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ON GROTHENDIECK–SERRE CONJECTURE CONCERNING PRINCIPAL BUNDLES

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Abstract

Let R be a regular local ring. Let $\mathbf{G}$ be a reductive group scheme over R. A well-known conjecture due to Grothendieck and Serre assertes that a principal $\mathbf{G}$-bundle over R is trivial, if it is trivial over the fraction field of R. In other words, if K is the fraction field of R, then the map of non-abelian cohomology pointed sets

$$\mathrm{H}^1_{\text{ét}}(R, \mathbf{G}) \to \mathrm{H}^1_{\text{ét}}(K, \mathbf{G}),$$

induced by the inclusion of R into K, has a trivial kernel. *The conjecture is solved in positive for all regular local rings contaning a field.* More precisely, if the ring R contains an infinite field, then this conjecture is proved in a joint paper due to R. Fedorov and I. Panin published in 2015 in Publications l'IHES. If the ring R contains a finite field, then this conjecture is proved in 2015 in a preprint due to I. Panin which can be found on preprint server Linear Algebraic Groups and Related Structures. A more structured exposition can be found in Panin's preprint of the year 2017 on arXiv.org.

This and other results concerning the conjecture are discussed in the present paper. We illustrate the exposition by many interesting examples. We begin with couple results for complex algebraic varieties and develop the exposition step by step to its full generality.

Contents

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1 Introduction

The conjecture was stated by J.-P. Serre in 1958 in so called constant case and by A. Grothendieck in 1968 in the general case. In this introduction we give couple results motivating the conjecture in the constant case. To do that recall some notation. Let X be an affine complex algebraic variety, smooth and irreducible. Let $\mathbb{C}[X]$ be the ring of regular functions on X and $f \in \mathbb{C}[X]$ be a non-zero function. Let $X_f := \{x \in X : f(x) \neq 0\}$. This open subset is called the principal open subset of X corresponding to the function f. This open subset X_f is itself is an affine algebraic variety and its ring of regular functions $\mathbb{C}[X_f]$ is the localization $\mathbb{C}[X]_f$ of the ring $\mathbb{C}[X]$ with respect to the element f. If A is a $\mathbb{C}[X]$-algebra, then we write A_f for the localization of A with respect to $f \in \mathbb{C}[X]$. We are ready now to formulate first result, which is due to Serre [1958].

Let A be a $\mathbb{C}[X]$-algebra, which is a free finitely generated $\mathbb{C}[X]$-module of rank n. Suppose that A is isomorphic to the matrix algebra $M_r(\mathbb{C}[X])$ locally for the complex topology on X. Suppose further that for a non-zero function $f \in \mathbb{C}[X]$ the $\mathbb{C}[X_f]$-algebras A_f and $M_r(\mathbb{C}[X_f])$ are isomorphic. Then for any point $x \in X$ there is a regular function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the $\mathbb{C}[X_g]$-algebras A_g and $M_r(\mathbb{C}[X_g])$ are isomorphic. In the other words, the $\mathbb{C}[X]$-algebras A and $M_r(\mathbb{C}[X])$ are isomorphic locally for the Zarisky topology on X.

Let us point out that the $\mathbb{C}[X]$-algebras A and $M_r(\mathbb{C}[X])$ are isomorphic locally for the complex topology on X by the assumption of the theorem. The theorem states that these $\mathbb{C}[X]$-algebras are isomorphic *locally for the Zarisky topology on X* provided that they are isomorphic over *a non-empty Zarisky open subset of X*.

It is easy to give many examples of $\mathbb{C}[X]$-algebras A, which are isomorphic to the matrix algebra $M_2(\mathbb{C}[X])$ locally for the complex topology, but which are not isomorphic to the matrix algebra $M_2(\mathbb{C}[X])$ locally for the Zarisky topology. These algebras can be found for instance among generalized quaternion algebras. Second result illustrating the conjecture is due to Ojanguren [1980].

Let X and $\mathbb{C}[X]$ be as above and let $a_i, b_i \in \mathbb{C}[X]$ be invertible functions on X, where $i \in \{1, ..., r\}$. Consider two quadratic spaces $P := \Sigma_{i=1}^{r} a_i T_i^2$ and $Q := \Sigma_{i=1}^{r} b_i T_i^2$ over $\mathbb{C}[X]$. Suppose for a non-zero function $f \in \mathbb{C}[X]$ these quadratic spaces are isomorphic over the ring $\mathbb{C}[X_f]$. Then the quadratic spaces P and Q are isomorphic locally for the Zarisky topology on X. In other words, for any point $x \in X$ there is a regular function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and quadratic spaces P and Q are isomorphic as quadratic spaces over $\mathbb{C}[X]_g$.

A very partial case of the above Serre's result can be formulated as follows. Let a and b two invertible regular functions on X. Consider so called generalized quaternion $\mathbb{C}[X]$-algebra A given by two generators u, v subjecting to defining relations $u^2 = a$, $v^2 = b$, $uv = -vu$. Suppose the $\mathbb{C}(X)$-algebra $A \otimes_{\mathbb{C}[X]} \mathbb{C}(X)$ is isomorphic to the matrix algebra $M_2(\mathbb{C}(X))$. Then the $\mathbb{C}[X]$-algebra A is isomorphic to the matrix algebra $M_2(\mathbb{C}[X])$ locally for the Zarisky topology on X.

The indicated results can be restated in terms of principal bundles for groups PGL_r, SO_r and PGL_2 respectively. It is pretty clear now that one can try to state a rather general theorem in terms of principal homogeneous spaces. That will be done in the next section.

2 Principal G-bundles

Recall some notion. Let $\mathbf{G}$ be a linear complex algebraic group, that is a closed subgroup of the general linear group $\mathrm{GL}_n(\mathbb{C})$ with respect to the Zarisky topology. Let X be as above. Let $(E, \nu : \mathbf{G} \times E \to E)$ be a pair such that E is a complex algebraic variety together with a regular map $p : E \to X$ and ν is a $\mathbf{G}$-action on E respecting to the map p, that is $p(\nu(g, e)) = p(e)$ for any $e \in E$ and $g \in \mathbf{G}$. We will write ge for $\nu(g, e)$.

A principal $\mathbf{G}$*-bundle over* X is a pair $(E, \nu : \mathbf{G} \times E \to E)$ above such that
1) the regular map $\mathbf{G} \times E \to E \times_X E$ taking (g, e) to (ge, e) is an isomorphism of algebraic varieties;
2) for any point $x \in X$ there are a neighborhood V of x in the complex topology on X and an isomorphism of complex holomorphic varieties $\varphi : E|_V := p^{-1}(V) \to \mathbf{G} \times V$ such that φ respects to the projection on V and φ respects to obvious $\mathbf{G}$-actions on both sides.

A morphism between two principal $\mathbf{G}$-bundles (E_1, ν_1) and (E_2, ν_2) is a morphism $\psi : E_1 \to E_2$ which respects as to the projections on the base X, so to the $\mathbf{G}$-actions. A trivialized $\mathbf{G}$-bundle is the $\mathbf{G}$-bundle $(\mathbf{G} \times X, \mu)$, where $\mu(g'(g, x)) = (g'g)x$. A trivial $\mathbf{G}$-bundle is a $\mathbf{G}$-bundle isomorphic to the trivialized one. Clearly, a $\mathbf{G}$-bundle (E, ν) over X is trivial if there is a section $s : X \to E$ of the projection $p : E \to X$. For a principal $\mathbf{G}$-bundle (E, ν) over X we often will write just E skipping ν from the notation. We will write often a $\mathbf{G}$-bundle for a principal $\mathbf{G}$-bundle.

Many examples of principal **G**-bundles are obtained by the following simple construction. Consider a closed embedding of algebraic groups $\mathbf{G} \subset \mathrm{GL}_n(\mathbb{C})$ and set $X = \mathbf{G}\backslash\mathrm{GL}_n(\mathbb{C})$ (the orbit variety of right cosets with respect to **G**). Then the pair $(\mathrm{GL}_n(\mathbb{C}), \nu : \mathbf{G} \times \mathrm{GL}_n(\mathbb{C}) \to GL_n(\mathbb{C}))$, where ν takes (g, h) to gh is a principal **G**-bundle over X. The fibres of the projection $p : \mathrm{GL}_n(\mathbb{C}) \to X$ are right cosets of $\mathrm{GL}_n(\mathbb{C})$ with respect to the subgroup **G**. This principal **G**-bundle is very often not trivial locally for the Zarisky topology. For instance, if **G** is the special orthogonal group $\mathrm{SO}_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$, then the above principal G-bundle *is not trivial locally for the Zarisky topology.*

A principal **G**-bundle E over X is not necessary trivial locally for the Zarisky topology on X, *however it is always trivial locally for the étale topology on* X. The latter means the following: there is a regular map $\pi : X' \to X$ of smooth algebraic varieties and a regular map $s' : X' \to E$ such that $p \circ s' = \pi$, π is surjective and for any point $x' \in X'$ the induced map of tangent spaces $T_{X',x'} \to T_{X,\pi(u)}$ is an isomorphism. In the other words, $\pi : X' \to X$ is a surjective étale regular map and the principal **G**-bundle $X' \times_X E$ over X' is trivial. Indeed, the regular map $(id_{X'}, s') : X' \to X' \times_X E$ is a section of the projection $pr_{X'}$.

We are ready now to state a very general result concerning principal **G**-bundles and extending the results from the introduction.

Theorem 2.1. Let **G** be a simple (or a semi-simple, or even a reductive) complex algebraic group. Let X be an affine complex algebraic variety, smooth and irreducible and let E_1, E_2 be two principal **G**-bundles over X. Suppose there is a non-zero regular function $f \in \mathbb{C}[X]$ such that the principal **G**-bundles $E_1|_{X_f}$ and $E_2|_{X_f}$ are isomorphic over X_f. Then the principal **G**-bundles E_1 and E_2 *are isomorphic locally for the Zarisky topology on* X.

In other words, for any point $x \in X$ there is a regular function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the principal **G**-bundles $E_1|_{X_g}$ and $E_2|_{X_g}$ are isomorphic over X_g.

Remark 2.2. Let E_2 be a trivial principal **G**-bundle, then this theorem states the following. If E_1 is trivial over a non-empty Zarisky open subset of X, then E_1 *is trivial locally for the Zarisky topology on* X.

If E_2 is a trivial , then this theorem is due to Colliot-Thélène and Ojanguren [1992].

The general case of the theorem is due to R. Fedorov and the speaker Fedorov and I. Panin [2015].

Remark 2.3. The case when E_2 is trivial corresponds to the so called ”constant” case of the conjecture. Indeed, in this case one principal **G**-bundle $E = E_1$ is given and the result is a theorem about that principal **G**-bundle. The general case of the theorem can not be reduced to an equivalent statement concerning a principal **G** for the group **G**.

Let us give now few examples illustrating Theorem 2.1. Those examples are partial cases of Theorem 2.1 for the projective linear groups PGL_2,PGL_n, for the exceptional group G_2 and for the special projective orthogonal PSO_{2n} and for the projective simplectic groups PSp_{2n} respectively.
(1) Let A_1, A_2 be two generalized quaternion $\mathbb{C}[X]$-algebras corresponding to pairs a_1, b_1 and a_2, b_2 respectively. Suppose for a non-zero function $f \in \mathbb{C}[X]$ the $\mathbb{C}[X_f]$-algebras $(A_1)_f$ and $(A_2)_f$ are isomorphic. Then the $\mathbb{C}[X]$-algebras A_1 and A_2 are isomorphic locally for the Zarisky topology on X.
(2) Let A_1 and A_2 be two algebras as in the Serre's theorem above. *They are called Azumaya* $\mathbb{C}[X]$*-algebras.* Suppose for a non-zero function $f \in \mathbb{C}[X]$ the $\mathbb{C}[X_f]$-algebras $(A_1)_f$ and $(A_2)_f$ are isomorphic. Then the $\mathbb{C}[X]$-algebras A_1 and A_2 are isomorphic locally for the Zarisky topology on X.
(3) Let a_1, b_1, c_1 be invertible functions in $\mathbb{C}[X]$ and let $\mathbb{O}(a_1, b_1, c_1)$ be a generalized octonion $\mathbb{C}[X]$-algebra. That is as a $\mathbb{C}[X]$-module $\mathbb{O}(a_1, b_1, c_1)$ is a free of rank 8 with a free basis $1, e_1, e_2, ..., e_7$. And the multiplication table is as follows: $e_1^2 = a_1, e_2^2 = b_1, e_3^2 = c_1$

$$e_4 = e_1e_2 = -e_2e_1, e_5 = e_2e_3 = -e_3e_2, e_6 = e_3e_4 = -e_4e_3, e_7 = e_4e_5 = -e_5e_4.$$

Let a_2, b_2, c_2 be invertible functions in $\mathbb{C}[X]$ and let $\mathbb{O}(a_2, b_2, c_2)$ be one more generalized octonion $\mathbb{C}[X]$-algebra. Suppose for a non-zero function $f \in \mathbb{C}[X]$ the $\mathbb{C}[X_f]$-algebras $\mathbb{O}(a_1, b_1, c_1)_f$ and $\mathbb{O}(a_2, b_2, c_2)_f$ are isomorphic. Then the $\mathbb{C}[X]$-algebras $\mathbb{O}(a_1, b_1, c_1)$ and $\mathbb{O}(a_2, b_2, c_2)$ are isomorphic locally for the Zarisky topology on X.
(4) Let (A_1, σ_1) be an Azumaya $\mathbb{C}[X]$-algebra with involution. That is A_1 is an Azumaya algebra and $\sigma_1 : A_1 \to A_1^{op}$ is an $\mathbb{C}[X]$-algebras isomorphism, where A_1^{op} is the opposite $\mathbb{C}[X]$-algebra. Let (A_2, σ_2) be one more Azumaya $\mathbb{C}[X]$-algebra with involution. Suppose for a non-zero function $f \in \mathbb{C}[X]$ the $\mathbb{C}[X_f]$-algebras $(A_1, \sigma_1)_f$ and $(A_2, \sigma_2)_f$ are isomorphic as $\mathbb{C}[X_f]$-algebras with involutions. That is there is a $\mathbb{C}[X_f]$-algebras isomorphism $\varphi : (A_1)_f \to (A_2)_f$ with $\sigma_2 \circ \varphi = \varphi \circ \sigma_1$. Then the $\mathbb{C}[X]$-algebras A_1 and A_2 are isomorphic locally for the Zarisky topology on X as $\mathbb{C}[X]$-algebras with involutions.

3 Non-constant case of the conjecture for complex algebraic varieties

We begin with few examples illustrating the conjecture in the non-constant case. Let X be as above an affine complex algebraic variety, smooth and irreducible.
1) Let a, b be invertible functions in $\mathbb{C}[X]$. Consider an equation

$$T_1^2 - aT_2^2 = b \tag{1}$$

If this equation has a solution over the field $\mathbb{C}(X)$ of rational functions on X, then for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the Equation (1) has a solution in $\mathbb{C}[X_g]$.
2) Let $t^n+a_{n-1}t^{n-1}+...+a_0 = F(t) \in \mathbb{C}[X]$ with $a_i \in \mathbb{C}[X]$ be a monic polinomial such that for any point $x \in X$ the polinomial $F_x(t) = t^n + a_{n-1}(x)t^{n-1} + ... + a_0(x) \in \mathbb{C}[t]$ has no multiple roots. Let $X' \subset X \times \mathbb{A}^1$ be a closed subvariety defined by the equation $\{F = 0\}$. The regular function ring $\mathbb{C}[X']$ is the factor ring $\mathbb{C}[X][t]/(F)$. It is a free rank n module over $\mathbb{C}[X]$. Therefore there is a norm map $N_{\mathbb{C}[X']/\mathbb{C}[X]} : \mathbb{C}[X']^\times \to \mathbb{C}[X]^\times$, which takes an element α to the element $det(m_\alpha)$, where $m_\alpha : \mathbb{C}[X'] \to \mathbb{C}[X']$ is a multiplication by α. Let $a \in \mathbb{C}[X]^\times$ be a unit. Suppose the equation

$$N_{\mathbb{C}(X')/\mathbb{C}(X)}(\alpha) = a \tag{2}$$

has a solution in $\mathbb{C}(X')$, then for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the equation $N_{\mathbb{C}[X'_g]/\mathbb{C}[X_g]}(\alpha) = a$ has a solution in $\mathbb{C}[X']_g$.
3) Let $a, b, c \in \mathbb{C}[X]$ be invertible functions. Consider an equation

$$T_1^2 - aT_2^2 - bT_3^2 + abT_4^2 = c \tag{3}$$

Suppose this equation has a solution over the field $\mathbb{C}(X)$. Then for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the Equation (3) has a solution in $\mathbb{C}[X_g]$.
4) Let A be an Azumaya $\mathbb{C}[X]$-algebra of rank n from the introduction. Let $Nrd : A \to \mathbb{C}[X]$ be the reduced norm map. It is a map such that for any point $x \in X$ the map $A/m_xA = M_r(\mathbb{C}) \to \mathbb{C}$ is the determinant. Let $a \in \mathbb{C}[X]^\times$. Suppose the equation

$$Nrd(\alpha) = a \tag{4}$$

has a solution in $A \otimes_{\mathbb{C}[X]} \mathbb{C}(X)$. Then for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the Equation (4) has a solution in A_g.
5) Let $Q = \Sigma_{i=1}^r b_i T_i^2$ be the quadratic space over $\mathbb{C}[X]$ from the introduction. Let $Q_{\mathbb{C}(X)}$ be the same quadratic space regarded over the field $\mathbb{C}(X)$. Let $a \in \mathbb{C}[X]^\times$ be a unite. Suppose a is a spinor norm over the field $\mathbb{C}(X)$, that is $a = Q(v_1) \cdot ... \cdot Q(v_{2n})$ for some vectors $v_1, ..., v_{2n} \in \mathbb{C}(X)^r$. Then a is a spinor norm locally for the Zarisky topology on X. In the other words, for any point x in X there is a function $g \in \mathbb{C}[X]$ not vanishing at x and there is a non-negative integer s and there are vectors $w_1, \dots, w_{2s} \in \mathbb{C}[X_g]^r$ such that $a = Q(w_1) \cdot \dots \cdot Q(w_{2s})$.
6) Let (A, σ) be an Azumaya $\mathbb{C}[X]$-algebra with involution and let the involution be othrogonal, that is σ corresponds to a quadratic space locally for the complex topology on X. We use the terminology of Definition 3.1 here. Let $\mathrm{SO}_{A,\sigma} \subset \mathrm{GL}_{1,A}$ be the special orthogonal X-group scheme of the Azumaya $\mathbb{C}[X]$-algebra with involution (A, σ). Let

$\pi : \mathrm{Spin}_{A,\sigma} \to \mathrm{SO}_{A,\sigma}$ be the corresponding spinor X-group scheme. The X-group scheme $\mathrm{SO}_{A,\sigma}$ is the factor of $\mathrm{Spin}_{A,\sigma}$ modulo an inolution ε. Consider an involution τ on the variety $\mathrm{Spin}_{A,\sigma} \times \mathbb{C}^\times$, which takes any point (g, z) to the point $(\varepsilon(g), -z)$. Let Γ^+ be the factor variety of $\mathrm{Spin}_{A,\sigma} \times \mathbb{C}^\times$ modulo the involution τ. It is easy to check that Γ^+ is an X-group scheme. The map $\mathrm{Spin}_{A,\sigma} \times \mathbb{C}^\times \xrightarrow{pr_{\mathbb{C}^\times}} \mathbb{C}^\times \xrightarrow{\uparrow 2} \mathbb{C}^\times$ induces a map $Sn : \Gamma^+ \to \mathbb{C}^\times$ called the spinor norm map. For any section $\alpha : X \to \Gamma^+$ of the projection $\Gamma^+ \to X$ set $Sn(\alpha) = Sn \circ \alpha$. Let $a \in \mathbb{C}[X]^\times$ be a unite. Consider an equation

$$Sn(\alpha) = a. \tag{5}$$

If this equation has a solution over the field $\mathbb{C}(X)$, then it has a solution locally for the Zarisky topology on X. If $A = M_r(\mathbb{C}[X])$ and the involution σ corresponds to the quadratic space Q as above, then this result is equivalent to the result from the previous example.

Definition 3.1. Let X be as above. A smooth X-group scheme consists of the following data $p : \mathbf{G} \to X, \mu : \mathbf{G} \times_X \mathbf{G} \to \mathbf{G}, i : \mathbf{G} \to \mathbf{G}, e : X \to \mathbf{G}$, where p, μ, i, e are a regular maps. The requirements are these ones: p is smooth, μ is associative, e is a two-sided neutral "element" of the composition law μ, i is the inverse "element" for the composition law μ, e is a section of p.

If $\mathbf{G}_0 \subset \mathrm{GL}_n(\mathbb{C})$ is a linear complex algebraic group, then $\mathbf{G} = X \times \mathbf{G}_0$ with the obvious regular maps $p = pr_X, \mu, i, e$ form an X-group scheme. We say that an X-group scheme $\mathbf{G}$ is *holomorphically isomorphic* to the X-group scheme $X \times \mathbf{G}_0$ locally for the complex topology, if for any point $x \in X$ there is a holomorphic isomorphism $\varphi : \mathbf{G}|_U = p^{-1}(U) \to U \times \mathbf{G}_0$, which respects to the projection on U and to all the group data on both sides.

An X-group scheme $\mathbf{G}$ is called *a reductive (respectively, semi-simple; respectively, simple)* if it is an affine complex algebraic variety and for certain complex algebraic reductive group $\mathbf{G}_0$ it is holomorphically isomorphic to the X-group scheme $X \times \mathbf{G}_0$ locally for the complex topology on X. Recall that $\mathbf{G}_0$ is *required to be connected* .

This definition in the case of smooth affine complex algebraic variety X coincides with the one from Demazure and Grothendieck [1970b][Exp. XIX, Defn.2.7]. The class of reductive group schemes contains the class of semi-simple group schemes which in turn contains the class of simple group schemes. This notion of a simple X-group scheme coincides with the notion of a simple semi-simple X-group scheme from Demazure and Grothendieck [ibid.][Exp. XIX, Def. 2.7 and Exp. XXIV, 5.3].

Definition 3.2. Let $\mathbf{G}$ be a reductive X-group scheme. *A principal* $\mathbf{G}$*-bundle over* X consists of data $(p : E \to X, \nu : \mathbf{G} \times_X E \to E)$ such that p is a smooth surjective

regular map, ν is a $\mathbf{G}$-action on E respecting to the projections on X and
1) the regular map $\mathbf{G} \times_X E \to E \times_X E$ taking (g,e) to (ge,e) is an isomorphism of algebraic varieties;
2) for any point $x \in X$ there are a neighborhood V of x in the complex topology on X and an isomorphism of complex holomorphic varieties $\varphi : E|_V := p^{-1}(V) \to \mathbf{G}|_V$ such that φ respects to the projection on V and φ respects to the obvious left $\mathbf{G}$-actions on both sides.

A principal $\mathbf{G}$-bundle E *is called trivial* if there is an isomorphism $E \to \mathbf{G}$ over X, which respects to the obvious left $\mathbf{G}$-action on both sides. It is easy to check that E *is trivial* if and only if there is a section $s : X \to E$ of the projection $p : E \to X$.

In the Equation (1) in this section the X-group scheme is defined by the equation $T_1^2 - aT_2^2 = 1$. In the example (2) the X-group scheme is defined by the equation $N_{\mathbb{C}[X']/\mathbb{C}[X]}(\alpha) = 1$. In the example (3) the X-group scheme is defined by the equation $T_1^2 - aT_2^2 - bT_3^2 + abT_4^2 = 1$. In the example (4) the X-group scheme is defined by the equation $Nrd(\alpha) = 1$. In the example (5) the X-group scheme is $Spin_Q$. In the example (6) the X-group scheme is $Spin_{A,\sigma}$. The corresponding principal homogeneous bundles are described in examples (1)-(6). All these and many other examples illustrating the conjecture are simple consequences or partial cases of the following general result.

Theorem 3.3 (Fedorov and I. Panin [2015]). Let X be as above in this section. Let $\mathbf{G}$ be a reductive X-group scheme and E be a principal $\mathbf{G}$-bundle. Suppose for a non-zero function f the principal $\mathbf{G}$-bundle $E|_{X_f}$ is trivial over X_f. Then it is trivial locally for the Zarisky topology on X. That is for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the principal $\mathbf{G}$-bundle $E|_{X_g}$ is trivial over X_g.

Remark 3.4. Let us point out that *the author still does not know* any proof of the result from the example (6), different from deriving it from Theorem 3.3. All results from examples (1)–(5) do have proofs avoiding any reference to Theorem 3.3.

Corollary 3.5 (of Theorem 3.3). Let X be as above in this section. Let $\mathbf{G}$ be a reductive X-group scheme and E_1, E_2 be two principal $\mathbf{G}$-bundles. Suppose for a non-zero $f \in \mathbb{C}[X]$ the principal $\mathbf{G}$-bundles $E_1|_{X_f}$ and $E_2|_{X_f}$ over X_f. Then the principal $\mathbf{G}$-bundles E_1 and E_2 are isomorphic locally for the Zarisky topology on X.

Indeed, consider an X-group scheme $\underline{\mathrm{Aut}}_{\mathbf{G}}(E_1)$ of the $\mathbf{G}$-bundle automorphisms and an X-scheme of principal $\mathbf{G}$-bundle isomorphisms $\underline{\mathrm{Iso}}_{\mathbf{G}}(E_1, E_2)$. The latter scheme is a principal $\underline{\mathrm{Aut}}_{\mathbf{G}}(E_1)$-bundle and the X-group scheme $\underline{\mathrm{Aut}}_{\mathbf{G}}(E_1)$ is a reductive X-group scheme isomorphic to the X-group scheme $\mathbf{G}$ locally for the complex topology on X. A principal $\mathbf{G}$-bundle isomorphism $\varphi : E_1|_{X_f} \to E_2|_{X_f}$ is a section of $\underline{\mathrm{Iso}}_{\mathbf{G}}(E_1, E_2)$ over X_f. Hence $\underline{\mathrm{Iso}}_{\mathbf{G}}(E_1, E_2)$ has sections locally for the Zarisky topology on X. Thus the principal $\mathbf{G}$-bundles E_1 and E_2 are isomorphic locally for the Zarisky topology on X.

Corollary 3.6 (of Theorem 3.3)**.** Let $\mathbf{G}_1, \mathbf{G}_2$ be two simple X-group schemes of the type G_2 (resp. F_4, resp. E_8). That is the fibres and $\mathbf{G}_1, \mathbf{G}_2$ over a point $x \in X$ are of the type G_2 (resp. F_4, resp. E_8). Suppose for a non-zero element $f \in \mathbb{C}[X]$ the X_f-group schemes $(\mathbf{G_1})|_{X_f}$ and $(\mathbf{G_2})|_{X_f}$ are isomorphic. Then the X-group schemes $\mathbf{G_1}$ and $\mathbf{G_2}$ are isomorphic locally for the Zarisky topology on X.

Indeed, consider an X-scheme $\underline{\mathrm{Iso}}_{X-gr-sch}(\mathbf{G_1}, \mathbf{G_2})$. There is a regular map

$$\mathbf{G}_1 \times_X \underline{\mathrm{Iso}}_{X-gr-sch}(\mathbf{G_1}, \mathbf{G_2}) \to \underline{\mathrm{Iso}}_{X-gr-sch}(\mathbf{G_1}, \mathbf{G_2})$$

given by $(g_1, \varphi) \mapsto \varphi \circ \mathrm{conj}(g_1)$. This map makes the X-scheme $\underline{\mathrm{Iso}}_{X-gr-sch}(\mathbf{G_1}, \mathbf{G_2})$ a principal $\mathbf{G_1}$-bundle over X. This principal $\mathbf{G_1}$-bundle has a section over X_f. Hence for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the X-scheme $\underline{\mathrm{Iso}}_{X-gr-sch}(\mathbf{G_1}, \mathbf{G_1})$ has a section over X_g. The latter means that the X_g-group schemes $(\mathbf{G_1})|_{X_g}$ and $(\mathbf{G_2})|_{X_g}$ are isomorphic.

4 The conjecture, main results and some corollaries

Assume that U is a regular scheme. Let $\mathbf{G}$ be a reductive U-group scheme, that is, $\mathbf{G}$ is affine and smooth as a U-scheme and, moreover, the geometric fibers of $\mathbf{G}$ are connected reductive algebraic groups (see Demazure and Grothendieck [1970b][Exp. XIX, Definition 2.7]). Recall that a U-scheme E with an action of $\mathbf{G}$ is called a principal $\mathbf{G}$-bundle over U, if $\mathbf{G}$ is faithfully flat and quasi-compact over U and the action is simply transitive, that is, the natural morphism $\mathbf{G} \times_U E \to E \times_U E$ is an isomorphism. It is well known that such a bundle is trivial locally in the étale topology but in general not in the Zariski topology. Grothendieck and Serre conjectured that if E is generically trivial, then it is locally trivial in the Zariski topology (see Serre [1958][Remarque, p. 31], Grothendieck [1958][Remarque 3, pp. 26–-27], and Grothendieck [1968a][Remarque 1.11.a]). More precisely, the following conjecture is widely attributed to them.

Conjecture 1. Let R be a regular local ring, let K be its field of fractions. Let $\mathbf{G}$ be a reductive group scheme over $U := Spec R$, let E be a principal $\mathbf{G}$-bundle. If E is trivial over $Spec K$, then it is trivial. That is $E(R) \neq \varnothing$.

Theorem 4.1 (Main)**.** If R is a regular local ring containing a field, then the above conjecture holds.

This theorem is proved by R. Fedorov and the author in Fedorov and I. Panin [2015] in the case, when R contains an infinite field. It is proved by the author in I. Panin [2017b], when R contains a finite field.

Corollary 4.2 (of Theorem 4.1)**.** Let R be a regular local ring containing a field and $\mathbf{G}$ be a reductive R-group scheme. Let E_1, E_2 be two principal $\mathbf{G}$-bundles. Suppose they are isomorphic over the fraction field of R. Then they are isomorphic.

The proof literally repeats the proof of Corollary 3.5.

Corollary 4.3 (of Theorem 4.1)**.** Let R be a regular local ring containing a field and $\mathbf{G}$ be a reductive R-group scheme. Let $\mu : \mathbf{G} \to \mathbf{T}$ be a group scheme morphism to an R-torus $\mathbf{T}$ such that μ is locally in the étale topology on $Spec R$ surjective. Assume further that the R-group scheme $\mathbf{H} := \mathrm{Ker}(\mu)$ is reductive. Let K be the fraction field of R. Then the group homomorphism

$$\mathbf{T}(R)/\mu(\mathbf{G}(R)) \to \mathbf{T}(K)/\mu(\mathbf{G}(K)).$$

is injective.

To derive this corollary from Theorem 4.1 consider a commutative diagram

$$\begin{array}{ccccc} \mathbf{G}(R) & \xrightarrow{\mu} & \mathbf{T}(R) & \xrightarrow{\nu} & H^1_{et}(R, \mathbf{H}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{G}(K) & \xrightarrow{\mu} & \mathbf{T}(K) & \xrightarrow{\nu} & H^1_{et}(K, \mathbf{H}) \end{array}$$

By Theorem 4.1 the right vertical arrow has trivial kernel. Now a simple diagram chase completes the proof. The latter corollary extends all the known results of this form proved in Colliot-Thélène and Ojanguren [1992], I. A. Panin and Suslin [1997], Zaĭnullin [2000], and Ojanguren, I. Panin, and Zainoulline [2004].

Corollary 4.4 (of Theorem 4.1)**.** Under the notation and the hypothesis of the previous corollary the following sequence is exact

$$\{1\} \to \mathbf{T}(R)/\mu(\mathbf{G}(R)) \to \mathbf{T}(K)/\mu(\mathbf{G}(K)) \xrightarrow{\Sigma\, res_p} \oplus_{ht(p)=1} \mathbf{T}(K)/\mathbf{T}(R_p) \cdot \mu(\mathbf{G}(K)) \to \{1\},$$

where p runs all hight 1 prime ideals in R and res_p is the obvious map.

The exactness at the term $\mathbf{T}(R)/\mu(\mathbf{G}(R))$ is due to the previous corollary. The surjectivity of the map $\Sigma\, res_p$ is due to Colliot-Thélène and Sansuc [1987]. The exactness at the middle term is proved in I. A. Panin [2016a], I. Panin [2017c].

There are two other very general results concerning the conjecture: due to Nisnevič [1977] and due to Colliot-Thélène and Sansuc [1987] (see the history of the topic).

5 History of the topic

History of the topic. — In his 1958 paper Jean-Pierre Serre asked whether a principal bundle is Zariski locally trivial, once it has a rational section (see Serre [1958][Remarque, p. 31]). In his setup the group is any algebraic group over an algebraically closed field. He gave an affirmative answer to the question when the group is PGL(n) (see Serre [ibid.][Prop. 18]) and when the group is an abelian variety (see Serre [ibid.][Lemme 4]). In the same year, Alexander Grothendieck asked a similar question (see Grothendieck [1958][Remarque 3, pp. 26–27]). A few years later, Grothendieck conjectured that the statement is true for any semi-simple group scheme over any regular local scheme (see Grothendieck [1965][Remarque 1.11.a]). Now by the Grothendieck–Serre conjecture we mean Conjecture 1 though this may be slightly inaccurate from historical perspective. Many results corroborating the conjecture are known.

Here is a list of known results in the same vein, corroborating the Grothendieck–Serre conjecture.

• The case when the group is PGL_n and the base field is algebraically closed is done by J.-P. Serre in 1958 in Serre [1958, Prop. 18].

• The case when the group scheme is PGL_n and the ring R is an arbitrary regular local ring is done by A. Grothendieck in 1968 in Grothendieck [1968a].

• The case when the local ring R contains a field of characteristic not 2 the group is SO_n over the ground field is done by M. Ojanguren in 1982 in Ojanguren [1980].

• The case of an arbitrary reductive group scheme over a discrete valuation ring or over a henselian ring is solved by Nisnevič [1977] in 1984.

• The case, where **G** is an arbitrary torus over a regular local ring, was settled by Colliot-Thélène and Sansuc [1987] in 1987.

• The case, when **G** is quasi-split reductive group scheme over arbitrary two-dimensional local rings, is solved by Nisnevich [1984] in 1989.

• The case, where the group scheme **G** comes from an infinite perfect ground field, solved by Colliot-Thélène and Ojanguren [1992] in 1992. As far as we know this work was inspired by the one Ojanguren [1980].

• The case, where the group scheme **G** comes from an arbitrary infinite ground field, solved by Raghunathan [1994, 1995] in 1994;

• O. Gabber announced in 1994 a proof for group schemes coming from arbitrary ground fields (including finite fields).

• For the group scheme $\mathrm{SL}_{1,A}$, where A is an Azumaya R-algebra and R contains a field the conjecture is solved by I. A. Panin and Suslin [1997] in 1998.

• For the unitary group scheme $\mathrm{U}^{\varepsilon}_{A,\sigma}$, where (A,σ) is an Azumaya R-algebra with involution R contains a field of characteristic not 2 the conjecture is solved by Ojanguren and I. Panin [2001] in 2001.

• For the special unitary group scheme $\mathrm{SU}_{A,\sigma}$, where (A,σ) is an Azumaya R-algebra with a unitary involution and R contains a field of characteristic not 2 the conjecture is solved by Zaĭnullin [2000] in 2001.

• For the spinor group scheme Spin_Q of a quadratic space Q over R containing a field of characteristic not 2 the conjecture is solved Ojanguren, I. Panin, and Zainoulline [2004] in 2004.

• Under an isotropy condition on $\mathbf{G}$ the conjecture is proved by A. Stavrova, N. Vavilov and the author in a series of preprints in 2009, published as papers in 2015 in I. Panin, A. Stavrova, and Vavilov [2015a] and in 2016 in I. A. Panin [2016a].

• The case of strongly inner simple adjoint group schemes of the types E_6 and E_7 is done by the second author, V. Petrov, A. Stavrova and the second author in 2009 in I. Panin, Petrov, and A. Stavrova [2009]. No isotropy condition is imposed there.

• The case, when $\mathbf{G}$ is of the type F_4 with trivial f_3-invariant and the field is infinite and perfect, is settled by Petrov and A. Stavrova [2009] in 2009.

• The case, when $\mathbf{G}$ is of the type F_4 with trivial g_3-invariant and the field is of characteristic zero, is settled by Chernousov [2010] in 2010;

• The conjecture is solved when R contains an infinite field, by R. Fedorov and the author in a preprint in 2013 and published in Fedorov and I. Panin [2015] in 2015.

• The conjecture is solved by the author in the case, when R contains a finite field in I. Panin [2015] (for a better structured proof see I. Panin [2017b]).

So, *the conjecture is solved in the case, when R contains a field*.

The case of mixed characteristic is widely open. Let us indicate two recent interesing preprints Fedorov [2015] and I. A. Panin and A. K. Stavrova [2016]. In Fedorov [2015] the conjecture is solved for a large class of regular local rings of mixed characteristic assuming that $\mathbf{G}$ splits. In I. A. Panin and A. K. Stavrova [2016] the conjecture is solved for any semi-local Dedekind domain providing that $\mathbf{G}$ is simple simply-connected and $\mathbf{G}$ contains a torus $\mathbb{G}_{m,R}$.

6 Sketch of the proof of Theorem 4.1 for an infinite field

To escape technicalities we suppose also that the field k below *is algebraically closed* (for instance, k is the complex numbers $\mathbb{C}$). And also suppose that the ring R is the semi-local ring $\mathcal{O}_{X,x_1,\dots,x_n}$ of finitely many closed points on a smooth affine k-variety X. There are two very general purity theorems I. A. Panin [2016a, Thm. 1.0.1, Thm. 1.0.2] which allows I. A. Panin [ibid., Thm. 1.0.3] to reduce Theorem 4.1 to the case, when the group scheme $\mathbf{G}$ is semi-simple simply-connected. Then using standard arguments as in I. Panin, A. Stavrova, and Vavilov [2015b] one can reduce the latter case to the case of simple and simply-connected group scheme $\mathbf{G}$ (*point out that this reduction requires to work with semi-local rings*). So, we will consider below only the case of simple and simply-connected group scheme $\mathbf{G}$. And for simplicity of notation we will suppose below the ring R is the local ring $\mathcal{O}_{X,x}$ of a closed point x on a smooth affine k-variety X.

Theorem 6.1 (I. Panin, A. Stavrova, and Vavilov [ibid.])**.** Let R be the local ring of a closed point on an irreducible smooth affine variety over the field k, set $U = \text{Spec } R$. Let $\mathbf{G}$ be a simple simply-connected group scheme over U (see Demazure and Grothendieck [1970a, Exp. XXIV, Sect. 5.3] for the definition). Let $\mathcal{G}$ be a principal $\mathbf{G}$-bundle over U which is trivial over the principal open subset $U_f \subset U$ for a non-zero $f \in R$. Then there exists a principal $\mathbf{G}$-bundle E_t over $\mathbb{A}^1_U$ and a monic polynomial $h(t) \in R[t]$ such that

(i) the $\mathbf{G}$-bundle E_t is trivial over $(\mathbb{A}^1_U)_h$,

(ii) the evaluation of E_t at $t = 0$ coincides with the original $\mathbf{G}$-bundle $\mathcal{G}$,

(iii) $\{1\} \times U \subset (\mathbb{A}^1_U)_h$.

Remark 6.2. If the field $k = \mathbb{C}$, then the principal $\mathbf{G}$-bundle E_t regarded as a topological principal $\mathbf{G}$-bundle for the complex topology is of the form $p^*(E_0)$ for a topological principal $\mathbf{G}$-bundle E_0, where $p : \mathbb{A}^1_U \to U$ is the projection. By the item (iii) and (ii) of the latter theorem the principal $\mathbf{G}$-bundle $\mathcal{G}$ *regarded as a topological principal* $\mathbf{G}$*-bundle is trivial.* Hence it is trivial even as the complex holomorphic principal $\mathbf{G}$-bundle.

However *these kind of arguments do not work in general for algebraic principal* $\mathbf{G}$*-bundles* since there are principal $\mathbf{G}$-bundles on $\mathbb{A}^1_U$ which do not come from U (see Fedorov [2016]). If $k = \mathbb{R}$ then there are many examples of principal $\mathbf{G}$-bundles on $\mathbb{A}^1_U$ which do not come from U. Those examples can be deduced from Knus, R. Parimala, and Sridharan [1981/82], Ojanguren, R. Parimala, and Sridharan [1983], S. Parimala [1978].

That is why we need in the following proposition and theorem.

Proposition 6.3 (Fedorov and I. Panin [2015])**.** Let k,R, U, $\mathbf{G}$, be the same as in Theorem 6.1. Let $Z \subset \mathbb{A}^1_U$ be a closed subscheme finite over U. Then there exists a closed subscheme Y in $\mathbb{A}^1_U$ such that Y is étale and finite over U, the Y-group scheme $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split and $Y \cap Z = \varnothing$.

Theorem 6.4 (Fedorov and I. Panin [ibid.])**.** Let k, R, U, $\mathbf{G}$, be the same as in Theorem 6.1. Let $Z \subset \mathbb{A}^1_U$ be a closed subscheme finite over U. Let $Y \subset \mathbb{A}^1_U$ be a closed subscheme étale and finite over U. Assume that $Y \cap Z = \varnothing$, and $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split.

Let E be a principal $\mathbf{G}$-bundle over $\mathbb{P}^1_U$ such that its restriction to $\mathbb{P}^1_U - Z$ is trivial. Then the restriction of E to $\mathbb{P}^1_U - Y$ is also trivial.

Derive now the simple simply-connected case (geometric) of Theorem 4.1 from these three statements. Let $\mathcal{G}$ be a principal $\mathbf{G}$-bundle over U which is trivial over the principal open subset $U_f \subset U$ for a non-zero $f \in R$. By Theorem 6.1 there are a monic polinomial $h(t) \in R[t]$ and a principal $\mathbf{G}$-bundle E_t over $\mathbb{A}^1_U$ such that (i) and (ii) hold. Since E_t is trivial over $(\mathbb{A}^1_U)_{h(t)}$ there is a principal $\mathbf{G}$-bundle E over $\mathbb{P}^1_U$ such that $E|_{\mathbb{P}^1_U - Z}$ is trivial and $E_{\mathbb{A}^1_U} = E_t$ (here $Z \subset \mathbb{A}^1_U \subset \mathbb{P}^1_U$ is the vanishing locus of $h(t) = 0$). By Proposition 6.3 there is a closed subscheme Y in $\mathbb{A}^1_U$ such that Y is étale and finite over U, the Y-group scheme $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split and $Y \cap Z = \varnothing$. By Theorem 6.4 the restriction of E to $\mathbb{P}^1_U - Y$ is trivial. Since U is local for each section $s : U \to \mathbb{A}^1_U$ of the projection $\mathbb{A}^1_U \to U$ either $s(U) \cap Y = \varnothing$ or $s(U) \cap Z = \varnothing$. In any case, the principal $\mathbf{G}$-bundle $s^*(E_t) = s^*(E)$ is trivial. By the item (ii) of Theorem 6.1 the original $\mathbf{G}$-bundle $\mathcal{G}$ is trivial.

First, we give a sketch of the proof of Theorem 6.1. Let k be the field. Let X be an affine k-smooth irreducible k-variety, and let x be a closed point in X. Let $U = Spec(\mathcal{O}_{X,x})$ and $\mathrm{f} \in k[X]$ be a non-zero function vanishing the point x. Let $\mathbf{G}_X$ be a simple simply-connected group scheme over X, $\mathbf{G}$ be its restriction to U.

Beginning with these data it is constructed in I. Panin, A. Stavrova, and Vavilov [2015b] a monic polinomial $h \in \mathcal{O}_{X,x}[t]$, a commutative diagram of schemes with the irreducible affine U-smooth variety Y

$$(6) \qquad \begin{array}{ccccc} (\mathbb{A}^1 \times U)_h & \xleftarrow{\tau_h} & Y_h := Y_{\tau^*(h)} & \xrightarrow{(p_X)|_{Y_h}} & X_f \\ \Big\downarrow{\scriptstyle inc} & & \Big\downarrow{\scriptstyle inc} & & \Big\downarrow{\scriptstyle inc} \\ (\mathbb{A}^1 \times U) & \xleftarrow{\tau} & Y & \xrightarrow{p_X} & X \end{array}$$

and a morphism $\delta : U \to Y$ subjecting to the following conditions:

(a) the left hand side square is *an elementary distinguished square* in the category of affine U-smooth schemes in the sense of Morel and Voevodsky [1999, Defn.3.1.3];

(b) $p_X \circ \delta = can : U \to X$, where can is the canonical morphism;

(c) $\tau \circ \delta = i_0 : U \to \mathbb{A}^1 \times U$ is the zero section of the projection $pr_U : \mathbb{A}^1 \times U \to U$;

(d) $h(1) \in \mathcal{O}_{X,x}[t]$ is a unit;

(e) for $p_U := pr_U \circ \tau$ there is a Y-group scheme isomorphism $\Phi : p_U^*(\mathbf{G}) \to p_X^*(\mathbf{G}_X)$ with $\delta^*(\Phi) = id_{\mathbf{G}}$.

Given this geometric result a proof of Theorem 6.1 run as follows. In general, $\mathbf{G}$ does not come from X. However we may assume, that $\mathbf{G}$ is a restriction to U of a simple and simply-connected X-group scheme $\mathbf{G}_X$, $\mathcal{G}$ is defined over X. Say, let $\mathcal{G}'$ be a principal $\mathbf{G}_X$-bundle on X with $\mathcal{G}'|_U = \mathcal{G}$ and such that $\mathcal{G}'$ is trivial over X_f for an $0 \neq f \in k[X]$ with $f(x) = 0$. In this case there are two reductive group schemes on Y. Namelly, $p_U^*(\mathbf{G})$ and $p_X^*(\mathbf{G}_X)$. By the property (b) they coincides when restricted to $\delta(U)$. By the property (e) the scheme Y is chosen such that two reductive group schemes $p_U^*(\mathbf{G})$ and $p_X^*(\mathbf{G}_X)$

on Y are isomorphic via an Y-group scheme isomorphism Φ and the restriction of Φ to $\delta(U)$ is the identity. Take $p_X^*(\mathcal{G}')$ and regard it as a principal $p_U^*(\mathbf{G})$-bundle using the isomorphism Φ. Denote that principal $p_U^*(\mathbf{G})$-bundle ${}_U\, p_X^*(\mathcal{G})$. It is trivial on Y_h, since $p_X^*(\mathcal{G}')$ is trivial on Y_h. Take the trivial $pr_U^*(\mathbf{G})$-bundle on $\mathbb{A}^1 \times U$ and glue it with ${}_U\, p_X^*(\mathcal{G})$ via an isomorphism over Y_h. That is possible by the condition (a). This way we get a principal $\mathbf{G}$-bundle $\mathcal{G}_t$ over $\mathbb{A}^1 \times U$ which is trivial over the open subscheme $(\mathbb{A}^1 \times U)_h$ and such that ${}_U\, p_X^*(\mathcal{G}) = \tau^*(\mathcal{G}_t)$. Clearly, it is the desired one. The polinomial h is the polinomial above. Whence the Theorem 6.1.

Secondly, we give a sketch of the proof of Proposition 6.3. Let $\mathfrak{B}$ be the U-scheme of Borel subgroup schemes of $\mathbf{G}$. It is a smooth projective U-scheme (see Demazure and Grothendieck [1970a, Cor. 3.5, Exp. XXVI]). Take an embedding of $\mathfrak{B}$ into a projective space $\mathbb{P}^N_U$ and intersect $\mathfrak{B}$ with appropriately chosen family of hyperplanes. Arguing as in the proof of Ojanguren and I. Panin [2001, Lemma 7.2], we get a scheme Y finite and étale over U and such that the Y-group scheme $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split. Since the field k is infinite and Y is finite étale over U, we can choose a closed U-embedding of Y in $\mathbb{A}^1_U$. We will identify Y with the image of this closed embedding. Since Y is finite over U, it is closed in $\mathbb{P}^1_U$. Applying to Y an appropriative affine U-transformation of $\mathbb{A}^1_U$ we get Y such that $Y \cap Z = \varnothing$. Whence the Proposition 6.3

The next result is a partial case of Theorem 9.6 of I. Panin, A. Stavrova, and Vavilov [2015b].

Proposition 6.5. Let U be as above and let $u \in U$ be its closed point. Let $\mathcal{E}$ be a $\mathbf{G}$-bundle over $\mathbb{P}^1_U$ such that $\mathcal{E}|_{\mathbb{P}^1_u}$ is a trivial $\mathbf{G}_u$-bundle. Assume that there exists a closed subscheme T of $\mathbb{P}^1_U$ finite over U such that the restriction of $\mathcal{E}$ to $\mathbb{P}^1_U - T$ is trivial. Then E is of the form: $\mathcal{E} = \mathrm{pr}^*(\mathcal{E}_0)$, where $\mathcal{E}_0$ is a principal $\mathbf{G}$-bundle over U and $\mathrm{pr} : \mathbb{P}^1_U \to U$ is the canonical projection.

If furthermore $T \cap \{\infty\} \times U = \varnothing$, then $\mathcal{E}$ is trivial.

Finally, we give a sketch of the proof of Theorem 6.4. Let $(Y^h, \pi : Y^h \to \mathbb{A}^1_U, s : Y \to Y^h)$ be the henselization of the pair $(\mathbb{A}^1_U, Y)$. Here $s : Y \to Y^h$ is the canonical closed embedding, see Fedorov and I. Panin [2015, Sect. 5.3] for more details. Let $in : \mathbb{A}^1_U \to \mathbb{P}^1_U$ be the embedding. Set $\dot{Y}^h := Y^h - s(Y)$. Note that as Y^h, so $\dot{Y}^h$ *are affine schemes*, see Fedorov and I. Panin [ibid., Sect. 5.3, Prop. 5.13]. Consider the following cartesian square of schemes

$$\begin{array}{ccc} \dot{Y}^h & \xrightarrow{\ j\ } & Y^h \\ \downarrow & & \downarrow{\scriptstyle in\circ\pi} \\ \mathbb{P}^1_U - Y & \xrightarrow{\ i\ } & \mathbb{P}^1_U \end{array} \tag{7}$$

As explained in Fedorov and I. Panin [2015, Prop. 5.15, Constr. 5.16] that square can be used to get $\mathbf{G}$-bundles on $\mathbb{P}^1_U$ beginning with a $\mathbf{G}$-bundle on $\mathbb{P}^1_U - Y$ and its trivialization over $\dot{Y}^h$. Let E' be a $\mathbf{G}$-bundle over $\mathbb{P}^1_U - Y$. Denote by $\mathrm{Gl}(E', \varphi)$ the $\mathbf{G}$-bundle over $\mathbb{P}^1_U$ obtained by gluing E' with the trivial $\mathbf{G}$-bundle $\mathbf{G} \times_U Y^h$ via a $\mathbf{G}$-bundle isomorphism $\varphi : \mathbf{G} \times_U \dot{Y}^h \to E'|_{\dot{Y}^h}$.

Similarly, set $Y_u := Y \times_U u$ and denote by Y^h_u the henselization of the pair $(\mathbb{A}^1_u, Y_u)$, let $s_u : Y_u \to Y^h_u$ be the closed embedding. Set $\dot{Y}^h_u := Y^h_u - s_u(Y_u)$. Let E'_u be a $\mathbf{G}_u$-bundle over $\mathbb{P}^1_u - Y_u$, where $\mathbf{G}_u := \mathbf{G} \times_U u$. Denote by $\mathrm{Gl}_u(E'_u, \varphi_u)$ the $\mathbf{G}_u$-bundle over $\mathbb{P}^1_u$ obtained by gluing E'_u with the trivial bundle $\mathbf{G}_u \times_u Y^h_u$ via a $\mathbf{G}_u$-bundle isomorphism $\varphi_u : \mathbf{G}_u \times_u \dot{Y}^h_u \to E'_u|_{\dot{Y}^h_u}$.

Note that the $\mathbf{G}$-bundle E can be presented in the form $\mathrm{Gl}(E', \varphi)$, where $E' = E|_{\mathbb{P}^1_U - Y}$. The idea is to show that

(∗) *There is $\alpha \in \mathbf{G}(\dot{Y}^h)$ such that the $\mathbf{G}_u$-bundle $Gl(E', \varphi \circ \alpha)|_{\mathbb{P}^1_u}$ is trivial (here α is regarded as an automorphism of the $\mathbf{G}$-bundle $\mathbf{G} \times_U \dot{Y}^h$ given by the right translation by the element α).*

If we find α satisfying condition (∗), then Proposition 6.5, applied to $T = Y \cup Z$, shows that the $\mathbf{G}$-bundle $\mathrm{Gl}(E', \varphi \circ \alpha)$ is trivial over $\mathbb{P}^1_U$. On the other hand, its restriction to $\mathbb{P}^1_U - Y$ coincides with the $\mathbf{G}$-bundle $E' = E|_{\mathbb{P}^1_U - Y}$. *Thus $E|_{\mathbb{P}^1_U - Y}$ is a trivial* ***G****-bundle*
.

To prove (∗), one should show that

(i) the bundle $E|_{\mathbb{P}^1_u - Y_u}$ is trivial;

(ii) each element $\gamma_u \in \mathbf{G}_u(\dot{Y}^h_u)$ can be written in the form $\gamma_u = \alpha|_{\dot{Y}^h_u}$ for a certain element $\alpha \in \mathbf{G}(\dot{Y}^h)$.

If we succeed to show (i) and (ii), then we proceed as follows. Present the $\mathbf{G}$-bundle E in the form $\mathrm{Gl}(E', \varphi)$ as above. Observe that

$$\mathrm{Gl}(E', \varphi)|_{\mathbb{P}^1_u} \cong \mathrm{Gl}_u(E'_u, \varphi_u),$$

where $E'_u := E'|_{\mathbb{P}^1_u - Y_u}$, $\varphi_u := \varphi|_{\mathbf{G}_u \times_u \dot{Y}^h_u}$.

Using property (i), find an element $\gamma_u \in \mathbf{G}_u(\dot{Y}^h_u)$ such that the $\mathbf{G}_u$-bundle $\mathrm{Gl}_u(E'_u, \varphi_u \circ \gamma_u)$ is trivial. For this γ_u find an element α as in (ii). Finally take the $\mathbf{G}$-bundle $\mathrm{Gl}(E', \varphi \circ \alpha)$. Then its restriction to $\mathbb{P}^1_u$ is trivial. Indeed, one has a chain of $\mathbf{G}_u$-bundle isomorphisms

$$\mathrm{Gl}(E', \varphi \circ \alpha)|_{\mathbb{P}^1_u} \cong \mathrm{Gl}_u(E'_u, \varphi_u \circ \alpha|_{\dot{Y}^h_u}) = \mathrm{Gl}_u(E'_u, \varphi_u \circ \gamma_u),$$

which is trivial by the very choice of γ_u. Thus, (∗) will be achieved.

Let us prove (i) and (ii). The $\mathbf{G}_u$-bundle E_u is trivial over $\mathbb{P}^1_u - Z_u$. The field $k(u) = k$ is algebraically closed. By a theorem Grothendieck–Harder Harder [1968, Satz 3.1, 3.4] there is an algebraic group morphism $\lambda : \mathbb{G}_{m,k(u)} \to \mathbf{G}_u$ such that the $\mathbf{G}_u$-bundle E_u on $\mathbb{P}^1_u$ is isomorphic to the one $\mathbf{G}_u \times_{\mathbb{G}_{m,k(u)}} \mathcal{O}(-1)$, where $\mathcal{O}(-1)$ is the $\mathbb{G}_{m,k(u)}$-bundle

$\mathbb{A}^2_{k(u)} - 0 \to \mathbb{P}^1_{k(u)}$. Since the field k is algebraically closed, hence $\mathbb{P}^1_u - Y_u$ is contained in an affine line $\mathbb{A}^1_u$. Thus the restrictions of $\mathcal{O}(-1)$ and E_u to $\mathbb{P}^1_u - Y_u$ are trivial. So, (i) is achieved.

To complete the proof it remains to achieve (ii). By our assumption on Y, the group scheme $\mathbf{G}_Y = \mathbf{G} \times_U Y$ is quasi-split. Thus we can and will choose a Borel subgroup scheme $\mathbf{B}^+$ in $\mathbf{G}_Y$.

Since Y is an affine scheme, by Demazure and Grothendieck [1970a, Exp. XXVI, Cor. 2.3, Th 4.3.2(a)] there is an opposite to $\mathbf{B}^+$ Borel subgroup scheme $\mathbf{B}^-$ in $\mathbf{G}_Y$. Let $\mathbf{U}^+$ be the unipotent radical of $\mathbf{B}^+$, and let $\mathbf{U}^-$ be the unipotent radical of $\mathbf{B}^-$.

We will write $\mathbf{E}$ for the functor, sending a Y-scheme T to the subgroup $\mathbf{E}(T)$ of the group $\mathbf{G}_Y(T) = \mathbf{G}(T)$ generated by the subgroups $\mathbf{U}^+(T)$ and $\mathbf{U}^-(T)$ of the group $\mathbf{G}_Y(T) = \mathbf{G}(T)$.

The functor $\mathbf{E}$ has the property that for every closed subscheme S in an affine Y-scheme T *the induced map* $\mathbf{E}(T) \to \mathbf{E}(S)$ *is surjective.* Indeed, the restriction maps $\mathbf{U}^\pm(T) \to \mathbf{U}^\pm(S)$ are surjective, since $\mathbf{U}^\pm$ are isomorphic to vector bundles as Y-schemes (see Demazure and Grothendieck [ibid., Exp. XXVI, Cor. 2.5]).

Recall that (Y^h, π, s) is the henselization of the pair $(\mathbb{A}^1_U, Y)$. Also, $in : \mathbb{A}^1_U \to \mathbb{P}^1_U$ is the embedding. Denote the projection $\mathbb{A}^1_U \to U$ by pr and the projection $\mathbb{A}^1_Y \to Y$ by pr_Y. It is proved in Fedorov and I. Panin [2015, Lemma 5.25] the following

Claim. There is a morphism $r : Y^h \to Y$ making the following diagram commutative

$$\begin{array}{ccc} Y^h & \xrightarrow{\ r\ } & Y \\ {\scriptstyle in\circ\pi}\downarrow & & \downarrow{\scriptstyle pr|_Y} \\ \mathbb{P}^1_U & \xrightarrow{\ pr\ } & U \end{array} \tag{8}$$

and such that $r \circ s = \mathrm{Id}_Y$.

We view Y^h as a Y-scheme via r. Thus various subschemes of Y^h also become Y-schemes. In particular, $\dot{Y}^h$ and $\dot{Y}^h_u$ are Y-schemes, and we can consider

$$\mathbf{E}(\dot{Y}^h) \subset \mathbf{G}(\dot{Y}^h) \ \text{ and } \ \mathbf{E}(\dot{Y}^h_u) \subset \mathbf{G}(\dot{Y}^h_u) = \mathbf{G}_u(\dot{Y}^h_u).$$

Since $\dot{Y}^h_u$ is an affine scheme corresponding to the direct product of few fields, $\mathbf{G}_u$ is *simply-connected and quasi-split*, hence one has an equality $\mathbf{G}_u(\dot{Y}^h_u) = \mathbf{E}(\dot{Y}^h_u)$. As indicated right above the diagram (7) the scheme $\dot{Y}^h$ is affine. Since $\dot{Y}^h_u$ is its closed subscheme the group homomorphism $\mathbf{E}(\dot{Y}^h) \to \mathbf{E}(\dot{Y}^h_u)$ is surjective. Thus the the group homomorphism $\mathbf{G}(\dot{Y}^h) \to \mathbf{G}(\dot{Y}^h_u) = \mathbf{G}_u(\dot{Y}^h_u)$ is surjective as well. We achieved the property (ii). Whence the Theorem 6.4. The sketch of the proof of the Theorem 4.1 is completed.

7 Sketch of the proof of Theorem 4.1 for a finite field

Let k be a finite field. Give a sketch of the proof of Theorem 4.1 in this case. The outline of the proof is the same. So, we will focus on crucial differences. There is a reduction to the case of simple and simply-connected group scheme $\mathbf{G}$ over the semi-local ring of finitely many closed points on a smooth affine variety X So, we will consider below only the case of simple and simply-connected group scheme $\mathbf{G}$. And for simplicity of notation we will suppose below the ring R is the local ring $\mathcal{O}_{X,x}$ of a closed point x on a smooth affine k-variety X.

The statement of Theorem 6.1 remains the same in the case of finite base field.

Theorem 7.1 (I. Panin [2017a]). Let R be the local ring of a closed point on an irreducible smooth affine variety over the finite field k, set $U = \operatorname{Spec} R$. Let $\mathbf{G}$ be a simple simply-connected group scheme over U. Let $\mathcal{G}$ be a principal $\mathbf{G}$-bundle over U which is trivial over the principal open subset $U_f \subset U$ for a non-zero $f \in R$. Then there exists a principal $\mathbf{G}$-bundle E_t over $\mathbb{A}^1_U$ and a monic polynomial $h(t) \in R[t]$ such that

(i) the $\mathbf{G}$-bundle E_t is trivial over $(\mathbb{A}^1_U)_h$,

(ii) the evaluation of E_t at $t = 0$ coincides with the original $\mathbf{G}$-bundle $\mathcal{G}$,

(iii) $h(1) \in R$ is invertible in R.

Proposition 6.3 one needs to replace with the following one

Proposition 7.2 (I. Panin [2017b]). Let k,R, U, $\mathbf{G}$, be the same as in Theorem 6.1 and k be the finite field. Let $Z \subset \mathbb{A}^1_U$ be a closed subscheme finite over U. Then there exists a closed subscheme Y in $\mathbb{A}^1_U$ such that Y is étale and finite over U,

(a) the Y-group scheme $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split,

(b) $\operatorname{Pic}(\mathbb{P}^1_u - Y_u) = 0$,

(c) $Y \cap Z = \varnothing$.

Theorem 6.4 one needs to replace with the following one

Theorem 7.3 (I. Panin [ibid.]). Let k, R, U, $\mathbf{G}$, be the same as in Theorem 6.1. Let $Z \subset \mathbb{A}^1_U$ be a closed subscheme finite over U. Let $Y \subset \mathbb{A}^1_U$ be a closed subscheme étale and finite over U. Assume that $Y \cap Z = \varnothing$, $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split and $\operatorname{Pic}(\mathbb{P}^1_u - Y_u) = 0$.

Let E be a principal $\mathbf{G}$-bundle over $\mathbb{P}^1_U$ such that its restriction to $\mathbb{P}^1_U - Z$ is trivial. Then the restriction of E to $\mathbb{P}^1_U - Y$ is also trivial.

The derivation of the simple simply-connected case (geometric) of Theorem 4.1 from these three statements remains the same as in the infinite field case above.

The proof of Theorem 7.1 is essentially more involved than the proof of Theorem 6.1 and is based on some new ideas (see I. Panin [2017a]).

Give a sketch of the proof of Proposition 7.2. For the closed point $u \in U$ choose a Borel subgroup $\mathbf{B}_u$ in $\mathbf{G}_u$. The latter is possible since the field $k(u)$ is finite. Let $\mathcal{B}$ be the U-scheme of Borel subgroup schemes of $\mathbf{G}$. It is a smooth projective U-scheme (see Demazure and Grothendieck [1970a, Cor. 3.5, Exp. XXVI]). The subgroup $\mathbf{B}_u$ in $\mathbf{G}_u$ is a $k(u)$-rational point b in the fibre of $\mathcal{B}$ over the point u.

Applying several many times Poonen's Bertini type theorem Poonen [2004, Thm. 1.2] find a closed subscheme Y' of $\mathcal{B}$ such that Y' is étale over U and the point b is in Y'. Clearly, the Y-group scheme $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split. To finish the proof of Proposition 7.2 it remains to find a closed embedding of Y' into $\mathbb{A}^1_U$ which satisfies properties (b) and (c).

However it might happen that there is no closed embedding of Y' into $\mathbb{A}^1_U$ at all. Indeed, if the number of $k(u)$-rational point on Y'_u *is strictly more* than the cardinality of the field $k(u)$, than there is no closed embedding of Y' into $\mathbb{A}^1_U$ at all. *To avoid this trouble we need in the following*

Lemma 7.4 (I. Panin [2017b]). Let U be as in the Proposition 7.2. Let $Z \subset \mathbb{A}^1_U$ be a closed subscheme finite over U. Let $Y' \to U$ be a finite étale morphism such that for the closed point u in U the fibre Y'_u of Y' over u contains a $k(u)$-rational point. Then there are finite field extensions k_1 and k_2 of the finite field k such that
(i) the degrees $[k_1 : k]$ and $[k_2 : k]$ are coprime,
(ii) $k(u) \otimes_k k_r$ is a field for $r = 1$ and $r = 2$,
(iii) the degrees $[k_1 : k]$ and $[k_2 : k]$ are strictly greater than any of the degrees $[k(z) : k(u)]$, where z runs over all closed points of Z,
(iv) there is a closed embedding of U-schemes $Y'' = ((Y' \otimes_k k_1) \sqcup (Y' \otimes_k k_2)) \xrightarrow{i} \mathbb{A}^1_U$,
(v) for $Y = i(Y'')$ one has $Y \cap Z = \varnothing$,
(vi) for the closed point u in U one has $\mathrm{Pic}(\mathbb{P}^1_u - Y_u) = 0$.

Finish now the proof of the proposition. The U-scheme Y' satisfies the hypotheses of Lemma 7.4. Take the closed subscheme Y of $\mathbb{A}^1_U$ as in the item (v) of the Lemma. For that specific Y the conditions (b) and (c) of the Proposition are obviously satisfied. The condition (a) is satisfied too, since it is satisfied already for the U-scheme Y'. The proposition follows.

Finally, we give a sketch of the proof of Theorem 7.3. It almost literally repeats the sketch of the proof of Theorem 6.4. The only difference is in checking the triviality of the bundle $E|_{\mathbb{P}^1_u - Y_u}$.

The $\mathbf{G}_u$-bundle E_u is trivial over $\mathbb{P}^1_u - Z_u$. The field $k(u)$ is finite and the $k(u)$-group $\mathbf{G}_u$ is quasi-split. By a theorem due to Harder Harder [1968, Satz 3.1, 3.4] there is an algebraic group morphism $\lambda : \mathbb{G}_{m,k(u)} \to \mathbf{G}_u$ such that the $\mathbf{G}_u$-bundle E_u on $\mathbb{P}^1$ is isomorphic to the one $\mathbf{G}_u \times_{\mathbb{G}_{m,k(u)}} \mathcal{O}(-1)$, where $\mathcal{O}(-1)$ is the $\mathbb{G}_{m,k(u)}$-bundle $\mathbb{A}^2_{k(u)} - 0 \to \mathbb{P}^1_{k(u)}$. Since $\mathrm{Pic}(\mathbb{P}^1_u - Y_u) = 0$, hence the restriction of $\mathcal{O}(-1)$ to $\mathbb{P}^1_u - Y_u$

is trivial. Thus, so is the restriction of E_u to $\mathbb{P}^1_u - Y_u$. Theorem 4.1 for the finite field case is proved.

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