

# THERMODYNAMIC FORMALISM METHODS IN ONE-DIMENSIONAL REAL AND COMPLEX DYNAMICS

FELIKS PRZYTYCKI

## Abstract

We survey some results on non-uniform hyperbolicity, geometric pressure and equilibrium states in one-dimensional real and complex dynamics. We present some relations with Hausdorff dimension and measures with refined gauge functions of limit sets for geometric coding trees for rational functions on the Riemann sphere. We discuss fluctuations of iterated sums of the potential  $-t \log |f'|$  and of radial growth of derivative of univalent functions on the unit disc and the boundaries of range domains preserved by a holomorphic map  $f$  repelling towards the domains.

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## 1 Thermodynamic formalism, introductory notions

Among founders of this theory are Sinaï [1972], Bowen [1975] and David Ruelle, who wrote in Ruelle [1978]: “thermodynamic formalism has been developed since G. W. Gibbs to describe [...] physical systems consisting of a large number of subunits”. In particular one considers a *configuration space*  $\Omega$  of functions  $\mathbb{Z}^n \rightarrow \mathbb{A}$  on the lattice  $\mathbb{Z}^n$  with interacting values in  $\mathbb{A}$  over its sites, e.g. “spin” values in the Ising model of ferromagnetism. One considers probability distributions on  $\Omega$ , invariant under translation, called *equilibrium states* for potential functions on  $\Omega$ .

Given a mapping  $f : X \rightarrow X$  one considers as a configuration space the set of trajectories  $n \mapsto (f^n(x))_{n \in \mathbb{Z}_+}$  or  $n \mapsto \Phi(f^n(x))_{n \in \mathbb{Z}_+}$  for a test function  $\Phi : X \rightarrow Y$ .

The following simple fact Bowen [1975, Lemma 1.1] and Ruelle [1978, Introduction], Przytycki and Urbański [2010, Introduction], resulting from Jensen’s inequality applied to the function logarithm, stands at the heart of thermodynamic formalism.

**Lemma 1.1** (Finite Variational Principle). *For given real numbers  $\phi_1, \dots, \phi_d$ , the function  $F(p_1, \dots, p_d) := \sum_{i=1}^n -p_i \log p_i + \sum_{i=1}^d p_i \phi_i$  defined on the simplex  $\{(p_1, \dots, p_d) : p_i \geq 0, \sum_{i=1}^d p_i = 1\}$  attains its maximum value  $P(\phi_1, \dots, \phi_d) = \log \sum_{i=1}^d e^{\phi_i}$  at and only at  $\hat{p}_j = e^{\phi_j} (\sum_{i=1}^d e^{\phi_i})^{-1}$ .*

We can read  $i \mapsto \phi_i, i = 1, \dots, d$  as a *potential* function and  $\hat{p}_i$  as the equilibrium probability distribution on the finite space  $\{1, \dots, d\}$ .  $P(\phi_1, \dots, \phi_d)$  is called the *pressure* or *free energy*, see Ruelle [1978].

Let  $f : X \rightarrow X$  be a continuous mapping of a compact metric space  $X$  and  $\phi : X \rightarrow \mathbb{R}$  be a continuous function (the potential). We define the *topological pressure* or free energy by

**Definition 1.2.**

$$(1-1) \quad P_{\text{var}}(f, \phi) = \sup_{\mu \in \mathcal{M}(f)} \left( h_{\mu}(f) + \int_X \phi \, d\mu \right),$$

where  $\mathcal{M}(f)$  is the set of all  $f$ -invariant Borel probability measures on  $X$  and  $h_{\mu}(f)$  is measure theoretical entropy. Sometimes we write  $\mathcal{M}(f, X)$ .

Recall that  $h_\mu(f) = \sup_{\mathbb{A}} \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{A \in \mathbb{A}^n} -\mu(A) \log \mu(A)$ , where the supremum is taken over finite partitions  $\mathbb{A}$  of  $X$ , where  $\mathbb{A}^n := \bigvee_{j=0, \dots, n} f^{-j} \mathbb{A}$ . Notice that this resembles the sum  $\sum_{i=1}^n -p_i \log p_i$  in [Lemma 1.1](#).

Topological pressure can also be defined in other ways, e.g. by (6-2), and then its equality to the one given by (1-1) is called the variational principle. This explains the notation  $P_{\text{var}}$ . Any  $\mu \in \mathcal{M}(f)$  for which the supremum in (1-1) is attained is called *equilibrium*, *equilibrium measure* or *equilibrium state*.

A model case is any map  $f : U \rightarrow \mathbb{R}^n$  of class  $C^1$ , defined on a neighbourhood  $U$  of a compact set  $X \subset \mathbb{R}^n$ , *expanding* (another name: *uniformly expanding* or *hyperbolic* in dimension 1) that is there exist  $C > 0, \lambda > 1$  such that for all positive integers  $n$  all  $x \in X$  and all  $v$  tangent to  $\mathbb{R}^n$  at  $x$ ,

$$(1-2) \quad \|Df^n(v)\| \geq C \lambda^n \|v\|,$$

and *repelling* that is every forward trajectory sufficiently close to  $X$  must be entirely in  $X$ . Not assuming the differentiability of  $f$  one uses the notion of *distance expanding* meaning the increase of distances under the action of  $f$  by a factor at least  $\lambda > 1$  for pairs of distinct points sufficiently close to each other. Repelling happens to be equivalent to the internal condition:  $f|_X$  being an open map, provided  $f$  is open on a neighbourhood of  $X$ , see [Przytycki and Urbański \[2010, Lemma 6.1.2\]](#). Then the classical theorem holds, here in the version from [Przytycki and Urbański \[ibid., Section 5.1\]](#):

**Theorem 1.3.** *Let  $f : X \rightarrow X$  be a distance expanding, topologically transitive continuous open map of a compact metric space  $X$  and  $\phi : X \rightarrow \mathbb{R}$  be a Hölder continuous potential. Then, there exists exactly one measure  $\mu_\phi \in \mathcal{M}(f, X)$ , called the Gibbs measure, satisfying*

$$(1-3) \quad C < \frac{\mu_\phi(f_x^{-n}(B(f^n(x), r_0)))}{\exp(S_n \phi(x) - nP(\phi))} < C^{-1}$$

where  $f_x^{-n}$  is the branch of  $f^{-n}$  mapping  $f^n(x)$  to  $x$  (locally making sense, since  $f$  is a local homeomorphism) and  $S_n \phi(x) := \sum_{j=0}^{n-1} \phi(f^j(x))$ .

The measure  $\mu_\phi$  is the only equilibrium state for  $\phi$ . It is equivalent to the unique  $\phi$ -conformal measure  $m_\phi$ , that is a forward quasi-invariant Borel probability measure  $m_\phi$  with Jacobian  $\exp -(\phi - P(\phi))$ . Moreover, the limit  $P(\phi) = P(f, \phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in f^{-n}(x_0)} \exp S_n \phi(x)$  exists and is equal to  $P_{\text{var}}(f, \phi)$  for every  $x \in X$ .

This  $P(\phi)$  is a normalizing quantity corresponding to  $P(\phi_1, \dots, \phi_d)$  in [Lemma 1.1](#) and the sum in the definition of  $P(\phi)$  corresponds to the so called *statistical sum* over the space  $\Omega_n$  of all admissible configurations over  $\{0, 1, \dots, n-1\}$ , as in the Ising model. Compare to the *tree pressure* defined in [Definition 6.2](#).

So  $\varsigma : \Sigma^d \rightarrow \Sigma^d$ , the shift to the left on the space  $\Sigma^d = \{(\alpha_0, \alpha_1, \dots) : \alpha_j \in \{1, \dots, d\}\}$ , defined by  $\varsigma((\alpha_n)) = (\alpha_{n+1})$ , is an example where [Theorem 1.3](#) holds. The sets  $f_x^{-n}(B(f^n(x), r_0))$  correspond to *cylinders* of fixed  $\{\alpha_j \in \{1, \dots, d\}, j = 0, \dots, n-1\}$ . One can impose an admissibility condition:  $\alpha_i \alpha_{i+1}$  admissible if the pair has the digit 1 attributed in a defining  $0,1$   $d \times d$  matrix. Then one calls the system a *one-sided topological Markov chain*.

The condition of openness of  $f$  can be replaced by a weaker one: the existence of a finite Markov partition, see [Przytycki and Urbański \[2010\]](#).

The existence of a conformal measure follows from the existence of a fixed point in the convex weakly\*-compact set of probability measures for the dual operator to the transfer (Perron-Frobenius-Ruelle) operator  $\mathcal{L}$  divided by the norm, where for  $u : X \rightarrow \mathbb{R}$  continuous one defines

$$(1-4) \quad \mathcal{L}(u)(x) := \sum_{y \in f^{-1}(x)} u(y) \exp \phi(y).$$

Indeed, for every Borel set  $Y \subset X$  on which  $f$  is injective, denoting by  $I_Y$  indicator function: 1 on  $Y$ , 0 outside  $Y$ , due to an approximation by continuous functions, one has for every finite Borel measure  $\nu$  on  $X$

$$(1-5) \quad (\mathcal{L}^*(\nu))(Y) = \int_X \mathcal{L}(I_Y) d\nu = \int_{f(Y)} \exp \phi \circ f|_Y^{-1} d\nu.$$

Hence the (positive) eigen-measure  $m_\phi$  has Jacobian for  $(f|_Y)^{-1}$  equal to  $\exp(\phi \circ f|_Y^{-1})/\lambda$ , hence  $f$  has Jacobian  $\exp(-\phi)$  multiplied by an eigenvalue  $\lambda := \exp P(\phi)$ .

The proof of the existence of an invariant Gibbs measure equivalent to  $m_\phi$  is harder. One first proves the existence of a positive eigenfunction  $u_\phi$  for  $\mathcal{L}$  and then defines  $\mu_\phi = u_\phi m_\phi$ . For a more complete introduction to this theory, see e.g. [Przytycki and Urbański \[ibid.\]](#).

## 2 Introduction to dimension 1

Thermodynamic formalism is useful for studying properties of the underlying space  $X$ . In dimension 1, for  $f$  real of class  $C^{1+\varepsilon}$  or  $f$  holomorphic, for an expanding repeller  $X$ , considering  $\phi = \phi_t := -t \log |f'|$  for  $t \in \mathbb{R}$ , [\(1-3\)](#) gives

$$(2-1) \quad \begin{aligned} \mu_{\phi_t}(f_x^{-n}(B(f^n(x), r_0))) &\approx \exp(S_n \phi(x) - nP(\phi_t)) \approx \\ &\text{diam } f_x^{-n}(B(f^n(x), r_0))^t \exp -nP(\phi_t). \end{aligned}$$

The latter follows from a comparison of the diameter with the inverse of the absolute value of the derivative of  $f^n$  at  $x$ , due to *bounded distortion*. Here, the symbol “ $\approx$ ” denotes that the mutual ratios are bounded by a constant.

When  $t = t_0$  is a zero of the function  $t \mapsto P(\phi_t)$ , this gives

$$(2-2) \quad \mu_{\phi_{t_0}}(B) \approx (\text{diam } B)^{t_0}$$

for all small balls  $B$  (the  $t_0$ -Ahlfors measure property). We obtain the so-called Bowen’s formula for Hausdorff dimension:

$$(2-3) \quad \text{HD}(X) = t_0.$$

Moreover, the Hausdorff measure of  $X$  in this dimension is finite and nonzero.

A model example of application is the proof of

**Theorem 2.1.** *For  $f_c(z) := z^2 + c$  for an arbitrary complex number  $c \neq 0$  sufficiently close to 0, the invariant Jordan curve  $J$  (Julia set for  $f_c$ ) is a fractal, i.e. has Hausdorff dimension bigger than 1.*

*Sketch of Proof.*  $t_0 > 1$  yields  $\text{HD}(J) = t_0 > 1$  by (2-2) (one does not need to use the invariance of  $\mu_{\phi_{t_0}}$ ).

The case  $t_0 = 1$  yields by (2-2) finite Hausdorff measure in dimension 1, i.e. the rectifiability of  $J$ . To conclude that  $J$  is a circle and  $c = 0$ , one can use ergodic invariant measures in the classes of harmonic ones on  $J$  from inside and outside. They must coincide. This relies on Birkhoff’s Ergodic Theorem, the heart of ergodic theory. This is an “echo” of the celebrated Mostov Rigidity Theorem. See [Sullivan \[1982\]](#) and [Przytycki and Urbański \[2010, Theorem 9.5.5\]](#).  $\square$

In dimension 1 (real or complex), we call  $c$  a critical point if the derivative  $f'(c) = 0$ . The set of critical points will be denoted by  $\text{Crit}(f)$ .

In this survey, we allow for the presence of critical points and concentrate mainly on two cases:

1. (Complex case)  $f$  is a rational mapping of degree at least 2 of the Riemann sphere  $\overline{\mathbb{C}}$ . We consider  $f$  acting on its Julia set  $K = J(f)$ .

For entire or meromorphic maps see e.g. [Barański, Karpińska, and Zdunik \[2009, 2012\]](#), compare Proposition 5.2.

2. (Real case)  $f$  is a *generalized multimodal map* defined on a neighbourhood  $U_K \subset \mathbb{R}$  of its compact invariant subset  $K$ . We assume that  $f \in C^2$ , is non-flat at all of its turning and inflection critical points, satisfies the *bounded distortion* property for iterates, abbr.

BD, see [Przytycki and Rivera-Letelier \[2014\]](#), is topologically transitive and has positive topological entropy on  $K$ .

We assume that  $K$  is a maximal invariant subset of a finite union of pairwise disjoint closed intervals  $\widehat{I} = I^1 \cup \dots \cup I^k \subset U_K$  whose endpoints are in  $K$ . (This maximality corresponds to the Darboux property, compare [Przytycki and Rivera-Letelier \[ibid., Appendix A\]](#) and [Misiurewicz and Szlenk \[1980, page 49\]](#).) We write  $(f, K) \in \mathcal{A}_+^{\text{BD}}$ , with the subscript  $+$  to mark positive entropy. In place of BD one can assume  $C^3$  (and write  $(f, K) \in \mathcal{A}_+^3$ ), and assume that all periodic orbits in  $K$  are hyperbolic repelling. Indeed, changing  $f$  outside  $K$  if necessary, one can get the corrected  $(f, K)$  in  $\mathcal{A}_+^{\text{BD}}$ .

Recall the notions concerning periodic orbits: *Parabolic* means  $f^n(p) = p$  with  $(f^n)'(p)$  being a root of unity. For  $|(f^n)'(p)| = 1$  the term *indifferent periodic* is used and for  $|(f^n)'(p)| > 1$  the term *hyperbolic repelling*. If  $|(f^n)'(p)| < 1$  the orbit is called *hyperbolic attracting*.

For the real setting, see [Przytycki and Rivera-Letelier \[2014\]](#), [Gelfert, Przytycki, and Rams \[2016\]](#) and [Przytycki \[2018\]](#). Examples are provided by basic sets in the spectral decomposition [de Melo and van Strien \[1993\]](#).

**Question.** Are there any other examples?

**Problem.** Generalize the real case theory, see further sections, to the piecewise continuous maps, that is allow the intervals  $I^j$  to have common ends (see [Hofbauer and Urbański \[1994\]](#) for some results in this direction).

In this survey, we compare equilibrium states to (refined) Hausdorff measures in the complex case. For the real case, we refer the reader to [Hofbauer and Keller \[1993\]](#) and the references therein.

### 3 Hyperbolic potentials

For general  $f : X \rightarrow X$  and  $\phi : X \rightarrow \mathbb{R}$  as in [Definition 1.2](#) the following conditions are of special interest [Inoquio-Renteria and Rivera-Letelier \[2012\]](#),

- 1)  $P(f, \phi) > \sup \phi$ ,
- 2)  $P(f^n, S_n \phi) > \sup_X S_n \phi$  for an integer  $n$ ,
- 3)  $P(f, \phi) > \sup_{\nu \in \mathcal{M}(f)} \int \phi d\nu$ ,
- 4) For each equilibrium state  $\mu$  for the potential  $\phi$ , the entropy  $h_\mu(f)$  is positive.

The conditions 2) – 4) are equivalent, see [Inoquio-Renteria and Rivera-Letelier \[ibid., Proposition 3.1\]](#). Potentials  $\phi$  satisfying them have been called in [Inoquio-Renteria and Rivera-Letelier \[ibid.\]](#) *hyperbolic*. The condition 1) has longer traditions, see [Denker and Urbański \[1991\]](#). The intuitive meaning is that no minority of trajectories carries the full pressure.

For every  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  rational of degree at least 2 and  $\phi : J(f) \rightarrow \mathbb{R}$  Hölder continuous, the following condition is also equivalent to 2)–4), see [Inoquio-Renteria and Rivera-Letelier \[2012\]](#):

5) For each ergodic equilibrium state  $\mu$  for  $\phi$ , the Lyapunov exponent  $\chi(\mu) := \int \log |f'| d\mu$  is positive, that is for  $\mu$ -a.e.  $x$ ,

$$\chi(\mu) = \chi(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log |(f^n)'(x)| > 0.$$

The conditions 2)-5) are also equivalent in the real case for  $(f, K) \in \mathcal{A}_+^{\text{BD}}$  or  $(f, K) \in \mathcal{A}_+^3$  and all periodic orbits hyperbolic repelling. The arguments in [Inoquio-Renteria and Rivera-Letelier \[ibid.\]](#) work. See also [Li and Rivera-Letelier \[2014b\]](#).

**Theorem 3.1.** *Let  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  be a rational mapping as above. If  $\phi$  is a Hölder continuous hyperbolic potential on  $J(f)$ , then there exists a unique equilibrium state  $\mu_\phi$ . For every Hölder  $u : J(f) \rightarrow \mathbb{R}$ , the Central Limit Theorem (abbr. CLT) for the sequence of random variables  $u \circ f^n$  and  $\mu_\phi$  holds.*

For a proof, see [Przytycki \[1990\]](#) and preceding [Denker and Urbański \[1991\]](#). To find this equilibrium one can iterate the transfer operator proving  $\mathcal{L}^n(\mathbb{1}) / \exp nP(f, \phi) \rightarrow u_\phi$ . The convergence is uniformly  $\exp -\sqrt{n}$  fast and the limit is Hölder continuous, [Denker, Przytycki, and Urbański \[1996\]](#). Finally, define  $\mu_\phi := u_\phi \cdot m_\phi$ , as at the end of [Section 1](#).

**Remark 3.2.** Given  $\mu_\phi$  a priori, an efficient way to study it is an inducing method, see [Szostakiewicz, Urbański, and Zdunik \[2015\]](#), i.e. the use of a return map  $A \ni x \mapsto f^{n(x)}(x) \in A$  for  $A$  and  $n(x)$  adequate to  $\mu_\phi$ . Then one proves even an exponential convergence (with any  $u$  Hölder in place of  $\mathbb{1}$ ), which yields exponential mixing, hence stochastic laws for  $u \circ f^n$  for Hölder  $u$ , e.g. CLT, LIL, compare [Sections 9](#) and [10](#). See also [Remark 5.4](#). The key feature is the exponential decay of  $\mu_\phi(A_n)$ , where  $A_n := \{x \in A : n(x) \geq n\}$ .

See also [Bruin and Todd \[2008\]](#) for the real case, and stronger [Li and Rivera-Letelier \[2014a\]](#) and [Li and Rivera-Letelier \[2014b\]](#) including also the complex case proving the exponential convergence to  $u_\phi$ , hence CLT and LIL. See also [Szostakiewicz, Urbański, and Zdunik \[2014\]](#) for endomorphisms  $f$  of higher dimensional complex projective space, where 1) is replaced by a stronger “gap” assumption.

## 4 Non-uniform hyperbolicity in real and complex dimension 1

Here we discuss a set of conditions, valid in both the real and complex situations. Below we concentrate on the case of complex rational maps with  $K = J(f)$ , only remarking differences in the real case.

(a) CE. *Collet-Eckmann condition*. There exist  $\lambda_{CE} > 1$  and  $C > 0$  such that for every critical point  $c$  in  $J(f)$ , whose forward orbit does not meet other critical points, for every  $n \geq 0$  we have

$$|(f^n)'(f(c))| \geq C\lambda_{CE}^n.$$

Moreover, there are no parabolic (indifferent) periodic orbits.

(b) CE2( $z_0$ ). *Backward or second Collet-Eckmann condition at  $z_0 \in J(f)$* . There exist  $\lambda_{CE2} > 1$  and  $C > 0$  such that for every  $n \geq 1$  and every  $w \in f^{-n}(z_0)$  (in a neighbourhood of  $K$  in the real case)

$$|(f^n)'(w)| \geq C\lambda_{CE2}^n.$$

(b') CE2. *The second Collet-Eckmann condition*. CE2(c) holds for all critical points  $c$  not in the forward orbit of any other critical point.

(c) TCE. *Topological Collet-Eckmann condition*. There exist  $M \geq 0, P \geq 1, r > 0$  such that for every  $x \in K$  there exists a strictly increasing sequence of positive integers  $n_j, j = 1, 2, \dots$ , such that  $n_j \leq P \cdot j$  and for each  $j$  (and discs  $B(\cdot)$  below understood in  $\overline{\mathbb{C}}$  or  $\mathbb{R}$ )

$$(4-1) \quad \#\{0 \leq i < n_j : \text{Comp}_{f^i(x)} f^{-(n_j-i)} B(f^{n_j}(x), r) \cap \text{Crit}(f) \neq \emptyset\} \leq M,$$

where in general  $\text{Comp}_z V$  means for  $z \in V$  the component of  $V$  containing  $z$ .

In the real case, one adds the condition that there are no parabolic periodic orbits, which is automatically true in the case of complex rational maps.

(d) ExpShrink. *Exponential shrinking of components*. There exist  $\lambda_{\text{Exp}} > 1$  and  $r > 0$  such that for every  $x \in K$ , every  $n > 0$  and every connected component  $W_n$  of  $f^{-n}(B(x, r))$  for the disc (interval)  $B(x, r)$  in  $\overline{\mathbb{C}}$  (or  $\mathbb{R}$ ), intersecting  $K$

$$(4-2) \quad \text{diam}(W_n) \leq \lambda_{\text{Exp}}^{-n}.$$

(e) LyapHyp (*Lyapunov hyperbolicity*). There is a constant  $\lambda_{\text{Lyap}} > 1$  such that the Lyapunov exponent  $\chi(\mu)$  of any ergodic measure  $\mu \in \mathcal{M}(f, K)$  satisfies  $\chi(\mu) \geq \log \lambda_{\text{Lyap}}$ .

(f) UHP. *Uniform Hyperbolicity on periodic orbits*. There exists  $\lambda_{\text{Per}} > 1$  such that every periodic point  $p \in K$  of period  $k \geq 1$  satisfies

$$|(f^k)'(p)| \geq \lambda_{\text{Per}}^k.$$

We distinguish LyapHyp as the most adequate among these conditions to carry the name (strong) non-uniform hyperbolicity.<sup>1</sup>

**Theorem 4.1.** *1. The conditions (c)–(f) and else (b) for some  $z_0$  are equivalent in the complex case. In the real case, the equivalence also holds under the assumption of weak isolation (see the definition below).*

*2. In the complex case, the suprema over all possible constants  $\lambda_{\text{Exp}}$ ,  $\lambda_{\text{CE2}}$  (supremum over all  $z_0$ ),  $\lambda_{\text{Per}}$  and  $\lambda_{\text{Lyap}}$  coincide.*

*3. Both CE and CE2 imply (c)–(f).*

*4. If there is only one critical point in the Julia set in the complex case or if  $f$  is  $S$ -unimodal on  $K = I$  in the real case, i.e. has just one turning critical point  $c$  and negative Schwarzian derivative on  $I \setminus \{c\}$ , then all conditions above are equivalent to each other.*

For more details, see [Przytycki, Rivera-Letelier, and Smirnov \[2003\]](#), [Rivera-Letelier \[2012\]](#) and [Przytycki and Rivera-Letelier \[2014\]](#).

**Definition 4.2.**  $(f, K) \in \mathcal{A}$  is said to be *weakly isolated* if there exists an open neighbourhood  $U$  of  $K$  in the domain of  $f$  such that for every  $f$ -periodic orbit  $O(p) \subset U$  is contained in  $K$ .

In the complex case, we can replace (4-1) by

$$\text{deg}\left(f^{n_j} \Big|_{\text{Comp}_x f^{-n_j}(B(f^{n_j}(x), r))}\right) \leq M'$$

for a constant  $M'$ . In the real case, this condition is weaker than (4-1) since  $f$  mapping  $W_{n+1}$  into  $W_n$  may happen not surjective. It can have folds, thus truncating backward trajectories of critical points acquired before when pulling back.

In the real case, the proof of  $\text{CE} \Rightarrow \text{TCE}$  can be found in [Nowicki and Przytycki \[1998\]](#). For the complex case, we refer the reader to [Przytycki and Rohde \[1998\]](#).

The implication  $\text{TCE} \Rightarrow \text{CE}$  was proved in the complex case in [Przytycki \[2000, Theorem 4.1\]](#). The proof used the idea of the “reversed telescope” by [Graczyk and Smirnov \[1998\]](#). In the real case, this implication was proved for  $S$ -unimodal maps in [Nowicki and Sands \[1998\]](#). In presence of more than one critical point this implication may be false, see [Przytycki, Rivera-Letelier, and Smirnov \[2003, Appendix C\]](#).

**Question.** Is this implication true for every  $(f, K) \in \mathcal{A}_+^{\text{BD}}$  with one critical point, provided it is weakly isolated? See [Definition 4.2](#). It seems that the answer is yes.

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<sup>1</sup>Then all Hölder continuous potentials are hyperbolic, see Condition 5) in [Section 3](#) and [Inoquio-Renteria and Rivera-Letelier \[2012\]](#).

Since the condition TCE is stated in purely topological terms (in the class of maps without indifferent periodic orbits), it is invariant under topological conjugacy. So we obtain the following immediate corollary.

**Corollary 4.3.** *All equivalent conditions listed above are invariant under topological conjugacies between  $(f, K)$ 's).*

Another proof of the topological invariance of CE in the complex case was provided in [Przytycki and Rohde \[1999\]](#) with the use of Heinonen and Koskela criterion for quasi-conformality, [Heinonen and Koskela \[1995\]](#).

Note that this topological invariance is surprising, as all the conditions except TCE are expressed in geometric-differential terms. I do not know how to express CE for unimodal maps of interval in the (topological-combinatorial) kneading sequence terms.

An important lemma used here has been an estimate of an average distance in the logarithmic scale of every orbit from  $\text{Crit}(f)$ , see [Denker, Przytycki, and Urbański \[1996\]](#). Namely

**Lemma 4.4.**

$$(4-3) \quad \sum_{j=0}^n -\log |f^j(x) - c| \leq Qn$$

for a constant  $Q > 0$  every  $c \in \text{Crit}(f)$ , every  $x \in K$  and every integer  $n > 0$ .  $\Sigma'$  means that we omit in the sum an index  $j$  of smallest distance  $|f^j(x) - c|$ .

An order of proving the equivalences in [Theorem 4.1](#) is  $\text{CE2}(z_0) \Rightarrow \text{ExpShrink} \Rightarrow \text{LyapHyp} \Rightarrow \text{UHP} \Rightarrow \text{CE2}(z_0)$  and separately  $\text{CE2}(z_0) \Leftrightarrow \text{TCE}$ . E.g. assumed UHP one proves  $\text{CE2}(z_0)$  by “shadowing”, compare the beginning of [Section 6](#).

## 5 Geometric pressure and equilibrium states

We go back to topological pressure, [Definition 1.2](#), but for  $\phi = -t \log |f'|$ ,  $t \in \mathbb{R}$  in the complex  $K = J(f)$  or real cases, where  $\phi$  can attain the values  $\pm\infty$  at the critical points of  $f$ . See the beginning of [Section 2](#). We call it the geometric pressure, because it is useful in studying of geometry of the underlying space, e.g. as in (2-3) via equilibrium states for all  $t$ .

The definition of  $P_{\text{var}}(f, -t \log |f'|)$  in [Definition 1.2](#) makes sense due to  $\chi(\mu) \geq 0$  for all  $\mu \in \mathcal{M}(f)$ , in particular due to the integrability of  $\log |f'|$ , see [Przytycki \[1993\]](#) and [Rivera-Letelier \[2012, Appendix A\]](#) for a simpler proof. We conclude that it is convex

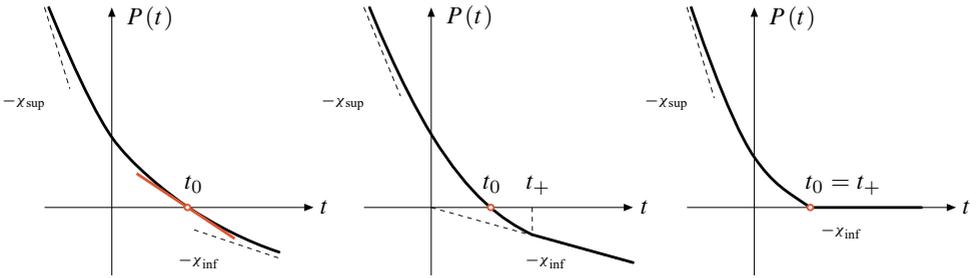


Figure 1: The geometric pressure: LyapHyp with  $t_+ = \infty$ , LyapHyp with  $t_+ < \infty$ , and non-LyapHyp. This Figure is taken from [Gelfert, Przytycki, and Rams \[2016\]](#), see notation in Remarks below.

and monotone decreasing. We start by defining a quantity occurring equal to  $P(t) = P_{\text{var}}(t) := P_{\text{var}}(f, -t \log |f'|)$ , to explain its geometric meaning, compare with [Section 2](#).

**Definition 5.1** (Hyperbolic pressure).

$$P_{\text{hyp}}(t) := \sup_{X \in \mathcal{H}(f, K)} P(f|_X, -t \log |f'|),$$

where  $\mathcal{H}(f, K)$  is defined as the space of all compact forward  $f$ -invariant (that is  $f(X) \subset X$ ) hyperbolic subsets of  $K$ , repellers in  $\mathbb{R}$ .

From this definition, it immediately follows that:

**Proposition 5.2.** (Generalized Bowen’s formula, compare (2-3)) *The first zero  $t_0$  of  $t \mapsto P_{\text{hyp}}(K, t)$  is equal to the hyperbolic dimension  $\text{HD}_{\text{hyp}}(K)$  of  $K$ , defined by  $\text{HD}_{\text{hyp}}(K) := \sup_{X \in \mathcal{H}(f, K)} \text{HD}(X)$ .*

For the discussion  $\text{HD}_{\text{hyp}}(J(f))$  vs  $\text{HD}(J(f))$ , see [Lyubich \[2014, Section 2.13\]](#).

Below we state [Theorem 5.3](#) proved in [Przytycki and Rivera-Letelier \[2011\]](#) in the complex setting and in [Przytycki and Rivera-Letelier \[2014\]](#) in the real setting. It extends [Bruin and Todd \[2009\]](#) and [Pesin and Senti \[2008\]](#) and [Iommi and Todd \[2010\]](#). See also impressive [Dobbs and Todd \[2015\]](#).

**Theorem 5.3.** 1. *Real case, [Przytycki and Rivera-Letelier \[2014\]](#). Let  $(f, K) \in \mathcal{A}_+^3$  and let all  $f$ -periodic orbits in  $K$  be hyperbolic repelling. Then  $P(t)$  is real analytic on the open interval bounded by the “phase transition parameters”  $t_-$  and  $t_+$ . For every  $t \in (t_-, t_+)$ , the domain where*

$$(5-1) \quad P(t) > \sup_{\nu \in \mathcal{M}(f)} -t \int \log |f'| d\nu,$$

there is a unique invariant equilibrium state. It is ergodic and absolutely continuous with respect to an adequate conformal measure  $m_{\phi_t}$  with the density bounded from below by a positive constant almost everywhere. If furthermore  $f$  is topologically exact on  $K$  (that is for every  $V$  an open subset of  $K$  there exists  $n \geq 0$  such that  $f^n(V) = K$ ), then this measure is mixing, has exponential decay of correlations and it satisfies the Central Limit Theorem for Lipschitz gauge functions.

2. Complex case, [Przytycki and Rivera-Letelier \[2011\]](#). The assertion is the same. One assumes a very weak expansion: the existence of arbitrarily small nice, or pleasant, couples and hyperbolicity away from critical points.

**Remarks.** 1)  $t_-$  and  $t_+$  are called the phase transition parameters. Since  $P(0) = h_{\text{top}}(f) > 0$ ,  $t_- < 0 < t_+$ , they need not exist; we say then they are equal to  $-\infty$  and/or  $+\infty$  respectively.  $P(t)$  is linear to the left of  $t_-$  and to the right of  $t_+$ , equal to  $t \mapsto -t\chi_{\text{sup}}$  where  $\chi_{\text{sup}} := \sup_v \chi(v)$  and  $t \mapsto -t\chi_{\text{inf}}$ , where  $\chi_{\text{inf}} := \inf_v \chi(v)$ , respectively. Of course,  $P(t)$  is not real-analytic at finite  $t_-$  and  $t_+$ .

2) For  $f(z) = z^2 - 2$ ,  $f : [-2, 2] \rightarrow [-2, 2]$  (the Tchebyshev polynomial), we have  $f(2) = 2$ ,  $f'(2) = 4$ ,  $\chi(l) = \log 2$ , where  $l$  is the normalized length measure. We have  $P(t) = \log 2 - t \log 2$  for  $t \geq -1$  and  $P(t) = -t \log 4$  for  $t \leq -1$ , so  $t_- = -1$ ,  $P(t)$  is non-differentiable at  $t_-$  and for  $t = -1$  there are two ergodic equilibrium states: Dirac at  $z = 2$  and  $l$ .

3) For any  $f$  non-LyapHyp,  $t_+ = t_0 < \infty$ . However  $t_+ < \infty$  can happen even for  $f$  LyapHyp, see [N. Makarov and Smirnov \[2003\]](#) and [Coronel and Rivera-Letelier \[2013, 2015\]](#).

4) Notice that the condition (5-1) is similar to the condition 3) from [Section 3](#). For  $f$  LyapHyp and  $t > t_+$ , no equilibrium state can exist, see [Inoquio-Renteria and Rivera-Letelier \[2012\]](#).

5) For real  $f$  as in [Theorem 5.3](#) satisfying LyapHyp and  $K = \widehat{I}$ , we have  $t_0 = 1$  and for  $-\log |f'|$  we conclude that a unique equilibrium state exists which is a.c.i.m. (that is: invariant absolutely continuous with respect to Lebesgue measure). In fact this assertions hold even for  $t = t_0 = t_+ = 1$  with very weak hyperbolicity properties e.g.  $|(f^n)'(f(c))| \rightarrow \infty$  for all  $c \in \text{Crit}(f)$ , see [Bruin, Rivera-Letelier, Shen, and van Strien \[2008\]](#) and [Shen and van Strien \[2014\]](#). For the complex case, see [Graczyk and Smirnov \[2009\]](#) and stronger [Rivera-Letelier and Shen \[2014\]](#).

**Remark 5.4.** In the proof of [Theorem 5.3](#), we use (compare with the [Remark 3.2](#)) a return map  $F(x) = f^{n(x)}$  to a “nice” (Markov) domain. However unlike in [Szostakiewicz, Urbański, and Zdunik \[2015\]](#), we do not use in the construction of this set the equilibrium measure  $\mu_\phi$  because we do not know a priori that it exists. The construction is geometric.  $F$  is an infinite Iterated Function System, more precisely the family of all branches of  $F^{-1}$

is, see [Mauldin and Urbański \[2003\]](#) and [Pesin \[2014\]](#) and references therein, expanding due to the “acceleration” from  $f$  to  $F$ . Then we consider an equilibrium state  $P$  for  $(F, \Phi)$  where  $\Phi(x) := \sum_{j=0}^{n(x)-1} \phi_t(f^j(x))$ , and consider an equivalent conformal measure. We propagate these measures to the Lai-Sang Young tower  $\{(x, j) : 0 \leq j < n(x)\}$  and project by  $(x, j) \mapsto f^j(x)$  to  $K$ .<sup>2</sup>

Stochastic properties of  $P$  stay preserved along the construction to  $\mu_\phi$ . The analyticity of  $P(t)$  follows from expressing  $P(t)$  as zero of a pressure for  $F$  with potential depending on two parameters and Implicit Function Theorem. The latter idea came from [Stratmann and Urbanski \[2003\]](#).

**Remark 5.5.** For probability measures  $\mu_n$  weakly\* convergent to some  $\widehat{\mu}$ , in presence of critical points  $\int \log |f'| d\mu_n$  need not converge to  $\int \log |f'| d\widehat{\mu}$ . Only upper semicontinuity holds. Therefore, for  $t > 0$ , the equilibrium states for  $t_n \rightarrow t$  need not converge to an equilibrium state for  $t$ . A priori, the free energy in the [Definition 1.2](#) can jump down. However, a modification of this method to prove existence of equilibria works, see [Dobbs and Todd \[2015\]](#).

Notice also that passing to a weak\*-limit with averages of Dirac measures on  $\{x, \dots, f^n(x)\}$  proves  $\limsup_{n \rightarrow \infty} \sup_{x \in K} \frac{1}{n} S_n(\log |f'|)(x) \leq \chi_{\max}$ . However an analogous inequality  $\liminf \dots \geq \chi_{\inf}$  is obviously false. These observations contribute to the understanding of Lyapunov spectrum.

**Remarks on the Lyapunov spectrum.** [Theorem 5.3](#) allows us to express the so-called dimension spectrum for Lyapunov exponents with the use of Legendre transform, that is for all  $\alpha > 0$  and  $\mathcal{L}(\alpha) := \{x \in K : \chi(x) = \alpha\}$

$$(5-2) \quad \text{HD}(\mathcal{L}(\alpha)) = \frac{1}{|\alpha|} \inf_{t \in \mathbb{R}} (P(t) + \alpha t).$$

An ingredient is **Mañé’s equality**

$$(5-3) \quad \text{HD}(\mu) = h_\mu(f) / \chi(\mu)$$

provided  $\chi(\mu) > 0$ , [Przytycki and Urbański \[2010\]](#), where  $\text{HD}(\mu) := \sup\{\text{HD}(X) : \mu(X) = 1\}$ , applied to  $\mu_{\phi_t}$ .

The equality (5-2) concerns regular  $x$ ’s, where  $\chi(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |(f^n)'(x)|$  exists. It is also possible to provide formulas or at least estimates for Hausdorff dimension of the sets of irregular points  $\mathcal{L}(\alpha, \beta) := \{x \in K : \underline{\chi}(x) = \alpha, \overline{\chi}(x) = \beta\}$  for lower and upper Lyapunov exponents where we replace  $\lim$  by  $\liminf$  and  $\limsup$  respectively. See

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<sup>2</sup>For applications to decide the existence or nonexistence of a finite a.c.i.m. for maps of interval with flat critical points or for entire or meromorphic maps depending on the  $P$ -integrability of the first return time, see papers by N. Dobbs, B. Skorulski, J. Kotus, G. Świątek.

Gelfert, Przytycki, and Rams [2010] and Gelfert, Przytycki, and Rams [2016] for this theory in complex and real settings.

However, these papers give no information about the size of sets with zero (upper) Lyapunov exponent. Note at least that if  $J(f) \neq \overline{\mathbb{C}}$  then  $\text{Leb}_2\{x \in J(f) : \bar{\chi}(x) > 0\} = 0$ . This is so because  $\bar{\chi}(x) > 0$  implies there exists  $\mathfrak{N} \subset \mathbb{Z}_+$  of positive upper density, such that for  $n \in \mathfrak{N}$ , (4-2) and (4-1) hold, see Levin, Przytycki, and Shen [2016, Section 3].

We do not know whether  $\chi(x) = -\infty$  can happen for  $x$  not pre-critical, except there is only one critical point in  $K$ , where  $\chi(x) > -\infty$  follows from (4-3), see Gelfert, Przytycki, and Rams [2010, Lemma 6].

For  $x$  being a critical value we can prove (in analogy to  $\chi(\mu) \geq 0$ ):

**Theorem 5.6** (Levin, Przytycki, and Shen [2016]). *If for a rational function  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  there is only one critical point  $c$  in  $J(f)$  and no parabolic periodic orbits, then  $\underline{\chi}(f(c)) \geq 0$ .*

For  $S$ -unimodal maps of interval this was proved by Nowicki and Sands [1998].

## 6 Other definitions of geometric pressure

**Definition 6.1** (safe). See Przytycki and Urbański [2010, Definition 12.5.7]. We call  $z \in K$  safe if  $z \notin \bigcup_{j=1}^{\infty} (f^j(\text{Crit}(f)))$  and for every  $\delta > 0$  and all  $n$  large enough  $B(z, \exp(-\delta n)) \cap \bigcup_{j=1}^n (f^j(\text{Crit}(f))) = \emptyset$ .

Notice that this definition implies that all points except at most a set of Hausdorff dimension 0, are safe.

**Definition 6.2** (Tree pressure). For every  $z \in K$  and  $t \in \mathbb{R}$  define

$$(6-1) \quad P_{\text{tree}}(z, t) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{f^n(x)=z, x \in K} |(f^n)'(x)|^{-t}.$$

Compare with  $P(f, \phi)$  from Theorem 1.3. Under suitable conditions, e.g. for  $z$  “safe” the limit exists, it is independent of  $z$  and equal to  $P(t)$ . See Przytycki [1999], Przytycki, Rivera-Letelier, and Smirnov [2003] and Przytycki and Urbański [2010] for the complex case and Przytycki and Rivera-Letelier [2014] and Przytycki [2018] for the real case.

A key is to extend all trajectories  $T_n(x) = \{x, \dots, z\}$  backward and forward by time  $m \ll n$  to get an Iterated Function System for  $f^{n+m}$  and to consider its limit set. Its trajectories for time  $n$  “shadow”  $T_n(x)$ . This proves  $P_{\text{tree}}(z, t) \leq P_{\text{hyp}}(t)$ . The opposite inequality is immediate.

(Similarly one proves  $P_{\text{var}}(t) \leq P_{\text{hyp}}(t)$ . Given  $\mu$  with  $\chi(\mu) > 0$  one captures a hyperbolic  $X$  by Pesin-Katok method.)

For a continuous potential  $\phi : X \rightarrow \mathbb{R}$ , consider

$$(6-2) \quad P_{\text{sep}}(f, \phi) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left( \sup_Y \sum_{y \in Y} \exp S_n \phi(y) \right),$$

where the supremum is taken over all  $(n, \varepsilon)$ -separated sets  $Y \subset X$ , that is such  $Y$  that for every distinct  $y_1, y_2 \in Y$ ,  $\rho_n(y_1, y_2) \geq \varepsilon$ , where  $\rho_n$  is the metric defined by  $\rho_n(x, y) = \max\{\rho(f^j(x), f^j(y)) : j = 0, \dots, n\}$ .

For  $\phi = -t \log |f'|$  for positive  $t$ , in presence of critical points for  $f$ ,  $P_{\text{sep}}$  is always equal to  $\infty$  by putting a point of a separated set at a critical point. So we replace it by the tree pressure. One can however use infimum over  $(n, \varepsilon)$ -spanning sets, thus defining  $\underline{P}_{\text{spanning}}(f, \phi)$ . This is a valuable notion, often coinciding with other pressures. See [Przytycki \[ibid.\]](#) for an outline of a respective theory. Let me mention only that this is equal to  $P(f, -t \log |f'|)$  for  $t > 0$  in the complex case if

**Definition 6.3.**  $f$  is *weakly backward Lyapunov stable* which means that for every  $\delta > 0$  and  $\varepsilon > 0$  for all  $n$  large enough and every disc  $B = B(x, \exp -\delta n)$  centered at  $x \in K$ , for every  $0 \leq j \leq n$  and every component  $V$  of  $f^{-j}(B)$  intersecting  $K$ , it holds that  $\text{diam } V \leq \varepsilon$ .

This holds for all rational maps with at most one critical point whose forward trajectory is in  $J(f)$  or is attracted to  $J(f)$ , due to [Theorem 5.6](#).

**Question.** Does backward weak Lyapunov stability hold for all rational maps?

Finally, *periodic pressure*  $P_{\text{Per}}$  is defined as  $P_{\text{tree}}$  with  $x \in \text{Per}_n$  (periodic of period  $n$ ) rather than  $f^n$ -preimages of  $z$ . In [Przytycki, Rivera-Letelier, and Smirnov \[2004\]](#), this was proved for rational  $f$  (see also [Binder, N. Makarov, and Smirnov \[2003\]](#) for a class of polynomials) on  $K = J(f)$  that  $P_{\text{Per}}(t) = P(t)$  provided

**Hypothesis H.** For every  $\delta > 0$  and all  $n$  large enough, if for a set  $\mathcal{P} \subset \text{Per}_n$  for all  $p, q \in \mathcal{P}$  and all  $i : 0 \leq i < n$   $\text{dist}(f^i(p), f^i(q)) < \exp -\delta n$ , then  $\#\mathcal{P} \leq \exp \delta n$ .

**Question.** Does this condition always hold? In particular, can large bunches of periodic orbits exist with orbits exponentially close to a Cremer fixed point?

## 7 Geometric coding trees, limit sets, Gibbs meets Hausdorff

The notion of geometric coding tree, g.c.t., already appeared in the work [Jakobson \[1978\]](#), where in the expanding case the finite-to-one property of the resulting coding was proved.

It was used later in Przytycki [1985, 1986] and in a full strength in Przytycki, Urbański, and Zdunik [1989, 1991] and papers following them. Similar graphs have since been constructed to analyse the topological aspects of non-invertible dynamics, see for instance Nekrashevych [2011] and Haïssinsky and Pilgrim [2009].

**Definition 7.1.** Let  $U$  be an open connected subset of the Riemann sphere  $\overline{\mathbb{C}}$ . Consider a holomorphic mapping  $f : U \rightarrow \overline{\mathbb{C}}$  such that  $f(U) \supset U$  and  $f : U \rightarrow f(U)$  is a proper map. Suppose that  $\text{Crit}(f)$  is finite. Consider an arbitrary  $z \in f(U)$ . Let  $z^1, z^2, \dots, z^d$  be some of the  $f$ -preimages of  $z$  in  $U$  with  $d \geq 2$ . Consider smooth curves  $\gamma^j : [0, 1] \rightarrow f(U)$ ,  $j = 1, \dots, d$ , joining  $z$  to  $z^j$  respectively (i.e.  $\gamma^j(0) = z, \gamma^j(1) = z^j$ ), intersections allowed, such that  $\gamma^j \cap f^n(\text{Crit}(f)) = \emptyset$  for every  $j$  and  $n > 0$ .

For every sequence  $\alpha = (\alpha_n)_{n=0}^\infty \in \Sigma^d$  (shift space with left shift map  $\zeta$  defined in Section 1) define  $\gamma_0(\alpha) := \gamma^{\alpha_0}$ . Suppose that for some  $n \geq 0$ , for every  $0 \leq m \leq n$ , and all  $\alpha \in \Sigma^d$ , curves  $\gamma_m(\alpha) : [0, 1] \rightarrow U$  are already defined. Suppose that for  $1 \leq m \leq n$  we have  $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\zeta(\alpha))$ , and  $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$ . Define the curves  $\gamma_{n+1}(\alpha)$  so that the previous equalities hold by taking respective  $f$ -preimages of curves  $\gamma_n$ . For every  $\alpha \in \Sigma^d$  and  $n \geq 0$  denote  $z_n(\alpha) := \gamma_n(\alpha)(1)$ .

The graph  $\mathcal{T} = \mathcal{T}(z, \gamma^1, \dots, \gamma^d)$  with the vertices  $z$  and  $z_n(\alpha)$  and edges  $\gamma_n(\alpha)$  is called a *geometric coding tree* with the root at  $z$ . For every  $\alpha \in \Sigma^d$  the subgraph composed of  $z, z_n(\alpha)$  and  $\gamma_n(\alpha)$  for all  $n \geq 0$  is called an *infinite geometric branch* and denoted by  $b(\alpha)$ . It is called *convergent* if the sequence  $\gamma_n(\alpha)$  is convergent to a point in  $\text{cl } U$ . We define the *coding map*  $z_\infty : \mathcal{D}(z_\infty) \rightarrow \text{cl } U$  by  $z_\infty(\alpha) := \lim_{n \rightarrow \infty} z_n(\alpha)$  on the domain  $\mathcal{D} = \mathcal{D}(z_\infty)$  of all such  $\alpha$ 's for which  $b(\alpha)$  is convergent.

Denote  $\Lambda := z_\infty(\mathcal{D}(z_\infty))$ . If the map  $f$  extends holomorphically to a neighbourhood of its closure  $\text{cl } \Lambda$  in  $\overline{\mathbb{C}}$ , then  $\Lambda$  is called a *quasi-repeller*, see Przytycki, Urbański, and Zdunik [1989].

A set formally larger than  $\text{cl } \Lambda$  is of interest, namely  $\widehat{\Lambda}$  being the set of all accumulation points of  $\{z_n(\alpha) : \alpha \in \Sigma^d\}$  as  $n \rightarrow \infty$ . If our g.c.t. is in  $\Omega$  being an RB-domain, see Section 8, or  $f$  is just  $R \circ g \circ R^{-1}$  defined only on  $\Omega$ , see Remarks below, then it is easy to see that  $\text{cl } \Lambda = \widehat{\Lambda}$ . I do not know how general this equality is.

**Remarks.** Given a Riemann map  $R : \mathbb{D} \rightarrow \Omega$  to a connected simply connected domain  $\Omega \subset \mathbb{C}$ , (i.e. holomorphic bijection) we can consider a branched covering map, say  $g(z) = z^d$  on  $\mathbb{D}$ , and  $f = R \circ g \circ R^{-1}$ . Then, chosen  $z \in \Omega$  and  $\gamma^j$  joining it with its preimages in  $\Omega$  (close to  $\text{Fr } \Omega$ ) we can consider the respective tree  $\mathcal{T}$ . Then instead of considering  $R$  and its radial limit  $\overline{R}$ , we can consider the limit (along branches)  $z_\infty : \Sigma^d \rightarrow \text{Fr } \Omega$ . This provides a structure of symbolic dynamics useful to verify stochastic laws.

This is especially useful if considered measures come from  $\partial\mathbb{D}$  via  $\overline{R}$ , rather than being some equilibrium states for potentials living directly on  $\text{Fr } \Omega$ . This is the case of harmonic measure  $\omega$  which is the  $\overline{R}_*$ -image of a length measure  $l$ . We can consider the lift of  $l$  to  $\Sigma^d$  via coding by the tree  $\mathcal{T}' = R^{-1}(\mathcal{T})$  and next its projection by  $(z_\infty)_*$  to  $\text{Fr } \Omega$ .

Our g.c.t.'s are always available in presence of adequate holomorphic  $f$ , even in the absence of  $\Omega$ , i.e. in the absence of a Riemann map. The tree with the coding it induces yields a discrete generalization/replacement of a Riemann map.

It was proved in [Przytycki and Skrzypczak \[1991\]](#) that  $\mathcal{D}$  is the whole  $\Sigma^d$  except a “thin” set. In particular, for a Gibbs measure  $\nu$  for a Hölder potential,  $z_\infty(\alpha)$  exists for  $\nu$ -a.e.  $\alpha$ , hence the push forward measure  $(z_\infty)_*(\nu)$  makes sense. Moreover, our codings  $\zeta_\infty$  are always “thin”-to-one. This is a discrete generalization of Beurling’s Theorem concerning the boundary behaviour of Riemann maps. “Thin” means of zero logarithmic capacity type, depending on the properties of the tree (the speed of the accumulation of  $\gamma^j$  by critical trajectories; the speed does not matter if we replace “thin” by zero Hausdorff dimension). In particular this coding preserves the entropies.

For appropriate  $\nu \in \mathcal{M}(\zeta, \Sigma^d)$  and  $\psi : \Sigma^d \rightarrow \mathbb{R}$  with  $\int \psi d\nu = 0$ , consider the asymptotic variance (of course one can consider spaces more general than  $\Sigma^d$ )

$$(7-1) \quad \sigma^2 = \sigma_\nu^2(\psi) := \lim_{n \rightarrow \infty} \frac{1}{n} \int (S_n \psi)^2 d\nu.$$

**Theorem 7.2.** *Let  $\Lambda$  be a quasi-repeller for a geometric coding tree for a holomorphic map  $f : U \rightarrow \overline{\mathbb{C}}$ . Let  $\nu$  be a  $\zeta$ -invariant Gibbs measure on  $\Sigma^d$  for a Hölder continuous real-valued function  $\phi$  on  $\Sigma^d$ . Assume  $P(\zeta, \phi) = 0$ . Consider  $\mu := (z_\infty)_*(\nu)$ .*

*Then, for  $\psi := -\text{HD}(\mu)(\log |f'| \circ z_\infty) - \phi$ , we have  $\int \psi d\nu = 0$ .*

*If the asymptotic variance  $\sigma^2 = \sigma_\nu^2(\psi)$  is positive, then there exists a compact  $f$ -invariant hyperbolic repeller  $X$  being a subset of  $\Lambda$  such that  $\text{HD}(X) > \text{HD}(\mu)$ . In consequence  $\text{HD}_{\text{hyp}}(\Lambda) > \text{HD}(\mu)$  (defined after (5-2)).*

*If  $\sigma^2 = 0$  then  $\psi$  is cohomologous to 0. Then for each  $x, y \in \text{cl } \Lambda$  not postcritical, if  $z = f^n(x) = f^m(y)$  for some positive integers  $n, m$ , the orders of criticality of  $f^n$  at  $x$  and  $f^m$  at  $y$  coincide. In particular all critical points in  $\text{cl } \Lambda$  are pre-periodic.*

The latter condition happens only in special situations, see e.g. [Theorem 7.3](#) below. See [Szostakiewicz, Urbański, and Zdunik \[2015\]](#) for more details;  $\phi$  lives there directly on  $J(f)$ , but it does not make substantial difference. See also [Section 10](#).

Given a mapping  $f : X \rightarrow X$ , given two functions  $u, v : X \rightarrow \mathbb{R}$  we call  $u$  cohomologous to  $v$  in class  $\mathcal{C}$  if there exists  $h : X \rightarrow \mathbb{R}$  belonging to  $\mathcal{C}$  such that  $u - v = h \circ f - h$ . An important [Przytycki, Urbański, and Zdunik \[1989, Lemma 1\]](#) says that  $\sigma^2 = 0$  above implies  $\psi$  cohomologous to 0 in  $L^2(\mu)$  and often in a smaller class depending on  $\psi$  (Livšic type rigidity).

Notice that  $\int \psi dv = -\text{HD}(\mu)\chi(\mu) - \int \phi dv = -h_\mu(f) - \int \phi dv = -h_\nu(\zeta) - \int \phi dv = P(\zeta, \phi) = 0$ . Now, to prove [Theorem 7.2](#) note  $2\chi(\mu) \geq h_\mu(f) = h_\nu(\zeta) > 0$ , see [Przytycki and Urbański \[2010, Ruelle’s inequality\]](#) (used also to 3) $\Rightarrow$ 5) in [Section 3](#)) and [Przytycki \[1985\]](#). So considering the natural extension of  $(\Sigma^d, \nu, \zeta)$  (here two-sided shift space) and Katok-Pesin theory, we find hyperbolic  $X$  with  $\text{HD}(X) \geq \text{HD}(\mu) - \varepsilon$  for an arbitrary  $\varepsilon > 0$ . Compare comments on shadowing in [Section 6](#).

- The positive  $\sigma^2$  yields by Central Limit Theorem large fluctuations of the sums  $\sum_{j=0}^{n-1} \psi \circ \zeta^j$  from  $n \int \psi dv$  (here 0), allowing to find  $X$  with  $\text{HD}(X) > \text{HD}(\mu)$ .

A special care is needed to get  $X \subset \Lambda$ , see [Przytycki \[2005\]](#) (originated in [Przytycki and Zdunik \[1994\]](#)).

The above fluctuations were used by A. Zdunik to prove for constant  $\phi$

**Theorem 7.3** ([Zdunik \[1990\]](#)). *Let  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  be a rational mapping of degree  $d \geq 2$ . If  $\sigma^2 > 0$ , then for  $\mu_{\max}(f)$  the measure of maximal entropy (equal  $\log d$ ),  $\text{HD}(J(f)) > \text{HD}(\mu_{\max}(f))$ . Otherwise,  $f$  is postcritically finite with a parabolic orbifold, [Milnor \[2006\]](#).*

She proved in fact the existence of a hyperbolic  $X \subset J(f)$  satisfying  $\text{HD}(X) > \text{HD}(\mu_{\max}(f))$ , hence  $\text{HD}_{\text{hyp}}(J(f)) > \text{HD}(\mu_{\max}(f))$ .

- In the  $\sigma^2 = 0$  case,  $v : J(f) \rightarrow \mathbb{R}$  satisfying the cohomology equation  $\log |f'| = v \circ f - v + \text{Const}$  on  $J(f)$  extends to a harmonic function beyond  $J(f)$  (Livšic rigidity) giving this equality on the union of real analytic curves containing  $J(f)$  (called *real case*) or to  $\overline{\mathbb{C}}$ . In [Theorem 7.2](#) on  $\Lambda$  and for the extension beyond, in [Theorem 7.3](#), the “orders” of growth of  $-\log |(f^n)'|$  at  $x$  and of  $-\log |(f^m)'|$  at  $y$  must by cohomology equation be equal to the “order” of growth of  $v$  at  $z$ , so they must coincide (a phenomenon “conjugated” to the presence of an invariant line field). This implies parabolic orbifold for [Theorem 7.3](#).

[Theorem 7.3](#) applied to a polynomial  $f$  with connected Julia set, by  $\text{HD}(\mu_{\max}(f)) = 1$  [Manning \[1984\]](#), implies the following Zdunik’s celebrated result:

**Theorem 7.4** ([Zdunik \[1990\]](#)). *For every polynomial  $f$  of degree at least 2, with connected Julia set, either  $J(f)$  is a circle or an interval or else it is fractal, namely  $\text{HD}(J(f)) > 1$ .*

## 8 Boundaries, radial growth, harmonic vs Hausdorff

For polynomials with connected Julia set the measure  $\mu_{\max}(f)$  coincides with harmonic measure  $\omega$  (viewed from  $\infty$ ). This led to another proof of [Theorem 7.4](#), especially the  $\sigma^2 = 0$  part, see [Zdunik \[1991\]](#), in the language of boundary behaviour of Riemann map and harmonic measure (compare also model [Theorem 2.1](#)).

**Theorem 7.4** has been strengthened from this point of view in [Przytycki \[2006\]](#), preceded by [Przytycki and Zdunik \[1994\]](#), as follows.

**Theorem 8.1.** *Let  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  be a rational map of degree at least 2 and  $\Omega$  be a simply connected immediate basin of attraction to an attracting periodic orbit (that is a connected component of the set attracted to the orbit, intersecting it). Then, provided  $f$  is not a finite Blaschke product in some holomorphic coordinates, or a two-to-one holomorphic factor of a Blaschke product,  $\text{HD}_{\text{hyp}}(\text{Fr } \Omega) > 1$ .*

The novelty was to show how to “capture” a large hyperbolic  $X$  in  $\text{Fr } \Omega$  in the case it was not the whole  $J(f)$ .

In fact the following “local” version of this theorem was proved in [Przytycki \[2006\]](#)

**Theorem 8.2.** *Assume that  $f$  is defined and holomorphic on a neighbourhood  $W$  of  $\text{Fr } \Omega$ , where  $\Omega$  is a connected simply connected domain in  $\overline{\mathbb{C}}$  whose boundary has at least 2 points. We assume that  $f(W \cap \Omega) \subset \Omega$ ,  $f(\text{Fr } \Omega) \subset \text{Fr } \Omega$  and  $\text{Fr } \Omega$  repels to the side of  $\Omega$ , that is  $\bigcap_{n=0}^{\infty} f^{-n}(W \cap \text{cl } \Omega) = \text{Fr } \Omega$ . Then either  $\text{HD}_{\text{hyp}}(\text{Fr } \Omega) > 1$  or  $\text{Fr } \Omega$  is a real-analytic Jordan curve or arc.*

$\Omega$  with  $f$  as above has been called an *RB-domain* (repelling boundary), introduced in [Przytycki \[1986\]](#) and [Przytycki, Urbański, and Zdunik \[1989\]](#). **Theorem 8.2** (at least the  $\sigma^2 > 0$  part) follows directly from **Theorem 7.2**. Let  $R : \mathbb{D} \rightarrow \Omega$  be a Riemann map and  $g : W' \rightarrow \mathbb{D}$  be defined by  $g := R^{-1} \circ f \circ R$  on  $W' = R^{-1}(W \cap \Omega)$ . We consider a g.c.t.  $\mathcal{T} = \mathcal{T}(z, \gamma^1, \dots, \gamma^d)$  with  $z$  and  $\gamma^j$  in  $W \cap \Omega$ , sufficiently close to  $\text{Fr } \Omega$  that the definition makes sense, and with  $d = \deg f|_{W \cap \Omega}$ , (the situation is the same as in [Remarks in Section 7](#) above, but the order of defining  $f$  and  $g$  is different). Consider the g.c.t.  $\mathcal{T}' = R^{-1}(\mathcal{T})$ . The function  $g$  extends holomorphically beyond the circle  $\partial \mathbb{D}$  and it is expanding. Hence  $\phi : \Sigma^d \rightarrow \mathbb{R}$  defined by  $\phi(\alpha) = -\log |g'| \circ (R^{-1}(z))_{\infty}(\alpha)$  for the tree  $\mathcal{T}'$  is Hölder continuous. Let  $\nu = \nu_{\phi}$ .

Note that here  $P(\phi) = 0$ , e.g. since by expanding property of  $g$  on  $\partial \mathbb{D}$  there exists  $\widehat{l} \in \mathcal{M}(g)$ , equivalent to length measure  $l$  (a.c.i.m.). Then  $\nu$  is the lift of  $\widehat{l}$  to  $\Sigma^d$  with the use of  $\mathcal{T}'$ . Note that our  $\mu = z_{\infty}(\nu)$  is equal to  $\widehat{\omega} = \overline{R}_*(\widehat{l})$  which is  $f$ -invariant, equivalent to harmonic measures  $\omega$  on  $\text{Fr } \Omega$  viewed from  $\Omega$ .

Note that  $\text{HD}(\widehat{\omega}) = 1$  due to Mañé’s equality, (5-3),  $h_{\widehat{\omega}}(f) = h_{\widehat{l}}(g)$ , see [Przytycki \[1985, 1986\]](#), and the equality of Lyapunov exponents  $\int \log |f'| d\widehat{\omega} = \int \log |g'| d\widehat{l} > 0$ . The latter equality holds due to the equality for almost every  $\zeta \in \partial \mathbb{D}$ :

$$(8-1) \quad \lim_{r \rightarrow 1} \frac{\log |(f^n)'(R(r\zeta))| - \log |(g^n)'(r\zeta)|}{\log(1-r)} = \lim_{r \rightarrow 1} \frac{-\log |R'(r\zeta)|}{\log(1-r)} = 0.$$

The first equality is proved using  $f \circ R = R \circ g$  in  $\mathbb{D}$ , first applying  $R$  close to  $\partial\mathbb{D}$ , next by iterating  $f$  applying  $R^{-1}$  well inside  $\Omega$ , finally iterating  $g$  back. The latter equality relies on the harmonicity of  $\log |R'|$  allowing to replace its integral on circles by its value at the origin. For details see [Przytycki \[1986\]](#). Remind however that in fact  $\text{HD}(\omega) = 1$  holds in general, see [N. G. Makarov \[1985\]](#).

The sketch of Proof of [Theorem 8.2](#) for  $\sigma^2 > 0$  is over. That  $\sigma^2 = 0$  implies the analyticity of  $\text{Fr}\Omega$  was already commented at the beginning of this Section.

## 9 Law of Iterated Logarithm refined versions

Applying Law of Iterated Logarithm (abbr. LIL) to  $\psi : \Sigma^d \rightarrow \mathbb{R}$  the fluctuations of  $S_n \psi$  from 0 which follow lead to, see [Przytycki, Urbański, and Zdunik \[1989\]](#) and [Przytycki and Urbański \[2010\]](#),

**Theorem 9.1.** *In the setting of [Theorem 7.2](#) if  $\sigma^2 = \sigma_v^2(\psi) > 0$ , for  $c(\mu) := \sqrt{2\sigma^2/\chi(\mu)}$ ,  $\kappa := \text{HD}(\mu)$  and  $\alpha_c(r) := r^\kappa \exp(c \sqrt{\log 1/r \log \log \log 1/r})$*

1)  $\mu \perp H_{\alpha_c}$ , that is singular with respect to the refined Hausdorff measure, [Przytycki and Urbański \[ibid., Section 8.2\]](#) for the gauge function  $\alpha_c$ , for all  $0 < c < c(\mu)$ ;

2)  $\mu \ll H_{\alpha_c}$ , that is absolutely continuous, for all  $c > c(\mu)$ .

Indeed, substituting in LIL  $n \sim (\log 1/r_n)/\chi(\mu)$  for  $r_n = |(f^n)'(z)|^{-n}$ , we get for  $\mu$ -a.e.  $z$

$$(9-1) \quad \limsup_{n \rightarrow \infty} \frac{\mu(B(z, r_n))}{\alpha_c(r_n)} = \infty \text{ for } 0 < c < c(\mu) \quad \text{and} \quad \dots = 0 \text{ for } c > c(\mu).$$

This is called the Refined Volume Lemma, [Przytycki, Urbański, and Zdunik \[1989, Section 4\]](#) and, the harder case:  $c > c(\mu)$ , [Przytycki, Urbański, and Zdunik \[1991, Section 5\]](#).

We can apply the assertion of [Theorem 9.1](#) for  $\mu = \widehat{\omega} \in \mathcal{M}(f, \text{Fr}\Omega)$  equivalent to a harmonic measure  $\omega$  as [Section 8](#).

This yields refined information about the radial growth of the derivative of Riemann maps, following the proof of [\(8-1\)](#):

**Theorem 9.2.** *Let  $\Omega$  be a simply connected RB-domain in  $\overline{\mathbb{C}}$  with non-analytic boundary and  $R : \mathbb{D} \rightarrow \Omega$  be a Riemann map. Then there exists  $c(\Omega) > 0$  such that for Lebesgue a.e.  $\zeta \in \partial\mathbb{D}$*

$$(9-2) \quad G^+(\zeta) := \limsup_{r \rightarrow 1} \frac{\log |R'(r\zeta)|}{\sqrt{\log(1/1-r) \log \log \log(1/1-r)}} = c(\Omega).$$

Similarly  $G^-(\zeta) := \liminf \dots = -c(\Omega)$ . Finally  $c(\Omega) = c(\widehat{\omega})$  in [Theorem 9.1](#).

In fact [Theorem 9.1](#) for  $\mu = \widehat{\omega}$  and [Theorem 9.2](#) hold for every connected, simply connected open  $\Omega \subset \mathbb{C}$ , together with  $c(\Omega) = c(\widehat{\omega})$ . No dynamics is needed. Of course one should add to both definitions  $\text{ess sup}$  over  $\zeta \in \partial\mathbb{D}$  and over  $z \in \text{Fr } \Omega$  (for  $c(z) = c(\omega)$  calculated from (9-1), see [Przytycki and Urbański \[2010, Th. 8.6.1\]](#) ) respectively, since in the absence of ergodicity these functions need not be constant. See [Garnett and Marshall \[2005, Th. VIII.2.1 \(a\)\]](#) and references to Makarov’s breakthrough papers therein, in particular [N. G. Makarov \[1985\]](#).

There is a universal Makarov’s upper bound  $C_M < \infty$  for all  $c(\Omega), c(\widehat{\omega})$ ’s in (9-2). The best upper estimate I found in literature is  $C_M \leq 1.2326$ , [Hedenmalm and Kayumov \[2007\]](#). I proved in [Przytycki \[1989\]](#) a much weaker estimate, using a natural method of representing  $\log |R'|$  by a series of weakly dependent random variables leading to a martingale on  $\partial\mathbb{D}$ , thus satisfying LIL. Unfortunately consecutive approximations resulted with looses in the final estimate.

For a holomorphic expanding repeller  $f : X \rightarrow X$  and a Hölder continuous potential  $\phi : X \rightarrow \mathbb{R}$ , the asymptotic variance for the equilibrium state  $\mu = \mu_{t_0\phi}$  for every  $t_0 \in \mathbb{R}$  satisfies Ruelle’s formula (see [Przytycki and Urbański \[2010\]](#)):

$$(9-3) \quad \sigma_\mu^2(\phi - \int \phi d\mu) = \left. \frac{d^2 P(t\phi)}{dt^2} \right|_{t=t_0}.$$

**Question.** Does (9-3) hold for all rational maps and hyperbolic potentials on Julia sets? For all simply connected RB-domains,  $f : \text{Fr } \Omega \rightarrow \text{Fr } \Omega$  and  $\mu = \widehat{\omega}$ ?

For a simply connected RB-domain  $\Omega$  for  $f$  and for  $\phi = -\log |f'|$ , if  $g(z) = z^d$  (e.g.  $\Omega$  being the basin of  $\infty$  for a polynomial  $f$ ), one considers the *integral means spectrum* depending only on  $\Omega$ ,

$$(9-4) \quad \beta_\Omega(t) := \limsup_{r \rightarrow 1} \frac{1}{|\log(1-r)|} \log \int_{\zeta \in \partial\mathbb{D}} |R'(r\zeta)|^t |d\zeta|$$

which happens to satisfy  $\beta_\Omega(t) = t - 1 + \frac{P(t\phi)}{\log d}$ , see e.g. [Przytycki and Urbański \[ibid., Eq. \(9.6.2.\)\]](#).

For  $t_0 = 0$  we have  $\mu = \widehat{\omega}$  and the left hand side of (9-3) can be written as  $(\frac{1}{2} \log d) \sigma^2(\log R')$ , see (7-1) and (8-1), where

$$\sigma^2(\log R') := \limsup_{r \rightarrow 1} \frac{\int_{\partial\mathbb{D}} |\log R'(t\zeta)|^2 |d\zeta|}{-2\pi \log(1-r)}.$$

So (9-3) changes to  $\sigma^2(\log R') = 2 \frac{d^2 \beta_\Omega(t)}{dt^2} \Big|_{t=0}$ , compare [Ivrii \[2016\]](#). It has an analytic, non-dynamical, meaning. It is also related to the Weil-Petersson metric, see [McMullen \[2008\]](#).

## 10 Accessibility

Let us recall the following theorem from Przytycki [1994].

**Theorem 10.1.** *Let  $\Lambda$  be a quasi-repeller for a geometric coding tree for a holomorphic map  $f : U \rightarrow \overline{\mathbb{C}}$ . Suppose that*

$$(10-1) \quad \text{diam}(\gamma_n(\alpha)) \rightarrow 0, \text{ as } n \rightarrow \infty$$

*uniformly with respect to  $\alpha \in \Sigma^d$ . Then every good  $q \in \widehat{\Lambda}$  (defined in Section 7) is a limit of a convergent branch  $b(\alpha)$ . So  $q \in \Lambda$ . In particular, this holds for every  $q$  with  $\chi(q) > 0$  and the local backward invariance (explained below).*

For the definition of “good”, see Przytycki [ibid., Definition 2.5]. It roughly says that there are many integers  $n$  (positive lower density) for which  $f^n$  properly map small domains  $D_{n,0}$  in  $U$  close to  $q$  onto large  $D_n \subset U$ , giving “telescopes”  $\text{Tel}_k$  with “traces”  $D_{n_k,0} \subset D_{n_{k-1},0} \subset \dots \subset D_{n_1,0} \subset D_0$ ; for each  $k$  the choices may be different. A part of this condition that  $D_{n,0} \subset U$  can be called a “local backward invariance” of  $U$  along the forward trajectory of  $q$ .

When  $U$  is an immediate basin of attraction of an attracting fixed point for a rational map  $f$  or just an RB-domain then this theorem asserts that  $q$  is an endpoint of a continuous curve in  $U$ . This is a generalization of the Douady-Eremenko-Levin-Petersen theorem where  $q$  is a repelling periodic point and the domain is completely invariant, e.g. basin of attraction to  $\infty$  for  $f$  a polynomial.

Due to this theorem we can prove that invariant measures of positive Lyapunov exponents lift to  $\Sigma^d$ . More precisely, the following holds:

**Corollary 10.2.** *Every non-atomic hyperbolic probability measure  $\mu$  (i.e.  $\chi(\mu) > 0$ ), on  $\widehat{\Lambda}$ , is the  $(z_\infty)_*$  image of a probability  $\zeta$ -invariant measure  $\nu$  on  $\Sigma^d$ , assumed (10-1),  $\mathcal{T}$  has no self-intersections and else  $\mu$ -a.e. local backward invariance of  $U$ . In particular,  $\nu$  exists for every RB-domain which is completely (i.e. backward) invariant.*

*Proof.* (the lifting part missing in Przytycki [ibid.] and Przytycki [2006]). By Theorem 10.1  $\mu$  is supported on  $\Lambda$  i.e. on  $z_\infty(\mathcal{D}(z_\infty))$ . The lift of  $\mu$  to  $\mu'$  on the pre-image  $\mathcal{B}'$  under  $z_\infty$  of the Borel  $\sigma$ -algebra of subsets of  $\Lambda$  can be extended to a  $\zeta$ -invariant  $\nu$  on  $\mathcal{B}$  the Borel  $\sigma$ -algebra of the subsets of  $\Sigma^d$  by using the fact that the set of at least triple points (limit points of at least three infinite branches of  $\mathcal{T}$ ) is countable, hence  $z_\infty^{-1}(x)$  of  $\mu$ -a.e.  $x$  contains at most 2 points. More precisely, let  $A_1$  be the set of points having one  $z_\infty$ -preimage,  $A_2$  two preimages. They are both  $f$ -invariant (except measure 0), so are their  $z_\infty$ -preimages  $A'_1$  and  $A'_2$  under  $\zeta$ . We extend  $\mu'$  by distributing conditional measures on the two points preimages of points in  $A_2$  half-half and Dirac on one point preimages. □

This allows to conclude [Theorem 9.1](#) (a part relying on CLT) and [Theorem 7.2](#) for equilibrium states for rational maps and Hölder potentials on  $J(f)$  by lifting  $\mu_\phi$  to  $\Sigma^d$  as in [Przytycki \[ibid.\]](#). However, this seems useless since the proof of CLT in [Przytycki \[ibid.\]](#) is done directly on  $J(f)$  (seemingly also for LIL, for which one should however refer to the proofs in [Przytycki, Urbański, and Zdunik \[1989\]](#)) and there are direct proofs of LIL in [Li and Rivera-Letelier \[2014b\]](#) and [Szostakiewicz, Urbański, and Zdunik \[2015\]](#).

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FELIKS PRZYTYCKI: INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8,  
00-656 WARSZAWA, POLAND

[feliksp@impan.gov.pl](mailto:feliksp@impan.gov.pl)

[feliksp@impan.pl](mailto:feliksp@impan.pl)