mathematical Society of Japan, Tokyo, pp. xii+209. MR: 2527432 (cit. on p. 1122).

Danny Calegari and Koji Fujiwara (2010). "Stable commutator length in word-hyperbolic groups". *Groups Geom. Dyn.* 4.1, pp. 59–90. MR: 2566301 (cit. on p. 1123).

Serge Cantat and Stéphane Lamy (2013). "Normal subgroups in the Cremona group". *Acta Math.* 210.1. With an appendix by Yves de Cornulier, pp. 31–94. MR: 3037611 (cit. on p. 1126).

Pierre-Emmanuel Caprace and Koji Fujiwara (2010). "Rank-one isometries of buildings and quasi-morphisms of Kac-Moody groups". *Geom. Funct. Anal.* 19.5, pp. 1296–1319. MR: 2585575 (cit. on p. 1125).

Pierre-Emmanuel Caprace and Michah Sageev (2011). "Rank rigidity for CAT(0) cube complexes". *Geom. Funct. Anal.* 21.4, pp. 851–891. MR: 2827012 (cit. on pp. 1125, 1127).

Ruth Charney and Harold Sultan (2015). "Contracting boundaries of CAT(0) spaces". *J. Topol.* 8.1, pp. 93–117. MR: 3339446 (cit. on p. 1125).

Marc Culler and Karen Vogtmann (1986). "Moduli of graphs and automorphisms of free groups". *Invent. Math.* 84.1, pp. 91–119. MR: 830040 (cit. on p. 1126).

F. Dahmani, V. Guirardel, and D. Osin (2017). "Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces". *Mem. Amer. Math. Soc.* 245.1156, pp. v+152. MR: 3589159 (cit. on p. 1126).

Thomas Delzant (2016). "Kähler groups, $\mathbb{R}$-trees, and holomorphic families of Riemann surfaces". *Geom. Funct. Anal.* 26.1, pp. 160–187. MR: 3494488 (cit. on p. 1124).

Cornelia Druţu and Mark Sapir (2005). "Tree-graded spaces and asymptotic cones of groups". *Topology* 44.5. With an appendix by Denis Osin and Mark Sapir, pp. 959–1058. MR: 2153979 (cit. on p. 1119).

H. Endo and D. Kotschick (2001). "Bounded cohomology and non-uniform perfection of mapping class groups". *Invent. Math.* 144.1, pp. 169–175. MR: 1821147 (cit. on p. 1123).

– (2007). "Failure of separation by quasi-homomorphisms in mapping class groups". *Proc. Amer. Math. Soc.* 135.9, pp. 2747–2750. MR: 2317948 (cit. on p. 1123).

David B. A. Epstein and Koji Fujiwara (1997). "The second bounded cohomology of word-hyperbolic groups". *Topology* 36.6, pp. 1275–1289. MR: 1452851 (cit. on pp. 1123, 1128).

Benson Farb, Alexander Lubotzky, and Yair N. Minsky (2001). "Rank-1 phenomena for mapping class groups". *Duke Math. J.* 106.3, pp. 581–597. MR: 1813237 (cit. on p. 1120).

Benson Farb and Dan Margalit (2012). *A primer on mapping class groups*. Vol. 49. Princeton Mathematical Series. Princeton, NJ: Princeton University Press, pp. xiv+472. MR: 2850125 (cit. on p. 1116).

Koji Fujiwara (1998). "The second bounded cohomology of a group acting on a Gromov-hyperbolic space". *Proc. London Math. Soc. (3)* 76.1, pp. 70–94. MR: 1476898 (cit. on p. 1128).

– (2000). "The second bounded cohomology of an amalgamated free product of groups". *Trans. Amer. Math. Soc.* 352.3, pp. 1113–1129. MR: 1491864 (cit. on p. 1128).

M. Gromov (1987). "Hyperbolic groups". In: *Essays in group theory*. Vol. 8. Math. Sci. Res. Inst. Publ. New York: Springer, pp. 75–263. MR: 919829 (cit. on p. 1105).

– (1993). "Asymptotic invariants of infinite groups". In: *Geometric group theory, Vol. 2 (Sussex, 1991)*. Vol. 182. London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, pp. 1–295. MR: 1253544 (cit. on p. 1109).

Thomas Haettel (n.d.). "Hyperbolic rigidity of higher rank lattices". arXiv: 1607.02004 (cit. on p. 1128).

Ursula Hamenstädt (2009). "Geometry of the mapping class groups. I. Boundary amenability". *Invent. Math.* 175.3, pp. 545–609. MR: 2471596 (cit. on p. 1110).

Michael Handel and Lee Mosher (n.d.). "Hyperbolic actions and 2nd bounded cohomology of subgroups of $Out(F_n)$. Part II: Finite lamination subgroups". arXiv: 1702.08050 (cit. on p. 1124).

– (2013). "The free splitting complex of a free group, I: hyperbolicity". *Geom. Topol.* 17.3, pp. 1581–1672. MR: 3073931 (cit. on p. 1128).

Pierre de la Harpe and Alain Valette (1989). "La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger)". *Astérisque* 175. With an appendix by M. Burger, p. 158. MR: 1023471 (cit. on p. 1125).

Sebastian Hensel, Piotr Przytycki, and Richard C. H. Webb (2015). "1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs". *J. Eur. Math. Soc. (JEMS)* 17.4, pp. 755–762. MR: 3336835 (cit. on p. 1117).

Yoshikata Kida (2008). "The mapping class group from the viewpoint of measure equivalence theory". *Mem. Amer. Math. Soc.* 196.916, pp. viii+190. MR: 2458794 (cit. on p. 1110).

Mustafa Korkmaz (2004). "Stable commutator length of a Dehn twist". *Michigan Math. J.* 52.1, pp. 23–31. MR: 2043394 (cit. on p. 1123).

Johanna Mangahas (2010). "Uniform uniform exponential growth of subgroups of the mapping class group". *Geom. Funct. Anal.* 19.5, pp. 1468–1480. MR: 2585580 (cit. on pp. 1118, 1121).

– (2013). "A recipe for short-word pseudo-Anosovs". *Amer. J. Math.* 135.4, pp. 1087–1116. MR: 3086070 (cit. on p. 1118).

J. F. Manning (2006). "Quasi-actions on trees and property (QFA)". *J. London Math. Soc. (2)* 73.1. With an appendix by N. Monod and B. Rémy, pp. 84–108. MR: 2197372 (cit. on pp. 1120, 1128).

Jason Fox Manning (2005). "Geometry of pseudocharacters". *Geom. Topol.* 9, 1147–1185 (electronic). MR: 2174263 (cit. on pp. 1109, 1128).

H. A. Masur and Yair N. Minsky (2000). "Geometry of the complex of curves. II. Hierarchical structure". *Geom. Funct. Anal.* 10.4, pp. 902–974. MR: 1791145 (cit. on pp. 1116–1119).

Howard A. Masur and Yair N. Minsky (1999). "Geometry of the complex of curves. I. Hyperbolicity". *Invent. Math.* 138.1, pp. 103–149. MR: 1714338 (cit. on p. 1116).

Yair N. Minsky (1996). "Quasi-projections in Teichmüller space". *J. Reine Angew. Math.* 473, pp. 121–136. MR: 1390685 (cit. on pp. 1116, 1126).

– (2006). "Curve complexes, surfaces and 3-manifolds". In: *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, pp. 1001–1033. MR: 2275633 (cit. on p. 1116).

Nicolas Monod (2006). "An invitation to bounded cohomology". In: *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, pp. 1183–1211. MR: 2275641 (cit. on p. 1128).

Lee Mosher, Michah Sageev, and Kevin Whyte (2003). "Quasi-actions on trees. I. Bounded valence". *Ann. of Math. (2)* 158.1, pp. 115–164. MR: 1998479 (cit. on p. 1121).

D. Osin (2016). "Acylindrically hyperbolic groups". *Trans. Amer. Math. Soc.* 368.2, pp. 851–888. MR: 3430352 (cit. on p. 1127).

John Roe (2003). *Lectures on coarse geometry*. Vol. 31. University Lecture Series. American Mathematical Society, Providence, RI, pp. viii+175. MR: 2007488 (cit. on pp. 1110, 1120).

Z. Sela (1997). "Acylindrical accessibility for groups". *Invent. Math.* 129.3, pp. 527–565. MR: 1465334 (cit. on p. 1126).

Jean-Pierre Serre (1980). *Trees*. Translated from the French by John Stillwell. Springer-Verlag, Berlin-New York, pp. ix+142. MR: 607504 (cit. on p. 1105).

Karen Vogtmann (2006). "The cohomology of automorphism groups of free groups". In: *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, pp. 1101–1117. MR: 2275637 (cit. on p. 1126).

Richard C. H. Webb (2015). "Combinatorics of tight geodesics and stable lengths". *Trans. Amer. Math. Soc.* 367.10, pp. 7323–7342. MR: 3378831 (cit. on p. 1120).

Guoliang Yu (1998). "The Novikov conjecture for groups with finite asymptotic dimension". *Ann. of Math. (2)* 147.2, pp. 325–355. MR: 1626745 (cit. on p. 1110).

Received 2017-11-21.

Koji Fujiwara (藤原　耕二)
Department of Mathematics
Kyoto University
Japan
kfujiwara@math.kyoto-u.ac.jp