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FUNCTIONAL TRANSCENDENCE AND ARITHMETIC APPLICATIONS

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Abstract

We survey recent results in functional transcendence theory, and give arithmetic applications to the André-Oort conjecture and other unlikely-intersection problems.

1 Introduction

The purpose of these notes is to give an introduction to the O-minimality approach to arithmetic geometric questions in the field now referred to as "Unlikely Intersections", as well as give a brief survey of recent results in the literature. We emphasize the functional transcendence results, which are both necessary for many interesting arithmetic applications, and which we hope are of independent interest. We include some recent results on variations of hodge structures, especially as we believe these are still in their infancy and could have fantastic developments in the near future. In §2, we give a survey of the types of functional transcendence statements that have stemmed from generalizing the Ax-Schanuel theorem. In §3 we introduce o-minimality and give a sort of users manual of the main results that have been useful for these sorts of applications. We then sketch the proofs of some functional transcendence results in §4. §5 is devoted to arithmetic applications, where we explain the recent developments in the André-Oort conjecture and the Zilber-Pink conjecture. We give some sketches of proofs, but we only try to convey the main ideas, rather then give complete arguments.

2 Transcendence Theory

2.1 Classical Results. Classical transcendental number theory is largely concerned with the algebraic properties of special values of special functions. We focus first on the case of exponentiation. There are the following fundamental classical results:

MSC2010: 11G10.

Theorem 2.1. *[Lindemann-Weirstrass] Let $x_1, \ldots, x_n \in \overline{\mathbb{Q}}$ be linearly independent over $\mathbb{Q}$. Then $e^{x_1}, \ldots, e^{x_n}$ are algebraically independent over $\overline{\mathbb{Q}}$.*

The above result implies, for example, that $\ln q$ is transcendental for every rational number q. Baker managed to prove a similar (though weaker) result for the more difficult case of ln:

Theorem 2.2. *[Baker [1975]] Let $x_1, \ldots, x_n \in \overline{\mathbb{Q}}$. If $\ln x_1, \ldots, \ln x_n$ are linearly independent over $\mathbb{Q}$ then they are also linearly independent over $\overline{\mathbb{Q}}$.*

In fact Baker proved a quantitative version of the above theorem. Both of the above results are encapsulated by the following conjecture of Schanuel, which seems to encapsulate all reasonable transcendence properties of the exponential function:

Conjecture 2.1. *Let $x_1, \ldots, x_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then*

$$\text{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n.$$

Note that in the case where the x_i are all in $\overline{\mathbb{Q}}$ one recovers the Lindemann-Weirstrass Theorem 2.1, and in the case where e^{x_i} are all in $\overline{\mathbb{Q}}$ one recovers Baker's Theorem 2.2. Schanuel's conjecture has immediate striking implications. For instance, if one takes $n = 2$ and $\{x_1, x_2\} = \{1, \pi i\}$ then an immediate corollary is that e, π are algebraically independent over $\mathbb{Q}$.

2.2 Functional Analogue.

While Schanuel's Conjecture 2.1 is still out of reach, one can get a lot more traction by considering a functional analogue. From a formal perspective, we replace the pair of fields $\mathbb{Q} \subset \mathbb{C}$ by the pair $\mathbb{C} \subset \mathbb{C}[[t_1, \ldots, t_m]]$. Then one has the following theorem due to Ax, referred to as the Ax-Schanuel theorem (see Ax [1971]):

Theorem 2.3. *Let $x_1, \ldots, x_n \in \mathbb{C}[[t_1, \ldots, t_m]]$ have no constant term and be such that they linearly independent over $\mathbb{Q}$ Then*

$$\text{tr.deg.}_{\mathbb{C}}\mathbb{C}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n + \text{rank}\left(\frac{\partial x_i}{\partial t_j}\right).$$

Example. Consider $n > m$ and let $x_i = t_i$ for $i \leq m$, and x_i to be linearly independent elements in $\mathbb{C}(t_1, \ldots, t_m)$ over $\mathbb{Q}$ for $i > m$. Then it follows from Theorem 2.3 that e^{x_i} are algebraically independent over $\mathbb{C}(t_1, \ldots, t_m, e^{t_1}, \ldots, e^{t_m})$. It immediately follows that the set $\{e^x, x \in \mathbb{C}(t_1, \ldots, t_m)\backslash\mathbb{C}\}$ is linearly independent over $\mathbb{C}(t_1, \ldots, t_m)$.

We pause to explain the extra term on the right hand side in Theorem 2.3. For the moment, suppose that the x_i are convergent power series in the t_j so that the x_i can be considered as functions in the t_j. Then the x_i can be thought of as a map from $\vec{x} : \mathrm{D}^m \to \mathbb{C}^n$ for some small disk D. Now by the implicit function theorem, the dimension of $\vec{x}(\mathrm{D}^m)$ is equal to rank $\left(\frac{\partial x_i}{\partial t_j}\right)$.. Thus, if x_n contributes one to rank $\left(\frac{\partial x_i}{\partial t_j}\right)$, we can consider x_n as a formal variable over $\mathbb{C}[[x_1, \ldots, x_{n-1}]]$ and then x_n, e^{x_n} are easily seen to be algebraically independent over all of $\mathbb{C}[[x_1, \ldots, x_{n-1}]]$.

2.3 Geometric Formulation. Note that the above suggests a geometric reformulation of the above result. Namely, let U be the image of the map $(\vec{x}, e^{\vec{x}}) : \mathrm{D}^m \to \mathbb{C}^n \times (\mathbb{C}^\times)^n$. Note that $U \subset \Gamma$ where Γ is the graph of the exponentiation map. Then the statement of Theorem 2.3 can be reinterpreted geometrically as follows:

- $\mathrm{tr.deg.}_{\mathbb{C}}\mathbb{C}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ is the dimension of the Zariski closure of U in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$
- rank $\left(\frac{\partial x_i}{\partial t_j}\right)$ is the dimension of U.
- The statement that x_i have no constant terms and are are linearly independent in $\mathbb{C}(t_1, \ldots, t_m)$ over $\mathbb{Q}$, implies that the projection of U to $\mathbb{C}^n$ contains the origin and is not contained in linear subspace defined over $\mathbb{Q}$. Equivalently, the projection of U to $(\mathbb{C}^\times)^n$ is not contained in a proper algebraic subgroup.

We thus have the following geometric reformulation of Ax-Schanuel (See Tsimerman [2015] for more details):

Theorem 2.4. *Let W be an irreducible algebraic variety in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$, and let U be an irreducible analytic component of $W \cap \Gamma$, where Γ is the graph of the exponentiation map. Assume that the projection of U to $(\mathbb{C}^\times)^n$ is not contained in a translate of any proper algebraic subgroup. Then*

$$\dim W = \dim U + n.$$

Even though it may seem that Theorem 2.3 is more general than the above due to the possibility of the x_i being non-convergent power series in the t_j, by formulating in terms differential fields and using the Seidenberg embedding theorem one can see that they are in fact equivalent. We list also an implication of the above theorem, as we will have use for it later. As it can be seen as analogous to the classical Lindemann-Weirstrass theorem, it has been dubbed the Ax-Lindemann-Weirstrass (or often just Ax-Lindemann) theorem by Pila:

Theorem 2.5. *Let W, V be irreducible algebraic varieties in $\mathbb{C}^n$, $(\mathbb{C}^\times)^n$ respectively such that $e^W \subset V$. Then there exists a translate S of an algebraic subgroup of $(\mathbb{C}^\times)^n$ such that $e^W \subset S \subset V$*

To deduce Theorem 2.5 one may apply the conclusion of Theorem 2.4 to the subvariety $W \times V$ of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$.

2.4 Generalizations to other geometric settings. Theorem 2.4 and Theorem 2.5 are stated in the context of the exponentiation map, but it is not hard to make formal generalizations to other settings. We describe now a recipe for generalizing to other contexts. One requires the following objects:

1. Two algebraic varieties $\widehat{D}, X$, an open subset $D \subset \widehat{D}$, and a holomorphic map $\pi : D \to X$. By convention, we define an algebraic subvariety of D to be an analytic component of $D \cap V$, where V is a subvariety of $\widehat{D}$.
2. A collection S of irreducible subvarieties of X called *weakly special* varieties such that their pre-image in D contains an irreducible algebraic component.

Given the above data, we may formulate an Ax-Schanuel conjecture as follows:

Conjecture 2.2. *(Ax-Schanuel for X) Let $\Gamma \subset D \times X$ be the graph of π. Let $W \subset X$ be an irreducible algebraic variety, and U an analytic component of $W \cap \Gamma$. Then if the projection of U to X does is not contained in any proper weakly special subvariety, we have*

$$\dim U = \dim W - \dim X.$$

We proceed to give some concrete examples of this principle.

2.4.1 Abelian and Semi-Abelian Varieties. Let X be a semi-abelian variety, in other words an extension of an Abelian variety by a torus. Let the dimension of the abelian part be a and of the toric part be t, and set $g = \dim X = a + t$. Then we may take $D = \widehat{D} = \mathbb{C}^g$ and write X as D/Λ for a discrete subgroup $\lambda \subset D$ of rank $2a + t$. Now we may take the weakly special varieties to be the cosets of algebraic subgroups - which are themselves necessarily semi-abelian subvarieties. Note that this case is a direct generalization of Theorem 2.4 which we may recover by setting $a = 0$. This case was settled by Ax [1972].

2.4.2 Shimura Varieties. Let S be a Shimura variety. We do not give precise definitions in this section, referring instead to surveys such as J. Milne [n.d.] and Moonen [1998] for more details. However, such varieties are naturally quotients of Symmetric spaces D by arithmetic groups, and moreover the spaces D can be identified as a quotient of real lie groups $D \cong G(\mathbb{R})/K$ for G a semisimple lie group defined over $\mathbb{Q}$, and K a maximal compact subgroup of G. This means that $G(\mathbb{R})$ acts on D, and the weakly special varieties can be characterized as the images of orbits $H(\mathbb{R}) \cdot v$, where $H \subset G$ is a semisimple lie subgroup defined over $\mathbb{Q}$ such that the orbit $H(\mathbb{R}) \cdot v$ is complex analytic. Thus, even though D and S do not form groups, the machinery of group theory is still very much present in this setting, though of course the groups are non abelian making this setting significantly more difficult then the abelian and semi-abelian case.

Example. For a positive integer n, one may take $D = \mathbb{H}^n, S = Y(1)^n$ and $\pi : D \to S$ be the j map, where $Y(1)$ is the (coarse) moduli space of complex elliptic curves and $\mathbb{H}$ is the usual upper-half plane. The weakly special shimura varieties V can be described very simply as being imposed by one of 2 types of conditions:

- One may impose a co-ordinate of S to be a constant
- One may insist that, for a fixed positive integer N, tow co-ordinates x_i, x_j of S correspond to elliptic curves which are related by a cyclic isogeny of degree N.

The above 2 operations may yield varieties that are not irreducible, so one should also be allowed to take irreducible components. This yields for a very nice combinatorial description of weakly pecial subvarieties which becomes significantly more complicated for other Shimura varieties, but in practise the explicitness of the description is rarely essential to proofs. In this context, the Theorem 2.5 was proven by Pila [2011] in his groundbreaking unconditional proof of André-Oort for $X(1)^n$, and the Theorem 2.4 was proven by Pila and Tsimerman [2016].

The general case of Theorem 2.5 was proven in Klingler, Ullmo, and Yafaev [2016] after partial progress was done by Ullmo and Yafaev [2014] for compact S, and by Pila and Tsimerman [2014] for $\mathfrak{A}_g$. The general case of Theorem 2.4 was recently announced by Mok, Pila, and Tsimerman [2017]. One may also generalize to *Mixed Shimura Varieties*, where Theorem 2.5 was proven by Gao [2017].

2.4.3 Hodge Structures. There is another generalization one may make, and that is to the setting of Hodge Structures. This setting is slightly more complicated than what we have defined so far and it doesn't exactly fit into our setup, for reasons we will describe. Nevertheless, it is important for arithmetic reasons that we will mention in later sections. For general background on Hodge Structures, we refer the reader to Voisin [2002]. We give a quick definition here.

Definition. *An integral hodge structure of weight m and dimension n consists of:*

- *A free abelian group L of dimension n*
- *An integer-valued non-degenerate quadratic form Q on L satisfying $Q(v, w) = (-1)^n Q(w, v)$*

- *A **Hodge decomposition** of complex vector spaces* $L \otimes \mathbb{C} = \oplus_{p+q=n} H^{p,q}$ *such that* $\overline{H^{p,q}} = H^{q,p}$ *and* $i^{p-q} Q(v, \bar{w})$ *is positive definite on* $H^{p,q}$.

The numbers $h^{p,q} := \dim H^{p,q}$ *are called the* Hodge numbers.

If one fixes a weight, dimension, hodge numbers, and isomorphism class of (L, Q) one obtains a complex analytic space X which parametrizes hodge structures up to isomorphism. Moreover, if one further fixes a basis for L, one obtains an open subset D of a complex Grassmanian manifold, and a holomorphic covering map $D \to X$ with Monodromy $G(\mathbb{Z})$, where $G = \mathrm{Aut}(L, Q)$. Moreover, $G(\mathbb{R})$ acts transitively on $D(\mathbb{R})$ with compact stabilizer, so we obtain a picture quite similar to the one which occurs for Shimura varieties. Indeed, by including some extra data one may recover all Shimura varieties as moduli of Hodge structures.

The reason that this setting presents significant additional complication is that for 'most' choices of Hodge numbers, Carlson and Toledo [2014] showed that X cannot be endowed with the structure of an algebraic variety. In our specific context, this makes formulating a transcendence conjecture difficult!

To resolve this problem, we note that a primary motivating reason for studying hodge structures is that for smooth projective varieties Y and a positive integer m, the cohomology group $H^m(Y, \mathbb{C})$ can naturally be given a hodge structure, with the integer lattice coming from Betti cohomology, the Hodge decomposition coming from Dolbeaut Cohomology, and the quadratic form coming from the cup product. This means that even though the moduli space X is not algebraic, for any family of smooth algebraic varieties over a base B such that the fibers have the right Hodge numbers, we get a *period map* $B \to X$. These period maps give us an algebraic structure to work with (namely, that on B) and so allows us to formulate a version of Theorem 2.4. Such a statement was conjectured by Klingler[1] in Klingler [n.d.], and was proven by Bakker and Tsimerman [2017]. The proof follows closely the structure of Mok, Pila, and Tsimerman [2017], with the primary difference being a new "volume-growth" inequality for moduli of Hodge structures in Griffiths-Transverse directions.

3 O-minimality

3.1 Definitions and Introduction. In the course of proving Theorem 2.4 as well as all its generalizations, one naturally deals with functions that are not algebraic. However, the set of all complex analytic functions can be too unwieldy, so it is natural to look for an intermediate category of functions in which to work. It turns out that one such theory which works particularly well for this class of problems is that of o-minimal structures.

[1] In fact, Klingler conjectures much more specific results

This allows us to work with enough functions to be able to talk about the transcendental covering maps in Theorem 2.4, while still maintaining many of the nice properties that algebraic functions possess.

For our purposes, a **Structure** $\mathcal{S}$ is a collection of sets $S_n \subset 2^{\mathbb{R}^n}$, where the elements of S_n are subsets of $\mathbb{R}^n$, such that the following properties hold:

- S_n is a boolean algebra
- $S_n \times S_m \subset S_{m+n}$
- If we let $\pi : \mathbb{R}^n \to \mathbb{R}^j$ be co-ordinate subspace projection, and $A \in S_n$, then $\pi(A) \in S_j$.
- The set $\{(x, x), x \in \mathbb{R}\}$ is in S_2, and the sets $\{(x, y, x + y), x, y \in \mathbb{R}\}, \{(x, y, xy), x, y, \in \mathbb{R}\}$ are in S_3.

We further say that $\mathcal{S}$ is *o-minimal* if S_1 consists precisely of finite unions of open intervals and points. We say that a set $Z \subset \mathbb{R}^n$ is *definable* in § if $Z \in S_n$, and we say that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is definable in $\mathcal{S}$ if its graph is.t turns out that o-minimal structures have a myriad of useful properties. For example, any set definable in an o-minimal structure has finitely many connected components, has a well-defined dimension, and is almost everywhere differentiable. For an introduction to the theory, see van den Dries [1998].

3.2 Examples of o-minimal structures. It follows from the definitions that the smallest possible structure is the structure $\mathbb{R}_{sa}$ which contains all semi-algebraic sets. It follows from the Tarski-Seidenberg theorem $\mathbb{R}_{sa}$ is o-minimal. It is highly non-trivial to prove that any enlargements are o-minimal.

- Gabrièlov [1968] prove that the structure $\mathbb{R}_{an}$, which is defined as the smallest structure containing all subanalytic functions is definable. Recall that a subanalytic function is a function $f : T \to \mathbb{R}$ where $T \subset \mathbb{R}^n$ is a compact ball such that f extends to an analytic function on an open neighbourhood of T.
- Building on work of Khovanskiĭ [1991], Wilkie [1996] proved that the structure $\mathbb{R}_{exp}$, which is defined as the smallest structure containing the graph of the **real** exponential function[2], is o-minimal.
- The strucure $\mathbb{R}_{an,exp}$ is defined to be the smallest structure containing $\mathbb{R}_{an}$ and $\mathbb{R}_{exp}$, and the o-minimality of this structure was shown by van den Dries and Miller

[2] The complex exponential function has countable pre-images, and so cannot be o-minimal.

[1994b]. Note that the structure generated by two o-minimal structures need not be o-minimal Rolin, Speissegger, and Wilkie [2003], so this is by no means a trivial theorem. The structure $\mathbb{R}_{an,exp}$ turns out to be large enough to encompass most functions that are needed for arithmetic applications, and so this is the structure we will ultimately work in.

3.3 Counting Rational Points in Definable Sets. One very common and useful heuristic in number theory is that sets contain "few" integer/rational points unless they have some kind of "reason" for doing so. As such, one would expect that Transcendental subvarieties contain few such points. However, one must be careful to avoid dealing with sets that are too unwieldy. For example, the graph of the function $\sin(\pi x)$ contains every integer point of the x-axis, and is quite transcendental. It turns out that for o-minimal sets such a thing can't happen, as was shown by Bombieri and Pila [1989]. To state their theorem, let us define the *height* of a rational number $x = a/b$ with $\gcd(a,b) = 1$ to be $H(x) = \max(|a|,|b|)$ and the height of a rational point $x = (x_1,\ldots,x_n)$ to be $H(x) = \max_i(H(x_i))$. For a subset $\mathbb{Z} \subset \mathbb{R}^n$ we define the counting function of Z by

$$N(Z,T) := \#\{x \in Z \cap \mathbb{Q}^n \mid H(x) \leq T\}.$$

Theorem 3.1. *Bombieri and Pila [ibid.] Let $Z \subset \mathbb{R}^2$ be a compact real-analytic transcendental curve. Then*

$$N(Z,T) = T^{o(1)}.$$

In other words, the number of points on Z grows subpolynomially.

The proof of the above theorem uses the determinant method, whereby one uses the rational points to form a determinant that has a lower bound stemming from arithmetic, and an upper bound stemming from geometry. One would like to generalize the above theorem to higher dimensions, but some care is required stemming from the fact that a transcendental surface could easily contain an algebraic curve, or even a line, and thus contain lots of rational points. As such, we define Z^{alg} to be the union of all semi-algebraic curves contained in Z. Then Pila and Wilkie [2006] prove the following higher dimensional generalization:

Theorem 3.2. *Pila and Wilkie [ibid.]*

Let $Z \subset \mathbb{R}^n$ be definable in an o-minimal structure. Then

$$N(Z - Z^{alg},T) = T^{o(1)}.$$

In other words, the number of points on $Z - Z^{alg}$ grows subpolynomially.

One may obtain the same quality bound for counting not only rational points, but algebraic points of a bounded degree over $\mathbb{Q}$. The proof of the above theorem proceeds roughly as follows: One applies the determinant method to show that the rational points in Z lie in the intersections $Z \cap V$ of Z with "few" hypersurfaces V of small degree. One then wishes to apply the theorem inductively on dimension. The key to doing this is a parametrization theorem which means that the intersections $Z \cap V$ can be parametrized by finitely many maps with uniformly bounded derivatives as V varies in the family of all hypersurfaces of a given degree. This is where the o-minimality is crucial to the argument.

3.4 Tame complex geometry. We first say a word about extending the notion of definability to sets that aren't subsets of $\mathbb{R}^n$. First, if Z is a subset of $\mathbb{C}^n$ one may use the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ to talk about the definability of Z. Moreover, we may talk about definable manifolds, by insisting on *finite* open covers that have an isomorphism onto a definable subset of $\mathbb{R}^m$ such that the transition maps are definable. In particular, this gives every complex algebraic variety the structure of a definable (in any structure) manifold by taking a finite affine open cover,.

Peterzil and Starchenko have a series of works Peterzil and Starchenko [2010] where they develop complex geometry in an o-minimal setting[3] where they prove tameness of complex analytic sets that are definable in an o-minimal setting. One particularly robust and applicable result to the arithmetic setting is the following version of Chows theorem, which works for any algebraic variety, not just proper varieties!

Theorem 3.3. *Peterzil and Starchenko [ibid.] Let V be a complex algebraic variety, and $S \subset V$ be a closed, complex-analytic subset, which is definable in an o-minimal structure. Then S is an algebraic subvariety of V.*

Note that complex analytic, closed subvarieties of proper algebraic varieties are definable in $\mathbb{R}_{sa}$, so Theorem 3.3 has the usual Chow theorem as an immediate consequence. Note that the above theorem is not stated in the above form in Peterzil and Starchenko [ibid.], but is easily deducible from those results. The deduction is spelled out in a few places, for example in Mok, Pila, and Tsimerman [2017].

3.5 Fundamental Domains and Definability of Uniformization Maps. In §1, we introduced the setting where we have transcendental Uniformization maps $\pi : D \to S$, where S, D are open subsets of complex algebraic varieties. One may hope for the maps π to be definable in $\mathbb{R}_{an,exp}$ with respect to the natural definable structure on D, S. However, the inverse images of points under these maps π are countable, discrete sets, and

[3] In fact, they in the more general setting of a totally-ordered field instead of $\mathbb{R}$

therefore cannot be definable in *any* o-minimal structure! Instead, what one does is restrict to a fundamental domain $\mathfrak{F} \subset D$. That is, one looks for a definable subset $\mathfrak{F}$ such that $\pi \mid \mathfrak{F}$ is an isomorphism onto S, and then asks whether that isomorphism is definable. Note that this is dependant on the fundamentalddomain that one chooses. Below we give some relevant examples:

- In the case of the exponential function $\pi : \mathbb{C}^n \to \mathbb{C}^{\times n}$, we use the fundamental domain
$$\mathfrak{F} = \{(z_1, \ldots, z_n), \Im(z_i) \in [0, 2\pi]\}.$$
In the real coordinates $z = x + iy$ the function e^z becomes $e^x \cos(y) + ie^x \sin(y)$. Now since we are restricting y to be in a bounded interval, $\cos y, \sin y$ restricted to this interval are definable in $\mathbb{R}_{an}$ and thus $\pi \mid \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

- In the case of an abelian variety, or indeed any case where S is compact, one may take for $\mathfrak{F}$ any bounded fundamental domain, and $\pi \mid \mathfrak{F}$ will be definable in $\mathbb{R}_{an}$.

- In the case of the j-function, $j : \mathbb{H} \to \mathbb{C}$, we may use the usual fundamental domain given by $\mathfrak{F} = \{z \in \mathbb{H}, \Re e(z) \in [-1/2, 1/2), |z| \geq 1\}$, and use the Laureny expansion of j in terms of $e^{2\pi iz}$ to see that $j \mid \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

- In the more general case of a Shimura variety $\pi : D \to S$, where $D \cong G(\mathbb{R})/K$ and $S = D/G(\mathbb{Z})$, one may use the Iwasawa decomposition $G = NAK$ to make a Siegel set for $G(\mathbb{R})$, and translate that to a fundamental domain $\mathfrak{F}$. It is substantially more difficult to show that $\pi \mid \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$. It was done for the moduli space $\mathfrak{A}_g$ of principally polarized abelian varieties by Peterzil and Starchenko [2013] and in general by Klingler, Ullmo, and Yafaev [2016]. This result was generalized to the Mixed Shimura case by Gao [2017].

- As mentioned before, the case $\pi : D \to X$ where D is a period domain and X parametrizes hodge structures of given hodge numbers is more difficult. Since X does not typically admit any algebraic structure it does come equipped with a definable structure either. Nevertheless, one may use any definable fundamental domain $\mathfrak{F}$ endow X with a definable $\mathbb{R}_{an,exp}$-structure. However, if we proceed with this setup, one wants the following natural property to be satisfied: Given a variation of hodge structures over an algebraic base B, one would like the period map $\psi : B \to X$ to be definable with respect to this structure. In forthcoming work of the author with Bakker, it is shown that if one uses a definable fundamental domain $\mathfrak{F}$ coming from a Siegel set, then the period maps are indeed definable.

4 Proofs of Functional Transcendence results

We attempt in this section to give a brief idea of how these results are proven. We first describe the proof of Theorem 2.5.

4.1 Ax-Lindemann Theorems. We restrict ourselves for expository purposes to the setting of a torus, and briefly explain how to adapt the methods to the setting of a Shimura variety. So suppose that $\pi : \mathbb{C}^n \to \mathbb{C}^{\times n}$ is the exponential map, and let $V \subset \mathbb{C}^n$, $W \subset \mathbb{C}^{\times n}$ be algebraic varieties such that $\pi(V) \subset W$. We pick our usual definable fundamental domain $\mathfrak{F}$, and we let $W' = (\pi \mid \mathfrak{F})^{-1}(W)$. Crucially for us W' is definable. The key idea of the proof lies in considering the following set:

$$I = \{t \in \mathbb{C}^n \mid (V + t) \cap \mathfrak{F} \subset W'.$$

It is easy to see that I is definable in $\mathbb{R}_{an,exp}$. Moreover, since $\pi^{-1}(W)$ is invariant under the monodromy group $\mathbb{Z}^n$, it follows that I contains all those elements $t \in \mathbb{Z}^n$ such that the V intersects $\mathfrak{F} - t$. There must be polynomially many of these elements (in fact, at least linearly many) and thus, by the Counting Theorem 3.2 we can conclude that I contains semialgebraic curves. It follows that there is a complex algebraic curve C such that $V + c \subset W'$ for all $c \in C$.

Now we may try to replace V by $V \cap V + C$ and use an induction argument on $\dim V - \dim W$. This will work unless V is invariant under C. If this is the case for all curves C, it would imply that the intersection of V with $\mathfrak{F} + t$ all 'look the same', or in other words that $\pi(V) = \pi(V \cap \mathfrak{F})$. But this implies that $\pi(V)$ is definable in $\mathbb{R}_{an,exp}$, and thus by the definable chows Theorem 3.3, we see that $\pi(V)$ is algebraic. The proof now follows from momodromy arguments.

To generalize the above to the context of Shimura varieties, one encounters two difficulties:

- One needs an additional argument to show that there are many elements $g \in G(\mathbb{Z})$ sich that V intersects $g \cdot \mathfrak{F}$. In the general case, one now uses a hyperbolic volume argument due to Hwang and To [2002], that says that in hyperbolic balls of radius r, the volume of a complex analytic ball must grow at least exponentially in r. By contrast, one can show using definability and siegel set arguments that the volume in any fundamental domain is bounded by a constant. Thus, any curve must pass through at least c^r fundamental domains of distance r away, and then one relates the distance of a fundamental domain $g \cdot \mathfrak{F}$ to the height of g.

- Additional care must be taken due to the non-abelian nature of the groups involved. This is indeed a difficulty, but by setting things up correctly the argument goes through.

4.1.1 Ax-Schanuel. One may use a similar setup to the above, but now we only know that $\dim(V \cap W') > \dim V + \dim W' - n$. Nontheless, we may still define

$$I = \{t \in \mathbb{C}^n \mid \dim((V+t) \cap \mathfrak{F} \cap W') = \dim(V \cap W')$$

and conclude that I contains many points. In fact, in this setting the argument can pushed through (see Tsimerman [2015]). However, in the Shimura setting, due to the extremely non-abelian nature of the groups involved and the fact that $V \cap \pi^{-1}(W)$ is not algebraic either in D, not once its pushedforward to X, the argument seems difficult to push through. However, there is a brilliant idea of Mok [n.d.] [4] which provides a great help in this context.

Mok realizes that by working in W, one can formulate the condition of the existence of V algebraically through a differential equation, even though the map π is extremely transcendental! This means that for a given W, if one V exists giving an excessively large dimension, then there must be a whole family of such V. With this extra freedom in hand to vary V, the argument goes through.

5 Arithmetic applications

We will discuss some problems that typically fall under the ”atypical intersection” umbrella.

5.1 Langs conjecture. Consider again the setting of the torus $X = \mathbb{C}^{\times n}$. The torsion points - points whose co-ordinates are all roots of unity - are distinguished algebraic points in X, and it is natural to ask which algebraic subvarieties contain infinitely many torsion points. In fact, it turns out to be a better (and essentially equivalent question) to ask which irreducible subvarieties contain a Zariski-dense set of torsion points. It is clear that subtori do, and in fact so do coset of a subtorus by a torsion point. The converse was conjectured by Lang [1966] and proven by Raynaud [1983a,b]:

Theorem 5.1. *Let $V \subset \mathbb{C}^{\times n}$ be an irreducible subvariety containing a Zariski dense set of torsion points. Then V is a coset of a subtorus of $\mathbb{C}^{\times n}$ by a torsion point.*

Recall that we defined the weakly special subvarieties to be cosets of subtori of X. The reason we used that strange terminology, is that we say that a *special subvariety* of X is translate of a torus by a torsion point. The above theorem is easily seen to be equivalent to the following statement: *The Zariski closure of an arbitrary union of special subvarieties is a finite union of special subvarieties*.

[4] In fact, Mok uses this idea in Mok [n.d.] to prove the Ax-Lindemann conjecture for all rank-1 quotients of hyperbolic space, even non-arithmetic ones!

The statement of Lang's conjecture can be generalized to the setting of abelian varieties almost verbatim, where we replace the word "subtorus" by "algebraic subgroup". The resulting conjecture is known as the Manin-Mumford conjecture, and was first proven by Raynaud [1983a,b]. Below we will give a proof of Lang's conjecture using the ideas we have developed in the previous section with o-minimality and functional transcendence, following Pila and Zannier [2008].

5.2 André-Oort conjecture. Let S be a shimura variety, and $\pi : D \to S$ be its covering by the corresponding symmetric space. There is a natural $\overline{\mathbb{Q}}$ structure on the variety S, and there are distingiushed $\overline{\mathbb{Q}}$ points on S called *CM points*. We call $V \subset S$ a *special subvariety* if V is a weakly special subvariety which contains at least one CM point, which will inn fact force V to contain a Zariski-dense set of CM points. If $S = Y(1)$ is the moduli space of elliptic curves, then the CM points correspond to those complex elliptic curves E with extra endomorphisms, so that $\mathbb{Z} \subsetneq \mathrm{End}(E)$. More generally, if $S = \mathfrak{A}_g$ is the moduli space of principally polarized abelian varieties of dimension g, than the CM points correspond to those Abelian varieties A such that the endomorphism algebras $\mathrm{End}(A) \otimes \mathbb{Q}$ contain a field of degree $2g$ over $\mathbb{Q}$. This can be intuitively thought of as saying that A has "as many symmetries as possible". Note that it is not immediately obvious that such abelian varieties are even defined over $\overline{\mathbb{Q}}$!

The André-Oort conjecture is the natural generalization of Lang's conjecture to this setting:

Conjecture 5.1. *Let $V \subset S$ be an irreducible subvariety containing a Zariski dense set of torsion points. Then V is a special subvariety.*

There has been much work on the AO conjecture. It was first proven unconditionally for $Y(1)^2$ by André, and later proven in generality but conditionally on the generalized Riemann hypothesis in Ullmo and Yafaev [2014], building on an idea of Edixhoven [2005] who handle the case of $Y(1)^2$ conditionally. Later, Pila adapted his method with Zannier to prove the AO conjecture unconditionally for $Y(1)^n$, and it was recently proven for $\mathfrak{A}_g$. The general case remains open

5.3 Crucial Ingredient: Galois Orbits.

5.3.1 Tori. A crucial ingredient in the Pila-Zannier approach to the special point problems that we have discussed is the ability to prove lower bounds for Galois orbits of special points. In the setting of Lang's conjecture, the special points are simply torsion points of the torus, so in this setting the relevant galois action is the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the roots of unity. By Class field theory for $\mathbb{Q}$, we know that this action is transitive on points of exact order n for each n, so we know precisely how large the Galois orbits are.

5.3.2 Abelian Varieties. In the setting of an Abelian variety A over $\mathbb{Q}$, it is much more difficult to get a handle on the action on the torsion points of A. Given a point P of order n, the lower bound that one needs is $[\mathbb{Q}(P) : \mathbb{Q}] \gg n^{\delta}$ for some positive integer $\delta > 0$. There are a few was to proceed here. The Galois action on torsion points of A gives rise to a morphism $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gsp}_{2d}(\widehat{\mathbb{Z}})$ where $d = \dim A$, and the action on torsion points can be read off easily from this action. It is conjectured that the image is open in the $\widehat{\mathbb{Z}}$ points of the Mumford-Tate group. While this is still open, it follows from work of Serre that that the image contains a power of the center, $(\widehat{\mathbb{Z}}^{\times})^m$, for some positive integer m. This immediately implies that the orbit of any torsion point is of size at least $|(\mathbb{Z}/n\mathbb{Z})^{\times m}| \gg n^{1-o(1)}$, which is sufficient.

There is also an analytic approach by D. W. Masser [1977] which yields this result in a form more suitable for studying families of Abelian varieties.

5.3.3 Shimura Varieties. In the setting where X is a Shimura variety and $p \in X$ is a CM point, one may use the theory of complex multiplication developed by Shimura, Taniyama, and others to relate the size of the Galois orbit of p to class groups of number fields. For example, if $X = X(1)$ and p corresponds to an elliptic curve E_p with endomorphism ring the ring of integers in $K = \mathbb{Q}(\sqrt{D})$ then the size of the Galois orbit of p is equal to the class number of K, which is asymptotic to $|D|^{\frac{1}{2}+o(1)}$. In general, there are two naturally associated tori S, T over $\mathbb{Q}$ such that the size of the Galois orbit of p is the image of the class group of S in the class group of T. Class groups of Tori can be defined naturally just as for number fields (see Shyr [1977]) and the sizes of the class groups satisfy an asymptotic Brauer-Siegel formula which gives us very precise control. However, the challenge comes from the fact that these isogenies can kill torsion of low order, and it is very difficult to obtain unconditional upper bounds on low-order torsion in class groups of number fields. In particular, it is a conjecture that for a number field K of discriminant D, fixed degree n over $\mathbb{Q}$, and a positive integer m that $|CL(K)[m]| = |D|^{o(1)}$, and yet one cannot in most cases even beat the trivial bound $|D|^{\frac{1}{2}+o(1)}$ given by Brauer-Siegel! For results in this direction see Bhargava, Shankar, Taniguchi, Thorne, Tsimerman, and Zhao [2017], Ellenberg and Venkatesh [2007]. However, we cannot even show that the class group of imaginary quadratic fields are not all mostly 5-torsion! If one assumes GRHOne can show something in this direction by using GRH to produce small split primes, and this is the primary reason that André-Oort is only known unconditionally under GRH. Nevertheless, one may push these methods to prove AO unconditionally for $\mathcal{A}_g$ for $g \leq 6$. See Tsimerman [2012],Ullmo and Yafaev [2015].

5.3.4 The case of $\mathcal{A}_g$. The required lower bounds were recently established for $X = \mathcal{A}_g$ using methods different from the above in Tsimerman [2018]. As a corollary, one derives the following result (which seems to be of the same level of difficulty), which was not preivously known:

Theorem 5.2. *For each positive integer g, there are finitely many CM points in $\mathcal{A}_g(\mathbb{Q})$.*

We briefly describe the proof. Let x be a CM point in $\mathcal{A}_g$. Then x occurs in a finite collection C of those CM points with the same endomorphism ring and CM type as x. In the case of elliptic curves $g = 1$ the set C is a single Galois orbit, but for larger g that is not usually the case. Moreover, the set C is acted on by the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the orbits all have the same size. So if x is defined over $\mathbb{Q}$, then all the points in C are defined over $\mathbb{Q}$. Moreover, all the points in C are isogenous. Now one uses a famous theorem of D. Masser and Wüstholz [1993]:

Theorem 5.3. *Let A, B be isogenous abelian varieties over some number field K. Then the degree of the smallest isogeny N between them satisfies $N \leq \max(h(A), [K : \mathbb{Q}])^{c_g}$ where $h(A)$ denotes the Faltings height of A, and $c_g > 0$ is a positive constant depending only on g.*

In other words, if two abelian varieties are isogenous, then there must exist an isogeny between them whose degree is not too large. Applying the above theorem to any two points in C and using our assumption that all the points in C are defined over $\mathbb{Q}$, it follows that they all have isogenies between them of degree at most $h(A)^{c_g}$. However, there are only polynomially many isogenies of degree N that one can take, so if $h(A)$ is sufficiently small one obtains a contradiction.

Now, in general heights of abelian varieties can be quite hard to get a handle on. However, for CM abelian varieties A, Colmez [1993] has a beautiful conjecture computing the Faltings height of A in terms of certain L-values at 1 of Artin representations. This conjecture combined with standard estimates on L functions implies the desired upper bound on $h(A)$. While the Colmez conjecture is still open in general, it was recently proven independently in Andreatta, Goren, Howard, and Madapusi Pera [2018], Yuan and Zhang [2018] that if one averages over a finite family of CM types, the Colmez conjecture is true. This finite average has minimal effect from an analytic standpoint, so is enough to complete the proof.

5.4 Proof of the André-oort conjecture. Once one has the required Galois orbit lower bounds and Functional Transcendence results at ones disposal, the proofs of the André-Oort conjecture and the Lang conjecture proceed among essentially identical lines, so we give them both at once using the language of special varieties. So suppose that $\pi : D \to X$

is our covering map, $V \subset X$ is an algebraic variety and V contains a Zariski-dense set of special points. It follows that V is defined over $\overline{\mathbb{Q}}$, and thus over a number field. For simplicity of exposition we assume that V is defined over $\mathbb{Q}$, though this minimally affects the proof. Let x_i be a sequence of CM points which is Zariski-dense in V. Let $\mathfrak{F} \subset D$ be a standard fundamental domain, and consider the pre-images y_i of the x_i under $\pi \mid \mathfrak{F}$. It turns out that the y_i are all defined over number fields of bounded degree over $\mathbb{Q}$. For example, in the case of Lang's conjecture, the pre-image under $z \to e^{2\pi i z}$ of the torsion points the rational numbers , and if one restricts to a suitable fundamental domain one obtains the rational numbers between 0 and 1. Moreover, the Galois lower bounds imply that the heights of these numbers satsify $H(y_i) \ll |\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})x_i|^\delta$ for some fixed positive constant δ. Since V contains all of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})x_i$, it follows that the pe-image W of V under π contains a lot of rational points. By the Counting Theorem 3.2 it follows that W must contain algebraic subvarieties containing all but finitely many of these special points. By Theorem 2.5 and its generalizations, it follows that these algebraic subvarieties must be pre-images of special varieties contained in V. We've thus succeeded in showing that all but finitely many special points in V are contained in higher dimensional special subvarieties. At this point, an induction argument using to finish the proof. We don't give the argument since it requires some deeper analysis using definability in o-minimal structures, and instead refer the interested reader to Pila and Zannier [2008] and Tsimerman [2015].

5.5 The Zilber-Pink Conjecture. Let us return to the setting of a Torus, and consider a proper subvariety $V \subset \mathbb{C}^{\times n}$. For any special subvariety (coset of a subtorus by a torsion point) T, naive dimension theory suggests that the dimension of $V \cap T$ is $\dim V + \dim T - n$. Thus, whenever $\dim(V \cap T) + \dim n > \dim V + \dim T$ we call $V \cap T$ an *unlikely intersection* for V. Notice that if T is a point, then V only intersects T if V contains T, in which case the intersection will be unlikely. Thus this concept generalizes the special point problems studied above. Of course, unlikely intersections can be easily constructed. For example, one may take V to be a subvariety of a subtorus. Then all of V is an unlikely intersection for V! More subtly, one may take any codimension ≥ 2 special variety T, take a codimension 1 subvariety $U \subset T$ and then arbitrarily take V to be another variety containing U as a divisor. However, one has the following conjecture, made by Bombieri-Masser-Zannier:

Conjecture 5.2. *Bombieri, D. Masser, and Zannier [1999] and Zilber [2002]*
Let $V \subset \mathbb{C}^{\times n}$ be a proper subvariety. Then there are finitely many unlikely intersections for V which are maximal *under inclusion.*

One may of course easily generalize to the setting of abelian varieties or (mixed) Shimura varieties, and it is in this general setting that the Zilber-Pink conjecture occurs. One may generalize the Pila-Zannier method to this setting, but this conjecture is substantially more difficult than the corresponding one for special point problems. For one thing, the required functional transcendence input is the Ax-Schanuel Theorem rather then the easier Ax-Lindemann theorem. However, this has been established in (essentially) complete generality so is no longer an obstruction. However, the lower bounds for Galois orbits that are required seem completely out of reach in general. In the André-Oort conjecture, we are interested in lower bounds of CM points, for which we have all the understanding provided by the theory of complex multiplication, and even here the problem is not solved. For the Zilber-Pink conjecture, one must understand Galois orbits of $V \cap T$, and these have no discernible special structure.

Nonetheless, there are impressive partial results. In the setting of a Torus, the result is known if V is a curve by Bombieri, D. Masser, and Zannier [1999]. In the shimura variety setting, It is proven by Habegger-Pila that if $C \subset Y(1)^n$ is a curve where the degrees of the n projections $C \to Y(1)$ are all different, then the Zilber-Pink conjecture holds. Their proof uses the Masswer-Wüstholz theorem in a very clever way to get prove the required Galois lower bounds. This result was recently partially generalized to certain curves in $\mathfrak{A}_g$ by Orr [2017].

5.6 Results on integral points. In recent, as of yet unpublished work, Lawrence-Venkatesh have come up with a new method by which to use transcendence results to prove powerful finiteness results concerning *integer points* on Varieties. There are sometimes called Shafarevich-type theorems after Shafarevich's theorem that there do not exist elliptic curves over $\operatorname{Spec}\mathbb{Z}$. Briefly, their idea is as follows. Consider a smooth, projective family $Y \to B$ over some algebraic variety $B/\mathbb{Q}$, such that some fibral cohomology group $H^n(Y_b, \mathbb{C})$ is non-zero and the corresponding period map $\psi : B \to X = D/\Gamma$ is not constant. Then one may use global results on Faltings to show that the Galois representations $\rho_b : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to H^n(Y_b, \mathbb{Q}_p)$ occur in finitely many isomorphism classes, as b varies over the integer points $B(\mathbb{Z})$. The reason one requires integer rather than rational points is so that one may control how many primes of bad reduction ρ_b has. Now consider a "p-adic lift" of ψ, which looks like $\tilde{\psi} : B(\mathbb{Q}_p) \to D(\mathbb{Q}_p)$.By results of p-adic hodge theory, the finiteness of Galois representations of the ρ_b implies that $\tilde{\psi}(B(\mathbb{Z}))$ is contained in an algebraic subvariety H of D. Now if $B(\mathbb{Z})$ is infinite, or Zariski-dense, one obtains a p-adic violation of a version of the Ax-Schanuel Theorem 2.4. In fact, this last part is not a complication, since one may formally deduce the p-adic Ax-Schanuel theorem from the complex version proven in Bakker and Tsimerman [2017]

Of course, we are skirting a myriad of complexities, but they can already prove the Mordell conjecture[5] using their methods - for which they do not require the Ax-Schanuel theorem, so it seems quite possible that this method has the potential to prove much more.

[5] The paper in its current form only handles certain cases, but they now claim the full result

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