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ALGEBRAIC SURFACES WITH MINIMAL BETTI NUMBERS

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Abstract

These are algebraic surfaces with the Betti numbers of the complex projective plane, and are called $\mathbb{Q}$-homology projective planes. Fake projective planes and the complex projective plane are smooth examples. We describe recent progress in the study of such surfaces, singular ones and fake projective planes. We also discuss open questions.

1 $\mathbb{Q}$-homology Projective Planes and Montgomery-Yang problem

A normal projective surface with the Betti numbers of the complex projective plane $\mathbb{CP}^2$ is called a *rational homology projective plane* or a $\mathbb{Q}$*-homology* $\mathbb{CP}^2$. When a normal projective surface S has only rational singularities, S is a $\mathbb{Q}$-homology $\mathbb{CP}^2$ if its second Betti number $b_2(S) = 1$. This can be seen easily by considering the Albanese fibration on a resolution of S.

It is known that a $\mathbb{Q}$-homology $\mathbb{CP}^2$ with quotient singularities (and no worse singularities) has at most 5 singular points (cf. Hwang and Keum [2011b, Corollary 3.4]). The $\mathbb{Q}$-homology projective planes with 5 quotient singularities were classified in Hwang and Keum [ibid.].

In this section we summarize progress on the Algebraic Montgomery-Yang problem, which was formulated by J. Kollár.

Conjecture 1.1 (Algebraic Montgomery–Yang Problem Kollár [2008]). *Let S be a $\mathbb{Q}$-homology projective plane with quotient singularities. Assume that $S^0 := S\backslash Sing(S)$ is simply connected. Then S has at most* 3 *singular points.*

This is the algebraic version of Montgomery–Yang Problem Fintushel and Stern [1987], which was originated from pseudofree circle group actions on higher dimensional sphere.

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Pseudofree circle group actions are those that have no points fixed by the entire circle group but that have isolated circles that are pointwise fixed by finite cyclic subgroups.

- Work over the field $\mathbb{C}$ of complex numbers, except a few remarks in positive characteristic and in differentiable case.

Definition 1.2. A normal projective surface S is called a $\mathbb{Q}$-homology $\mathbb{CP}^2$ if it has the same Betti numbers as $\mathbb{CP}^2$, i.e. $(b_0, b_1, b_2, b_3, b_4) = (1, 0, 1, 0, 1)$.

Examples 1.3.

1. If S is smooth, then $S = \mathbb{CP}^2$ or a fake projective plane(fpp).
2. If S is singular and has only A_1-singularities, then S is isomorphic to the quadric cone $(xy = z^2)$ in $\mathbb{CP}^3$, i.e., the weighted projective plane $\mathbb{CP}^2(1, 1, 2)$ (cf. Keum [2010]).
3. Cubic surfaces with $3A_2$ in $\mathbb{CP}^3$ (such surfaces can be shown to be isomorphic to $(w^3 = xyz)$).
4. If S has A_2-singularities only, then S has $3A_2$ or $4A_2$ and $S = \mathbb{CP}^2/G$ or fpp$/G$, where $G \cong \mathbb{Z}/3$ or $(\mathbb{Z}/3)^2$.
5. If S has A_1 or A_2-singularities only, then $S = \mathbb{CP}^2(1, 2, 3)$ or one of the above (see Keum [2015] for details).

In this section, S has at worst quotient singularities. Then it is easy to see that

- S is a $\mathbb{Q}$-homology $\mathbb{CP}^2$ if and only if $b_2(S) = 1$.
- A minimal resolution of S has $p_g = q = 0$.

1.1 Trichotomy: K_S ample, $-$ample, numerically trivial.

Let S a $\mathbb{Q}$-hom $\mathbb{CP}^2$ with quotient singularities, $f : S' \to S$ a minimal resolution. The canonical class K_S falls in one of the three cases.

1. $-K_S$ is ample (log del Pezzo surfaces of Picard number 1): their minimal resolutions have Kodaira dimension $\kappa(S') = -\infty$; typical examples are $\mathbb{CP}^2/G$, $\mathbb{CP}^2(a, b, c)$.
2. K_S is numerically trivial (log Enriques surfaces of Picard number 1): their minimal resolutions have Kodaira dimension $\kappa(S') = -\infty, 0$.

3. K_S is ample: : their minimal resolutions have Kodaira dimension $\kappa(S') = -\infty, 0, 1, 2$; typical examples are fake projective planes and their quotients, suitable contraction of a suitable blowup of some Enriques surface or $\mathbb{CP}^2$.

The following problem was raised by J. Kollár [2008].

Problem 1.4. Classify all $\mathbb{Q}$-homology projective planes with quotient singularities.

1.2 The Maximum Number of Quotient Singularities. Let S be a $\mathbb{Q}$-homology $\mathbb{CP}^2$ with quotient singularities. From the orbifold Bogomolov–Miyaoka–Yau inequality (Sakai [1980], Miyaoka [1984], and Megyesi [1999]), one can derive that S has at most 5 singular points (see Hwang and Keum [2011b, Corollary 3.4]). Many examples with 4 or less singular points were provided (Brenton [1977] and Brenton, Drucker, and Prins [1981]), but no example with 5 singular points until D. Hwang and the author characterised the case with 5 singular points.

Theorem 1.5 (Hwang and Keum [2011b])**.** *Let S be a $\mathbb{Q}$-homology projective plane with quotient singularities. Then S has at most 4 singular points except the following case which is supported by an example: S has 5 singular points of type $3A_1 + 2A_3$.*
In the exceptional case the minimal resolution of S is an Enriques surface.

In fact, our proof assumed the condition that K_S is nef to apply the orbifold Bogomolov–Miyaoka–Yau inequality. On the other hand, the case where $-K_S$ is ample was dealt with by Belousov [2008, 2009]. He proved that log del Pezzo surfaces of Picard number 1 with quotient singularities have at most 4 singular points.

Corollary 1.6. *The following hold true.*

1. *$\mathbb{Q}$-cohomology projective planes with quotient singularities have at most 4 singular points except the case given in the above theorem.*

2. *$\mathbb{Z}$-homology projective planes with quotient singularities have at most 4 singular points.*

Here, a $\mathbb{Q}$-cohomology projective plane is a normal projective complex surface having the same $\mathbb{Q}$-cohomology ring with $\mathbb{CP}^2$. A $\mathbb{Q}$-homology projective plane with quotient singularities is a $\mathbb{Q}$-cohomology projective plane.

The problem of determining the maximum number of singular points on $\mathbb{Q}$-homology projective planes with quotient singularities is related to the algebraic Montgomery–Yang problem (Montgomery and Yang [1972] and Kollár [2008]).

Remark 1.7. (1) Every $\mathbb{Z}$-cohomology $\mathbb{CP}^2$ with quotient singularities has at most 1 singular point, and, if it has, then the singularity must be of type E_8 (Bindschadler and Brenton [1984]).

(2) If a $\mathbb{Q}$-homology projective plane S is allowed to have rational singular points (quotient singular points are rational, but the converse does not holds), then there is no bound for the number of singular points. In fact, there are $\mathbb{Q}$-homology projective planes with an arbitrary number of rational singular points. Such examples can be constructed by modifying Example 5 from Kollár [2008]: take a minimal ruled surface $\mathbb{F}_e \to \mathbb{P}^1$ with negative section E, blow up m distinct fibres into m strings of 3 rational curves (-2)—(-1)—(-2), then contract the proper transform of E with the m adjacent (-2)-curves, and also the m remaining (-2)-curves. If $e = -E^2$ is sufficiently larger than m, then the big singular point is rational and yields a $\mathbb{Q}$-homology $\mathbb{CP}^2$ with $m+1$ rational singular points.

(3) In char 2 there is a rational example of Picard number 1 with seven A_1-singularities.

Theorem 1.8 (The orbifold BMY inequality)**.** *Let S be a normal projective surface with quotient singularities. Assume that K_S is nef. Then*

$$K_S^2 \le 3e_{orb}(S).$$

Theorem 1.9 (The weak oBMY inequality, Keel and McKernan [1999])**.** *Let S be a normal projective surface with quotient singularities. Assume that $-K_S$ is nef. Then*

$$0 \le e_{orb}(S).$$

Here the orbifold Euler characteristic is defined by

$$e_{orb}(S) := e(S) - \sum_{p \in Sing(S)} \left(1 - \frac{1}{|\pi_1(L_p)|}\right)$$

1.3 Smooth $\mathbf{S}^1$-actions on $\mathbf{S}^m$. Let $\mathbf{S}^1$ be the circle group. Consider a faithful $\mathcal{C}^\infty$ action of $\mathbf{S}^1$ on the m-dimensional sphere $\mathbf{S}^m$

$$\mathbf{S}^1 \subset Diff(\mathbf{S}^m).$$

The identity element $1 \in \mathbf{S}^1$ acts as the identity on $\mathbf{S}^m$. Each diffeomorphism $g \in \mathbf{S}^1$ is homotopic to the identity on $\mathbf{S}^m$. By Lefschetz Fixed Point Formula,

$$e(Fix(g)) = e(Fix(1)) = e(\mathbf{S}^m).$$

If m is even, then $e(\mathbf{S}^m) = 2$ and such an action has fixed points, so is not pseudofree. Pseudofree circle actions are those that have no points fixed by the entire circle group but that have isolated circles that are pointwise fixed by finite cyclic subgroups.

Assume $m = 2n - 1$ odd.

Definition 1.10. A faithful $\mathcal{C}^\infty$-action of $\mathbf{S}^1$ on $\mathbf{S}^{2n-1}$

$$\mathbf{S}^1 \times \mathbf{S}^{2n-1} \to \mathbf{S}^{2n-1}$$

is called *pseudofree* if it is free except for finitely many orbits whose isotropy groups $\mathbb{Z}/a_1\mathbb{Z}, \dots, \mathbb{Z}/a_k\mathbb{Z}$ have pairwise prime orders.

1.4 Pseudofree $\mathbf{S}^1$-actions on $\mathbf{S}^{2n-1}$.

Example 1.11 (Linear action)**.** For $a_1, ..., a_n$ pairwise prime

$$\mathbf{S}^1 \times \mathbf{S}^{2n-1} \to \mathbf{S}^{2n-1}$$

$$(\lambda, (z_1, z_2, ..., z_n)) \mapsto (\lambda^{a_1} z_1, \lambda^{a_2} z_2, ..., \lambda^{a_n} z_n)$$

where

$$\mathbf{S}^{2n-1} = \{(z_1, z_2, ..., z_n) : |z_1|^2 + |z_2|^2 + ... + |z_n|^2 = 1\} \subset \mathbb{C}^n$$
$$\mathbf{S}^1 = \{\lambda : |\lambda| = 1\} \subset \mathbb{C}.$$

In such a linear action

- $$\mathbf{S}^{2n-1}/\mathbf{S}^1 \cong \mathbb{CP}^{n-1}(a_1, a_2, ..., a_n);$$
- The orbit of the i-th coordinate point $e_i = (0, ..., 0, 1, 0..., 0) \in \mathbf{S}^{2n-1}$ is an exceptional orbit iff $a_i \geq 2$;
- The orbit of a non-coordinate point of $\mathbf{S}^{2n-1}$ is NOT exceptional;
- This action has at most n exceptional orbits;
- The quotient map $\mathbf{S}^{2n-1} \to \mathbb{CP}^{n-1}(a_1, a_2, ..., a_n)$ is a Seifert fibration.

For Seifert fibrations we recall the following.

1. For $n = 2$ Seifert [1933] showed that every pseudofree $\mathbf{S}^1$-action on $\mathbf{S}^3$ is diffeomorphic to a diagonal one and hence has at most 2 exceptional orbits.

2. For $n = 4$ Montgomery and Yang [1972] showed that given arbitrary collection of pairwise prime positive integers $a_1, \dots, a_k$, there is a pseudofree $\mathbf{S}^1$-action on a homotopy $\mathbf{S}^7$ whose exceptional orbits have exactly those orders.

3. Petrie [1975] generalised the result of Montgomery-Yang for all $n \geq 5$.

Conjecture 1.12 (Montgomery–Yang problem, Fintushel and Stern [1987]). *A pseudo-free $\mathbf{S}^1$-action on $\mathbf{S}^5$ has at most 3 exceptional orbits.*

The problem has remained unsolved since its formulation.

- Pseudo-free $\mathbf{S}^1$-actions on a manifold Σ have been studied in terms of the pseudofree orbifold $\Sigma/\mathbf{S}^1$ (see e.g., Fintushel and Stern [1985, 1987]).
- The orbifold $X = \mathbf{S}^5/\mathbf{S}^1$ is a 4-manifold with isolated singularities whose neighborhoods are cones over lens spaces corresponding to the exceptional orbits of the $\mathbf{S}^1$-action.
- Easy to check that X is simply connected and $H_2(X, \mathbb{Z})$ has rank 1 and intersection matrix $(1/a_1 a_2 \cdots a_k)$.
- An exceptional orbit with isotropy type $\mathbb{Z}/a$ has an equivariant tubular neighborhood which may be identified with $\mathbb{C} \times \mathbb{C} \times \mathbf{S}^1$ with a $\mathbf{S}^1$-action

  $$\lambda \cdot (z, w, u) = (\lambda^r z, \lambda^s w, \lambda^a u)$$

  where r and s are relatively prime to a.

The following 1-1 correspondence was known to Montgomery–Yang, Fintushel–Stern, and revisited by Kollár [2005, 2008].

Theorem 1.13. *There is a one-to-one correspondence between:*

1. *Pseudofree $\mathbf{S}^1$-actions on $\mathbb{Q}$-homology 5-spheres Σ with $H_1(\Sigma, \mathbb{Z}) = 0$.*
2. *Compact differentiable 4-manifolds M with boundary such that*
   (a) $\partial M = \bigcup_i L_i$ *is a disjoint union of lens spaces* $L_i = \mathbf{S}^3/\mathbb{Z}_{a_i}$,
   (b) *the orders a_i's are pairwise prime,*
   (c) $H_1(M, \mathbb{Z}) = 0$,
   (d) $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$.

Furthermore, Σ is diffeomorphic to $\mathbf{S}^5$ iff $\pi_1(M) = 1$.

1.5 Algebraic Montgomery-Yang Problem. This is the Montgomery-Yang Problem when the pseudofree orbifold $\mathbf{S}^5/\mathbf{S}^1$ attains a structure of a normal projective surface.

Conjecture 1.14 (Kollár [2008]). *Let S be a $\mathbb{Q}$-homology $\mathbb{CP}^2$ with at worst quotient singularities. If the smooth part S^0 has $\pi_1(S^0) = \{1\}$, then S has at most 3 singular points.*

What happens if the condition $\pi_1(S^0) = \{1\}$ is replaced by the weaker condition $H_1(S^0, \mathbb{Z}) = 0$?

Remark 1.15. If $H_1(S^0, \mathbb{Z}) = 0$, then
(1) K_S cannot be numerically trivial;
(2) it follows from the oBMY that $|Sing(S)| \le 4$.

There are infinitely many examples S with

$$H_1(S^0, \mathbb{Z}) = 0, \ \ \pi_1(S^0) \neq \{1\}, \ \ |Sing(S)| = 4,$$

obtained from the classification of surface quotient singularities by Brieskorn [1967/1968].

Example 1.16 (Brieskorn quotients). Let $I_m \subset GL(2, \mathbb{C})$ be the $2m$-ary icosahedral group $I_m = \mathbb{Z}_{2m}.\mathfrak{A}_5$,

$$1 \to \mathbb{Z}_{2m} \to I_m \to \mathfrak{A}_5 \subset PSL(2, \mathbb{C}).$$

The action of I_m on $\mathbb{C}^2$ extends naturally to $\mathbb{CP}^2$. Then

$$S := \mathbb{CP}^2/I_m$$

is a $\mathbb{Q}$-homology $\mathbb{CP}^2$ such that

- $-K_S$ ample;
- S has 4 quotient singularities: one non-cyclic singularity of type I_m (the image of the origin $O \in \mathbb{C}^2$) and 3 cyclic singularities of order $2, 3, 5$ (on the image of the line at infinity);
- $\pi_1(S^0) = \mathfrak{A}_5$, hence $H_1(S^0, \mathbb{Z}) = 0$.

1.6 Progress on Algebraic Montgomery-Yang Problem. Algebraic Montgomery-Yang problem holds true if S has at least 1 non-cyclic singular point.

Theorem 1.17 (Hwang and Keum [2011a]). *Let S be a $\mathbb{Q}$-homology $\mathbb{CP}^2$ with quotient singular points, not all cyclic, such that $\pi_1(S^0) = \{1\}$. Then $|Sing(S)| \le 3$.*

More precisely

Theorem 1.18. *Let S be a $\mathbb{Q}$-homology $\mathbb{CP}^2$ with 4 or more quotient singular points, not all cyclic, such that $H_1(S^0, \mathbb{Z}) = 0$. Then S is isomorphic to a Brieskorn quotient.*

Remark 1.19. In Hwang and Keum [2011a], though the proof was correct, a wrong conclusion was made in the statement (4) of Theorem 3 that the smooth part S^0 of such a surface S is deformation equivalent to the smooth part of a Brieskorn quotient. A corrected statement was given in Hwang and Keum [2009].

More Progress on the Algebraic Montgomery-Yang Problem:

Theorem 1.20 (Hwang and Keum [2013, 2014])**.** *Let S be a $\mathbb{Q}$-homology $\mathbb{CP}^2$ with cyclic singularities such that $H_1(S^0, \mathbb{Z}) = 0$. If either S is not rational or $-K_S$ is ample, then $|Sing(S)| \leq 3$.*

The Remaining Case of the Algebraic Montgomery-Yang Problem:

S is a $\mathbb{Q}$-homology $\mathbb{CP}^2$ such that

1. S has cyclic singularities only,
2. S is a rational surface with K_S ample.

Looking at the adjunction formula

$$K_{S'} = \pi^* K_S - \sum D_p,$$

where $S' \to S$ is a resolution, one sees that K_S, though ample, is "smaller than $\sum D_p$" so that no positive multiple of $K_{S'}$ is effective. Such surfaces were given by

Keel and McKernan [1999];
Kollár [2008]: infinite series of examples with $|Sing(S)| = 2$;
Hwang and Keum [2012]: infinite series of examples with $|Sing(S)| = 1, 2, 3$.
Urzúa and Yáñez [2016]: characterization of Kollár surfaces.

There is no known example with $|Sing(S)| = 4$, even if the condition $H_1(S^0, \mathbb{Z}) = 0$ is removed.

Problem 1.21. Is there a $\mathbb{Q}$-homology projective plane S which is a rational surface with K_S ample and $|Sing(S)| = 4$?

Another interesting line of research is to obtain surfaces with quotient singularities with small volume. Alexeev and Liu [2016] has constructed a surface S with log terminal singularities (quotient singularities) and ample canonical class that has $K_S^2 = 1/48983$ and a log canonical pair (S, B) with a nonempty reduced divisor B and ample $K_S + B$ that has $(K_S + B)^2 = 1/462$, both examples signifcantly improve known record.

1.7 Cascade structure on rational $\mathbb{Q}$-homology projective planes. A rational $\mathbb{Q}$-homology $\mathbb{CP}^2$ is obtained by blow-ups and downs from a "basic surface" which is a rational minimal surface with certain configuration of curves.
In the case where $-K_S$ ample, all basic surfaces have been classified by Hwang [n.d.].

1.8 Gorenstein $\mathbb{Q}$-homology projective planes. These are $\mathbb{Q}$-homology projective planes with ADE-singular points (i.e., rational double points).

Let R be the singularity type, i.e., the corresponding root sublattice of the cohomology lattice of S', the minimal resolution of S.
Since S is Gorenstein, rank(R) is bounded.

$$1 + \mathrm{rank}(R) = b_2(S') = 10 - K_{S'}^2 = 10 - K_S^2 \leq 10,$$

$$\mathrm{rank}(R) \leq 9$$

with equality iff K_S is numerically trivial iff S' is an Enriques surface.

With D. Hwang and H. Ohashi, we classified all possible singularity types of Gorenstein $\mathbb{Q}$-homology projective planes. There are 58 types total.

Theorem 1.22 (Hwang, Keum, and Ohashi [2015])**.** *The singularity type R of a Gorenstein $\mathbb{Q}$-homology $\mathbb{CP}^2$ is one of the following:*

(1) K_S *±ample* (27 *types*)*:*
A_8, A_7, D_8, E_8, E_7, E_6, D_5, A_4, A_1;
$A_7 \oplus A_1$, $A_5 \oplus A_2$, $A_5 \oplus A_1$, $2A_4$, $A_2 \oplus A_1$, $D_6 \oplus A_1$, $D_5 \oplus A_3$, $2D_4$, $E_7 \oplus A_1$, $E_6 \oplus A_2$;
$A_5 \oplus A_2 \oplus A_1$, $2A_3 \oplus A_1$, $A_3 \oplus 2A_1$, $3A_2$, $D_6 \oplus 2A_1$;
$2A_3 \oplus 2A_1$, $4A_2$, $D_4 \oplus 3A_1$,

(2) K_S *numerically trivial* (31 *types*)
A_9, D_9;
$A_8 \oplus A_1$, $A_7 \oplus A_2$, $A_5 \oplus A_4$, $D_8 \oplus A_1$, $D_6 \oplus A_3$, $D_5 \oplus A_4$, $D_5 \oplus D_4$, $E_8 \oplus A_1$, $E_7 \oplus A_2$, $E_6 \oplus A_3$;
$A_7 \oplus 2A_1$, $A_6 \oplus A_2 \oplus A_1$, $A_5 \oplus A_3 \oplus A_1$, $A_5 \oplus 2A_2$, $2A_4 \oplus A_1$, $3A_3$, $D_7 \oplus 2A_1$, $D_6 \oplus A_2 \oplus A_1$, $D_5 \oplus A_3 \oplus A_1$, $2D_4 \oplus A_1$, $E_7 \oplus 2A_1$, $E_6 \oplus A_2 \oplus A_1$;

$A_5 \oplus A_2 \oplus 2A_1$, $A_4 \oplus A_3 \oplus 2A_1$, $2A_3 \oplus A_2 \oplus A_1$, $A_3 \oplus 3A_2$, $D_6 \oplus 3A_1$, $D_4 \oplus A_3 \oplus 2A_1$; $2A_3 \oplus 3A_1$.

The 27 types with $-K_S$ ample were classified by Furushima [1986], Miyanishi and Zhang [1988], Ye [2002]. Our method uses only lattice theory, different from theirs.

Among the 31 types with $K_S \equiv 0$, 29 types are supported by Enriques surfaces with finite automorphism group. Enriques surfaces with $|Aut| < \infty$ must have finitely many (-2)-curves, and were classified by Nikulin and Kondō [1986] into 7 families, two families 1-dimensional and five 0-dimensional. Schütt [2015] has constucted explicitly, for each of the 31 types, the moduli space of such Enriques surfaces, all 1-dimensional.

Remark 1.23. In positive characteristic the case with $K_S \equiv 0$ has been classified by M. Schütt [2016, 2017], which build on recent deep results of Katsura and Kondō [2018], Martin [2017], and Katsura, Kondō, and Martin [2017], also on a unpublished work of Dolgachev-Liedtke.

1.9 The differentiable case. Let M be a smooth, compact 4-manifold whose boundary components are spherical, that is, lens spaces $L_i = \mathbf{S}^3$. One can then attach cones to each boundary component to get a 4-dimensional orbifold S. As in the algebraic case, there is a minimal resolution $f : S' \to S$, where S' is a smooth, compact 4-manifold without boundary.

To each singular point $p \in S$ (the vertex of each cone), we assign a uniquely defined class $\mathfrak{D}_p = \sum(a_j E_j) \in H^2(S', \mathbb{Q})$ such that

$$\mathfrak{D}_p \cdot E_i = 2 + E_i^2$$

for each component E_i of $f^{-1}(p)$. We always assume that S and S' satisfy the following two conditions:

(1) S is a $\mathbb{Q}$-homology $\mathbb{CP}^2$, i.e. $H^1(S, \mathbb{Q}) = 0$ and $H^2(S, \mathbb{Q}) \cong \mathbb{Q}$.

(2) The intersection form on $H^2(S', \mathbb{Q})$ is indefinite, and is negative definite on the subspace generated by the classes of the exceptional curves of f.

If there is a class $K_{S'} \in H^2(S', \mathbb{Q})$ satisfying both the Noether formula

$$K_{S'}^2 = 10 - b_2(S')$$

and the adjunction formula

$$K_{S'} \cdot E + E^2 = -2$$

for each exceptional curve E of f, we call it a *formal canonical class of* S', and the class $K_{S'} + \sum \mathfrak{D}_p \in H^2(S', \mathbb{Q})$ a *formal canonical class of* S.

Theorem 1.24 (Hwang and Keum [2011b] Theorem 8.1)**.** *Let M, S, and S' be the same as above satisfying the conditions* (1) *and* (2)*. Assume that S' admits a formal canonical class $K_{S'}$. Assume further that*

$$K_{S'}^2 - \sum_{p \in Sing(S)} \mathfrak{D}_p^2 \leq 3e_{orb}(S).$$

Then M has at most 4 boundary components except the following two cases:
M has 5 boundary components of type $3A_1 + 2A_3$ or $4A_1 + D_5$.

Note that the assumptions in Theorem 1.24 all hold for algebraic $\mathbb{Q}$-homology projective planes with quotient singularities such that the canonical divisor is nef.

Theorem 1.25 (Hwang and Keum [ibid.] Theorem 8.2)**.** *Let M, S, and S' be the same as above satisfying the conditions* (1) *and* (2)*. Assume that S' admits a formal canonical class $K_{S'}$. Assume further that*

$$0 \leq e_{orb}(S).$$

Then M has at most 5 boundary components. The bound is sharp.

The assumptions in Theorem 1.25 all hold for algebraic $\mathbb{Q}$-homology projective planes with quotient singularities.

1.10 The symplectic case. If S is a symplectic orbifold, then S' is a symplectic manifold and the symplectic canonical class $K_{S'}$ gives a formal canonical class.

Problem 1.26. Is there a Bogomolov–Miyaoka–Yau type inequality for symplectic 4-manifolds?
Is there an orbifold Bogomolov–Miyaoka–Yau type inequality for symplectic orbifolds?

The following question is also interesting in view of Sasakian geometry.

Problem 1.27 (Muñoz, Rojo, and Tralle [2016])**.** There does not exist a Kähler manifold or a Kähler orbifold with $b_1 = 0$ and $b_2 \geq 2$ having b_2 disjoint complex curves all of genus $g \geq 1$ which generate $H_2(S, \mathbb{Q})$.

A ruled surface over a curve of genus g has two disjoint curves, the negative section and a general section, of genus g, but has $b_1 \neq 0$ if $g \geq 1$.

2 Fake projective planes

A compact complex surface with the same Betti numbers as the complex projective plane is called *a fake projective plane* if it is not biholomorphic to the complex projective plane.

A fake projective plane has ample canonical divisor, so it is a smooth proper (geometrically connected) surface of general type with geometric genus $p_g = 0$ and self-intersection of canonical class $K^2 = 9$ (this definition extends to arbitrary characteristic.) The existence of a fake projective plane was first proved by Mumford [1979] based on the theory of 2-adic uniformization, and later two more examples by Ishida and Kato [1998] 1998) in a similar method. Keum [2006] gave a construction of a fake projective plane with an order 7 automorphism, which is birational to an order 7 cyclic cover of a Dolgachev surface. This surface and Mumford fake projective plane belong to the same class, in the sense that their fundamental groups are both contained in the same maximal arithmetic subgroup of the isometry group of the complex 2-ball.

Fake projective planes have Chern numbers $c_1^2 = 3c_2 = 9$ and are complex 2-ball quotients by Aubin [1976] and Yau [1977]. Such ball quotients are strongly rigid by Mostow's rigidity theorem (Mostow [1973]), that is, determined by fundamental group up to holomorphic or anti-holomorphic isomorphism. Fake projective planes come in complex conjugate pairs by Kulikov and Kharlamov [2002] and have been classified as quotients of the two-dimensional complex ball by explicitly written co-compact torsion-free arithmetic subgroups of $\mathrm{PU}(2,1)$ by Prasad and Yeung [2007, 2010] and Cartwright and Steger [2010, n.d.]. The arithmeticity of their fundamental groups was proved by Klingler [2003]. There are exactly 100 fake projective planes total, corresponding to 50 distinct fundamental groups. Cartwright and Steger also computed the automorphism group of each fake projective plane X, which is given by $\mathrm{Aut}(X) \cong N(X)/\pi_1(X)$, where $N(X)$ is the normalizer of $\pi_1(X)$ in its maximal arithmetic subgroup of $\mathrm{PU}(2,1)$. In particular $\mathrm{Aut}(X) \cong \{1\}$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or G_{21} where $\mathbb{Z}_n$ is the cyclic group of order n and G_{21} is the unique non-abelian group of order 21. Among the 50 pairs exactly 33 admit non-trivial automorphisms: 3 pairs have $\mathrm{Aut} \cong G_{21}$, 3 pairs have $\mathrm{Aut} \cong \mathbb{Z}_3^2$ and 27 pairs have $\mathrm{Aut} \cong \mathbb{Z}_3$.

For each pair of fake projective planes Cartwright and Steger [n.d.] also computed the torsion group

$$H_1(X,\mathbb{Z}) = \mathrm{Tor}(H^2(X,\mathbb{Z})) = \mathrm{Tor}(\mathrm{Pic}(X))$$

which is the abelianization of the fundamental group. According to their computation exactly 29 pairs of fake projective planes have a 3-torsion in $H_1(X,\mathbb{Z})$.

In this section we summarize recent progress on these fascinating objects.

2.1 Picard group of a fake projective plane. Since $p_g(X) = q(X) = 0$, the long exact sequence induced by the exponential sequence gives $\mathrm{Pic}(X) = H^2(X,\mathbb{Z})$. By the universal coefficient theorem, $\mathrm{Tor}H^2(X,\mathbb{Z}) = \mathrm{Tor}H_1(X,\mathbb{Z})$. This implies that

$$\mathrm{Pic}(X) = H^2(X,\mathbb{Z}) \cong \mathbb{Z} \times H_1(X,\mathbb{Z}).$$

Two ample line bundles with the same self-intersection number on a fake projective plane differ by a torsion.

It can be shown (cf. Keum [2017, Lemma 1.5]) that if a fake projective plane X has no 3-torsion in $H_1(X, \mathbb{Z})$ (21 pairs of fake projective planes satisfy this property), then the canonical class K_X is divisible by 3 and has a unique cube root, i.e., a unique line bundle L_0 up to isomorphism such that $3L_0 \equiv K_X$.

By a result of Kollár [1995, p. 96] the 3-divisibility of K_X is equivalent to the liftability of the fundamental group to $\mathrm{SU}(2,1)$. Except 4 pairs of fake projective planes the fundamental groups lift to $\mathrm{SU}(2,1)$ (Prasad and Yeung [2010, Section 10.4], Cartwright and Steger [2010, n.d.]). In the notation of Cartwright and Steger [2010], these exceptional 4 pairs are the 3 pairs in the class $(\mathcal{C}_{18}, p = 3, \{2\})$, whose automorphism groups are of order 3, and the one in the class $(\mathcal{C}_{18}, p = 3, \{2I\})$, whose automorphism group is trivial. There are fake projective planes with a 3-torsion and with canonical class divisible by 3 Cartwright and Steger [n.d.].

2.2 Quotients of fake projective planes. Let X be a fake projective plane with a nontrivial group G acting on it. In Keum [2008], all possible structures of the quotient surface X/G and its minimal resolution were classified:

Theorem 2.1 (Keum [ibid.]). *1. If $G = \mathbb{Z}_3$, then X/G is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{3}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 3$.*

2. If $G = \mathbb{Z}_3^2$, then X/G is a $\mathbb{Q}$-homology projective plane with 4 singular points of type $\frac{1}{3}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 1$.

3. If $G = \mathbb{Z}_7$, then X/G is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{7}(1,5)$ and its minimal resolution is a $(2,3)$- or $(2,4)$- or $(3,3)$-elliptic surface.

4. If $G = \mathbb{Z}7 : \mathbb{Z}3 = G_{21}$, then X/G is a $\mathbb{Q}$-homology projective plane with 4 singular points, where three of them are of type $\frac{1}{3}(1,2)$ and one of them is of type $\frac{1}{7}(1,5)$, and its minimal resolution is a $(2,3)$- or $(2,4)$- or $(3,3)$-elliptic surface.

A fake projective plane is a nonsingular $\mathbb{Q}$-homology projective plane, hence every quotient is again a $\mathbb{Q}$-homology projective plane. An (a,b)-elliptic surface is a relatively minimal elliptic surface over $\mathbb{P}^1$ with $c_2 = 12$ having two multiple fibres of multiplicity a and b respectively. It has Kodaira dimension 1 if and only if $a \geq 2, b \geq 2, a + b \geq 5$. It is an Enriques surface iff $a = b = 2$. It is rational iff $a = 1$ or $b = 1$. All (a, b)-elliptic surfaces have $p_g = q = 0$, and by van Kampfen theorem its fundamental group is the cyclic group $\mathbb{Z}_d$ (see Dolgachev [2010]), where d is the greatest common divisor of a and b. A simply connected (a, b)-elliptic surface is called a Dolgachev surface.

Remark 2.2. The possibility of $(3, 3)$-elliptic surface was further removed by the computation of Cartwright-Steger. In Cartwright and Steger [n.d.] they also computed the fundamental group of each quotient X/G, which is by the result of Armstrong [1968] isomorphic to the quotient group of the augmented fundamental group $\langle \pi_1(X), G' \rangle$ by the normal subgroup generated by elements with nonempty fixed locus on the complex 2-ball, where G' is a lifting of G onto the ball. According to their computation, $\pi_1(X/G) = \{1\}$ or $\mathbb{Z}_2$ if $G = \mathbb{Z}_7$.

2.3 Vanishing theorem for some fake projective planes. For an ample line bundle M on a fake projective plane X, M^2 is a square integer. When $M^2 \geq 9$, $H^0(X, M) \neq 0$ if and only if $M \not\cong K_X$. This follows from the Riemann-Roch and the Kodaira vanishing theorem. When $M^2 \leq 4$, $H^0(X, M)$ may not vanish, though no example of non-vanishing so far has been known. If it does not vanish, then it gives an effective curve of small degree. The non-vanishing of $H^0(X, M)$ is equivalent to the existence of certain automorphic form on the 2-ball.

Theorem 2.3 (Keum [2017])**.** *Let X be a fake projective plane with* $\mathrm{Aut}(X) \cong \mathbb{Z}_7 : \mathbb{Z}_3$. *Then for every $\mathbb{Z}_7$-invariant ample line bundle M with $M^2 = 4$ we have the vanishing*

$$H^0(X, M) = 0.$$

In particular, for each line bundle L with $L^2 = 1$

$$H^0(X, 2L) = 0.$$

Remark 2.4.

1. A fake projective plane with $\mathrm{Aut}(X) \cong \mathbb{Z}_7 : \mathbb{Z}_3$ has only 2-torsions Cartwright and Steger [n.d.], more precisely

$$H_1(X, \mathbb{Z}) = \mathbb{Z}_2^3,\ \mathbb{Z}_2^4,\ \mathbb{Z}_2^6.$$

2. Thus K_X of such a surface has a unique cube root L_0.

3. For such a surface two ample line bundles with the same self-intersection number differ by a 2-torsion. If $M^2 = m^2$, then $M \equiv mL_0 + t$ for a 2-torsion t, hence $2M \equiv 2mL_0$ and is invariant under every automorphism.

4. The above theorem in Keum [2017] was stated only for the case $M = 2L$, but the proof used only the invariance of M under the order 7 automorphism.

5. If $M = 2L_0 + t$ is invariant under an automorphism iff so is t.

6. By Catanese and Keum [2018] the $\mathbb{Z}_7$ action on $H_1(X, \mathbb{Z})$ fixes no 2-torsion element in the case of $H_1(X, \mathbb{Z}) \cong \mathbb{Z}_2^3,\ \mathbb{Z}_2^6$, and one in the case of $H_1(X, \mathbb{Z}) \cong \mathbb{Z}_2^4$.

Theorem 2.5 (Keum [2017])**.** *Let X be a fake projective plane with* $\mathrm{Aut}(X) \cong \mathbb{Z}_3^2$*. Then for every* $\mathrm{Aut}(X)$*-invariant ample line bundle M with $M^2 = 4$ we have the vanishing*

$$H^0(X, M) = 0.$$

In particular, for the cubic root L_0 of K_X

$$H^0(X, 2L_0) = 0.$$

Remark 2.6. 1. A fake projective plane with $\mathrm{Aut}(X) \cong \mathbb{Z}_3^2$ has

$$H_1(X, \mathbb{Z}) = \mathbb{Z}_{14},\ \mathbb{Z}_7,\ \mathbb{Z}_2^2 \times \mathbb{Z}_{13}.$$

2. Thus K_X of such a surface has a unique cube root L_0.

3. For such a surface two ample line bundles with the same self-intersection number differ by a torsion.

4. The above theorem in Keum [ibid.] was stated only for the case $M = 2L_0$, but the proof used only the invariance of M under $\mathrm{Aut}(X) \cong \mathbb{Z}_3^2$.

5. If $M = 2L_0 + t$ is invariant under an automorphism iff so is t.

6. By Catanese and Keum [2018] no torsion element is $\mathbb{Z}_3^2$-invariant in the case of $H_1(X, \mathbb{Z}) \cong \mathbb{Z}_7,\ \mathbb{Z}_2^2 \times \mathbb{Z}_{13}$, and only the unique 2-torsion is $\mathbb{Z}_3^2$-invariant in the case of $H_1(X, \mathbb{Z}) \cong \mathbb{Z}_{14}$.

Both proofs used the structure of the quotients of X given in the previous subsection. The key idea of proof is that if $H^0(X, M) \neq 0$, then $\dim H^0(X, 2M) \geq 4$, contradicting the Riemann-Roch which yields $\dim H^0(X, 2M) = 3$.

2.4 Exceptionl collections of line bundles. Let $D^b(coh(W))$ denote the bounded derived category of coherent sheaves on a smooth variety W. It is a triangulated category. An object E in a triangulated category is called exceptional if $Hom(E, E[i]) = \mathbb{C}$ if $i = 0$, and $= 0$ otherwise. A sequence $E_1, \ldots, E_n$ of exceptional objects is called an exceptional sequence if $Hom(E_j, E_k[i]) = 0$ for any $j > k$, any i. When W is a smooth surface with $p_g = q = 0$, every line bundle is an exceptional object in $D^b(coh(W))$.

Let X be a fake projective plane and L be an ample line bundle with $L^2 = 1$. The three line bundles

$$2L,\ L,\ \mathcal{O}_X$$

form an exceptional sequence if and only if $H^j(X, 2L) = H^j(X, L) = 0$ for all j. Write

$$D^b(coh(X)) = \langle 2L,\ L,\ \mathcal{O}_X,\ \mathcal{A}\rangle$$

where $\mathcal{A}$ is the orthogonal complement of the admissible triangulated subcategory generated by $2L,\ L,\ \mathcal{O}_X$. Then the Hochshield homology

$$HH_*(\mathcal{A}) = 0.$$

This can be read off from the Hodge numbers. In fact, the Hochshield homology of X is the direct sum of Hodge spaces $H^{p,q}(X)$, and its total dimension is the sum of all Hodge numbers. The latter is equal to the topological Euler number $c_2(X)$, as a fake projective plane has Betti numbers $b_1(X) = b_3(X) = 0$.

The Grothendieck group $K_0(X)$ has filtration

$$K_0(X) = F^0K_0(X) \supset F^1K_0(X) \supset F^2K_0(X)$$

with

$$F^0K_0(X)/F^1K_0(X) \cong CH^0(X) \cong \mathbb{Z},$$

$$F^1K_0(X)/F^2K_0(X) \cong \mathrm{Pic}(X),$$

$$F^2K_0(X) \cong CH^2(X).$$

If the Bloch conjecture holds for X, i.e. if $CH^2(X) \cong \mathbb{Z}$, then $K_0(\mathcal{A})$ is finite.

Corollary 2.7. *Let X be a fake projective plane with $\mathrm{Aut}(X) \cong \mathbb{Z}_7 : \mathbb{Z}_3$ or $\mathbb{Z}_3^2$. Let L_0 be the unique cubic root of K_X. Then the three line bundles*

$$\mathcal{O}_X,\ -L_0,\ -2L_0$$

form an exceptional collection on X. If t is a $\mathbb{Z}_7$- or $\mathbb{Z}_3^2$-invariant torsion line bundle, then the three line bundles

$$\mathcal{O}_X,\ -(L_0 + t),\ -2L_0$$

form another exceptional collection.

Such a torsion line bundle t exists for only one pair of fake projective planes in each case $\mathrm{Aut}(X) \cong \mathbb{Z}_7 : \mathbb{Z}_3$ or $\mathbb{Z}_3^2$ by Catanese and Keum [2018] (see Remark 2.4, Remark 2.6).

This is equivalent to that $H^i(X, 2L_0) = H^i(X, L_0) = H^i(X, L_0 + t) = 0$ for all i, hence follows from Theorem 2.3 and 2.5. Indeed, since L_0 is a cubic root of K_X, these vanishings are equivalent to the vanishing $H^0(X, 2L_0) = H^0(X, 2L_0 + t) = 0$. This confirms, for fake projective planes with enough automorphisms, the conjecture raised by Galkin, Katzarkov, Mellit, and Shinder [2013] that predicts the existence of an exceptional sequence of length 3 on every fake projective plane. Disjoint from our cases, Fakhruddin [2015] confirmed the conjecture for the case of three 2-adically uniformized fake projective planes found by Mumford [1979] and Ishida and Kato [1998].

2.5 Bicanonical map of fake projective planes. By Reider's theorem Reider [1988] (see Barth, Hulek, Peters, and Van de Ven [2004] for a slightly refined version) on adjoint linear systems the bicanonical system $|2K_X|$ of a ball quotient X is base point free, thus it defines a morphism.

If the ball quotient X has $\chi(X) \geq 2$, then $K_X^2 \geq 9\chi(X) \geq 10$, and since a ball quotient cannot contain a curve of geometric genus 0 or 1, the bicanonical map embeds X unless X contains a smooth genus 2 curve C with $C^2 = 0$, and $CK_X = 2$.

In the case $\chi(X) = 1$, for instance if we have a fake projective plane, we are below the Reider inequality $K_X^2 \geq 10$, and the question of the very-ampleness of the bicanonical system is interesting.

Conjecture 2.8. *For each fake projective plane its bicanonical map is an embedding into* $\mathbb{P}^9$.

Every fake projective plane X with automorphism group of order 21 cannot contain an effective curve with self-intersection 1 (Theorem 2.3), as was first proved in Keum [2013] (see Keum [2017], also Galkin, Katzarkov, Mellit, and Shinder [2015]). Thus by applying I. Reider's theorem, one sees that the bicanonical map of such a fake projective plane is an embedding into $\mathbb{P}^9$ (see also Di Brino and Di Cerbo [2018]).

Including these 3 pairs of fake projective planes, for 10 pairs the conjecture has been confirmed by Catanese and Keum [2018]. For nine pairs this follows from the vanishing result of Keum [2013, 2017], Catanese and Keum [2018]. For one pair we do not have the vanishing theorem, and the surface possesses either none or 3 curves D with $D^2 = 1$. But even in the latter case we manage to prove the very-ampleness of the bicanonical system.

2.6 Explicit equations of fake projective planes. It has long been of great interest since Mumford to find equations of a projective model of a fake projective plane.

In a recent joint work Borisov and Keum [2017, 2018] we find equations of a projective model (the bicanonical image) of a conjugate pair of fake projective planes by studying the geometry of the quotient of such surface by an order seven automorphism. The equations are given explicitly as 84 cubics in 10 variables with coefficients in the field $\mathbb{Q}[\sqrt{-7}]$. The complex conjugate equations define the bicanonical image of the complex conjugate of the surface,

This pair has the most geometric symetries among the 50 pairs, in the sense that it has the large automorphism group $G_{21} = \mathbb{Z}_7 : \mathbb{Z}_3$ and the $\mathbb{Z}_7$-quotient has a smooth model of a $(2, 4)$-elliptic surface which is not simply connected. For several pairs of fake projective planes including this pair the bicanonical map gives an embedding into the 9-dimensional projective space.

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