PROC. INT. CONG. OF MATH. – 2018
Rio de Janeiro, Vol. 3 (2131–2154)

QUANTITATIVE ALMOST REDUCIBILITY AND ITS APPLICATIONS

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Abstract

We survey the recent advances of almost reducibility and its applications in the spectral theory of one dimensional quasi-periodic Schrödinger operators.

1 Quasi-periodic operators, cocycles and systems

1.1 One dimensional quasi-periodic Schrödinger operators. One dimensional quasi-periodic discrete time Schrödinger operators are operators defined on $l^2(\mathbb{Z})$ as

$$(H_{V,\omega,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\omega)u_n, \quad \forall n \in \mathbb{Z}, \tag{1-1}$$

where $\theta \in \mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$ is called *phase*, $V : \mathbb{T}^d \to \mathbb{R}$ is called *potential*, rationally independent $\omega \in \mathbb{T}^d$ is called *frequency*(when ω is one dimensional, we will replace it by α to respect the traditional notation in literatures). The simplest but the most important special case is the *almost Mathieu operators*(AMO), i.e., the three-parameter family:

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)u_n, \quad \forall n \in \mathbb{Z}. \tag{1-2}$$

Due to the rich backgrounds in quantum physics, quasi-periodic Schrödinger operators especially the almost Mathieu operators have been extensively studied Last [2005]. In 1980's, there was an almost periodic flu which already swept the world Simon [1982]. In 2000's, people found that one can use ideas from the dynamical systems (mainly linear co-cycles) to study the operators (1-1), and many important progresses have been made since then (Avila [2008, 2015a], Avila and Krikorian [2006], Avila and Jitomirskaya [2009], and Puig [2004]). This survey will focus on how almost reducibility is used to give a systematical study of various delicate spectral properties of (1-1).

This work was partially supported by NNSF of China (11471155) and 973 projects of China (2014CB340701).
MSC2010: primary 37C55; secondary 37J40, 47B36.

It is well known that $H = H_{V,\omega,\theta}$ is a bounded self-adjoint operator, its spectrum $\Sigma_{V,\omega,\theta}$ is a compact perfect subset of $\mathbb{R}$ which is independent of θ if ω is rationally independent. In the following, when no danger of confusion, we sometimes simply denote $H_{V,\omega,\theta}$ and $\Sigma_{V,\omega,\theta}$ by H and Σ.

Given an operator $H = H_{V,\omega,\theta}$ and a $\phi \in l^2(\mathbb{Z})$, we define a measure μ^ϕ on Σ such that

$$\langle \phi, f(H)\phi\rangle = \int f(E)d\mu^\phi(E),$$

holds for any $f \in C^0(\Sigma)$, $d\mu = d\mu^{e_0} + d\mu^{e_1}$ is called the *spectral measure* of H. And the *integrated density of states* (IDS) $N : \mathbb{R} \to [0,1]$ of H is defined as

$$N(E) := \int_{\mathbb{T}} d\mu(-\infty, E]\, d\theta. \tag{1-3}$$

$N(E)$ is always monotone and continuous no matter what $d\mu$ is. Any bounded connected component of $\mathbb{R}\backslash\Sigma$ is called a *spectral gap* of the operator $H_{V,\omega,\theta}$. By Gap-Labelling Theorem (Johnson and Moser [1982]), there is a unique $k \in \mathbb{Z}^d$ such that $N(E) = \langle k, \omega\rangle \mod \mathbb{Z}$ for all E in a gap. In other words, the spectral gaps can be labelled by $k \in \mathbb{Z}^d$. We denote by $G_k(V) = (E_k^-(V), E_k^+(V))$ the gap with labelling k. If $G_k(V)$ is not empty for all k, we say *all gaps of H are open*. When Σ is a Cantor set, we say the operator H has *Cantor spectrum*.

The continuous time quasi-periodic Schrödinger operators $\mathcal{L} = \mathcal{L}_{q,\omega,\theta}$ are defined on $L^2(\mathbb{R})$ as

$$(\mathcal{L}_{q,\omega,\theta}y)(t) = -y''(t) + q(\theta + \omega t)y(t) \tag{1-4}$$

where $q : \mathbb{T}^d \to \mathbb{R}$, and $\omega \in \mathbb{T}^d$ is rationally independent. It is known that $\mathcal{L}_{q,\omega,\theta}$ is self-ajoint and unbounded, its spectrum is an unbounded perfect subset of $\mathbb{R}$ independent of θ. All concepts above for the discrete time Schrödinger operators can be defined similarly for the continuous time quasi-periodic Schödinger operators.

The spectrum and spectral measure are two central subjects in spectral theory. For the spectrum, people are mainly interested in the Lebesgue measure of Σ, Cantor spectrum, homogeneity of the spectrum, opening gaps and gap estimates. For the spectral measure, people are interested in the nature of the measure: when it is absolute continuous, when it is singular continuous or pure point; if it is pure point, when it has Anderson localization (pure point with exponential decay eigenfunctions) or dynamical localization. What is the modulus of the continuity of IDS and the spectral measure? If the operator contains parameters, the phase transition is also an important issue.

1.2 Quasi-periodic cocycles and quasi-periodic linear systems. Note that $(u_n)_{n\in\mathbb{Z}}$ is a formal solution of the eigenvalue equation $H_{V,\omega,\theta}u = Eu$ if and only if it satisfies

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S_E^V(\theta + n\omega) \cdot \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \tag{1-5}$$

where

$$S_E^V(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{R}).$$

The dynamical systems (ω, S_E^V) defined on $\mathbb{T}^d \times \mathbb{R}^2$ by

$$(\theta, v) \mapsto (\theta + \omega, S_E^V(\theta)v) \tag{1-6}$$

are called Schrödinger cocycles.

In general, an analytic quasi-periodic linear cocycle (ω, A) on $\mathbb{T}^d$ with coefficients in Lie group $\mathbf{G}$ (its Lie algebra will be denoted by $\mathbf{g}$) is defined by

$$\begin{aligned} \mathbb{T}^d \times \mathbb{R}^N &\to \mathbb{T}^d \times \mathbb{R}^N \\ (\theta, v) &\mapsto (\theta + \omega, A(\theta) \cdot v). \end{aligned} \tag{1-7}$$

where $A \in C^\omega(\mathbb{T}^d, \mathbf{G})$, $\mathbf{G}$ will be usually taken as $GL(N,\mathbb{R})$, $Sp(2N,\mathbb{R})$. The iterate of the cocycle is defined as

$$\mathcal{A}_n(\theta) := \begin{cases} A(\theta + (n-1)\omega) \cdots A(\theta + \omega)A(x), & n \geq 0 \\ A^{-1}(\theta + n\omega)A^{-1}(\theta + (n+1)\omega) \cdots A^{-1}(\theta - \omega), & n < 0. \end{cases}$$

Let $\lambda_1(\theta), \lambda_2(\theta), \cdots, \lambda_N(\theta)$ be the singular values of $\mathcal{A}_n(\theta)$. By Oseledets theory,

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} \ln \lambda_i(\theta)\, d\theta, \quad i = 1, \cdots, N$$

exsit and are same for almost all θ. λ_i's are called *Lyapunov exponents* of (1-7), among them $L(\omega, A) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}^d} \ln \|\mathcal{A}_n(\theta)\| d\theta$ is the largest. When ω has been fixed, we simply write $L(A)$ for $L(\omega, A)$. If $A(\theta)$ are in $SL(2,\mathbb{R})$, the two Lyapunov exponents are $\pm L(\omega, A)$. If furthermore $A(\theta) \in C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$, Avila [2015a] proved that

$$\omega(A) := \lim_{\epsilon\to 0} \frac{L(A_\epsilon) - L(A)}{2\pi\epsilon}, \quad \text{where } A_\epsilon = A(\theta + i\epsilon),$$

exists and moreover it is an integer. The quantity $\omega(A)$, called *acceleration*, plays an important role in Avila's global theory of one frequency analytic quasi-periodic Schrödinger cocycles (Avila [ibid.]).

The cocycle (1-7) is said to be *uniformly hyperbolic* if there exists a continuous splitting $\mathbb{R}^N = E^s(\theta) \oplus E^u(\theta)$ such that for every $n \geq 0$,

$$|\mathfrak{A}_n(\theta)\, v| \leq Ce^{-cn}|v|, \;\; v \in E^s(\theta),$$
$$|\mathfrak{A}_n(\theta)^{-1} v| \leq Ce^{-cn}|v|, \;\; v \in E^u(\theta + n\omega),$$

for some constants $C, c > 0$. Moreover, this splitting is invariant, i.e.,

$$A(\theta)E^s(\theta) = E^s(\theta+\omega), \quad A(\theta)E^u(\theta) = E^u(\theta+\omega), \quad \forall\, \theta \in \mathbb{T}^d.$$

A cocycle is said to be *non-uniformly hyperbolic* if it is not uniformly hyperbolic and all the Lyapunov exponents are not zero.

If $G = SL(2,\mathbb{R})$, another dynamical quantity, the rotation number can be defined. Assume that $A \in C(\mathbb{T}^d, SL(2,\mathbb{R}))$ is homotopic to the identity. It introduces the projective skew-product $F_A \colon \mathbb{T}^d \times \mathbb{S}^1 \to \mathbb{T}^d \times \mathbb{S}^1$ with

$$F_A(\theta, w) := \left(\theta + \omega,\, \frac{A(\theta)v}{|A(\theta)v|}\right),$$

which is also homotopic to the identity. Thus we can lift F_A to a map $\widetilde{F}_A \colon \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R}$ of the form $F_A(\theta, y) = (\theta + \omega, y + \psi_\theta(y))$, where for every $\theta \in \mathbb{T}^d$, ψ_θ is $\mathbb{Z}$-periodic. The map $\psi \colon \mathbb{T}^d \times \mathbb{T} \to \mathbb{R}$ is called a *lift* of A. Let μ be any probability measure on $\mathbb{T}^d \times \mathbb{R}$ which is invariant by $\widetilde{F}_A$, and whose projection on the first coordinate is given by Lebesgue measure. The number

$$\rho(\omega, A) := \int_{\mathbb{T}^d \times \mathbb{R}} \psi_\theta(y)\, d\mu(\theta, y) \bmod \mathbb{Z}$$

which depends neither on the lift ψ nor on the measure μ, is called the *fibered rotation number* of (ω, A) (see Herman [1983] and Johnson and Moser [1982] for more details). It is known that $\rho(\omega, A) \in [0, \frac{1}{2}]$.

The continuous counterpart of quasi-periodic cocycles is the quasi-periodic linear systems, i.e., the ordinary differential equations

$$\dot{x} = A(\theta)x, \quad \dot{\theta} = \omega, \tag{1-8}$$

where A is assumed to be in a Lie algebra $\mathfrak{g}$. The eigenvalue equations of continuous quasi-periodic Schrödinger operator

$$(\mathfrak{L}_{q,\omega,\theta}\, y)(t) = -y''(t) + q(\theta + \omega t)y(t) = Ey(t) \tag{1-9}$$

are equivalent to the linear systems

$$\begin{cases} \dot{x} = V_{E,q}(\theta)x \\ \dot{\theta} = \omega \end{cases} \tag{1-10}$$

where

$$V_{E,q}(\theta) = \begin{pmatrix} 0 & 1 \\ q(\theta) - E & 0 \end{pmatrix} \in sl(2, \mathbb{R}).$$

The Poincaré map of the flow of (1-8) in fact defines a quasi-periodic cocycle. In converse, quasi-periodic cocycles close to constant can be embedded into the flow of quasi-periodic linear systems (You and Zhou [2013]). So there are paralell concepts, methods and theories for cocycles and systems.

Uniform hyperbolicity, the Lyapunov exponents, the rotation number and the acceleration are important concepts and quantities in the study of the dynamics of quasi-periodic cocycles and quasi-periodic linear systems. The central problems include positivity, continuity and regularity of the Lyapunov exponents, absolute continuity and Hölder continuity of the rotation number. Avila's acceleration is an important new index, its relation with dynamics and spectral theory has not been sufficiently explored.

1.3 Relations between operators and dynamical systems. For simplicity, the Lyapunov exponent, the rotation number and the IDS of (1-6) or (1-10) will be simply denoted by $L(E), \rho(E)$ and $N(E)$ when V and ω are fixed. The spectral theory of (1-1) (respectively (1-4)) are closely related to the dynamics of the one parameter family Schödinger cocycles (1-6) (respectively (1-10)) where the energy $E \in \mathbb{R}$ serves as parameter. Full understanding of the one parameter family of dynamical systems (1-6) or (1-10) would lead to a full understanding of the spectral theory of the Schrödinger operators (1-1) or (1-4).

There are some classical relationships between the spectrum of (1-1) (respectively (1-4)) and the dynamics of (1-6) (respectively (1-10)). It is known that $E \notin \Sigma$ if and only if the corresponding Schrödinger cocycle $(\omega, S_E^V(\cdot))$ is uniformly hyperbolic. IDS is the average of the spectral measure, which relates transparently to the rotation number by the formula $N(E) = 1 - 2\rho(E)$ and relates to the Lyapunov exponent through the Thouless formula

$$L(E) = \int \log |E - E'| \, dN(E').$$

Moreover, by Kotani's theory (Kotani [1984]), the absolutely continuous spectrum is the essential closure of the energies E such that (ω, S_E^V) has zero Lyapunov exponent. To obtain more precise information of the spectrum and the spectral measure, we need another tool: almost reducibility, which has been proved to be very powerful. In this survey, we will emphasize the applications of almost reducibility in the study of the spectral theory of the quasi-periodic Schrödinger operators.

2 Almost Reducibility

An analytic cocycle (ω, A) defined in (1-7) is said to be *reducible* if it can be conjugated to a constant cocycle, i.e., there exist $B \in C^\omega(2\mathbb{T}^d, \mathbf{G})$ and $C \in \mathbf{G}$ such that

$$B(\cdot + \omega)^{-1} A(\cdot) B(\cdot) = C.$$

Similarly, an analytic quasi-periodic linear system defined in (1-8) is said to be reducible if there exist $B \in C^\omega(2\mathbb{T}^d, \mathbf{G})$, $C \in \mathbf{g}$ such that

$$\partial_\omega B + BA - CB = C.$$

There are obstructions to the reducibility. The first obstruction is the presence of non-uniformly hyperbolicity. The second obstruction comes from the arithmetic condition on ω. Usually, reducibility requires that ω is Diophantine, i.e.,

$$\min_{l \in \mathbb{Z}} |\langle k, \omega \rangle - l| > \frac{\gamma^{-1}}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^d, \tag{2-1}$$

with fixed $\gamma, \tau > 1$. Here (γ, τ) are called the Diophantine constants of ω. Denote by $DC(\gamma, \tau)$ the set of all (γ, τ)-type Diophantine ω and $DC = \cup_{\gamma,\tau>1} DC(\gamma, \tau)$ (DC is of full measure). If $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, let $\frac{p_n}{q_n}$ be the $n-$th continued fraction convergents of irrational α, then we define

$$\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}.$$

$\beta(\alpha)$ measures how Liouvillean α is. Obviously if $\alpha \in DC$, then $\beta(\alpha) = 0$.

A quasi-periodic cocycle is in general not reducible when α is Liouvillean. A weaker concept than reducibility is almost reducibility. An analytic cocycle (ω, A) is said to be *almost reducible* if there exist a sequence of conjugations $B_n \in C^\omega_{h_n}(2\mathbb{T}^d, \mathbf{G})$, a sequence of constant matrices $A_n \in \mathbf{G}$ and a sequence of $F_n \in C^\omega_{h_n}(\mathbb{T}^d, \mathbf{g})$ converging to zero in C^ω topology, such that B_n conjugates (ω, A) to $(\omega, A_n e^{F_n(\cdot)})$. (ω, A) is called *weak almost reducible* if $h_n \to 0$, and *strong almost reducible* if $h_n \to h' > 0$. Some problems in the spectral theory needs strong almost reducibility and even require very precise estimates on B_n and F_n. Almost reducibility with more precise estimates is refered to as *quantitative almost reducibility*. For applications in the spectral theory of Schrödinger operators, the most interesting cases are $\mathbf{G} = SL(2, \mathbb{R})$ and $\mathbf{g} = sl(2, \mathbb{R})$.

Almost reducibility is useful and important since it prescribes a domain of applicability of local theories of cocycles close to constant (Avila [2010]). Due to its importance in the theory of dynamical systems and the spectral theory of quasi-periodic Schrödinger operators, reducibility has received much attention.

2.1 Perturbative reducibility. The rotation number of quasi-periodic linear system or quasi-periodic cocycle plays an important role in reducibility theory and its application to the the spectrum theory. We say that the rotation number ρ is rational with respect to (w.r.t. for short) ω if $\rho = \frac{1}{2}\langle k_0, \omega\rangle$ for some $k_0 \in \mathbb{Z}^2$, and to be Diophantine w.r.t. ω, with constants $\gamma, \tau > 1$, if

$$\min_{l\in\mathbb{Z}} |\langle k, \omega\rangle - 2\rho - l| \geq \frac{\gamma^{-1}}{|k|^\tau}, \quad k \in \mathbb{Z}^2.$$

We denote by $DC_\omega(\gamma, \tau)$ the set of all such ρ. It is well known that the union $DC_\omega = \cup_{\gamma,\tau>1} DC_\omega(\gamma, \tau)$ is a full measure subset of $\mathbb{R}$.

The reducibility of quasi-periodic linear systems (1-10) and its applications in the spectral theory were initiated by Dinaburg and Sinaĭ [1975], based on classical KAM theory, they proved that if q is analytic and sufficiently small, then (1-10) is reducible for $\rho(E) \in DC_\omega(\gamma, \tau)$. Dinaburg and Sinai's reducibility result implies the existence of absolutely continuous spectrum of the Schrödinger operator (1-4). The first breakthrough was due to (Eliasson [1992]), who proved the following:

Theorem 2.1. *Eliasson [ibid.] Suppose that $\omega \in DC(\tau, \gamma)$ and q is analytic and sufficiently small, then* (1-10) *is weak almost reducible for all E. Moreover* (1-10) *is reducible if $\rho(E) \in DC_\omega$ or rational w.r.t ω.*

The proof in Eliasson [ibid.] uses a crucial resonance-cancelation technique which was introduced by Moser and Pöschel [1984] earlier. Eliasson [1992] work has profound impact: Theorem 2.1 can describe the dynamical behavior for all parameters E, while the classical KAM theory can only describe a positive measure set of E. Theorem 2.1 implies that, when the potential is analytic and small, the spectral measure of (1-9) is purely absolutely continuous for all phases θ, which shows that almost reducibility could play an important role in the study of the spectrum of quasi-periodic Schrödinger operators. Moreover, the later non-perturbative and quantitative versions of Theorem 2.1 have been found useful in the study of Cantor spectrum, gap estimates, Anderson localization, Hölder continuity of IDS and even more, which we will review separately in the following sections.

Theorem 2.1 holds for more general quasi-periodic cocycles $A \in C^\omega(\mathbb{T}^d, \mathbf{G})$ with A close to some constant (Chavaudret [2013] and Krikorian [1999b,a]), even for finite smooth case (Cai, Chavaudret, You, and Zhou [2017]). We remark that all the above mentioned results are *perturbative*, i.e., the smallness of q depends on the frequencies ω through the Diophantine constants (γ, τ). The perturbative reducibility result is optimal when $d \geq 2$ in the discrete case and $d > 2$ in the continuous case as shown by a counterexample of Bourgain [2002]. However, when $d = 2$ in the continuous case and $d = 1$ in the discrete case, one can expect more. In the following we shall restrict our attention to these cases.

2.2 Non-perturbative reducibility. The non-perturbative reducibility means that the smallness of the perturbation does not depend on the Diophantine constants (γ, τ) of α. The non-perturbative reducibility was first proved by Puig [2006] for Schrödinger cocycles $(\alpha, S_E^V(\cdot))$ with one frequency $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. However, the proof, which is based on Aubry duality (Aubry and André [1980] and Gordon, Jitomirskaya, Last, and Simon [1997] and Anderson localization results of Bourgain and Jitomirskaya [2002a], doesn't work for the continuous linear systems. Hou and You [2012] gave a non-perturbative version of Theorem 2.1 in the continuous case.

Theorem 2.2. *Hou and You [ibid.] Let $h > 0$ and $\omega = (\alpha, 1)$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Consider*

$$\begin{cases} \dot{x} = (A + F(\theta))x \\ \dot{\theta} = \omega \end{cases} \tag{2-2}$$

with $A \in sl(2,\mathbb{R})$ and $F \in C_h^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$. Then there exists $\delta = \delta(A,h) > 0$ depending on A, h but not on α, such that system (2-2) is weak almost reducible if $\sup_{|Im\theta|<h} |F(\theta)| < \delta$. Moreover (2-2) is reducible if ω is Diophantine and $\rho(E) \in DC_\omega$ or rational w.r.t ω.

Remark 2.1. *By an embedding theorem of You and Zhou [2013], one sees that the same result in Theorem 2.2 holds for $SL(2,\mathbb{R})$ cocycles with one frequency.*

We remark that Theorem 2.2 works for any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ not merely Diophantine frequency, its proof is based on KAM and Floquet theory. Before Hou and You [2012], Avila, Fayad, and Krikorian [2011] proved that for any analytic $SL(2,\mathbb{R})$ cocycles (α, A) which is close to constant, for any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, the cocycles are analytic rotations reducible (analytic conjugacy to a cocycle with values in $SO(2,\mathbb{R})$) for full measure rotation number, their proof is based on "algebraic conjugacy trick" which was first developed by Fayad and Krikorian [2009].

2.3 Strong almost reducibility and Quantitative almost reducibility. We remark that in the above mentioned almost reducibility results, the convergence to constants occurs on analytic strips of width going to zero. Breakthrough belongs to Avila and Jitomirskaya [2010]. Based on almost localization and Aubry duality, Avila and Jitomirskaya [ibid.] gave a non-perturbative strong almost reducibility result for Schrödinger cocycles with a single frequency $\alpha \in DC$ and small potentials. Avila [2008] generalized the result to $\beta(\alpha) = 0$ with much more delicate estimates. Chavaudret [2013] proved a strong almost reducibility result in the local regime for multiple Diophantine frequencies. However, as we mentioned, in spectral applications, we need quite delicate quantitative estimates, while Chavaudret's estimates are not enough to give interesting consequence in applications. Recently, Leguil, You, Zhao, and Zhou [2017] gave another strong almost reducibility result with more precise estimates. As a consequence, several interesting spectral applications were obtained. We will explain the applications in Section 3.

2.4 Global reducibility. In all the results above, we assume that the cocycle or system is close to a constant. For cocycles not close to a constant, Kotani's theory (Kotani [1984]) essentially asserts that there is an almost surely dichotomy between non-uniform hyperbolicity and L^2 rotations-reducibility of the cocycles (α, S_E^V). L^2-conjugation can be further proved to be smooth by a renormalization scheme developed by Avila and Krikorian [2006, 2015] and Krikorian [2004]. Thus for any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and for $a.e. E \in \mathbb{R}$, (α, S_E^V) is either analytically rotations reducible or non-uniformly hyperbolic (Avila and Krikorian [2006] and Avila, Fayad, and Krikorian [2011]).

The real breakthrough is Avila's global theory for one frequency analytic $\mathrm{SL}(2,\mathbb{R})$ cocycles Avila [2015a]. Avila classified $(\alpha, A) \in \mathbb{R}\backslash\mathbb{Q}\times C^\omega(\mathbb{T}, SL(2,\mathbb{R}))$, which is not uniformly hyperbolic, into three classes according to the Lyapunov exopents and acceleration: supercritical, subcritical and critical. A cocycle (α, A) is supercritical, if $L(\alpha, A) > 0$, it is called subcritical, if $L(\alpha, A(z)) = 0$ for $|\Im z| \leq \delta$, it is called critical otherwise. We say that H is acritical if (α, S_E^v) is not critical for all $E \in \Sigma$. The main result in Avila's global theory is the following:

Theorem 2.3. *Avila [ibid.] Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then for a (measure-theoretically) typical analytic potential $V \in C^\omega(\mathbb{T}, \mathbb{R})$, the operator $H_{V,\alpha,\theta}$ is acritical.*

Avila's global theory is crucial in the study of the spectral theory of Schrödinger operators, especially when the potential is not small but the corresponding Schrödinger cocycles are still in the subcritical region. The cornerstone in Avila's global theory is his *Almost Reducibility Conjecture(ARC)*: subcriticity implies almost reducibility, which has been solved completely by Avila [2010, n.d.(b)].

Theorem 2.4 (Avila [2010, n.d.(b)]). *Let $A \in C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$. Then (α, A) is strong almost reducible if it is subcritical.*

However, we remark that sometimes Avila's global almost reducibility does not contain sufficient estimates on the conjugations(since it deals with global cocycle directly). In applications, we have to cook it with finer local quantitative almost reducibility results.

3 Applications to quasi-periodic Schrödinger operators

Quasi-periodic Schrödinger operators are mathematical models for many subjects in quantum physics including quantum Hall effect and the nature of quasi-crystal. It is also a subject to test the power of mathematical theories and methods, thus has attracted much attentions, which we refer to survey articles of Damanik [2017], Jitomirskaya [2007], Marx and Jitomirskaya [2017], Last [2005], and Simon [1982]. In this survey we will only present some of them which are closely related to the theory of almost reducibility, readers are invited to consult the former references for other interesting results.

3.1 Spectrum of quasi-periodic Schrödinger operators.

The spectrum Σ is one of most important objects in the spectral theory of quasi-periodic Schrödinger operators.

3.1.1 Cantor spectrum.

Cantor spectrum was conjectured to be a generic phenomenon for one dimensional almost periodic Schrödinger operator (Problem 6 of Simon [1982]). There are few exceptions in this case (the so called finite gap potentials). In one frequency case, there is no counter-example with big potential so far, but the recent work of Goldstein-Schlag-Voda shows that finite gap happens for multi-frequency case (Goldstein, Schlag, and Voda [2017]).

However, to prove the existence of Cantor spectrum is not an easy task. Eliasson [1992] proved that for any given $\omega \in DC(\gamma,\tau)$, $H_{V,\omega,\theta}$ has Cantor spectrum for *generic* small analytic potentials. His proof is based on Moser-Pöschel argument (Moser and Pöschel [1984]) and the fact: if $\rho(E)$ is rational w.r.t ω, then (ω, S_E^V) is reducible. Eliasson's proof applies to the cocycle case, however his proof is not constructive, which can not provide any concrete example.

In the discrete case, the situation is better. Goldstein and Schlag proved that for any fixed non-constant analytic potential, in the supercritical region, the spectrum is a Cantor set for almost all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ (Goldstein and Schlag [2011]). In C^0-topology, Avila-Bochi-Damanik proved Cantor spectrum for any fixed totally irrational vector $\omega \in \mathbb{T}^d$ and generic $V \in C^0(\mathbb{T}^d,\mathbb{R})$ (Avila, Bochi, and Damanik [2009]). In the case of C^k-topology ($1 \le k \le \infty$ or even in analytic category), based on Avila [2011] and Goldstein and Schlag [2011], one can prove that for generic $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and generic $V \in C^k(\mathbb{T},\mathbb{R})$, the spectrum of $H_{V,\omega,\theta}$ is a Cantor set, one can consult footnote 1 of J. Wang, Zhou, and Jäger [2016] for the outline of this proof.

The most remarkable progress is for almost Mathieu operator. AMO $H_{\lambda,\alpha,\theta}$ has been conjectured for a long time to have Cantor spectrum for irrational α (consult a 1964 paper of Azbel [1964]). This conjecture has been dubbed the Ten Martini Problem by Simon [2000], after an offer of Kac [n.d.] in 1981. Since when it was posed, Ten Martini Problem became the central problem in the spectral theory of quasi-periodic Schödinger operators and attracted a lot of attentions. Finally, it was completely solved by Avila and Jitomirskaya [2009] (see the references therein for partial advances). One main ingredient of the proof is the reducibility: the cocycle cann't be reducible to rotations for all E in an interval.

We should also mention the works by Sinaĭ [1987] and Y. Wang and Z. Zhang [2017] where the Cantor spectrum was proved for $H_{\lambda V,\alpha,\theta}$ with sufficiently large λ and C^2 cosine-like V. So far, those are the only known concrete operators having Cantor spectrum. Particularly, there is no concrete examples of Cantor spectrum in the continuous time case. In a forthcoming paper, we will give more concrete examples of Cantor spectrum for both discrete and continuous quasi-periodic Schrödinger operators based on reducibility (Hou, Shan, and You [n.d.]).

3.1.2 All gaps are open. Certainly, one should not expect that all gaps are open for all analytic quasi-periodic Schrödinger operators because of, as we mentioned before, the existence of the finite gap potentials. But AMO is special. Motivated by Hofstadter's numerical result (Hofstadter [1976]), Kac [n.d.] raised the well-known question: Are all possible spectral gaps of AMO open? The problem was named as the Dry Ten Martini Problem by Simon [1982]. The Dry Ten Martini Problem was also named as Ten Martini Conjecture by physicists, which has particular importance in quantum physics such as the Integer Quantum Hall effect. Some works in physics have been done under the assumption that the Dry Ten Martini Problem is true, see i.g. Osadchy and J. E. Avron [2001].

Obviously, Dry Ten Martini Problem automatically implies Ten Martini Problem. Certainly people want to solve this original problem of Kac. In the last thirty years, substantial progresses were made by Choi, Elliott, and Yui [1990], Puig [2004], Avila and Jitomirskaya [2009, 2010]. However, the problem has not been completely solved for any fixed λ.

Recently, Avila, You, and Zhou [2016] gave a complete answer to the Dry Ten Martini Problem for the non-critical case $\lambda \neq 1$ by quantitative almost reducibility.

Theorem 3.1. *Avila, You, and Zhou [ibid.] $H_{\lambda,\alpha,\theta}$ has all spectral gaps open for all irrational α and all $\lambda \neq 1$.*

The strategy of Puig's proof Puig [2004] is to prove that (α, S_E^{λ}) is reducible but it cann't be reduced to (α, Id) if $N_{\lambda,\alpha}(E)$ is rational w.r.t α. Developing this idea, Avila and Jitomirskaya solved the problem for $\alpha \in DC$, and $\lambda \neq 1$ (Avila and Jitomirskaya [2010]). However, if $\beta(\alpha) > 0$ as in our case, one cann't expect that the cocycle is still reducible. However, we can show that the cocycle can not be almost reducible to (α, Id) with fast decay of $\|B_n\|_h \|F_n\|_h$. Once we have this, Theorem 3.1 can be proved by a modified Moser-Pöschel argument. We finally remark that the Dry Ten Martini problem has not been completely solved for $\lambda = 1$.

3.1.3 Estimate of the spectral gaps. Recall G_k, the spectral gaps with labelling k. The well-known Dry Ten Martini Problem asks whether G_k is empty or not for AMO. Further problem is: how big the gaps are? More precisely, can we give any lower bound or upper bound for G_k? For AMO, Leguil, You, Zhao, and Zhou [2017] proved the following result by quantitative almost reducibility:

Theorem 3.2. *Leguil, You, Zhao, and Zhou [ibid.] For $\alpha \in \mathrm{DC}$, and for any $0 < \xi < 1$, there exist constants $C = C(\lambda, \alpha, \xi) > 0$, $\tilde{C} = \tilde{C}(\lambda, \alpha)$, such that for all $k \in \mathbb{Z}\backslash\{0\}$,*

$$\tilde{C}\lambda^{\tilde{\xi}|k|} \leq |G_k(\lambda)| \leq C\lambda^{\xi|k|}, \qquad \text{if} \quad 0 < \lambda < 1,$$

$$\tilde{C}\lambda^{-\tilde{\xi}|k|} \leq |G_k(\lambda)| \leq C\lambda^{-\xi|k|}, \qquad \text{if} \quad 1 < \lambda < \infty,$$

where $\tilde{\xi} > 1$ is a numerical constant, $|G_k(\lambda)|$ denotes the length of $G_k(\lambda)$.

For general analytic potential, Leguil, You, Zhao, and Zhou [ibid.] also proved that $|G_k(V)| \leq \varepsilon_0^{\frac{2}{3}} e^{-r|k|}$ for all $k \in \mathbb{Z}^d\backslash\{0\}$ and any $r \in (0, h)$ if $\sup_{|\Im x|<h} |V(x)| < \varepsilon_0$ is small enough and $\omega \in \mathrm{DC}$ and $V \in C_h^{\omega}(\mathbb{T}^d, \mathbb{R})$. Before Leguil, You, Zhao, and Zhou [ibid.], Damanik and Goldstein [2014] have shown that $|G_k(V)| \leq \epsilon_0 e^{-\frac{h}{2}|k|}$. We remark that the proof in Damanik and Goldstein [ibid.] is based on the localization argument, which cannot be directly applied to the discrete case, while the proof in Leguil, You, Zhao, and Zhou [2017] is based on reducibility, so it works equally well both for the continuous time operators and discrete time operators. For more history on the study of the upper bounds, one may consult Leguil, You, Zhao, and Zhou [ibid.] and the references therein.

In methodology, for estimating of the spectral gaps we need to analyze the behavior of Schrödinger cocycle at the edge points of the spectral gaps, where the cocycles are reducible to constant parabolic cocycles. The crucial points for the gap estmate are the proof of the exponential decay of the off-diagonal element of the parabolic matrix and the exponential growth of the conjugacy with respect to the labelling k. Furthermore, in order to prove the decay rate to be uniform with respect to the labelling k, we need the strong quantitative almost reducibility result, i.e. the cocycle is almost reducible in a fixed band, with precise estimates on the conjugations and the off-diagonal element of the (reduced) parabolic matrix (Leguil, You, Zhao, and Zhou [ibid.]).

3.1.4 Homogeneous spectrum. we say that Σ is μ-homogenuous if for any $E \in \Sigma$ and any $0 < \epsilon \leq \operatorname{diam}\Sigma$, we have $|\mathcal{S} \cap (E - \epsilon, E + \epsilon)| > \mu\epsilon$ for $\mu > 0$. We say that $H_{V,\alpha,\theta}$ has homogenuous spectrum if Σ is homogenuous. The homogeneity of the spectrum plays an essential role in the inverse spectral theory of almost periodic potentials(refer to the fundamental work of Sodin and Yuditskii [1995, 1997]).

The exponential decay of the spectral gaps can be used to prove the homogeneity of the spectrum.

Theorem 3.3 (Leguil, You, Zhao, and Zhou [2017]). *Let $\alpha \in$ SDC [1]. For a (measure-theoretically) typical analytic potential $V \in C^{\omega}(\mathbb{T}, \mathbb{R})$, the spectrum $\Sigma_{V,\alpha}$ is μ-homogeneous for some $0 < \mu < 1$. Especially, the spectrum is homogenuous for small analytic potentials.*

Homogeneity of the spectrum $\Sigma_{V,\alpha}$ in the subcritical regime is derived from the upper bounds of the spectral gaps and Hölder continuity of the IDS (Leguil, You, Zhao, and Zhou [ibid.]). While the homogeneity in supercritical region was proved by Damanik, Goldstein, Schlag, and Voda [2015]. Together with Theorem 2.3, one sees that the homogeneity of the spectrum is a typical phenomenon for analytic Schrödinger operators when α is strong Diophantine. We remark that, at least in the subcritical region, strong Diophantine is not necessary (Leguil, You, Zhao, and Zhou [2017]), however some kind of arithmetic property is necessary. After Leguil, You, Zhao, and Zhou [ibid.], Avila, Last, Shamis, and Zhou [n.d.] proved that there exists a dense set of Liouvillean frequencies α such that $\Sigma_{\lambda,\alpha}$ of AMO is not homogeneous.

3.2 The spectral measure, IDS and Lyapunov exponent.

The nature of the spectral measure, IDS and Lyapunov exponent are central subjects in the spectral theory of quasi-periodic Schrödinger operators, while the study of Lyapunov exponents is also a central subject in smooth dynamical systems.

3.2.1 Anderson localization.

There are two important results concerning Anderson localization in supercritical regime. The first result belongs to Jitomirskaya [1999], who proved that for almost Mathieu operator, $H_{\lambda,\alpha,\theta}$ has Anderson localization for a.e. θ if $|\lambda| > 1$ and $\alpha \in DC$. Another result belongs to Bourgain and Goldstein [2000], who proved that up to a typical perturbation of the frequency, Anderson localization holds through the supercritical regime. Comparing the two results above, the result of Jitomirskaya [1999] is for fixed frequency and typical phases (depending on the frequency), while Bourgain and Goldstein [2000] is for fixed phase and typical frequencies (depending on the phase). Both results are proved by the positivity of the Lyapunov exponent, which is classical method for studying the pure point spectrum of the Schrödinger operators.

At first glance, the reducibility has no business with the Anderson localization spectrum since the point spectrum corresponds to the non-uniformly hyperbolicity which is definitely not almost reducible. Surprisingly, one can use reducibility to study the point spectrum, even Anderson localization and dynamical localization. This idea was first

[1] SDC is the set of strong Diophantine numbers, i.e., there exist $\gamma, \tau > 0$ such that

$$\inf_{j \in \mathbb{Z}} |n\alpha - j| \geq \frac{\gamma}{|n|(\log|n|)^{\tau}}, \quad \forall\, n \in \mathbb{Z}\backslash\{0\}. \tag{3-1}$$

appeared in You and Zhou [2013], and completely built in Avila, You, and Zhou [2017]. The bridge is the Aubry duality (Aubry and André [1980] and Gordon, Jitomirskaya, Last, and Simon [1997]): Suppose that the quasi-periodic Schrödinger operator (1-1) with one frequency has an analytic quasi-periodic Bloch wave $u_n = e^{2\pi i n\varphi}\overline{\psi}(n\alpha+\phi)$ for some $\overline{\psi} \in C^\omega(\mathbb{T},\mathbb{C})$ and $\varphi \in [0,1)$, then the Fourier coefficients of $\overline{\psi}(\theta)$ satisfy the following long range operator:

$$(\widehat{H}_{V,\alpha,\varphi}x)_n = \sum_{k\in\mathbb{Z}} V_k x_{n-k} + 2cos2\pi(\varphi+n\alpha)x_n = Ex_n, \tag{3-2}$$

where V_k is the Fourier coefficients of $V(\theta)$. The converse is also true. We remark that the almost Mathieu family $\{H_{2\lambda\cos,\alpha,\theta}\}_{\lambda>0}$ is self-dual. The reducibility of $(\alpha, S_E^V(\theta))$ will provide analytic quasi-periodic Bloch waves of the operator (1-1) and thus will provide eigenfunctions for its dual operator, so the general philosophy is that the full measure reducibility of $(\alpha, S_E^V(\theta))$ will imply Anderson localization of the dual operator $\widehat{H}_{V,\alpha,\varphi}$ for almost every phases. Let us mention a recent work by Avila, You, and Zhou [2017] for the almost Mathieu operators as an example.

Theorem 3.4. *Avila, You, and Zhou [ibid.] If $|\lambda| > e^{\beta(\alpha)}$, then $H_{\lambda,\alpha,\theta}$ has Anderson localization for a.e. θ.*

Note that $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for all θ if $|\lambda| < 1$ (Avila [2008], Avila and Damanik [2008], Avila and Jitomirskaya [2010], and Jitomirskaya [1999]), and $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for all θ if $1 \le |\lambda| < e^{\beta(\alpha)}$ (Avila, You, and Zhou [2017], J. Avron and Simon [1982], and Gordon [1976]). Now one sees the sharp phase transition scenario for three types of the spectral measure for a.e. θ, and solves a conjecture of Jitomirskaya [1995], which is the corrected version of a conjecture by Aubry and André [1980]. We remark that based on localization method, before Avila, You, and Zhou [2017], Avila and Jitomirskaya [2009] proved that $H_{\lambda,\alpha,\theta}$ has Anderson localization for a.e. θ, if $|\lambda| > e^{16\beta(\alpha)/9}$. More recently, Jitomirskaya and Liu [n.d.] proved a refined result of Avila, You, and Zhou [2017] with precise description on the localized phases and the hierarchical structure of eigenfunctions. We also refer to Jitomirskaya and S. Zhang [2015] for another interesting phase transition result, valid for general analytic potentials.

For the proof of Theorem 3.4, a new criterion (which reveals the fact that nice asymptotical distribution of the eigenfunctions implies Anderson localization) for establishing the purity of the point spectrum was developed in Avila, You, and Zhou [2017], which applies to general ergodic family of operators (one may consult Jitomirskaya and Kachkovskiy [2016] for another proof but with same spirit). Compared with traditional localization argument, the trade off is the loss of precise arithmetic control on the localization phases.

However, by this approach, we indeed establish a kind of equivalence between full measure reducibility of Schrödinger cocycles and Anderson localization of the dual operators. The methods developed in Avila, You, and Zhou [2017] has further applications. For example, it can be used to study the spectral properties at the transition line $\lambda = e^{\beta}$ (Avila, Jitomirskaya, and Zhou [2018]). In a forthcoming paper, we even show that it can be used to study the dynamical localization of the long-range operator and a family of Schrödinger operators on $l^2(\mathbb{Z}^d)$. Finally, we remark that it has been an open question for a long time whether in the supercritical regime, $H_{V,\alpha,\theta}$ with fixed Diophantine α has Anderson localiztion for a.e. phase (consult Eliasson [1997] for partial advances). Our method might provide a way to study this problem when the potential is a trigonometric polynomial. In this case, Equation (3-2) naturally defines a $2d$-dimensional cocycles, while the full measure reducibility is easy to establish, the difficulty remains in the proof is the purity of the point spectrum.

3.2.2 Absolutely continuous spectrum. Absolute continuity of the spectral measure is a traditional territory of reducibility. If the cocycles (α, S_E^V) are reducible for positive measure of $E \in \Sigma$, then the operator has absolutely continuous spectrum (Dinaburg and Sinaĭ [1975]). Based on Theorem 2.1, Eliasson [1992] proved directly that the spectral measure $H_{V,\alpha,\theta}$ is absolutely continuous spectrum if the potential V is small enough and $\alpha \in DC$. Recently, Avila [2008] gave a new understanding of Eliasson's result based on Gilbert-Person's subordinacy theory Gilbert and Pearson [1987]: $\mu_{\alpha,\theta,V}|_{\mathfrak{B}}$ is absolutely continuous for all $\theta \in \mathbb{R}$ where $\mathfrak{B}$ is the set of $E \in \mathbb{R}$ such that the cocycle (α, S_E^V) is bounded. Thus to obtain purely absolutely continuous spectrum, one only needs to show that $\mu_{\alpha,\theta,V}(\Sigma\backslash\mathfrak{B}) = 0$. Note that by reducibility, we can prove that $\mathfrak{A}_n$ are bounded for almost all E in the spectrum Σ. However the bounds are not uniform, the proof of $\mu_{\alpha,\theta,V}(\Sigma\backslash\mathfrak{B}) = 0$ relies on the measure estimate of $E \in \Sigma$ for any given bound. Based on this idea, Avila [2008] proved purely absolutely continuous spectrum for general one frequency analytic Schödinger operators if the potential is small and $\beta(\alpha) = 0$. Recent breakthrough also belongs to Avila [n.d.(b)], he shows that almost reducibility actually implies pure absolutely continuous spectrum. Together with Theorem 2.3 and formly mentioned Bourgain-Goldstein's result (Bourgain and Goldstein [2000]), it implies that typical one frequency analytic Schödinger operators don't have singular continuous spectrum.

Concerning the absolutely continuous spectrum, we also mention the well known Kotani-Last conjecture (Kotani and Krishna [1988]). It says that if an one-dimensional ergodic Schrödinger operator has absolutely continuous spectrum, then its potential is almost periodic. By periodic approximation, Avila [2015b] constructed non-almost periodic Schrödinger operators with absolutely continuous spectrum both for the discrete and continuous cases. Another independent work was due to Volberg and Yuditskii [2014], they constructed, by inverse spectral theory, counter-examples in the discrete case (see also the example Damanik and Yuditskii [2016] for the continuous case). The reducibility theory can also provide another approach to construct counterexamples in the continuous case (You and Zhou [2015]). The idea is that reducibility theory and subordinacy theory ensures the existence of ac spectrum, while time scaling make the potential non-almost periodic.

3.2.3 Continuity of Lyapunov exponent and IDS. By Thouless formula and the non-negativity of $L(E)$, one knows that $N(E)$ is always Log-Hölder continuous and that the Hölder continuity of $L(E)$ is equivalent to the Hölder continuity of $N(E)$. IDS is the average of the spectral measure, in general it is more regular than the spectral measure, in fact it is always continuous. However, behavior of the Lyapunov exponents of quasi-periodic cocycles is very complicated. They could be discontinuous in the space of smooth $SL(2,\mathbb{R})$ cocycles (Bochi [2002] and Furman [1997] for C^0 case, Y. Wang and You [2013] for smooth case). Different from the smooth case, the Lyapunov exponent is alway continuous in the space of analytic $SL(2,\mathbb{C})$ cocycles (Bourgain [2005b], Bourgain and Jitomirskaya [2002b], and Jitomirskaya, Koslover, and Schulteis [2009]), even in the space of higher dimensional $GL(d,\mathbb{C})$ cocycles (Avila, Jitomirskaya, and Sadel [2014]). The continuity of Lyapunov exponents implies that the set of the cocycles with positive Lyapunov exponent is open in analytic topology. Together with the denseness result by Avila [2011], one knows that the set of quasi-periodic cocycles with positive Lyapunov exponent is open and dense in analytic topology, but this result is not true in the space of smooth quasi-periodic cocycles (Y. Wang and You [2015]).

One could expect the Hölder continuity in analytic case when the freqencies satisfy some arithmetic conditions. In the supercritical region, Goldstein and Schlag [2001] proved the Hölder continuity of $L(E)$ if $V(x)$ is analytic and α is strong Diophantine. You and S. Zhang [2014] generalized Goldstein-Schlag's result to all Diophantine α and some weaker Liouvillean α, which shows that the Diophantine condition on ω is not necessary for the Hölder continuity of $L(E)$. However, some kind of arithmetic assumptions on α is neccessary (Bourgain [2005a]). Recently, Avila, Last, Shamis, and Zhou [n.d.] proved that IDS of $H_{\lambda,\alpha,\theta}$ is not even weak Hölder if α is extremely Liouvillean.

For Diophantine frequency, the modulus of the Hölder continuity is not very clear so far. It is already known that it can not be better than $\frac{1}{2}$-Hölder. In the supercritical regime, and if furthermore the potential V is in a small L^∞ neighborhood of a trigonometric polynomial of degree d, then the IDS is $(\frac{1}{2d}-\epsilon)$-Hölder for all $\epsilon > 0$ (Goldstein and Schlag [2008]), and it is exactly $\frac{1}{2}$-Hölder for AMO (Avila [2008] and Avila and Jitomirskaya [2010]). However, $(\frac{1}{2d}-\epsilon)$-Hölder continuity is surely not optimal. By reducibility argument, we conjecture that the modulus of Hölder continuity of $L(E)$ is at least $\frac{1}{2N}$, where N is the acceleration of the Schrödinger cocycle $(\alpha, S_E^V(\cdot))$.

For small analytic potentials, the reducibility argument was used by Hadj Amor [2009] to prove the $\frac{1}{2}$-Hölder continuity of the IDS and the Lyapunov exponent if ω is Diophantine. However when dealing with subcritical regime, her approach does not work since the estimates need explicit dependence on the parameters. In fact, when reducing the global potential to local regimes by Avila's global theory, the explicit dependence of the parameters is lost. Based on Thouless's formula, Avila and Jitomirskaya [2010] developed a new understanding of the problem. In order to prove that IDS is $\frac{1}{2}$ Hölder, it is sufficient to prove

$$L(E+i\epsilon) - L(E) \leq \epsilon^{1/2},$$

which relates to the growth of cocycles $\|\mathfrak{A}_n\|_{C^0}$. Thus in the almost reducible scheme, one only needs to estimate the C^0 norm of $\|B_n\|$ and $\|F_n\|$. By this method and Avila's global theory, one can show in the subcritical region for $\beta(\alpha)=0$, then IDS is $\frac{1}{2}$-Hölder continuous (Avila [2008], Avila and Jitomirskaya [2010], Avila [n.d.(b)], and Leguil, You, Zhao, and Zhou [2017]). The method also works for finite smooth potentials, recently Cai, Chavaudret, You, and Zhou [2017] proved the $\frac{1}{2}$-Hölder continuity of IDS for operators with finite smooth small potentials and Diophantine frequency.

3.2.4 Positivity of Lyapunov exponent. The positivity of $L(E)$ is also a big issue. Actually, it is difficult to compute. Herman [1983] proved that, by the subharmonicity method, $L(E) \geq \ln|\lambda|$ for almost Mathieu operator $H_{\lambda,\alpha,\theta}$ with $|\lambda|>1$. By continuity of Lyapunov exponent (Bourgain and Jitomirskaya [2002b]), it was further proved that

$$L(E) = \max\{0, \ln|\lambda|\},$$

for $E \in \Sigma$ (consult Avila [2015a] for another elegant proof). Herman's subharmonicity trick also works for trigonometric polynomials $\lambda V(x)$ with large λ (Herman [1983]). The generalization to arbitrary one-frequency nonconstant real analytic potentials was given by Sorets and Spencer [1991], who proved that if $|\lambda| \geq \lambda_0$, then

$$L(E) \geq \ln|\lambda| - C, \tag{3-3}$$

Here, C and λ depend on V but not on α (consult Bourgain [2005b], Bourgain and Goldstein [2000], Duarte and Klein [2014], Goldstein and Schlag [2001], and Z. Zhang [2012] for simplified proofs and generalizations) .

Compared with the very precise estimate of $L(E)$ for the almost Mathieu operators, the formula (3-3) is too rough. Based on a generalized Thouless formula by Haro and Puig [2013] and the large deviation theorem for identically singular cocycles, Duarte and Klein [n.d.] showed that

$$L(E) = \log\lambda + \int |V(\theta) - \frac{E}{\lambda}|\, d\theta + O(e^{-c(\ln|\lambda|)^b}),$$

where $c > 0$ and $0 < b < 1$. Han and Marx [2018] further improved the bound to

$$L(E) = \log\lambda + \int |V(\theta) - \frac{E}{\lambda}|\, d\theta + O(|\lambda|^{-\frac{2}{2N+1}}),$$

where $N = N(V)$ is a large number. The proof of Han and Marx [ibid.] relies on estimating the acceleration of the cocycle which is defined by Avila [2015a]. In a forthcoming paper, we will show that, based on almost reducibility and Aubry duality,

$$L(E) = \log\lambda + \int |V(\theta) - \frac{E}{\lambda}|\, d\theta + O(\lambda^{-\frac{1}{2d}}),$$

for trigonometric polynomial potentials of order d and sufficiently large λ.

Acknowledgments. The author would like to thank Qi Zhou for his careful reading of the manuscript and many valuable suggestions.

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Received 2017-11-27.

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