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COMPLEXITY PROBLEMS IN ENUMERATIVE COMBINATORICS

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Abstract

We give a broad survey of recent results in enumerative combinatorics and their complexity aspects.

Introduction

The subject of Enumerative Combinatorics is both classical and modern. It is classical as the basic counting questions go back millennia, yet it is modern in the use of a large variety of the latest ideas and technical tools from across many areas of mathematics. The remarkable successes from the last few decades have been widely publicized, yet they come at a price, as one wonders if there is anything left to explore. In fact, are there enumerative problems which cannot be resolved with existing technology? In this paper we present many challenges in the field from the Computational Complexity point of view, and describe how recent results fit into the story.

Let us first divide the problems into three major classes. This division is not as neat at it may seem as there are problems which fit into multiple or none of the classes, especially if they come from other areas. Still, it would provide us with a good starting point.

(1) **Formula.** Let $\mathcal{P}$ be a set of combinatorial objects, think of trees, words, permutations, Young tableaux, etc. Such objects often come with a parameter n corresponding to the size of the objects. Let $\mathcal{P}_n$ be the set of objects of size n. Find a formula for $|\mathcal{P}_n|$.

(2) **Bijection.** Now let $\mathcal{P}$ and $\mathcal{Q}$ be two sets of (possibly very different) combinatorial objects. Say, you know (or at least suspect) that $|\mathcal{P}_n| = |\mathcal{Q}_n|$. Find an explicit bijection $\varphi : \mathcal{P}_n \to \mathcal{Q}_n$.

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(3) **Combinatorial interpretation.** Now suppose there is an integer sequence $\{a_n\}$ given by a formula. Say, you know (or at least suspect) that $a_n \geq 0$ for all n. Find a combinatorial interpretation of a_n, i.e. a set of combinatorial objects $\mathcal{P}$ such that $|\mathcal{P}_n| = a_n$.

People in the area are well skilled in both resolving and justifying these problems. Indeed, a formula is a good thing to have in case one needs to compute $|\mathcal{P}_n|$ explicitly for large n, find the asymptotics, gauge the structural complexity of the objects, etc. A bijection between a complicated set $\mathcal{P}$ and a simpler set $\mathcal{Q}$ is an even better thing to have, as it allows one to better understand the nature of $\mathcal{P}$, do a refined counting of $\mathcal{P}_n$ with respect to various statistics, generate elements of $\mathcal{P}_n$ at random, etc. Finally, a combinatorial interpretation is an excellent first step which allows one to proceed to (1) and then (2), or at least obtain some useful estimates for a_n.

Here is the troubling part, which comes in the form of inquisitive questions in each case:

(1′) What is a formula? What happens if there is no formula? Can you prove there isn't one? How do you even formalize the last question if you don't know the answer to the first?

(2′) There are, obviously, $|\mathcal{P}_n|!$ bijections $\varphi : \mathcal{P}_n \to \mathcal{Q}_n$, so you must want a particular one, or at least one with certain properties? Is there a "canonical" bijection, or at least the one you want best? What if there isn't a good bijection by whatever measure, can you prove that? Can you even formalize that?

(3′) Again, what do you do in the case when there isn't a combinatorial interpretation? Can you formally prove a negative result so that others stop pursuing these problems?

We have a few formal answers to these questions, at least in some interesting special cases.[1] As the reader will see, the complexity approach does bring some clarity to these matters. But to give the answers we first need to explain the nature of combinatorial objects in each case, and to review the literature. That is the goal of this survey.

1 What is a formula?

1.1 Basic examples. We start with the Fibonacci numbers Sloane [n.d., A000045]:

(1-1) $$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = F_1 = 1$$

(1-2) $$F_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}$$

[1] Due to space limitations, we address (3) and (3′) in the full version of the paper.

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n + (-\phi)^{-n}\right), \quad \text{where} \quad \phi = \frac{1+\sqrt{5}}{2} \tag{1-3}$$

$$F_n = \left(A^n\right)_{2,2}, \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{1-4}$$

Equation (1-1) is usually presented as a definition, but can also be used to compute F_n in $poly(n)$ time. Equation (1-2) is useful to place Fibonacci numbers in the hierarchy of integer sequences (see below). Equation (1-3) is useful to obtain asymptotics, and equation (1-4) gives a fast algorithm for computing F_n (by repeated squaring). *The moral*: there is no one notion of a "good formula", as different equations have different uses.

Let us consider a few more plausible formula candidates:

$$D_n = [[n!/e]], \quad \text{where } [[x]] \text{ denotes the nearest integer} \tag{1-5}$$

$$C_n = [t^n]\, \frac{1-\sqrt{1-4t}}{2t} \tag{1-6}$$

$$E_n = n! \cdot [t^n]\, y(t), \quad \text{where } 2y' = 1 + y^2, \ y(0) = 1 \tag{1-7}$$

$$T_n = (n-1)! \cdot [t^n]\, z(t), \quad \text{where } z = t\, e^{t\, e^{t\, e^{t\, e^{\cdot^{\cdot^{\cdot}}}}}} \tag{1-8}$$

Here D_n is the number of *derangements* (fixed-point-free permutations in S_n), C_n is the *Catalan number* (the number of binary trees with n vertices), E_n is the *Euler number* (the number of *alternating permutations* $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots$ in S_n), T_n is the *Cayley number* (the number of spanning trees in K_n), and $[t^n]\, F(t)$ denotes the coefficient of t^n in $F(t)$.

In each case, there are better formulas for applications:

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \tag{1-9}$$

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{1-10}$$

$$E_n = n! \cdot [t^n]\, y(t), \quad \text{where } y(t) = \tan(t) + \sec(t) \tag{1-11}$$

$$T_n = n^{n-2} \tag{1-12}$$

In all four cases, the corresponding formulas are equivalent by mathematical reasoning. Whether or not you accept (1-5)–(1-8)as formulas, it is their meaning that's important, not their form.

Finally, consider the following equations for the *number of partitions* $p(n)$, and *n-th prime number* p_n:

$$p(n) = [t^n] \prod_{i=1}^{\infty} \frac{1}{1-t^i} \tag{1-13}$$

(1-14)

$$p_n = \sum_{m=2}^{n^2} m \Big\langle 1 + \Big| n - \langle \gamma_m \rangle \sum_{r=2}^{m} \langle \gamma_r \rangle \Big| \Big\rangle, \text{ where } \langle x \rangle := \left\lfloor \frac{1}{x} \right\rfloor, \ \gamma_r := \sum_{d=1}^{\sqrt{r}} \left\lfloor \frac{\lfloor r/d \rfloor}{r/d} \right\rfloor.$$

Equation (1-13) is due to Euler (1748), and had profound implications in number theory and combinatorics, initiating the whole area of *partition theory* (see e.g. R. Wilson and Watkins [2013]). Equation (1-14) is from Tsangaris [2007]. Esthetic value aside, both equations are largely unhelpful for computing purposes and follow directly from definitions. Indeed, the former is equivalent to the standard counting algorithm (*dynamic programming*), while the latter is an iterated divisibility testing in disguise.

In summary, we see that the notion of "good formula" is neither syntactic nor semantic. One needs to make a choice depending on the application.

1.2 Wilfian formulas.

In his pioneer 1982 paper Wilf [1982], Wilf proposed to judge a formula from the complexity point of view. He suggested two definitions of "good formulas" for computing an integer sequence $\{a_n\}$:

(W1) There is an algorithm which computes a_n in time *poly*(n).

(W2) There is an algorithm which computes a_n in time $o(a_n)$.

In the literature, such algorithms are called sometimes *Wilfian formulas*. Note that (W1) is aimed to apply for sequences $\{a_n\}$ of at most exponential growth $a_n = \exp O(n^c)$, while (W2) for $\{a_n\}$ of at most polynomial growth (see e.g. Garrabrant and Pak [2017] and Flajolet and Sedgewick [2009] for more on growth of sequences).

Going over our list of examples we conclude that (1-1), (1-2), (1-4), (1-9), (1-10) and (1-12) are all transparently Wilfian of type (W1). Equations (1-3), (1-6), (1-7) and (1-11) are Wilfian of type (W1) in a less obvious but routine way (see below). Equations (1-3) and (1-5) do give rise to ad hoc *poly*(n) algorithms, but care must be applied when dealing with irrational numbers. E.g., one must avoid circularity, such as when computing $\{p_n\}$ by using the *prime constant* $\sum_n 1/2^{p_n}$, see Sloane [n.d., A051006]. Finally, equation (1-8) is not Wilfian of type (W1), while (1-14) is not Wilfian of type (W2).

Let us add two more notions of a "good formula" in the same spirit, both of which are somewhat analogous but more useful than (W2):

(W3) There is an algorithm which computes a_n in time *poly*$(\log n)$.

(W4) There is an algorithm which computes a_n in time $n^{o(1)}$.

Now, for a *combinatorial sequence* $\{a_n\}$ one can ask if there is a Wilfian formula. In the original paper Wilf [1982] an explicit example is given:

Conjecture 1.1 (Wilf). *Let a_n be the number of unlabeled graphs on n vertices. Then $\{a_n\}$ has no Wilfian formula of type* (W1).

See Sloane [n.d., A000088] for this sequence. Note that by the classical result Erdős and Rényi [1963] (see also Babai [1995, §1.6]), we have $a_n \sim 2^{\binom{n}{2}}/n!$, so the problem is not approximating a_n, but computing it exactly. For comparison, the sequence $\{c_n\}$ of the number of connected (labeled) graphs does have a Wilfian formula:

$$c_n = 2^{\binom{n}{2}} - \frac{1}{n}\sum_{k=1}^{n-1} k \binom{n}{k} 2^{\binom{n-k}{2}} c_k$$

(see Sloane [n.d., A001187] and Harary and Palmer [1973, p. 7]).

The idea behind Conjecture 1.1 is that the *Pólya theory* formulas (see e.g. Harary and Palmer [ibid.]) are fundamentally not Wilfian. We should mention that we do not believe the conjecture in view of Babai's recent quasipolynomial time algorithm for Graph Isomorphism Babai [2016]. While the connection is indirect, it is in fact conceivable that both problems can be solved in *poly*(n) time.

Open Problem 1.2. *Let $\pi(n)$ denote the number of primes $\leq n$. Does $\{\pi(n)\}$ have a Wilfian formula of type* (W4)*?*

The *prime-counting function* $\pi(n)$ has a long history. Initially Wilf asked about formula of type (W2), and such formula was found in Lagarias, V. S. Miller, and Odlyzko [1985]. Note that even the parity of $\pi(n)$ is hard to compute Tao, Croot, and Helfgott [2012].

1.3 Complexity setting and graph enumeration.

Let $\mathcal{P}_n$ denote the set of certain *combinatorial objects* of size n. This means one can decide if $X \in \mathcal{P}_n$ in time *poly*(n). The problem of computing $a_n := |\mathcal{P}_n|$ is in #EXP because the input n has *bit-length* $O(\log n)$.[2] This is a counting version of the decision problem NEXP.

[2] To bring the problem into the (usual) polynomial hierarchy, the input n should be given in *unary*, cf. Goldreich [2008] and Moore and Mertens [2011].

For example, let $a_n = |\mathcal{P}_n|$ be the set of (labeled) planar 3-regular 3-connected graphs on n vertices. Graphs in $\mathcal{P}_n$ are exactly graphs of simple 3-dimensional polytopes. Since testing each property can be done in $poly(n)$ time, the decision problem is naturally in NEXP, and the counting problem is in #EXP. In fact, the decision problem is trivially in P, since such graphs exist for all even $n \geq 4$ and don't exist for odd n. Furthermore, Tutte's formula for the number of rooted plane triangulations gives a simple product formula for a_n, and thus can be computed in $poly(n)$ time, see Tutte [1998, Ch. 10].

On the one hand, counting the number of non-Hamiltonian graphs in $\mathcal{P}_n$ is not naturally in #EXP, since testing non-Hamiltonicity is co-NP-complete in this case Garey, Johnson, and Tarjan [1976]. On the other hand, the corresponding decision problem (the existence of such graphs) is again in P by Tutte's disproof of Tait's conjecture, see Tutte [1998, Ch. 2].

Note that GRAPH ISOMORPHISM is in P for trees, planar graphs and graphs of bounded degree, see e.g. Babai [1995, §6.2]. The discussion above suggests the following counterpart of Wilf's Conjecture 1.1.

Conjecture 1.3. *Let a_n be the number of unlabeled plane triangulations with n vertices, b_n the number of 3-connected planar graphs with n vertices, and t_n the number of unlabeled trees with n vertices. Then $\{a_n\}$, $\{b_n\}$ and $\{t_n\}$ can be computed in* poly(n) *time.*

We are very optimistic about this conjecture. For triangulations and trees, there is some recent evidence in Kang and Sprüssel [2018] and the theory of species Bergeron, Labelle, and Leroux [1998], respectively. See also Noy, Requilé, and Rué [2018] for further positive results on enumeration of (labeled) planar graphs.

Denote by a_n the number of 3-regular labeled graphs on $2n$ vertices. The sequence $\{a_n\}$ can be computed in polynomial time via the following recurrence relation, see Sloane [n.d., A002829].

(1-15)
$$\begin{aligned}
3(3n-7)(3n-4)\cdot a_n \;=\; & 9(n-1)(2n-1)(3n-7)(3n^2-4n+2)\cdot a_{n-1} \\
& + (n-1)(2n-3)(2n-1)(108n^3-441n^2+501n-104)\cdot a_{n-2} \\
& + 2(n-2)(n-1)(2n-5)(2n-3)(2n-1)(3n-1)(9n^2-42n+43)\cdot a_{n-3} \\
& - 2(n-3)(n-2)(n-1)(2n-7)(2n-5)(2n-3)(2n-1)(3n-4)(3n-1)\cdot a_{n-4}
\end{aligned}$$

Conjecture 1.4. *Fix $k \geq 1$ and let a_n be the number of unlabeled k-regular graphs with n vertices. Then $\{a_n\}$ can be computed in poly(n) time.*

For $k = 1, 2$ the problem is elementary, but for $k = 3$ is related to enumeration of certain 2-groups (cf. Luks [1982]).

Consider now the problem of computing the number $f(m,n)$ of triangulations of an integer $[m \times n]$ grid (see Figure 1). This problem is a distant relative of Catalan numbers C_n in (1-10) which Euler proved counts the number of triangulations of a convex $(n+2)$-gon, see R. P. Stanley [2015], and is one of the large family of triangulation problems, see De Loera, Rambau, and Santos [2010]. Kaibel and Ziegler prove in Kaibel and Ziegler [2003] that $f(m,n)$ can be computed in *poly*(n) time for every fixed m, but report that their algorithm is expensive even for relatively small m and n (see Sloane [n.d., A082640]).

Question 1.5. *Can* $\{f(n,n)\}$ *can be computed in* poly(n) *time?*

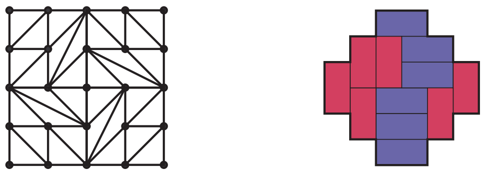

Figure 1: Grid triangulation of $[5 \times 5]$ and a domino tiling.

1.4 Computability setting and polyomino tilings.

Let a_n be the number of domino tilings on a $[2n \times 2n]$ square. Kasteleyn and Temperley–Fisher classical determinant formula (1961) for the number of perfect matchings of planar graphs gives a *poly*(n) time algorithm for computing $\{a_n\}$, see e.g. Kenyon [2004] and Lovász and Plummer [1986]. This foundational result opens the door to potential generalizations, but, unfortunately, most of them turn out to be computationally hard.

First, one can ask about computing the number b_n of 3-dimensional domino tilings of a $[2n \times 2n \times 2n]$ box. Or how about the seemingly simpler problem of counting he number c_n of 3-dimensional domino tilings of a "slim" $[2 \times n \times n]$ box? We don't know how to solve either problem, but both are likely to be difficult. The negative results include #P-completeness of the counting problem for general and slim regions Pak and Yang [2013] and Valiant [1979], and topological obstacles, see Freire, Klivans, Milet, and Saldanha [2017] and Pak and Yang [2013, Prop. 8.1].

Consider now a fixed finite set $\mathbf{T} = \{\tau_1, \dots, \tau_k\}$ of general polyomino tiles on a square grid: $\tau_i \subset \mathbb{Z}^2$, $1 \le i \le k$. To tile a region $\Gamma \subset \mathbb{Z}^2$, one must cover it with copies of the tiles without overlap. These copies must be parallel translations of τ_i (rotations and reflections are not allowed). There exist NP-complete tileability problems even for a fixed set of few small tiles. We refer to Pak [2003] for short survey of the area.

For a fixed $\mathbf{T}$, denote by $g(m,n)$ the number of tilings of $[m \times n]$ with $\mathbf{T}$. Is $g(m,n)$ computable in polynomial time? The following conjecture suggests otherwise.

Conjecture 1.6. *There exists a finite set of tiles* $\mathbf{T}$ *such that counting the number of tilings of* $[n \times n]$ *with* $\mathbf{T}$ *is* #EXP*-complete.*

In fact, until we started writing this survey, we always believed this result to be known, only to realize that standard references such as van Emde Boas [1997] fall a bit short. Roughly, one needs to embed a #EXP-complete language into a counting tilings problem of a rectangle. This may seem like a classical idea (see e.g. Moore and Mertens [2011, §5.3.4, §7.6.5]), which worked well for many related problems. For example, the RECTANGULAR TILEABILITY asks: given a finite set of tiles $\mathbf{T}$, does there exists integers m and n, such that $\mathbf{T}$ tiles $[m \times n]$.

Theorem 1.7 (Yang [2014])**.** *The* RECTANGULAR TILEABILITY *problem is undecidable.*

In the proof, Yang embeds the HALTING PROBLEM into RECTANGULAR TILEABILITY. So can one embed a NEXP-complete problem into tileability of $[m \times n]$ rectangle? The answer is *yes* if $\mathbf{T}$ is allowed to be part of the input. In fact, even Levin's original 1973 paper introducing NP-completeness proposed this approach L. A. Levin [1973]. The following result should come as a surprise, perhaps.

Theorem 1.8 (Lam [2008])**.** *Given* T*, the tileability of* $[m \times n]$ *can be decided in* $O(\log m + \log n)$ *time.*

The proof is nonconstructive; it is based on *Hilbert's Basis Theorem* and the algebraic approach by F. W. Barnes. A combination of Theorem 1.7 and Theorem 1.8 implies that the constant implied by the $O(\cdot)$ notation is not computable as a function of $\mathbf{T}$. Roughly, we do know that a linear time algorithm exists, but given $\mathbf{T}$ it is undecidable to find it. Theorem 1.8 also explains why Conjecture 1.6 remains open – most counting results in the area use parsimonious reductions (think bijections between solutions of two problems), and in this case a different approach is required.

2 Classes of combinatorial sequences

2.1 Algebraic and D-algebraic approach. Combinatorial sequences $\{a_n\}$ are traditionally classified depending on the algebraic properties of their GFs

$$A(t) = \sum_{n=0}^{\infty} a_n t^n.$$

We list here only four major classes:

Rational: $A(t) = P(t)/Q(t)$, for some $P, Q \in \mathbb{Z}[t]$,
Algebraic: $c_0 A^k + c_1 A^{k-1} + \ldots + c_k = 0$, for some $k \in \mathbb{N}$, $c_i \in \mathbb{Z}[t]$,
D-finite: $c_0 A + c_1 A' + \ldots + c_k A^{(k)} = b$, for some $k \in \mathbb{N}$, $b, c_i \in \mathbb{Z}[t]$,
D-algebraic: $Q\big(t, A, A, \ldots, A^{(k)}\big) = 0$, for some $k \in \mathbb{N}$, $Q \in \mathbb{Z}[t, x_0, x_1, \ldots, x_k]$.

Note that rational GFs are exactly those $\{a_n\}$ which satisfy linear recurrence:

$$c_0 a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}, \text{ for some } k \in \mathbb{N},\ c_i \in \mathbb{Z}.$$

Such sequences $\{a_n\}$ are called *C-recursive*. For example, Fibonacci numbers satisfy (1-1) and have GF $(1 - t - t^2)^{-1}$. Similarly, *Catalan numbers* have algebraic GF by (1-6). D-finite GFs (also called *holonomic*) are exactly those $\{a_n\}$ which satisfy polynomial recurrence

$$c_0(n) a_n = c_1(n) a_{n-1} + \ldots + c_k(n) a_{n-k}, \text{ for some } k \in \mathbb{N}, c_i \in \mathbb{Z}[n].$$

Such sequences $\{a_n\}$ are called *P-recursive*. Examples include $\{n!\}$, derangement numbers $\{D_n\}$ by (1-9), the number of 3-regular graphs by (1-15), and the numbers $\{r_n\}$ of involutions in S_n, which satisfy $r_n = r_{n-1} + (n-1)r_{n-2}$, see Sloane [n.d., A000085]. Finally, *D-algebraic* GFs (also called *ADE* and *hyperalgebraic*) include Euler numbers by the equation (1-7).

Theorem 2.1 (see e.g. R. P. Stanley [1999], Ch. 6)**.**

Rational $\subset$ *Algebraic* $\subset$ *D-finite* $\subset$ *D-algebraic.*

Here only the inclusion *Algebraic* $\subset$ *D-finite* is nontrivial. The following observation explains the connection to the subject.

Proposition 2.2. *Sequences with* D-algebraic *GFs have Wilfian formulas of type* (W1).

In other words, if one wants to show that a sequence does not have a Wilfian formula, then proving that it is *D-transcendental*, i.e. non-D-algebraic, is a good start.[3] Unfortunately, even proving that a sequence is non-P-recursive is often challenging (see below).

Example 2.3 (Bell numbers)**.** Let B_n denotes the number of *set partitions* of $\{1, \ldots, n\}$, see R. P. Stanley [ibid.] and Sloane [n.d., A000110]. Let

$$y(t) = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}, \qquad z(t) = \sum_{n=0}^{\infty} B_n t^n$$

[3] To simplify exposition and for the lack of better terminology, here and in the future we refer to sequences by the properties of their GFs.

be the exponential and ordinary GFs of Bell numbers, respectively. On the one hand, we have:

$$y(t) = e^{e^t - 1}, \qquad y'' y - (y')^2 - y' y = 0.$$

Thus, $y(t)$ is D-algebraic, and the proposition implies that $\{B_n\}$ can be computed in $poly(n)$ time. On the other hand, $z(t)$ is D-transcendental by Klazar's theorem Klazar [2003].

This also implies that $y(t)$ is not D-finite. Indeed, observe by definition, that if a sequence $\{a_n\}$ is P-recursive, then so is $\{n! a_n\}$, which implies the result by taking $a_n = B_n/n!$ (cf. Lipshitz and Rubel [1986]). Of course, there is a more direct way to prove that $y(t)$ is not D-finite by repeated differentiation or via the asymptotics, see below. This suggests the following advanced generalization of Klazar's theorem.

Open Problem 2.4 (P.–Yeliussizov)**.** *Suppose $\{a_n/n!\}$ is D-algebraic but not P-recursive. Does this imply that $\{a_n\}$ is D-transcendental?*

Before we proceed to more combinatorial examples, let us mention that D-transcendental GFs are the subject of *Differential Galois Theory*, which goes back to Liouville, Lie, Picard and Vessiot in the 19th century (see e.g. Ritt [1950]), and continues to be developed van der Put and Singer [1997]. Some natural GFs are known to be D-transcendental, e.g. $\Gamma(z)$, $\zeta(z)$, etc., but there are too few methods to prove this in most cases of interest. Here are some of our favorite open problems along these lines, unapproachable with existing tools.

Conjecture 2.5. *$\sum_{n\ge1} p_n t^n$ and $\sum_{n\ge1} \pi(n) t^n$ are D-transcendental.*

Here p_n is n-th prime, $\pi(n)$ is the number of primes $\le n$, as above. Both GFs are known to be non-D-finite, as shown by Flajolet, Gerhold, and Salvy [2004/06] by asymptotic arguments. The authors quip: "*Almost anything is non-holonomic unless it is holonomic by design*". Well, maybe so. But the same applies for D-transcendence where the gap between what we believe and what we can prove is much wider. The reader should think of such open problems as irrationality of $e + \pi$ and $\zeta(5)$, and imagine a similar phenomenon in this case.

Conjecture 2.6. *$\displaystyle\sum_{n\ge0} t^{n^3}$ is D-transcendental.*

This problem should be compared with Jacobi's 1848 theorem that $\sum_{n\ge0} t^{n^2}$ is D-algebraic. To understand the difference, the conjecture is saying that there are no good formulas governing the number of ways to write n as a sum of k cubes, for any k, the kind of formulas that exist for sums of two, four and six squares, see Hardy and Wright [2008, §XX].

2.2 Asymptotic tools. The following result is the best tool we have for proving that a combinatorial sequence is not P-recursive. Note that deriving such asymptotics can be very difficult; we refer to Flajolet and Sedgewick [2009] and Pemantle and M. C. Wilson [2013] for recent comprehensive monographs on the subject.

Theorem 2.7. *Let $\{a_n\}$ be a P-recursive sequence, s.t. $a_n \in \mathbb{Q}$, $C_1^n < a_n < C_2^n$ for some $C_2 > C_1 > 0$ and all $n \geq 1$. Then:*

$$a_n \sim \sum_{i=1}^{m} K_i \, \lambda_i^n \, n^{\alpha_i} \, (\log n)^{\beta_i} \, ,$$

where $K_i \in \mathbb{R}_+$, $\lambda_i \in \overline{\mathbb{Q}}$, $\alpha_i \in \mathbb{Q}$, and $\beta_i \in \mathbb{N}$.

The theorem is a combination of several known results Garrabrant and Pak [2017]. Briefly, the generating series $\mathcal{A}(t)$ is a G-function in a sense of Siegel (1929), which by the works of André, Bombieri, Chudnovsky, Dwork and Katz, must satisfy an ODE which has only regular singular points and rational exponents. We then apply the Birkhoff–Trjitzinsky claim/theorem, which in the regular case has a complete and self-contained proof in Flajolet and Sedgewick [2009] (see Theorem VII.10 and subsequent comments).

Example 2.8 (Euler numbers E_n)**.** Recall that

$$E_n \sim \frac{4}{\pi} \left(\frac{2}{\pi}\right)^n n!$$

(see e.g. Flajolet and Sedgewick [ibid., p. 7]). Then $\{E_n\}$ is not P-recursive, since otherwise $E_n/n! \sim K\lambda^N$ with a transcendental exponent $\lambda = (2/\pi) \notin \overline{\mathbb{Q}}$.

Example 2.9 (n-th prime p_n)**.** Following Flajolet, Gerhold, and Salvy [2004/06], recall that $p_n = n \log n + n \log \log n + O(n)$. Observe that the *harmonic number* h_n is P-recursive by definition:

$$h_n = h_{n-1} + \frac{1}{n} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} = \log n + O(1).$$

Then $\{p_n\}$ is not P-recursive, since otherwise so is

$$p_n - n h_n = n \log \log n + O(n),$$

which is impossible by Theorem 2.7.

2.3 Lattice walks. Let $\Gamma = (V, E)$ be a graph and let $v_0, v_1 \in V$ be two fixed vertices. Let a_n be the number of walks $v_0 \to v_1$ in Γ of length n. This is a good model which leads to many interesting sequences. For example, Fibonacci number F_n is the number of walks $1 \to 1$ of length n in the graph on $\{1, 2\}$, with edges $(1, 1)$, $(1, 2)$ and $(2, 1)$.

For general finite graphs we get C-recursive sequences $\{a_n\}$ with rational GFs. For the usual walks $0 \to 0$ on $\mathbb{N}$ we get Catalan numbers $a_{2n} = C_n$ as in (1-10), while for ± 1 walks in $\mathbb{Z}$ we get $a_{2n} = \binom{2n}{n}$, both algebraic sequences. Similarly, for $(0, \pm 1)$, $(\pm 1, 0)$ walks in $\mathbb{Z}^2$, we get $a_{2n} = \binom{2n}{n}^2$, which is P-recursive but not algebraic. In higher dimensions or for more complicated graphs, there is no such neat formula.

Theorem 2.10. *Let $S \subset \mathbb{Z}^d$ be a fixed finite set of steps, and let a_n be the number of walks $O \to O$ in $\mathbb{Z}^d$ of length n, with steps in S. Then $\{a_n\}$ is P-recursive.*

This result is classical and follows easily from R. P. Stanley [1999, §6.3]. It suggests that to obtain more interesting sequences one needs to look elsewhere. Notably, one can consider natural lattice walks on some portion of $\mathbb{Z}^d$. There is a tremendous number of results in the literature, remarkable both in scope and beauty.

In recent years, M. Bousquet-Mélou and her coauthors initiated a broad study of the subject, and now have classified all walks in the first quadrant which start and end at the origin O, and have a fixed set S of steps with both coordinates in $\{0, \pm 1\}$. There are in principle $2^8 - 1 = 255$ such walks, but some of them are trivial and some are the same up to symmetries. After the classification was completed, some resulting sequences are proved algebraic (say, *Kreweras walks* and *Gessel walks*), very surprisingly so, some are D-finite (not a surprise given Theorem 2.10), some are D-algebraic (this required development of new tools), and some are D-transcendental (it is amazing that this can be done at all).

Example 2.11 (Case 16)**.** Let $S = \{(1, 1), (-1, -1), (-1, 0), (0, -1)\}$, and let a_n be the number of walks $O \to O$ in the first quadrant of length n, with steps in S, see Sloane [n.d., A151353]. It was shown in Bostan, Raschel, and Salvy [2014, Case 16] that

$$a_n \sim K \lambda^n n^\alpha,$$

where $\lambda \approx 3.799605$ is a root of $x^4 + x^3 - 8x^2 - 36x - 11 = 0$, and $\alpha \approx -2.318862$ satisfies $c = -\cos(\pi/\alpha)$, and c is a root of

$$y^4 - \frac{9}{2} y^3 + \frac{27}{4} y^2 - \frac{35}{8} y + \frac{17}{16} = 0$$

Since $\alpha \notin \mathbb{Q}$, Theorem 2.7 implies that $\{a_n\}$ is not P-recursive.

We refer to Bousquet-Mélou [2006] and Bousquet-Mélou and Mishna [2010] for a comprehensive overview of the background and early stages of this far reaching project, and to Bernardi, Bousquet-Mélou, and Raschel [2017] and Bostan, Bousquet-Mélou, Kauers, and Melczer [2016] for some recent developments which are gateways to references. Finally, let us mention a remarkable recent development Dreyfus, Hardouin, Roques, and Singer [2017], which proves D-transcendence for many families of lattice walks. Let us single out just one of the many results in that paper:

Theorem 2.12 (Dreyfus, Hardouin, Roques, and Singer [ibid.], Thm. 5.8)**.** *Sequence* $\{a_n\}$ *defined in Example 2.11 is D-transcendental.*

In conclusion, let us mention that $\{a_n\}$ can be computed in polynomial time straight from definition using dynamic programming, since the number of points reachable after n steps is $poly(n)$. This leads us to consider walks with constraints or graphs of superpolynomial growth.

Conjecture 2.13. *Let* a_n *denotes the number of self-avoiding walks* $O \to O$ *in* $\mathbb{Z}^2$ *of length* n*. Then sequence* $\{a_n\}$ *has no Wilfian formula of type* (W1).

We refer to Guttmann [2009] for an extensive investigation of *self-avoiding walks* and its relatives, and the review of the literature.

2.4 Walks on Cayley graphs. Let $G = \langle S \rangle$ be a finitely generated group G with a generating set S. Let $a_n = a_n(G, S)$ be the number of words in S of length n equal to 1; equivalently, the number of walks $1 \to 1$ of length n, in the Cayley graph $\Gamma = \Gamma(G, S)$. In this case $\{a_n\}$ is called the *cogrowth sequence* and its GF $A(t)$ the *cogrowth series*. They were introduced by Pólya in 1921 in probabilistic context of random walks on graphs, and by Kesten in the context of amenability Kesten [1959].

The cogrowth sequence $\{a_n\}$ is C-recursive if only if G is finite Kouksov [1998]. It is algebraic for the dihedral group Humphries [1997], for the free group Haiman [1993] and for free products of finite groups Kuksov [1999], all with standard generators. The cogrowth sequence is P-recursive for many abelian groups Humphries [1997], and for the *Baumslag-Solitar groups* $G = \mathrm{BS}(k, k)$ in the standard presentation $\mathrm{BS}(k, \ell) = \langle x, y \mid x^k y = y x^\ell \rangle$, see Elder, Rechnitzer, Janse van Rensburg, and Wong [2014].

Theorem 2.14 (Garrabrant and Pak [2017])**.** *Sequence* $\{a_n(G, S)\}$ *is not P-recursive for all* symmetric $S = S^{-1}$*, and the following classes of groups* G*:*

(1) *virtually solvable groups of exponential growth with finite Prüfer rank,*
(2) *amenable linear groups of superpolynomial growth,*
(3) *groups of weakly exponential growth*

$$A e^{n^\alpha} < \gamma_{G,S}(n) < B e^{n^\beta},$$

where $A, B > 0$*, and* $0 < \alpha, \beta < 1$*,*
(4) *the Baumslag–Solitar groups* $\mathrm{BS}(k, 1)$*,* $k \geq 2$*,*
(5) *the lamplighter groups* $L(d, H) = H \wr \mathbb{Z}^d$*, where* H *is a finite abelian group and* $d \geq 1$.

Since $G \simeq \mathbb{Z} \ltimes \mathbb{Z}^2$ with a free action of $\mathbb{Z}$, is linear of exponential growth, by (2) we obtain a solution to the question originally asked by Kontsevich, see R. Stanley [2014].

Corollary 2.15 (Garrabrant and Pak [2017])**.** *There is a linear group* G *and a symmetric generating set* S*, s.t. the sequence* $\{a_n(G, S)\}$ *is not P-recursive.*

The proof of Theorem 2.14 is a combination of many results by different authors. For example, for $G = \mathrm{BS}(k, 1)$, $k \geq 2$, and every symmetric $\langle S \rangle = G$, there exist $C_1, C_2 > 0$ which depend on S, s.t.

$$|S|^n \, e^{-C_1 \sqrt[3]{n}} \, \leq \, a_n(G, S) \, \leq \, |S|^n \, e^{-C_2 \sqrt[3]{n}} \,, \tag{2-1}$$

see Woess [2000, §15.C]. The result now follows from Theorem 2.7.

It may seem from Theorem 2.14 that the properties of $\{a_n(G, S)\}$ depend only on G, but that is false. In fact, for $G = F_k \times F_\ell$ there are generating sets with both P-recursive and non-P-recursive sequences Garrabrant and Pak [2017]. For groups in the theorem, this is really a byproduct of probabilistic tools used in establishing the asymptotics such as (2-1). In fact, the probabilities of return of the random walk $a_n(G, S)/|S|^n$ always have the same growth under *quasi-isometry*, see e.g. Woess [2000].[4]

In a forthcoming paper Garrabrant and Pak [n.d.] we construct an explicit but highly artificial non-symmetric set $S \subset F_k \times F_\ell$ with D-transcendental cogrowth sequence. In Kassabov and Pak [n.d.] we use the tools in Kassabov and Pak [2013] to prove that groups have an uncountable set of *spectral radii*

$$\rho(G, S) \, := \, \lim_{n \to \infty} a_n(G, S)^{1/n} \, .$$

Since the set of D-algebraic sequence is countable, this implies the existence of D-transcendental Cayley graphs with symmetric S, but such proof is nonconstructive.

Open Problem 2.16. *Find an explicit construction of* $\Gamma(G, S)$ *when* S *is symmetric, and* $\{a_n(G, S)\}$ *is D-transcendental.*

Sequences $\{a_n\}$ have been computed in very few special cases. For example, for $\mathrm{PSL}(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ with the natural symmetric generating set, the cogrowth series

[4] While the leading term in the asymptotics remains the same, lower order terms can change for different S, see Woess [2000, §17.B].

$A(t)$ is computed in Kuksov [1999]:

$$A(t) = \frac{(1+t)\big(-t+t^2-8t^3+3t^4-9t^5+(2-t+6t^2)\sqrt{\mathcal{R}(t)}\big)}{2(1-3t)(1+3t^2)(1+3t+3t^2)(1-t+3t^2)},$$
$$\text{where } \mathcal{R}(t) = 1-2t+t^2-6t^3-8t^4-18t^5+9t^6-54t^7+81t^8.$$

There are more questions than answers here. For example, can cogrowth sequence be computed for nilpotent groups?

Before we conclude, let us note that everywhere above we are implicitly assuming that G either has a faithful rational representation, e.g. $G = \mathrm{BS}(k,1)$ as in (4) above, or more generally has the *word problem* solvable in polynomial time (cf. Lipton and Zalcstein [1977]). The examples include the *Grigorchuk group* $\mathbb{G}$, which is an example of 3, see Grigorchuk and Pak [2008] and the lamplighter groups $L(d,H)$ as in (5). Note that in general the word problem can be superpolynomial or even unsolvable, see e.g. C. F. Miller I. [1992], in which case $\{a_n\}$ is no longer a combinatorial sequence.

2.5 Partitions. Let $p(n)$ be the number of integer partitions of n, as in (1-13). We have the *Hardy–Ramanujan formula*:

$$p(n) \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{\frac{2n}{3}}} \quad \text{as } n \to \infty. \tag{2-2}$$

(see e.g. Flajolet and Sedgewick [2009, p. VIII.6]). Theorem 2.7 implies that $\{p(n)\}$ is not P-recursive. On the other hand, it is known that

$$F(t) := \sum_{n=0}^{\infty} p(n)\,t^n = \prod_{i=1}^{\infty} \frac{1}{1-t^i}$$

satisfies the following ADE:

$$4F^3F'' + 5t\,F^3F''' + t^2F^3F^{(4)} - 16F^2(F')^2 - 15t\,F^2F'F''$$
$$- 39t^2F^2(F'')^2 + 10t\,F(F')^3 + 12t^2F(F')^2F'' + 6t^2(F')^4 = 0$$

(cf. Zagier [2008]). A quantitative version of Proposition 2.2 then implies that $\{p(n)\}$ can be computed in time $O^*(n^{4.5})$, where O^* indicates $\log n$ terms. For comparison, the dynamic programming takes $O(n^{2.5})$ time, where $O(\sqrt{n})$ comes as the cost of addition. Similarly, *Euler's recurrence* famously used by MacMahon (1915) to compute $p(200)$, gives an $O(n^2)$ algorithm:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \ldots$$

(cf. Calkin, Davis, James, Perez, and Swannack [2007]). There is also an efficient implementation Johansson [2012] based on the Hardy–Ramanujan–Rademacher sharp asymptotic formula which extends (2-2) to $o(1)$ additive error. It would be interesting to analyze this algorithm perhaps using Lehmer's estimates used in DeSalvo and Pak [2015].

Now, for a subset $\mathcal{Q} \subseteq \{1, 2, \ldots\}$, denote by $p_{\mathcal{Q}}(n)$ the number of partitions of n into parts in $\mathcal{Q}$. The dynamic programming algorithm is easy to generalize to every $\{p_{\mathcal{Q}}(n)\}$ where the membership $a \in ?\,\mathcal{Q}$ can be decided in $poly(\log a)$ time, giving a Wilfian formula of type (W1). This is polynomially optimal for partitions into primes Sloane [n.d., A000607] or squares Sloane [ibid., A001156], but not for sparse sequences.

Proposition 2.17. *Let $\mathcal{Q} = \{a_1, a_2, \ldots\}$, such that $a_k \geq c^k$, for some $c > 1$ and all $k \geq 1$. Then $p_{\mathcal{Q}}(n) = n^{O(\log n)}$.*

Thus, $p_{\mathcal{Q}}(n)$ as in the proposition could in principle have a Wilfian formula of type (W3). Notable examples include the number $q(n)$ of *binary partitions* (partitions of n into powers of 2), see Sloane [ibid., A000123], *partitions into Fibonacci numbers* Sloane [ibid., A003107], and *s-partitions* defined as partitions into $\{1, 3, 7, \ldots, 2^k - 1, \ldots\}$ Sloane [ibid., A000929].

Theorem 2.18 (Pak and Yeliussizov [n.d.]). *Let $\mathcal{Q} = \{a_1, a_2, \ldots\}$, and suppose a_k / a_{k-1} is an integer ≥ 2, for all $k > 1$. Then $\{p_{\mathcal{Q}}(n)\}$ can be computed in time* $\operatorname{poly}(\log n)$.

This covers binary partitions, partitions into factorials Sloane [n.d., A064986], etc. We conjecture that partitions into Fibonacci numbers and s-partitions also have Wilfian formulas of type (W3). Cf. N. Robbins [1996] for an algorithm for partitions into distinct Fibonacci numbers. Other partitions functions such as partitions into Catalan numbers Sloane [n.d., A033552] and partitions into partition numbers Sloane [ibid., A007279], could prove less tractable. We should mention that connection between algebraic properties of GFs and complexity goes only one way:

Theorem 2.19. *The sequence $\{q(n)\}$ of the number of binary partitions is D-transcendental.*

This follows from the *Mahler equation*

$$Q(t) - t\,Q(t) - Q(t^2) = 0, \quad \text{where} \quad Q(t) = \sum_{n=0}^{\infty} q(n) t^n,$$

see Dreyfus, Hardouin, and Roques [2015]. We conjecture that $\{a_n\}$ and $\{b_n\}$ from Conjecture 1.3 satisfy similar functional equations, and are also D-transcendental.

2.6 Pattern avoidance. Let $\sigma \in S_n$ and $\omega \in S_k$. Permutation σ is said to *contain* the *pattern* ω if there is a subset $X \subseteq \{1, \ldots, n\}$, $|X| = k$, such that $\sigma|_X$ has the same relative order as ω. Otherwise, σ is said to *avoid* ω.

Fix a set of patterns $\mathcal{F} \subset S_k$. Denote by $\mathsf{A}_n(\mathcal{F})$ the number of permutations $\sigma \in S_n$ *avoiding* all patterns $\omega \in \mathcal{F}$. The sequence $\{\mathsf{A}_n(\mathcal{F})\}$ is the fundamental object of study in the area of *pattern avoidance*, extensively analyzed from analytic, asymptotic and combinatorial points of view.

The subject was initiated by MacMahon (1915) and Knuth (1973), who showed that $\mathsf{A}_n(123) = \mathsf{A}_n(213) = C_n$, the n-th Catalan number (1-10). The *Erdős–Szekeres theorem* (1935) on longest increasing and decreasing subsequences in a permutation can also be phrased in this language: $\mathsf{A}_n(12\cdots k, \ell\cdots 21) = 0$, for all $n > (k-1)(\ell-1)$.

To give a flavor of subsequent developments, let us mention a few more of our most favorite results. Simion–Schmidt (1985) proved $\mathsf{A}_n(123, 132, 213) = F_{n+1}$, the Fibonacci numbers. Similarly, Shapiro–Stephens (1991) proved $\mathsf{A}_n(2413, 3142) = S(n)$, the Schröder numers Sloane [n.d., A006318]. The celebrated Marcus–Tardos theorem Marcus and Tardos [2004] states that $\{\mathsf{A}_n(\omega)\}$ is at most exponential, for all $\omega \in S_k$, with a large base of exponent for random $\omega \in S_k$ Fox [2013]. We refer to Kitaev [2011], Klazar [2010], and Vatter [2015] for many results on the subject, history and background.

The *Noonan–Zeilberger conjecture* Noonan and Zeilberger [1996], first posed as a question by Gessel [1990], stated that the sequence $\{\mathsf{A}_n(\mathcal{F})\}$ is P-recursive for all $\mathcal{F} \subset S_k$. It was recently disproved:

Theorem 2.20 (Garrabrant and Pak [2015]). *There is $\mathcal{F} \subset S_{80}$, $|\mathcal{F}| < 30,000$, such that $\{\mathsf{A}_n(\mathcal{F})\}$ is not P-recursive.*

We extend this result in a forthcoming paper Garrabrant and Pak [n.d.], where we construct a D-transcendent pattern avoiding sequence $\{\mathsf{A}_n(\mathcal{F})\}$, for some $\mathcal{F} \subset S_{80}$. Both proofs involve embedding of Turing Machines into the problem modulo 2. We also prove the following result on complexity of counting pattern avoiding permutations, our only result forbidding Wilfian formulas:

Theorem 2.21 (Garrabrant and Pak [2015]). *If* $\mathsf{EXP} \neq \oplus\mathsf{EXP}$*, then* $\mathsf{A}_n(\mathcal{F}) \bmod 2$ *cannot be computed in* $\mathrm{poly}(n)$ *time.*

In other words, counting parity of pattern avoiding permutations is likely hard. We conjecture that $\mathsf{A}_n(\mathcal{F})$ is #EXP-complete, but we are not very close to proving this.

Theorem 2.22 (Garrabrant and Pak [ibid.]). *The problem whether $\mathsf{A}_n(\mathcal{F}) = \mathsf{A}_n(\mathcal{F}')$* mod 2 *for all n, is undecidable.*

The theorem implies that in some cases even a large amount of computational evidence in pattern avoidance is misleading. For example, there exists two sets of patterns $\mathcal{F}, \mathcal{F}' \in S_k$, so that the first time they have different parity is for $n >$ tower of 2s of length 2^k.

Finally, let us mention an ongoing effort to find a small set of patterns $\mathcal{F}$, so that $\{\mathsf{A}_n(\mathcal{F})\}$ is not P-recursive. Is one permutation enough? It is known that $\{\mathsf{A}_n(1342)\}$ is algebraic Bóna [1997], while $\{\mathsf{A}_n(1234)\}$ is P-recursive Gessel [1990]. One of the most challenging problems is to analyze $\{\mathsf{A}_n(1324)\}$, the only 4-pattern remaining. The asymptotics obtained experimentally in Conway, Guttmann, and Zinn-Justin [2018] based on the values for $n \leq 50$, suggests:

$$\mathsf{A}_n(1324) \sim B\,\lambda^n\,\mu^{\sqrt{n}}\,n^{\alpha},$$

where $\lambda = 11.600 \pm 0.003$, $\mu = 0.0400 \pm 0.0005$, $\alpha = -1.1 \pm 0.1$. If true, Theorem 2.7 would imply that $\{\mathsf{A}_n(1324)\}$ is not P-recursive. While this remains out of reach, the following problem could be easier.

Open Problem 2.23. *Can $\{\mathsf{A}_n(1324)\}$ be computed in poly(n) time?*[5] *More generally, can one find a single permutation π such that $\{\mathsf{A}_n(\pi)\}$ cannot be computed in poly(n) time? Is the computation of $\{\mathsf{A}_n(\pi)\}$ easier or harder for* random *permutations $\pi \in S_k$?*

3 Bijections

3.1 Counting and sampling via bijections. There is an ocean of bijections between various combinatorial objects. They have a variety of uses: to establish a theorem, to obtain refined counting, to simplify the proof, to make the proof amenable for generalizations, etc. Last but not least, some especially beautiful bijections are often viewed as a piece of art, an achievement in its own right, a result to be taught and admired.

From the point of view of this survey, bijections $\varphi : \mathcal{A}_n \to \mathcal{B}_n$ are simply algorithms which require complexity analysis. There are two standard applications of such bijections. First, their existence allows us to reduce counting of $\{|\mathcal{A}_n|\}$ to counting of $\{|\mathcal{B}_n|\}$. For example, the classical *Prüfer's algorithm* allows counting of spanning trees in K_n, reducing it to Cayley's formula (1-12).

Second and more recent application is to *random sampling* of combinatorial objects. Oftentimes, one of the sets has a much simpler structure which allows (nearly) uniform sampling. To compare the resulting algorithm with other competing approaches one then needs a worst case and/or average case analysis of the complexity of the bijection.

[5] In 2005, Doron Zeilberger expressed doubts that $\mathsf{A}_{1000}(1324)$ can be computed even by Hashem. This sentiment has been roundly criticized on both mathematical and theological grounds (see Steingrímsson [2013]).

Of course, most bijections in the literature are so straightforward that their analysis is elementary, think of the Prüfer's algorithm or the classical "plane trees into binary trees" bijection de Bruijn and Morselt [1967]. But this is also what makes them efficient. For example, the bijections for planar maps are amazing in their elegance, and have some important applications to statistical physics; we refer to Schaeffer [2015] for an extensive recent survey and numerous references.

Finally, we should mention a number of *perfect sampling* algorithms, some of which in the right light can also be viewed as bijections. These include most notably general techniques such as *Boltzmann samplers* Duchon, Flajolet, Louchard, and Schaeffer [2004] and *coupling from the past* D. A. Levin, Peres, and Wilmer [2017]. Note also two beautiful ad hoc algorithms: *Wilson's LERW* D. B. Wilson [1996] and the *Aldous–Broder algorithm* for sampling uniform spanning trees in a graph (both of which are highly nontrivial already for K_n), see e.g. D. A. Levin, Peres, and Wilmer [2017].

3.2 Partition bijections. Let $q(n)$ denote the number of *concave partitions* defined by $\lambda_i - \lambda_{i+1} \geq \lambda_{i+1} - \lambda_{i+2}$ for all i. Then $\{q(n)\}$ can be computed in *poly*(n) time. To see this, recall *Corteel's bijection* between convex partitions and partitions into triangular numbers Sloane [n.d., A007294]. We then have:

$$\sum_{n=1}^{\infty} q(n)t^n \;=\; \prod_{k=2}^{\infty} \frac{1}{1-t^{\binom{k}{2}}}\,,$$

see Canfield, Corteel, and Hitczenko [2001]. This bijection can be described as a linear transformation which can be computed in polynomial time Corteel and Savage [2004] and Pak [2004a]. More importantly, the bijections allow random sampling of concave partitions, leading to their limit shape Canfield, Corteel, and Hitczenko [2001] and DeSalvo and Pak [2016].

On the opposite extreme, there is a similar *Hickerson's bijection* between s-partitions and partitions with $\lambda_i \geq 2\lambda_{i+1}$ for all $i \geq 1$, see Canfield, Corteel, and Hitczenko [2001] and Pak [2004a]. Thus, both sets are equally hard to count, but somehow this makes the problem more interesting.

The Garsia and Milne [1981] celebrated *involution principle* combines the Schur and Sylvester's bijections in an iterative manner, giving a rather complicated bijective proof of the *Rogers–Ramanujan identity*:

$$1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)} \;=\; \prod_{i=0}^{\infty} \frac{1}{(1-t^{5i+1})(1-t^{5i+4})}\,. \tag{3-1}$$

To be precise, they constructed a bijection $\Psi_n : \mathcal{P}_n \to \mathcal{Q}_n$, where $\mathcal{P}$ is the set of partitions into parts $\lambda_i \geq \lambda_{i+1} + 2$, and $\mathcal{Q}$ is the set of partitions into parts $\pm 1 \bmod 5$. In Pak

[2006, §8.4.5] we conjecture that Ψ_n requires $\exp n^{\Omega(1)}$ iterations in the worst case. Partial evidence in favor of this conjecture is our analysis of O'Hara's bijection in Konvalinka and Pak [2009], with a $\exp \Omega(\sqrt[3]{n})$ worst case lower bound. On the other hand, the iterative proof in Boulet and Pak [2006] for (3-1) requires only $O(n)$ iterations.

3.3 Plane partitions and Young tableaux. Denote by $sp(n)$ the number of 3-*dimensional* (or *solid*) partitions. MacMahon famously proved in 1912 that

$$\sum_{n=0}^{\infty} sp(n)t^n = \prod_{k=1}^{\infty} \frac{1}{(1-t^k)^k},$$

which gives a *poly*(n) time algorithm for computing $\{sp(n)\}$. A variation on the celebrated *Hillman-Grassl* and *RSK* bijections proves this result and generalizes it Pak [2006]. The application to sampling of this bijection have been analyzed in Bodini, Fusy, and Pivoteau [2010]. On the other hand, there is a strong evidence that the RSK-based algorithms cannot be improved. While we are far from proving this, let us note that in Pak and Vallejo [2010] we show linear time reductions between all major bijections in the area, so a speedup of one of them implies a speedup of all.

A remarkable *Krattenthaler's bijection* allows enumerations of solid partitions which fit into $[n \times n \times n]$ box Krattenthaler [1999]. This bijection is based on top of the *NPS algorithm*, which has also been recently analyzed Neumann and Sulzgruber [2015] and Schneider and Sulzgruber [2017]. Curiously, there are no analogous results in $d \geq 4$ dimensions, making counting such d-dimensional partitions an open problem (cf. Govindarajan [2013]).

3.4 Complexity of bijections. Let us now discuss the questions (2′) in the introduction, about the nature of bijections $\varphi : \mathcal{P}_n \to \mathcal{Q}_n$ from an algorithmic point of view.

If we think of φ as a map, we would want both φ and φ^{-1} to be computable in polynomial time. If that's all we want, it is not hard to show that such φ can always be constructed whenever there is a polynomial time algorithm to compute $|\mathcal{P}_n| = |\mathcal{Q}_n|$. For example, the dynamic programming plus the *divide and conquer* allows a construction of *poly*(n) time bijection $\varphi_n : \mathcal{P}_n \to \mathcal{Q}_n$, proving Rogers–Ramanujan identity (3-1). Since such construction would require prior knowledge of $|\mathcal{P}_n| = |\mathcal{Q}_n|$, from a combinatorial point of view this is unsatisfactory.

Alternatively, one can think of a bijection as an algorithm which computes a given map φ_n as above in *poly*(n) time. This is a particularly unfriendly setting as one would essentially need to prove new *lower bounds* in complexity. Worse, we proved in Konvalinka and Pak [2009] that in some cases O'Hara's algorithm requires superpolynomial time, while the map given by the algorithm can be computed in *poly*(n) time using *integer programming*. Since this is the only nice bijective proof of the *Andrews identities* that we know (see Pak [2006]), this suggests that either we don't understand the nature of these identities or have a very restrictive view on what constitutes a combinatorial bijection. Or, perhaps, the complexity approach is simply inapplicable in this combinatorial setting.

There are other cases of unquestionably successful bijections which are inferior to other algorithms from complexity point of view. For example, stretching the definitions a bit, Wilson's LERW algorithm for generating random (directed) spanning trees requires exponential time on directed graphs D. B. Wilson [1996], while a straightforward algorithm based on the *matrix-tree theorem* is polynomial, of course.

Finally, even when the bijection is nice and efficient, it might still have no interesting properties, so the only application is the proof of the theorem. One example is an iterative bijection for the Rogers–Ramanujan identity (3-1) which is implied by the proof in Boulet and Pak [2006]. We don't know if it respects any natural statistics which would imply a stronger result. Thus, we left it in the form of a combinatorial proof to make the underlying algebra clear.

3.5 Probabilistic/asymptotic approach. Suppose both sets of combinatorial objects $\mathcal{P}_n$ and $\mathcal{Q}_n$ have well-defined *limit shapes* π and ω, as $n \to \infty$. Such limits exists for various families of trees Drmota [2009], graphs Lovász [2012], partitions DeSalvo and Pak [2016], permutations Hoppen, Kohayakawa, Moreira, Ráth, and Menezes Sampaio [2013], solid partitions Okounkov [2016], Young tableaux Romik [2015], etc.[6] For a sequence $\{\varphi_n\}$ of bijections $\varphi_n : \mathcal{P}_n \to \mathcal{Q}_n$, one can ask about the *limit bijection* $\Phi : \pi \to \omega$, defined as $\lim_{n\to\infty} \varphi_n$. We can then require that Φ satisfies certain additional properties, This is the approach taken in Pak [2004b] to prove the following result:

Theorem 3.1. *The Rogers–Ramanujan identity* (3-1) *has no* geometric bijection.

Here the *geometric bijections* are defined as compositions of certain piecewise $\mathrm{GL}(2, \mathbb{Z})$ maps acting on Ferrers diagrams, which are viewed as subsets of $\mathbb{Z}^2$. We first prove that the limits of such bijections are *asymptotically stable*, i.e. act piecewise linearly on the limit shapes. The rest of the proof follows from existing results on the limit shapes π and ω on both sides of (3-1), which forbid a piecewise linear map $\Phi : \pi \to \omega$, see DeSalvo and Pak [2016].

The next story is incomplete, yet the outlines are becoming clear. Let $\mathrm{ASM}(n)$ be the number of *alternating sign matrices* of order n, defined as the number of $n \times n$ matrices where every row has entries in $\{0, \pm 1\}$, with row and column sums equal to 1, and all signs alternate in each row and column. Let $\mathrm{FSLT}(n)$ be the number of the *fully symmetric*

[6] Here the notion of a "limit shape" is used very loosely as it means very different things in each case.

lozenge tilings, defined as lozenge tilings of the regular $2n$-hexagon with the full group of symmetries D_6. Such tilings are in easy bijection with solid partitions which fit into $[2n \times 2n \times 2n]$ box, have full group of symmetries S_3, and are self-complementary within the box (cf. Section 3.3). Finally, let $\mathrm{TSPP}(n)$ be the number of *triangular shifted plane partitions* defined as plane partitions $(b_{ij})_{1 \le i \le j}$ of shifted shape $(n-1, n-2, \ldots, 1)$, and entries $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

The following identity is justly celebrated:

$$\mathrm{ASM}(n) = \mathrm{FSLT}(n) = \mathrm{TSPP}(n) = \frac{1!\,4!\,7!\,\cdots\,(3n-2)!}{n!\,(n+1)!\,\cdots\,(2n-1)!} \tag{3-2}$$

Here the second equality is known to have a bijective proof Mills, D. P. Robbins, and Rumsey [1986]. Finding bijective proof of the third equality is a major open problem. See Bressoud [1999] and Krattenthaler [2016] for the history of the problem and Sloane [n.d., A005130] for further references.

Claim 3.2. *The equality* $\mathrm{ASM}(n) = \mathrm{FSLT}(n)$ *has no* geometric bijection.

We now know (conjecturally) what the *frozen regions* in each case are: the circle for FSLTs and a rather involved sextic equation for ASMs. The latter is an ingenuous conjecture in Colomo and Pronko [2010] (see also Colomo and Sportiello [2016]), while the former is a natural conjecture about the *Arctic Circle* which remains when the symmetries are introduced (cf. Panova [2015]).[7] We are not sure in this case what do we mean by a "geometric bijection". But any natural definition should imply that the two shapes are incompatible. It would be interesting to formalize this even before both frozen regions are fully established.

There is another aspect of this asymptotic approach, which allows us to distinguish between different equinumerous collections of combinatorial objects with respect to some (transitive) notions of a "good" (canonical) bijection, and thus divide them into equivalence classes. This method would allow us to understand the nature of these families and ignore superficial differences within the same class.

The prototypical example of this is a collection of over 200 objects enumerating Catalan numbers R. P. Stanley [2015], but there are other large such collections: for Motzkin numbers, Schröder numbers, Euler numbers (1-11), etc. A natural approach would be to use the symmetry properties or the topology, but such examples are rare (see, however, Armstrong, Stump, and Thomas [2013] and West [1995] for two "canonical" bijections between Catalan objects).

[7] While the frozen region hasn't been established for FSLTs, it is known that if exists it must be a circle (Greta Panova, personal communication).

In Miner and Pak [2014], we studied the limit averages of permutation matrices corresponding to $\mathsf{A}_n(\mathfrak{F})$. We showed that the limit surfaces corresponding to $\mathsf{A}_n(123)$ and $\mathsf{A}_n(213)$ are quite different, even though their sizes are Catalan numbers (see also Hoffman, Rizzolo, and Slivken [2017] and Madras and Pehlivan [2016]). This partly explains a well known phenomenon: there are *nine*(!) different bijections between these two families described in Kitaev [2011], each with its own special properties – there is simply no "canonical" bijection in this case. See also Dokos and Pak [2014] for the analysis of another interesting family of Catalan objects.

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