EXT-ANALOGUES OF BRANCHING LAWS

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Abstract

We consider the Ext-analogues of branching laws for representations of a group to its subgroups in the context of p-adic groups.

Contents

1 Introduction

Considering the restriction of representations of a group G to one of its subgroups H, say of $G = \mathrm{SO}_{n+1}(F)$ to $H = \mathrm{SO}_n(F)$ for a non-archimedean local field F has been a very

fruitful direction of research especially through its connections to questions on period integrals of automorphic representations, cf. Gan, Gross, and Prasad [2012] for the conjectural theory both locally and globally. The question for local fields amounts to understanding $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$ for irreducible admissible representations π_1 of $\mathrm{SO}_{n+1}(F)$, and π_2 of $\mathrm{SO}_n(F)$. The first result proved about this is the multiplicity one property which says that this space is at most one dimensional, cf. Aizenbud, Gourevitch, Rallis, and Schiffmann [2010], Sun and Zhu [2012]. It may be mentioned that before the full multiplicity one theorem was proved, even finite dimensionality of the space was not known. With multiplicity one theorem proved, one then goes on to prove more precise description of the set of irreducible admissible representations π_1 of $\mathrm{SO}_{n+1}(F)$, and π_2 of $\mathrm{SO}_n(F)$ with $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2] \neq 0$. These have now become available in a series of papers due to Waldspurger, and Moeglin-Waldspurger, cf. J.-L.Waldspurger [2010], J.-L.Waldspurger [2012b], J.-L.Waldspurger [2012a], Moeglin and J.-L.Waldspurger [2012]. There is also a recent series of papers by Beuzart-Plessis on similar questions for unitary groups, cf. Beuzart-Plessis [2016], Beuzart-Plessis [2015], Beuzart-Plessis [2014].

Given the interest in the space $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$, it is natural to consider the related spaces $\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$, and in fact homological algebra methods suggest that the simplest answers are not for these individual spaces, but for the alternating sum of their dimensions: $\mathrm{EP}[\pi_1, \pi_2] = \sum_{i=0}^{\infty}(-1)^i \dim \mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$; these hopefully more manageable objects –certainly more flexible– when coupled with vanishing of higher Ext's (when available) may give theorems about $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$. We hasten to add that before we can define $\mathrm{EP}[\pi_1, \pi_2]$, $\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$ needs to be proved to be finite dimensional for π_1 and π_2 finite length admissible representations of $\mathrm{SO}_{n+1}(F)$ and $\mathrm{SO}_n(F)$ respectively, and also proved to be 0 for i large. Vanishing of $\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$ for large i is a well-known generality to which we will come to later. Towards a proof of finite dimensionality of Ext^i in this case, to be made by an inductive argument on n later in the paper, we note that unlike $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$, where we will have no idea how to prove finite dimensionality if both π_1 and π_2 are cuspidal, exactly this case we can handle apriori, for $i > 0$, as almost by the very definition of cuspidal representations, they are both projective and injective objects in the category of smooth representations. Recently, there is a very general finiteness theorem for $\mathrm{Ext}^i[\pi_1, \pi_2]$ (for spherical varieties) in Aizenbud and Sayag [2017]. However, we have preferred to give our own older approach via Bessel models which intervene when analyzing principal series representations of $\mathrm{SO}_{n+1}(F)$ when restricted to $\mathrm{SO}_n(F)$. As a bonus, this approach gives explicit answers about Euler-Poincaré characteristics.

Thinking about Ext-analogues suggest interchanging the roles of π_1 and π_2 in analogy with the known relationship, $\mathrm{EP}[V_1, V_2^{\vee}] = \mathrm{EP}[V_2, V_1^{\vee}]$ when V_1 and V_2 are finite

length representations on the same group, and allows one to consider submodules as in $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_2, \pi_1]$, and more generally, $\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_2, \pi_1]$.

Based on various examples, a clear picture seems to be emerging about $\mathrm{Ext}^i_H[\pi_1, \pi_2]$. For example, we expect that when π_1 and π_2 are tempered, $\mathrm{Ext}^i_H[\pi_1, \pi_2]$ is nonzero only for $i = 0$. On the other hand we expect that $\mathrm{Ext}^i_H[\pi_2, \pi_1]$ is typically zero for $i = 0$ (so no wonder branching is usually not considered as a subrepresentation!), and shows up only for i equals the split rank of the center of the Levi from which π_2 arises through parabolic induction of a supercuspidal representation; in fact $\mathrm{Ext}^i_H[\pi_2, \pi_1]$ is zero beyond the split rank of the center of this Levi by generalities, so $\mathrm{Ext}^i_H[\pi_1, \pi_2]$ is typically nonzero only for $i = 0$, whereas $\mathrm{Ext}^i_H[\pi_2, \pi_1]$ is nonzero only for the largest possible i. We make precise some of these suggestions during the course of the paper, and discuss some examples as evidence for the suggested conjectures made here.

In the process of relating $\mathrm{Ext}^i[\pi_1, \pi_2]$ with $\mathrm{Ext}^i[\pi_2, \pi_1]$, we were led to a duality theorem for a general reductive group which turned out to be a consequence of the work in P.Schneider and U.Stuhler [1997]. It is the subject matter of Section 8. As an example, calculation of $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \pi_3]$ which was part of author's work in Prasad [1990], and simple calculations about $\mathrm{EP}_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \pi_3]$ allow the calculation of $\mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \pi_3]$, and then by the duality theorem, we are able to analyze $\mathrm{Hom}_{\mathrm{PGL}_2(F)}[\pi_3, \pi_1 \otimes \pi_2]$ (irreducible submodules of the tensor product).

In the archimedean case, several papers of T. Kobayashi, see e.g. Kobayashi [1994], do study the restriction problem for $(\mathfrak{g}, K)$-modules in the sense of sub-modules but the analogous restriction problem in the sense of sub-modules seems to be absent in the p-adic case. Proposition 5.1 and Proposition 5.3 suggest that $\mathrm{Hom}_H[\pi_2, \pi_1] = 0$ whenever π_1 is an irreducible tempered representation of G (assumed to be simple) unless H has compact center, and π_2 is a supercuspidal representation of it.

To summarize the main results of the paper, we might mention Theorem 4.2 giving a complete understanding of $\mathrm{EP}_{\mathrm{GL}_n(F)}[\pi_1, \pi_2]$ for π_1 and π_2 finite length representations of $\mathrm{GL}_{n+1}(F)$ and $\mathrm{GL}_n(F)$ respectively. Theorem 6.1 proves $\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$ to be finite dimensional for π_1 and π_2 finite length representations of $\mathrm{SO}_{n+1}(F)$ and $\mathrm{SO}_n(F)$ respectively, and as Corollary 6.3 of the proof, gives a good understanding of $\mathrm{EP}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$ when π_1 is a principal series. We formulate as Conjecture 5.1 the vanishing of $\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2]$ for $i > 0$ for generic representations, and Conjecture 7.1 suggests that the integral formula discovered by Waldspurger in the papers J.-L.Waldspurger [2010] and J.-L.Waldspurger [2012b] are actually for Euler-Poincaré characteristic of general finite length representations in the spirit of Kazhdan orthogonality. In Section 9 we suggest that all nontrivial Ext's have some 'geometric' origin.

2   Preliminaries

Given a connected reductive F-group G, we make the usual abuse of notation to also denote by G the locally compact totally disconnected group $G(F)$ of F-rational points of the algebraic group G. We denote by $\mathcal{R}(G)$ the abelian category of smooth representations of G over $\mathbb{C}$. The abelian category $\mathcal{R}(G)$ has enough projectives and enough injectives, e.g. for any compact open subgroup K of G, $\mathrm{ind}_K^G(\mathbb{C})$ is a projective object in $\mathcal{R}(G)$, and $\mathrm{Ind}_K^G(\mathbb{C})$ is an injective object in $\mathcal{R}(G)$ (we use throughout the paper ind for compactly supported induction and Ind for induction without compact support condition); in fact these projective objects and their direct summands, and their smooth duals as injective objects suffice for all considerations in the paper. Since $\mathcal{R}(G)$ has enough projectives and enough injectives, it is meaningful to talk about $\mathrm{Ext}_G^i[\pi_1, \pi_2]$ as the derived functors of $\mathrm{Hom}_G[\pi_1, \pi_2]$.

For reductive p-adic groups G considered in this paper, it is known that $\mathrm{Ext}_G^i[\pi, \pi']$ is zero for any two smooth representations π and π' of G when i is greater than the F-split rank of G. This is a standard application of the projective resolution of the trivial representation $\mathbb{C}$ of G provided by the building associated to G. For another proof of this, and for finite dimensionality of $\mathrm{Ext}_G^i[\pi, \pi']$, see Proposition 2.9 below.

For two smooth representations π and π' of G one can consider the Euler-Poincaré pairing $\mathrm{EP}_G[\pi, \pi']$ between π and π' defined by

$$\mathrm{EP}_G[\pi, \pi'] = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}_G^i[\pi, \pi'].$$

For this definition to make sense, we must prove that $\mathrm{Ext}_G^i[\pi, \pi']$ are finite-dimensional vector spaces over $\mathbb{C}$ for all integers i. An obvious remark which will be tacitly used throughout this paper is that if

$$0 \to \pi_1 \to \pi \to \pi_2 \to 0,$$

is an exact sequence of smooth G-modules, and if any two of the $\mathrm{EP}_G[\pi_1, \pi']$, $\mathrm{EP}_G[\pi, \pi']$, $\mathrm{EP}_G[\pi_2, \pi']$, make sense, then so does the third (finite dimensionality of the Ext groups, and zero beyond a stage), and

$$\mathrm{EP}_G[\pi, \pi'] = \mathrm{EP}_G[\pi_1, \pi'] + \mathrm{EP}_G[\pi_2, \pi'].$$

This remark will be used to break up representations π or π' in terms of simpler objects for which EP can be proved to make sense by reducing to smaller groups via some form of Frobenius recoprocity.

The following proposition summarizes some key properties of the Euler-Poincaré pairing, see P.Schneider and U.Stuhler [1997] for the proofs (part 2.1 (4) is known only in characteristic zero).

Proposition 2.1. *Let π and π' be finite-length, smooth representations of a reductive p-adic group G. Then:*

1. $\mathrm{EP}[\pi_1^\vee, \pi_2]$ *is a symmetric, $\mathbb{Z}$-bilinear form on the Grothendieck group of finite-length representations of G.*

2. EP *is locally constant. (A family $\{\pi_\lambda\}$ of representations on a fixed vector space V is said to* vary continuously *if all $\pi_\lambda|_K$ are all equivalent for some compact open subgroup K, and the matrix coefficients $\langle \pi_\lambda v, \tilde{v} \rangle$ vary continuously in λ.)*

3. $\mathrm{EP}_G[\pi, \pi'] = 0$ *if π or π' is induced from any proper parabolic subgroup in G.*

4. $\mathrm{EP}_G[\pi, \pi'] = \int_{C_{ellip}} \Theta(c) \bar{\Theta}'(c) \, dc$, *where Θ and Θ' are the characters of π and π' assumed to have the same unitary central character, and dc is a natural measure on the set C_{ellip} of regular elliptic conjugacy classes in G. (Note that if G has noncompact center, then both sides of this equality are zero; the right hand side being zero as there are no regular elliptic elements in G in that case, and the left hand side being zero by a simple argument.)*

Several assertions about Hom spaces can be converted into assertions about Ext^i. The following generality allows one to do so.

Proposition 2.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two abelian categories, and $\mathfrak{F}$ a functor from $\mathfrak{A}$ to $\mathfrak{B}$, and $\mathfrak{G}$ a functor from $\mathfrak{B}$ and $\mathfrak{A}$. Assume that $\mathfrak{G}$ is a left adjoint of $\mathfrak{F}$, i.e., there is a natural equivalence of functors:*

$$\mathrm{Hom}_{\mathfrak{B}}[X, \mathfrak{F}(Y)] \cong \mathrm{Hom}_{\mathfrak{A}}[\mathfrak{G}(X), Y].$$

Then,

1. *If $\mathfrak{F}$ and $\mathfrak{G}$ are exact functors, then $\mathfrak{F}$ maps injective objects of $\mathfrak{A}$ to injective objects of $\mathfrak{B}$, and $\mathfrak{G}$ maps projective objects of $\mathfrak{B}$ to projective objects of $\mathfrak{A}$.*

2. *If $\mathfrak{F}$ and $\mathfrak{G}$ are exact functors, then $\mathrm{Ext}^i_{\mathfrak{B}}[X, \mathfrak{F}(Y)] \cong \mathrm{Ext}^i_{\mathfrak{A}}[\mathfrak{G}(X), Y].$*

Proof. Part (1) of the Proposition follows directly from definitions; see Bernstein [1992], Proposition 8. For part (2), it suffices to note that if

$$\cdots P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0,$$

is a projective resolution of an object X in $\mathfrak{B}$, then by part (1) of the Proposition,

$$\cdots \mathcal{G}(P_n) \to \mathcal{G}(P_{n-1}) \to \cdots \to \mathcal{G}(P_1) \to \mathcal{G}(P_0) \to \mathcal{G}(X) \to 0,$$

is a projective resolution of $\mathcal{G}(X)$. Therefore part (2) of the proposition follows from the adjointness relationship between $\mathcal{F}$ and $\mathcal{G}$. $\square$

The following is a direct consequence of Frobenius reciprocity combined with Proposition 2.2.

Proposition 2.3. *Let H be a closed subgroup of a p-adic Lie group G. Then,*

1. *The restriction of any smooth projective representation of G to H is a projective object in $\mathcal{R}(H)$, and $\mathrm{Ind}_H^G U$ is an injective representation of G for any injective representation U of H.*

2. *For any smooth representation U of H, and V of G,*

$$\mathrm{Ext}_G^i[V, \mathrm{Ind}_H^G U] \cong \mathrm{Ext}_H^i[V, U].$$

Note that for any two smooth representations U, V of G,

$$\mathrm{Hom}_G[U, V^\vee] \cong \mathrm{Hom}_G[V, U^\vee],$$

where $U^\vee, V^\vee$ are the smooth duals of U, V respectively. Therefore we have adjoint functors as in Proposition 2.2 with $\mathcal{F} = \mathcal{G}$ to be the smooth dual from the category of smooth representations of a p-adic group G to its opposite category. By Proposition 2.2, it follows that the smooth dual of a projective object in $\mathcal{R}(G)$ is an injective object in $\mathcal{R}(G)$, and further we have the following proposition.

Proposition 2.4. *For a p-adic Lie group G, let U and V be two smooth representations of G. Then,*

$$\mathrm{Ext}_G^i[U, V^\vee] \cong \mathrm{Ext}_G^i[V, U^\vee].$$

Since the smooth dual of $\mathrm{ind}_H^G(U)$ is $\mathrm{Ind}_H^G(U^\vee)$ (for normalized induction), the previous two propositions combine to give:

Proposition 2.5. *For H a closed subgroup of a p-adic Lie group G, let U be a smooth representation of H, and V a smooth representation of G. Then,*

$$\mathrm{Ext}_H^i[V, U^\vee] \cong \mathrm{Ext}_G^i[V, \mathrm{Ind}_H^G(U^\vee)] \cong \mathrm{Ext}_G^i[\mathrm{ind}_H^G U, V^\vee].$$

For smooth representations U, V, W of G, the canonical isomorphism,

$$\mathrm{Hom}_G[V \otimes U, W^\vee] \cong \mathrm{Hom}_G[U \otimes W, V^\vee],$$

translates into the following proposition by Proposition 2.2.

Proposition 2.6. *For a p-adic Lie group G, and U, V, W smooth representations of G, there are canonical isomorphisms,*

$$\mathrm{Ext}_G^i[V \otimes U, W^\vee] \cong \mathrm{Ext}_G^i[U \otimes W, V^\vee],$$

in particular

$$\mathrm{Ext}_G^i[V \otimes U, \mathbb{C}] \cong \mathrm{Ext}_G^i[V, U^\vee].$$

Proposition 2.2 with the form of Frobenius reciprocity for Jacquet modules, implies the following proposition.

Proposition 2.7. *For P a parabolic subgroup of a reductive p-adic group G with Levi decomposition $P = MN$, the Jacquet functor $V \to V_N$ from $\mathcal{R}(G)$ to $\mathcal{R}(M)$ takes projective objects to projective objects, and for $V \in \mathcal{R}(G)$, $U \in \mathcal{R}(M)$, we have (using normalized parabolic induction and normalized Jacquet module),*

$$\mathrm{Ext}_G^i[V, \mathrm{Ind}_P^G U] \cong \mathrm{Ext}_M^i[V_N, U].$$

The proof of the following proposition is exactly as the proof of the earlier proposition. This proposition will play an important role in setting-up an inductive context to prove theorems on a group G in terms of similar theorems for subgroups.

Proposition 2.8. *For P a (not necessarily parabolic) subgroup of a reductive p-adic group G with Levi decomposition $P = MN$, let ψ be a character of N normalized by M. Then for any irreducible representation μ of M, one can define a representation of P, denoted by $\mu \cdot \psi$ which when restricted to M is μ, and when restricted to N is ψ. For any smooth representation V of G, let $V_{N,\psi}$ be the twisted Jacquet module of V with respect to the character ψ of N which is a smooth representation of M. Then,*

$$\mathrm{Ext}_G^i[\mathrm{ind}_P^G(\mu \cdot \psi), V^\vee] \cong \mathrm{Ext}_M^i[V_{N,\psi}, \mu^\vee].$$

The following much deeper result than these earlier results follows from the so called *Bernstein's second adjointness theorem*.

Theorem 2.1. *For P a parabolic subgroup of a reductive p-adic group G with Levi decomposition $P = MN$, let U be a smooth representation of M thought of as a representation of P, and V a smooth representation of G. Let $P^- = MN^-$ be the parabolic*

opposite to $P = MN$. Then we have (using normalized parabolic induction and normalized Jacquet module),

$$\mathrm{Ext}^i_G[\mathrm{Ind}^G_P U, V] \cong \mathrm{Ext}^i_M[U, V_{N-}].$$

As a sample of arguments with the Ext groups, we give a proof of the following basic proposition.

Proposition 2.9. *Suppose that V is a smooth representation of G of finite length, and that all of its irreducible subquotients are subquotients of representations induced from supercuspidal representations of a Levi factor of the standard parabolic subgroup $P = MU$ of G, defined by a subset Θ of the set of simple roots for a maximal split torus of G. Then if V' is a finite length smooth representation of G, $\mathrm{Ext}^i_G[V, V']$ and $\mathrm{Ext}^i_G[V', V]$ are finite dimensional vector spaces over $\mathbb{C}$. If V' is any smooth representation of G, $\mathrm{Ext}^i_G[V, V'] = \mathrm{Ext}^i_G[V', V] = 0$ for $i > d(M) = d - |\Theta|$ where d is the F-split rank of G. Further, $\mathrm{EP}_G[\pi, \pi'] = 0$ if both π, π' are finite length representations of G, and π or π' is induced from a proper parabolic subgroup of G.*

Proof. This is Corollary III.3.3 in P.Schneider and U.Stuhler [1997]. Since this is elementary enough, we give another proof.

We begin by noting that tensoring V by the resolution of $\mathbb{C}$ by projective modules in $\mathcal{R}(G)$ afforded by the building associated to G gives a projective resolution of V, but one which is not finitely generated as a G-module even if V is irreducible, and therefore proving finite dimensionality of $\mathrm{Ext}^i_G[V, V']$ requires some work. The resolution given by the building at least proves that these are 0 beyond the split rank of G. Our proof below first proves the assertions on $\mathrm{Ext}^i_G[V, V']$ if V or V' is a full principal series $\mathrm{Ind}^G_P \rho$ where ρ is a cuspidal representation of M, and then handles all subquotients by a standard *dévissage*.

Fix a surjective map $\phi : M \to \mathbb{Z}^{d(M)}$ with kernel M^ϕ which is sometimes called the subgroup of M generated by compact elements.

Let ρ be a cuspidal representation of M. Therefore ρ restricted to M^ϕ, which is $[M, M]$ up to a compact group, is an injective module, and hence $\mathrm{Ext}^i_{M^\phi}[V_N, \rho] = 0$ for $i > 0$. By Frobenius reciprocity, combined with the spectral sequence associated to the normal subgroup M^ϕ of M with quotient $\mathbb{Z}^d$, it follows that:

$$\mathrm{Ext}^i_G[V, \mathrm{Ind}^G_P \rho] \cong \mathrm{Ext}^i_M[V_N, \rho]$$
$$\cong H^i(\mathbb{Z}^{d(M)}, \mathrm{Hom}_{M^\phi}[V_N, \rho]).$$

This proves that $\mathrm{Ext}^i_G[V, \mathrm{Ind}^G_P \rho] = 0$ for $i > d(M)$ for any smooth representation V of G, and that $\mathrm{Ext}^i_G[V, \mathrm{Ind}^G_P \rho]$ are finite dimensional for V of finite length.

Similarly, by the second adjointness theorem, it follows that $\mathrm{Ext}_G^i[\mathrm{Ind}_P^G(\rho), V'] = 0$ for $i > d(M)$, and that $\mathrm{Ext}_G^i[\mathrm{Ind}_P^G(\rho), V']$ are finite dimensional for V' a finite length representation in $\mathfrak{R}(G)$.

Having proved properties of $\mathrm{Ext}_G^i[V, \mathrm{Ind}_P^G \rho]$ and $\mathrm{Ext}_G^i[\mathrm{Ind}_P^G(\rho), V']$, the rest of the proposition about $\mathrm{Ext}_G^i[V, V']$ follows by *dévissage* by writing an irreducible representation V of G as a quotient of a principal series $Ps = \mathrm{Ind}_P^G \rho$, and using conclusions on the principal series to make conclusions on V. This part of the argument is very similar to what we give in Lemma 6.1, so we omit it here. □

3 Kunneth Theorem

In this section, we prove a form of the Kunneth theorem which we will have several occasions to use. A version of Kunneth's theorem is there in Raghuram [2007] assuming, however, finite length conditions on both E_1 and E_2 which is not adequate for our applications.

During the course of the proof of the Kunneth Theorem, we will need to use the following most primitive form of Frobenius reciprocity.

Lemma 3.1. *Let K be an open subgroup of a p-adic group G. Let E be a smooth representation of K, and F a smooth representation of G. Then,*

$$\mathrm{Hom}_G[\mathrm{ind}_K^G E, F] \cong \mathrm{Hom}_K[E, F].$$

Theorem 3.1. *Let G_1 and G_2 be two p-adic groups. Let E_1, F_1 be any two smooth representations of G_1, and E_2, F_2 be any two smooth representations of G_2. Then assuming that G_1 is a reductive p-adic group, and E_1 has finite length, we have*

$$\mathrm{Ext}_{G_1 \times G_2}^i[E_1 \boxtimes E_2, F_1 \boxtimes F_2] \cong \bigoplus_{i=j+k} \mathrm{Ext}_{G_1}^j[E_1, F_1] \otimes \mathrm{Ext}_{G_2}^k[E_2, F_2].$$

Proof. If P_1 is a projective module for G_1, and P_2 a projective module for G_2, then $P_1 \boxtimes P_2$ is a projective module for $G_1 \times G_2$.

Let

$$\cdots \to P_1 \to P_0 \to E_1 \to 0,$$

$$\cdots \to Q_1 \to Q_0 \to E_2 \to 0,$$

be a projective resolution for E_1 as a G_1-module, and a projective resolution for E_2 as a G_2-module.

It follows that the tensor product of these two exact sequences:

$$\cdots \to P_1 \boxtimes Q_0 + P_0 \boxtimes Q_1 \to P_0 \boxtimes Q_0 \to E_1 \boxtimes E_2 \to 0,$$

is a projective resolution of $E_1 \boxtimes E_2$. Therefore, $\mathrm{Ext}^i_{G_1 \times G_2}[E_1 \boxtimes E_2, F_1 \boxtimes F_2]$ can be calculated by taking the cohomology of the chain complex $\mathrm{Hom}_{G_1 \times G_2}[\bigoplus_{i+j=k} P_i \boxtimes Q_j, F_1 \boxtimes F_2]$.

It is possible to choose a projective resolution of E_1 by $P_i = \mathrm{ind}^{G_1}_{K_i} W_i$ for finite dimensional representations W_i of compact open subgroups K_i of G_1. The existence of such a projective resolution is made possible through the construction of an equivariant sheaf on the Bruhat-Tits building of G_1 associated to the representation E_1, cf. P.Schneider and U.Stuhler [1997]. this is the step which needs G_1 to be reductive, and also requires the admissibility of E_1.

Since W_i are finite dimensional, we have the isomorphism

$$\mathrm{Hom}_{K \times G_2}[W_i \boxtimes Q_j, F_1 \boxtimes F_2] \cong \mathrm{Hom}_K[W_i, F_1] \otimes \mathrm{Hom}_{G_2}[Q_j, F_2],$$

therefore,

$$
\begin{aligned}
\mathrm{Hom}_{G_1 \times G_2}[P_i \boxtimes Q_j, F_1 \boxtimes F_2] &= \mathrm{Hom}_{G_1 \times G_2}[\mathrm{ind}^{G_1}_K(W_i) \boxtimes Q_j, F_1 \boxtimes F_2] \\
&\cong \mathrm{Hom}_{K \times G_2}[W_i \boxtimes Q_j, F_1 \boxtimes F_2] \\
&\cong \mathrm{Hom}_K[W_i, F_1] \otimes \mathrm{Hom}_{G_2}[Q_j, F_2] \\
&\cong \mathrm{Hom}_{G_1}[P_i, F_1] \otimes \mathrm{Hom}_{G_2}[Q_j, F_2].
\end{aligned}
$$

Thus we are able to identify the chain complex $\mathrm{Hom}_{G_1 \times G_2}[\bigoplus_{i+j=k} P_i \boxtimes Q_j, F_1 \boxtimes F_2]$ as the tensor product of the chain complexes $\mathrm{Hom}_{G_1}[P_i, F_1]$ and $\mathrm{Hom}_{G_2}[Q_j, F_2]$. Now the abstract Kunneth theorem which calculates the cohomology of the tensor product of two chain complexes in terms of the cohomology of the individual chain complexes completes the proof of the theorem.   □

4   Branching laws from $\mathrm{GL}_{n+1}(F)$ to $\mathrm{GL}_n(F)$

We begin by recalling the following basic result in this context, cf. Prasad [1993].

Theorem 4.1. *Given an irreducible generic representation π_1 of $\mathrm{GL}_{n+1}(F)$, and an irreducible generic representation π_2 of $\mathrm{GL}_n(F)$,*

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = \mathbb{C}.$$

The aim of this section is to prove the following theorem which can be considered as the Euler-Poincaré version of Theorem 4.1.

Theorem 4.2. *Let π_1 be an admissible representation of $\mathrm{GL}_{n+1}(F)$ of finite length, and π_2 an admissible representation of $\mathrm{GL}_n(F)$ of finite length. Then, $\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2]$ are*

finite dimensional vector spaces over $\mathbb{C}$, *and*

$$\mathrm{EP}_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = \dim \mathrm{Wh}(\pi_1) \cdot \dim \mathrm{Wh}(\pi_2),$$

where $\mathrm{Wh}(\pi_1)$, *resp.* $\mathrm{Wh}(\pi_2)$, *denotes the space of Whittaker models for* π_1, *resp.* π_2, *with respect to fixed non-degenerate characters on the maximal unipotent subgroups in* $\mathrm{GL}_{n+1}(F)$ *and* $\mathrm{GL}_n(F)$.

The proof of this theorem will be accomplished using some results of Bernstein and Zelevinsky regarding the structure of representations of $\mathrm{GL}_{n+1}(F)$ restricted to the mirabolic subgroup.

Denote by E_n the mirabolic subgroup of $\mathrm{GL}_{n+1}(F)$ consisting of matrices whose last row is equal to $(0, 0, \cdots, 0, 1)$ and let N_{n+1} be the group of upper triangular unipotent matrices in $\mathrm{GL}_{n+1}(F)$. We will be using subgroups $\mathrm{GL}_i(F)$ of $\mathrm{GL}_{n+1}(F)$ for $i \leq n+1$ always sitting at the upper left corner of $\mathrm{GL}_{n+1}(F)$. We fix a nontrivial character ψ_0 of F and let ψ_{n+1} be the character of N_{n+1} given by

$$\psi_{n+1}(u) = \psi_0(u_{1,2} + u_{2,3} + \cdots + u_{n,n+1}).$$

For a representation π of $\mathrm{GL}_{n+1}(F)$, let

$$\pi^i = \text{ the } i\text{-th derivative of } \pi,$$

which is a representation of $\mathrm{GL}_{n+1-i}(F)$. It will be important for us to note that π^i are representations of finite length of $\mathrm{GL}_{n+1-i}(F)$ if π is of finite length for $\mathrm{GL}_{n+1}(F)$.

To recall the definition of π^i, let $R_{n+1-i} = \mathrm{GL}_{n+1-i}(F) \cdot V_i$ be the subgroup of $\mathrm{GL}_{n+1}(F)$ consisting of matrices

$$\begin{pmatrix} g & v \\ 0 & z \end{pmatrix}$$

with $g \in \mathrm{GL}_{n+1-i}(F)$, $v \in M_{(n+1-i)\times i}$, $z \in N_i$. If the character ψ_i of N_i is extended to V_i by extending it trivially across $M_{(n+1-i)\times i}$, then we have

$$\pi^i = \pi_{V_i, \psi_i},$$

where π_{V_i, ψ_i} is the twisted Jacquet module of π, i.e., the maximal quotient of π on which V_i operates via the character ψ_i.

Here is a generality from Bernstein and Zelevinsky [1976], §3.5.

Proposition 4.1. *Any smooth representation* Σ *of* E_n *has a natural filtration by* $E = E_n$ *modules*

$$0 = \Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_{n+1} = \Sigma$$

such that

$$\Sigma_{i+1}/\Sigma_i = \text{ind}_{R_i}^{E_n}(\Sigma^{n+1-i} \otimes \psi_{n+1-i}) \quad \textit{for } i = 0, \cdots, n,$$

where $R_i = \text{GL}_i(F) \cdot V_{n+1-i}$ *is the subgroup of* $\text{GL}_{n+1}(F)$ *consisting of*

$$\begin{pmatrix} g & v \\ 0 & z \end{pmatrix}$$

with $g \in \text{GL}_i(F)$, $v \in M_{i \times (n+1-i)}$, $z \in N_{n+1-i}$, *and the character* ψ_{n+1-i} *on* N_{n+1-i} *is extended to* V_{n+1-i} *by extending it trivially across* $M_{i \times (n+1-i)}$.

The proof of the following proposition is a direct consequence of Proposition 2.8.

Proposition 4.2. *For a smooth representation* π_1 *of* $\text{GL}_{n+1}(F)$, *and* π_2 *of* $\text{GL}_n(F)$,

$$\text{Ext}_{\text{GL}_n(F)}^j[\text{ind}_{R_i}^{\text{GL}_n(F)}(\pi_1^{n-i+1} \otimes \psi_{n-i}), \pi_2^\vee] = \text{Ext}_{\text{GL}_i(F)}^j[\pi_2^{n-i}, (\pi_1^{n-i+1})^\vee].$$

Proof. Since $\text{GL}_n(F) \cdot R_i = E_n$, $\text{GL}_n(F) \cap R_i = \text{GL}_i(F) \cdot V_{n-i}$, the restriction of $\pi_1^{n+1-i} \otimes \psi_{n+1-i}$ from R_i to $\text{GL}_n(F) \cap R_i = \text{GL}_i(F) \cdot V_{n-i}$ is $\pi_1^{n+1-i} \otimes \psi_{n-i}$ for any $i, 0 \le i \le n$. Therefore, the proposition follows from Proposition 2.8.   □

Lemma 4.1. *For any two smooth representations* V_1, V_2 *of* $\text{GL}_n(F)$, $n \ge 1$, *of finite length,*

$$\text{EP}_{\text{GL}_n(F)}[V_1, V_2] = 0.$$

Proof. It suffices to prove the lemma assuming that both V_1 and V_2 are irreducible representations of $\text{GL}_n(F)$. If the two representations V_1, V_2 were irreducible, and had different central characters, then clearly $\text{Ext}_{\text{GL}_n(F)}^i[V_1, V_2] = 0$ for all integers i. On the other hand, we know by Proposition 2.1(b) that for representations V_1 and V_2 of finite length of $\text{GL}_n(F)$, $\text{EP}_{\text{GL}_n(F)}[V_1, V_2]$ is constant in a connected family, so denoting by ν the character $\nu(g) = |\det g|$ of $\text{GL}_n(F)$, we have $\text{EP}_{\text{GL}_n(F)}[V_1, V_2] = \text{EP}_{\text{GL}_n(F)}[\nu^s \cdot V_1, V_2]$ for all $s \in \mathbb{C}$. Choosing s appropriately, we can change the central character of $\nu^s \cdot V_2$ to be different from V_1, and hence $\text{EP}_{\text{GL}_n(F)}[V_1, V_2] = \text{EP}_{\text{GL}_n(F)}[\nu^s \cdot V_1, V_2] = 0$.   □

Proof of Theorem 4.2: Since the Euler-Poincaré characteristic is additive in exact sequences, it suffices to calculate $\text{EP}_{\text{GL}_n(F)}[\text{ind}_{R_i}^{\text{GL}_n(F)}(\pi_1^{n-i} \otimes \psi_{n-i+1}), \pi_2]$ which by Proposition 2.8 is $\text{EP}_{\text{GL}_i(F)}[(\pi_2^{n-i})^\vee, (\pi_1^{n-i+1})^\vee]$, which by Lemma 4.1 above is 0 unless $i = 0$. (Note that in $\text{EP}_{\text{GL}_i(F)}[(\pi_2^{n-i})^\vee, (\pi_1^{n-i+1})^\vee]$, both the representations involved are admissible representations of $\text{GL}_i(F)$.) For $i = 0$, note that we are dealing with $\text{GL}_0(F) = 1$, and the representations involved are $(\pi_1^\vee)^{n+1}$ and $(\pi_2^\vee)^n$, which are nothing but the space of Whittaker models of $\pi_1^\vee$ and $\pi_2^\vee$. Since for representations V_1, V_2 of the group $\text{GL}_0(F) = 1$, $\text{EP}[V_1, V_2] = \dim \text{Hom}[V_1, V_2] = \dim V_1 \cdot \dim V_2$, this completes the proof of the theorem.

Remark 4.1. One knows, cf. Prasad [1993], that there are irreducible generic representations of $\mathrm{GL}_3(F)$ which have the trivial representation of $\mathrm{GL}_2(F)$ as a quotient; similarly, there are irreducible nongeneric representations of $\mathrm{GL}_3(F)$ with irreducible generic representations of $\mathrm{GL}_2(F)$ as a quotient. For such pairs (π_1, π_2) of representations, it follows from Theorem 4.2 that $\mathrm{EP}_{\mathrm{GL}_2(F)}[\pi_1, \pi_2] = 0$, whereas $\mathrm{Hom}_{\mathrm{GL}_2(F)}[\pi_1, \pi_2] \neq 0$. Therefore, for such pairs (π_1, π_2) of irreducible representations, we must have $\mathrm{Ext}^i_{\mathrm{GL}_2(F)}[\pi_1, \pi_2] \neq 0$, for some $i > 0$.

5   Conjectural vanishing of Ext groups for generic representations

The following conjecture seems to be at the root of why the simple and general result of previous section on Euler-Poincaré characteristic translates into a simple result about Hom spaces for generic representations. The author has not managed to prove it in any generality. There is a recent preprint by Chan and Savin, cf. Chan and Savin [2017] dealing with some cases of this conjecture.

Conjecture 5.1. *Let π_1 be an irreducible generic representation of $\mathrm{GL}_{n+1}(F)$, and π_2 an irreducible generic representation of $\mathrm{GL}_n(F)$. Then,*

$$\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = 0, \quad \text{for all} \quad i > 0.$$

Remark 5.1. By Remark 4.1, one cannot remove the genericity condition for either π_1 or π_2 in the above conjecture. In particular, one cannot expect that a generic representation of $\mathrm{GL}_{n+1}(F)$ when restricted to $\mathrm{GL}_n(F)$ is a projective representation in $\Re(\mathrm{GL}_n(F))$ although this is the case for supercuspidal representations of $\mathrm{GL}_{n+1}(F)$. The paper Chan and Savin [ibid.] proves that the part of the Steinberg representation of $\mathrm{GL}_{n+1}(F)$ (denoted St_{n+1}) in the Iwahori component of the Bernstein decomposition for $\Re(GL_n(F))$ is a projective module. There is no doubt then that St_{n+1} when restricted to $\mathrm{GL}_n(F)$ is a projective representation in $\Re(\mathrm{GL}_n(F))$, therefore

$$\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\mathrm{St}_{n+1}, \pi_2] = 0 \quad \text{for} \quad i > 0$$

for any irreducible representation π_2 of $\mathrm{GL}_n(F)$. As a consequence, it will follow from the duality theorem of Schneider-Stuhler, cf. Theorem 8.1 below, that St_{n+1} contains no irreducible submodule of $\mathrm{GL}_n(F)$.

Towards checking the validity of this conjecture in some cases, note that by Theorem 4.1 and Theorem 4.2, under the hypothesis of the conjecture,

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = 1, \quad \text{and} \quad \mathrm{EP}[\pi_1, \pi_2] = 1.$$

It follows that if we already knew that $\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = 0, \ i > 1$, then we will also know that, $\mathrm{Ext}^1_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = 0$, and the conjecture will be proved for such representations.

It is easy to see that if π_1 or π_2 is cuspidal, then $\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = 0$ for $i > 1$. We do one slightly less obvious case when π_1 arises as a subquotient of a principal series representation induced from a cuspidal representation of a maximal parabolic in $\mathrm{GL}_{n+1}(F)$.

It follows from Corollary III.3.3(i) of P. Schneider and U. Stuhler [1997] that if π_1 arises from a cuspidal representation of a maximal parabolic in $\mathrm{GL}_{n+1}(F)$, it has a projective resolution of length 1 in the category $\mathcal{R}_\chi(G)$ of smooth representations of $G = \mathrm{GL}_{n+1}(F)$ with central character χ. It is easy to see that a projective module in $\mathcal{R}_\chi(\mathrm{GL}_{n+1}(F))$ when considered as a representation of $\mathrm{GL}_n(F)$ is a projective module in $\mathcal{R}(\mathrm{GL}_n(F))$, cf. Proposition 3.2 in M. Nori and Prasad [2017]. This proves vanishing of $\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = 0$ for $i > 1$, hence also of $\mathrm{Ext}^1_{\mathrm{GL}_n(F)}[\pi_1, \pi_2]$.

This takes care of $G = \mathrm{GL}_{n+1}(F)$ for $n+1 \leq 3$, except that for $\mathrm{GL}_3(F)$ if both π_1 and π_2 arise as components of principal series representations induced from their Borel subgroups then there is a possibility of having nontrivial $\mathrm{Ext}^2_{\mathrm{GL}_2(F)}[\pi_1, \pi_2]$. By the duality theorem of Schneider-Stuhler, cf. Theorem 8.1 below, we will have, $\mathrm{Hom}_{\mathrm{GL}_2(F)}[D\pi_2, \pi_1] \neq 0$ (where $D\pi_2$ is the Aubert-Zelevinsky involution of π_2). The following proposition takes care of this.

Proposition 5.1. *Let π_1 be an irreducible generic representation of $\mathrm{GL}_3(F)$, and π_2 any irreducible representation of $\mathrm{GL}_2(F)$ which is not a twist of the Steinberg representation of $\mathrm{GL}_2(F)$. Then*

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}[\pi_2, \pi_1] = 0.$$

We will not prove this proposition here but discuss two propositions which deal with all but a few cases of the proposition above. The cases left out by the next two propositions can be handled by the Mackey restriction of an explicit principal series (especially using that different inducing data can give rise to the same principal series).

Proposition 5.2. *Let $H_1 \subset H$ be p-adic groups with $Z = F^\times$ contained in the center of H with $Z \cap H_1 = \{1\}$. Suppose μ is a smooth representation of H_1, and π_2 an irreducible admissible representation of H. Then $\mathrm{Hom}_H[\pi_2, \mathrm{ind}^H_{H_1}(\mu)] = 0$.*

Proof. Note that for each $x \in H/H_1$, restriction of functions from H to $(Z = F^\times) \cdot x$ gives rise to $F^\times$-equivariant maps

$$\mathrm{ind}^H_{H_1}(\mu) \longrightarrow \mathcal{S}(F^\times),$$

which can be assumed to be nonzero for any $f \in \mathrm{ind}^H_{H_1}(\mu)$ by choosing $x \in H/H_1$ appropriately.

Therefore if $\mathrm{Hom}_H[\pi_2, \mathrm{ind}_{H_1}^H(\mu)] \neq 0$ choosing $x \in H/H_1$ appropriately, we get a nonzero map from $\pi_2 \longrightarrow \mathcal{S}(F^\times)$ which is ω-equivariant where ω is the central character of π_2. Since $\mathcal{S}(F^\times)$ has no functions on which $F^\times$ operates by a character, the proof of the proposition is complete. □

This lemma when combined with the Bernstein-Zelevinsky filtration in Proposition 4.1 has the following as an immediate consequence.

Proposition 5.3. *Let π_1 be any smooth representation of $\mathrm{GL}_{n+1}(F)$ of finite length, and π_2 any irreducible representation of $\mathrm{GL}_n(F)$. Then if*

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}[\pi_2, \pi_1] \neq 0,$$

then π_2 appears a submodule of $J_{n,1}(\pi_1)$ where $J_{n,1}$ denotes the un-normalized Jacquet module with respect to the $(n,1)$ parabolic in $\mathrm{GL}_{n+1}(F)$ considered as a module for $\mathrm{GL}_n(F) \subset \mathrm{GL}_n(F) \times \mathrm{GL}_1(F)$; in particular, if $J_{n,1}(\pi_1) = 0$, then there are no nonzero $\mathrm{GL}_n(F)$-submodules in π_1.

Proof. The proof of the proposition is an immediate consequence of the observation that the Bernstein-Zelevinsky filtration in Proposition 4.1 when restricted to $\mathrm{GL}_n(F)$ gives rise to representations of $\mathrm{GL}_n(F)$ induced from subgroups $H_i \subset \mathrm{GL}_n(F)$ with $H_i \cap \{Z(\mathrm{GL}_n(F)) = F^\times\} = \{1\}$ except in the case when $H_i = \mathrm{GL}_n(F)$ which corresponds to the Jacquet module $J_{n,1}(\pi_1)$. □

6 Finite dimensionality of Ext groups

In this section we prove the finite dimensionality of Ext-groups in the case of $\mathrm{SO}_n(F) \subset \mathrm{SO}_{n+1}(F)$. The proof will have an inductive structure, and will involve Bessel models in the inductive step, so we begin by recalling the concept of Bessel models.

Let $V = X + D + W + Y$ be a quadratic space over the non-archimedean local field F with X and Y totally isotropic subspaces of V in duality with each other under the underlying bilinear form, D an anisotropic line in V, and W a quadratic subspace of V. Suppose that the dimension of X is k; fix a complete flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \cdots, e_k \rangle = X$ of isotropic subspaces in X. Let $P = MU$ be the parabolic subgroup in $\mathrm{SO}(V)$ stabilizing this flag, with $M = \mathrm{GL}_1(F)^k \times \mathrm{SO}(D+W)$. For $W \subset V$ a codimension $2k+1$ subspace as above, the subgroup $\mathrm{SO}(W) \cdot U$ which is uniquely defined up to conjugacy by $\mathrm{SO}(V)$ makes frequent appearance in this work, as well as in other works on classical groups. We call this subgroup as the *Bessel subgroup*, and denote it as $\mathrm{Bes}(V, W) = \mathrm{SO}(W) \cdot U$.

Let $P_X = M_X \cdot U_X$ be the maximal parabolic of $\mathrm{SO}(V)$ stabilizing X. We have $M_X \cong \mathrm{GL}(X) \cdot \mathrm{SO}(W + D)$, and U_X sits in the exact sequence,

$$1 \to \Lambda^2 X \to U_X \to X \otimes (D + W) \to 1.$$

Let $\ell : U \to F$ be a linear form such that

1. its restriction to each of the simple root spaces in $\mathrm{GL}(X)$ defined by the flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \cdots, e_k \rangle = X$ of isotropic subspaces in X is non-trivial;

2. its restriction to the unipotent radical of the parabolic $P_X = M_X U_X$ in $\mathrm{SO}(V)$ stabilizing X is trivial on the subgroup of U_X which is $\Lambda^2 X$;

3. and on the quotient of U_X by $\Lambda^2 X$ which can be identified to $(D + W) \otimes X$, ℓ is given by the tensor product of a linear form on $D + W$ which is trivial on W, and a linear form on X which is trivial on the subspace $\langle e_1, e_2, \cdots, e_{k-1} \rangle$.

Composing the linear form $\ell : U \to F$ with a nontrivial character $\psi_0 : F \to \mathbb{C}^\times$, we get a character $\psi : U \to \mathbb{C}^\times$. This character $\psi : U \to \mathbb{C}^\times$ depends only on $W \subset V$ a nondegenerate subspace of V of odd codimension, such that the quadratic space V/W is split, and is independent of all choices made along the way (including that of the character ψ_0). The character ψ of U is invariant under $\mathrm{SO}(W)$. For any representation σ of $\mathrm{SO}(W)$, $\mathrm{Bes}(V, W) = \mathrm{SO}(W) \cdot U$ comes equipped with the representation which is σ on $\mathrm{SO}(W)$, and ψ on U; since ψ is fixed when considering representations of $\mathrm{Bes}(V, W) = \mathrm{SO}(W) \cdot U$, we denote this representation of $\mathrm{Bes}(V, W) = \mathrm{SO}(W) \cdot U$ as σ itself or sometimes as $\sigma \otimes \psi$.

The Bessel models of a smooth representation π of $\mathrm{SO}(V)$ are irreducible admissible representations σ of $\mathrm{SO}(W)$ such that the following isomorphic spaces are nonzero:

$$\mathrm{Hom}_{\mathrm{Bes}(V,W)}[\pi, \sigma] \cong \mathrm{Hom}_{\mathrm{SO}(V)}\left[\pi, \mathrm{Ind}_{\mathrm{Bes}(V,W)}^{\mathrm{SO}(V)}(\sigma) \right]$$

$$\cong \mathrm{Hom}_{\mathrm{SO}(V)}\left[\mathrm{ind}_{\mathrm{Bes}(V,W)}^{\mathrm{SO}(V)}(\sigma^\vee), \pi^\vee \right].$$

When W is a codimension one subspace of V, then $\mathrm{Bes}(V, W) = \mathrm{SO}(W)$, and the notion of a Bessel model is simply that of restriction from $\mathrm{SO}(V)$ to $\mathrm{SO}(W)$, whereas when $\dim(W) = 0, 1$, then the notion of a Bessel model is nothing but that of the Whittaker model (for a particular character of the maximal unipotent subgroup of $\mathrm{SO}(V)$ if $\dim(W) = 1$).

We can define the higher Ext versions of the Bessel models as one of the following isomorphic spaces:

$$\mathrm{Ext}^i_{\mathrm{Bes}(V,W)}[\pi, \sigma] \cong \mathrm{Ext}^i_{\mathrm{SO}(V)} \left[\pi, \mathrm{Ind}^{\mathrm{SO}(V)}_{\mathrm{Bes}(V,W)}(\sigma) \right]$$

$$\cong \mathrm{Ext}^i_{\mathrm{SO}(V)} \left[\mathrm{ind}^{\mathrm{SO}(V)}_{\mathrm{Bes}(V,W)}(\sigma^\vee), \pi^\vee \right].$$

The following proposition whose proof we will omit is analogous to that of Theorem 15.1 of Gan, Gross, and Prasad [2012]. It allows one to prove finite dimensionality of $\mathrm{Ext}^i_{\mathrm{Bes}(V,W_0)}[\pi, \sigma]$ if we know the finite dimensionality of $\mathrm{Ext}^i_{\mathrm{SO}(W)}[\pi, \sigma']$ where W is a codimension one subspace in the quadratic space V, and σ' an irreducible representation of $\mathrm{SO}(W)$.

Proposition 6.1. *Let $W \subset V$ be a nondegenerate quadratic subspace of codimension 1 over a non-archimedean local field F. Suppose that*

$$W = Y_k \oplus W_0 \oplus Y_k^\vee$$

and

$$V = Y_k \oplus V_0 \oplus Y_k^\vee,$$

with Y_k and $Y_k^\vee$ isotropic subspaces and $W_0 \subset V_0$ nondegenerate quadratic spaces with W_0 a subspace of codimension one in V_0. Let $P_W(Y_k)$ be the parabolic in $\mathrm{SO}(W)$ stabilizing Y_k with Levi subgroup

$$M = \mathrm{GL}(Y_k) \times \mathrm{SO}(W_0)$$

For an irreducible supercuspidal representation τ of $\mathrm{GL}(Y_k)$ and an irreducible admissible representation π_0 of $\mathrm{SO}(W_0)$, let

$$\tau \rtimes \pi_0 = \mathrm{Ind}^{\mathrm{SO}(W)}_{P_W(Y_k)}(\tau \boxtimes \pi_0)$$

be the corresponding (un-normalized) principal series representation of $\mathrm{SO}(W)$. Let π be an irreducible admissible representation $\mathrm{SO}(V)$ which does not belong to the Bernstein component associated to $(\mathrm{GL}(Y_k) \times \mathrm{SO}(V_0), \tau \boxtimes \mu)$ for any irreducible representation μ of $\mathrm{SO}(V_0)$. Then

$$\mathrm{Ext}^i_{\mathrm{SO}(W)}[\pi, \tau \rtimes \pi_0] \cong \mathrm{Ext}^i_{\mathrm{Bes}(V,W_0)}[\pi, \pi_0].$$

Corollary 6.1. *With the notation as above, if $\mathrm{Ext}^i_{\mathrm{SO}(W)}[\pi, \tau \rtimes \pi_0]$ are finite dimensional, then so are $\mathrm{Ext}^i_{\mathrm{Bes}(V,W_0)}[\pi, \pi_0]$.*

Proof. It suffices to observe that given π_0, there is a representation π of $\mathrm{SO}(V)$ which does not belong to the Bernstein component associated to $(\mathrm{GL}(Y_k) \times \mathrm{SO}(V_0), \tau \boxtimes \mu)$ for any irreducible representation μ of $\mathrm{SO}(V_0)$. $\square$

We now come to the proof of finite dimensionality of the Ext groups.

Theorem 6.1. *Let $V = X + D + W + Y$ be as at the beginning of the section, a quadratic space over the non-archimedean local field F with W a quadratic subspace of codimension $2k + 1$. Then for any irreducible admissible representation π of $\mathrm{SO}(V)$ and irreducible admissible representation σ of $\mathrm{SO}(W)$, $\mathrm{Ext}^i_{\mathrm{Bes}(V,W)}[\pi, \sigma]$ are finite dimensional vector spaces over $\mathbb{C}$ for all $i \geq 0$.*

Proof. The proof of this theorem will be by induction on the dimension of V. We thus assume that for any quadratic spaces $\mathcal{W} \subset \mathcal{V}$ with $\dim(\mathcal{V}) < \dim(V)$ (with $\mathcal{V}/\mathcal{W}$ a split quadratic space of odd dimension), and for any irreducible admissible representation π of $\mathrm{SO}(\mathcal{V})$ and irreducible admissible representation σ of $\mathrm{SO}(\mathcal{W})$,

$$\mathrm{Ext}^i_{\mathrm{Bes}(\mathcal{V},\mathcal{W})}[\pi, \sigma]$$

are finite dimensional vector spaces over $\mathbb{C}$ for all $i \geq 0$.

We begin by proving the theorem for a principal series representation of $\mathrm{SO}(V)$ induced from an irreducible representation of a maximal parabolic subgroup. By the previous proposition, we need only prove the finite dimensionality of $\mathrm{Ext}^i_{\mathrm{SO}(V')}[\pi, \pi']$ where V' is a codimension one subspace of V, and π' is an irreducible, admissible representation of $\mathrm{SO}(V')$.

Much of the proof below closely follows the paper Moeglin and J.-L.Waldspurger [2012], where they have to do much harder work to precisely analyze $\mathrm{Hom}_{\mathrm{SO}(V')}[\pi, \pi']$.

Assume that the dimension of V is $n + 1$, and that V' is a subspace of dimension n. Let $V = X + V_0 + Y$ with X and Y totally isotropic subspaces of V of dimension m, and in perfect pairing with each other. Let P be the maximal parabolic subgroup of $\mathrm{SO}(V)$ stabilizing X. Let $M = \mathrm{GL}(X) \times \mathrm{SO}(V_0)$ be a Levi subgroup of P, $\pi_0 \otimes \sigma_0$ an irreducible representation of M realized on the space $E_{\pi_0} \otimes E_{\sigma_0}$, and $\pi = \pi_0 \rtimes \sigma_0$ the corresponding principal series representation of $\mathrm{SO}(V)$. Denote by E_π the space of function on $\mathrm{SO}(V)$ with values in $E_{\pi_0} \otimes E_{\sigma_0}$ verifying the usual conditions under left translation by $P(F)$ for defining the principal series representation $\pi = \pi_0 \rtimes \sigma_0$ of $\mathrm{SO}(V)$.

To understand the restriction of the principal series $\pi = \pi_0 \rtimes \sigma_0$ to $\mathrm{SO}(V')$, we need to analyze the orbits of $\mathrm{SO}(V')$ on $P(F) \backslash \mathrm{SO}(V)$. To every $g \in P(F) \backslash \mathrm{SO}(V)$, one can associate an isotropic subspace $g^{-1}(X)$ of V. Let $\mathcal{U}$ be the set of $g \in P(F) \backslash \mathrm{SO}(V)$ such that $\dim(g^{-1}(X) \cap V') = m - 1$, and let $\mathcal{X}$ be the set of $g \in P(F) \backslash \mathrm{SO}(V)$ such that $\dim(g^{-1}(X) \cap V') = m$. Then $\mathcal{U}$ is an open subset of $P(F) \backslash \mathrm{SO}(V)$ which is a single orbit under $\mathrm{SO}(V')$, and $\mathcal{X}$ is a closed subset of $P(F) \backslash \mathrm{SO}(V)$ which is a single orbit under $\mathrm{SO}(V')$ unless n is even, and $n = 2m$ in which case there are two orbits in $\mathcal{X}$ under $\mathrm{SO}(V')$.

Denote by $E_{\pi,\mathcal{U}}$ the subspace of functions in E_π with support in $\mathcal{U}$, and denote by $E_{\pi,\mathcal{X}}$ the space $E_\pi / E_{\pi,\mathcal{U}}$. The spaces $E_{\pi,\mathcal{U}}$ and $E_{\pi,\mathcal{X}}$ are invariant under $\mathrm{SO}(V')$, and

we have an exact sequence of $SO(V')$-modules,

$$0 \to E_{\pi,\mathfrak{u}} \to E_\pi \to E_{\pi,\mathfrak{x}} \to 0.$$

To prove the finite dimensionality of Ext groups $\mathrm{Ext}^i_{SO(V')}[E_\pi, \pi']$, it suffices to prove similar finite dimensionality theorems for the Ext groups involving the $SO(V')$-modules $E_{\pi,\mathfrak{u}}$ and $E_{\pi,\mathfrak{x}}$. We analyze the two terms separately.

For analyzing $E_{\pi,\mathfrak{x}}$, we assume (after conjugation by $SO(V)$) that both X and Y are contained in V'. Thus, $V' = X + V'_0 + Y$ with $V'_0 = V' \cap V_0$. It can be seen that,

$$E_{\pi,\mathfrak{x}} = \pi_0|.|^{1/2} \times \sigma_0|_{SO(V'_0)}.$$

By Theorem 2.1 (the second adjointness theorem of Bernstein),

$$\mathrm{Ext}^i_{SO(V')}[E_{\pi,\mathfrak{x}}, \pi'] = \mathrm{Ext}^i_M[\pi_0|.|^{1/2} \boxtimes \sigma_0|_{SO(V'_0)}, \pi'_{N-}],$$

where $M = GL(X) \times SO(V'_0)$.

The proof of the finite dimensionality of $\mathrm{Ext}^i_{SO(V')}[E_{\pi,\mathfrak{x}}, \pi']$ now follows from the induction hypothesis according to which the theorem was supposed to be known for $SO(V'_0) \subset SO(V_0)$, besides the fact that π'_{N-}, the Jacquet module with respect to the opposite parabolic $P^- = MN^-$ is an admissible representation of $M = GL(X) \times SO(V'_0)$, hence has a finite filtration by tensor product of irreducible representations of $GL(X)$ and $SO(V'_0)$, and then an application of the Kunneth theorem.

We now move on to $E_{\sigma,\mathfrak{u}}$. In this case, after conjugation by $SO(V)$, we will be in the situation,

$$V' = X' + D' + V_0 + Y',$$

where X' and Y' are totally isotropic subspaces of V' of dimension $(m-1)$, and $D', V_0, X'+Y'$ are non-degenerate quadratic spaces. Let $X' = \{e_1, \cdots, e_{m-1}\}$, and $Y' = \{f_1, \cdots, f_{m-1}\}$. Let $V'_k = X'_{k-1} + D' + V_0 + Y'_{k-1}$, where $X'_{k-1} = \{e_{m-k+1}, \cdots, e_{m-1}\}$, and $Y'_{k-1} = \{f_{m-k+1}, \cdots, f_{m-1}\}$, each of dimension $(k-1)$ for $k = 1, \cdots, m$, and let $G'_k = SO(V'_k)$.

Using the filtration of the representation π_0 of $GL(X)$ restricted to its mirabolic subgroup given by Proposition 4.1 in terms of the derivatives $\pi_0^k = \Delta^k(\pi_0)$, Moeglin and J.-L.Waldspurger [ibid.] obtain a filtration,

$$0 = \mu_{m+1} \subset \mu_m \subset \mu_{m-1} \cdots \subset \mu_1 = E_{\sigma,\mathfrak{u}},$$

with,

$$\mu_k/\mu_{k+1} \cong \Delta^k(\pi_0) \rtimes \mu'_k,$$

as modules for $SO(V')$; the representation $\Delta^k(\pi_0) \rtimes \mu'_k$ is a principal series representation of $SO(V')$ induced from a parabolic $P_k = M_k N_k$ with Levi subgroup $M_k = GL_{m-k}(F) \times G'_k$, and where

$$\mu'_k = \text{ind}_{\text{Bes}(V'_k, V_0)}^{G'_k}(\sigma_0),$$

By Theorem 2.1,

$$\text{Ext}^i_{SO(V')}[\mu_k/\mu_{k+1}, \pi'] = \text{Ext}^i_{SO(V')}[\Delta^k(\pi_0) \rtimes \mu'_k, \pi'] \cong \text{Ext}^i_{M_k}[\Delta^k(\pi_0) \boxtimes \mu'_k, \pi'_{N_k^-}],$$

with $M_k = GL_{m-k}(F) \times G'_k$, a Levi subgroup in $SO(V')$. Once again Kunneth theorem implies the finite dimensionality of the Ext groups,

$$\text{Ext}^i_{M_k}[\Delta^k(\pi_0) \boxtimes \mu'_k, \pi'_{N_k^-}].$$

Having proved the theorem for principal series representations of $SO(V)$ induced from maximal parabolics, the next lemma proves the theorem in general. □

Lemma 6.1. *Let V be a quadratic space over the non-archimedean local field F with W a quadratic subspace of codimension 1. If for any principal series representation Ps of $SO(V)$ induced from a maximal parabolic and any irreducible admissible representation σ of $SO(W)$, $\text{Ext}^i_{SO(W)}[Ps, \sigma]$ are finite dimensional vector spaces over $\mathbb{C}$ for all $i \geq 0$, then for any irreducible representation π of $SO(V)$ and any irreducible admissible representation σ of $SO(W)$, $\text{Ext}^i_{SO(W)}[\pi, \sigma]$ are finite dimensional vector spaces over $\mathbb{C}$ for all $i \geq 0$.*

Proof. By Proposition 2.3, if π is a supercuspidal representation of $SO(V)$, its restriction to $SO(W)$ is a projective object in $\mathcal{R}(SO(W))$. Therefore $\text{Ext}^i_{SO(W)}[\pi, \sigma]$ are zero for $i > 0$, and $\text{Ext}^0_{SO(W)}[\pi, \sigma] = \text{Hom}_{SO(W)}[\pi, \sigma]$ is finite dimensional.

Assume now that π is not a supercuspidal representation, and write π as a quotient of a principal series representation induced from a representation of a maximal parabolic subgroup of $SO(V)$. We thus have an exact sequence,

$$0 \to \lambda \to Ps \to \pi \to 0.$$

This gives rise to a long exact sequence,

$$0 \to \text{Hom}_{SO(W)}[\pi, \sigma] \to \text{Hom}_{SO(W)}[Ps, \sigma] \to \text{Hom}_{SO(W)}[\lambda, \sigma] \to$$
$$\to \text{Ext}^1_{SO(W)}[\pi, \sigma] \to \text{Ext}^1_{SO(W)}[Ps, \sigma] \to \text{Ext}^1_{SO(W)}[\lambda, \sigma] \to \text{Ext}^2_{SO(W)}[\pi, \sigma] \to \cdots$$

Since we know that all Hom spaces in the above exact sequence are finite dimensional, and also $\text{Ext}^1_{SO(W)}[Ps, \sigma]$ is given to be finite dimensional, we get the finite dimensionality of $\text{Ext}^1_{SO(W)}[\pi, \sigma]$ for any irreducible representation π of $SO(V)$. This implies finite

dimensionality of $\text{Ext}^1_{\text{SO}(W)}[\pi, \sigma]$ for any finite length representation π of $\text{SO}(V)$. Armed with this finite dimensionality of $\text{Ext}^1_{\text{SO}(W)}[\pi, \sigma]$ for any finite length representation π of $\text{SO}(V)$, and with the knowledge that $\text{Ext}^2_{\text{SO}(W)}[Ps, \sigma]$ is given to be finite dimensional, we get the finite dimensionality of $\text{Ext}^2_{\text{SO}(W)}[\pi, \sigma]$ for any irreducible representation π of $\text{SO}(V)$, and similarly we get the finite dimensionality of $\text{Ext}^i_{\text{SO}(W)}[\pi, \sigma]$ for any irreducible representation π of $\text{SO}(V)$, and any $i \geq 0$. $\qquad\square$

The proof of Theorem 6.1 uses Theorem 3.1 (Kunneth theorem) for representations of $\text{GL}_m(F) \times G'_k$. Since for any two irreducible representations V, V' of $\text{GL}_m(F)$, $\text{EP}_{\text{GL}_m(F)}[V, V'] = 0$ unless $m = 0$ cf. Lemma 4.1, we obtain the following corollary of the proof of the theorem.

Corollary 6.2. *For a principal series representation $\pi = \pi_0 \rtimes \sigma_0$ of $\text{SO}(V)$ where σ_0 is an admissible representation of $\text{SO}(W)$, and π' is an admissible representation of $\text{SO}(V')$ where $W \subset V' \subset V$ with V' a nondegenerate codimension 1 subspace of the quadratic space V with $\dim(V') - \dim(W) = 2m - 1$,*

$$\text{EP}_{\text{SO}(V')}[\pi, \pi'] = \text{EP}_{\text{Bes}(V', W)}[\pi', \sigma_0] \cdot \dim \text{Wh}(\pi_0).$$

Definition: A finite length representation π of a classical group will be called a *full principal series* if it is irreducible and supercuspidal, or is of the form $\pi = \pi_0 \rtimes \sigma_0$ with both π_0 and σ_0 irreducible, and σ_0 supercuspidal.

The following corollary is a consequence of the previous corollary together with the fact that if σ is a cuspidal representation of $\text{SO}(W)$, then $\sigma \otimes \psi$ is an injective module for $\text{Bes}(V, W)$.

Corollary 6.3. *Let π be a finite length representation of $\text{SO}(V)$, and π' of $\text{SO}(V')$ where $V' \subset V$ is a nondegenerate codimension 1 subspace of the quadratic space V. Assume that π is a full principal series, and π' is an irreducible representation of $\text{SO}(V')$. Then, $\text{EP}_{\text{SO}(V')}[\pi, \pi']$ is either 0 or 1. If $\pi = \pi_0 \rtimes \sigma_0$ of $\text{SO}(V)$ where σ_0 is an admissible representation of $\text{SO}(W)$ with $\dim W \leq 1$, $\text{EP}_{\text{SO}(V')}[\pi, \pi'] = \dim \text{Wh}(\pi) \cdot \dim \text{Wh}(\pi')$ (if $\dim W = 1$, $\text{Wh}(\pi')$ is for a particular character of a maximal unipotent subgroup of $\text{SO}(V')$).*

Remark 6.1. In the previous corollary, we see a large number of cases when the Euler-Poincaré characteristic is 0 or 1. Is there a multiplicity one result for EP, or for Ext^i?

7 An integral formula of Waldspurger, and a conjecture on E-P

In this section we review an integral formula of Waldspurger which we then propose to be the integral formula for the Euler-Poincaré pairing for $\text{EP}_{\text{Bes}(V, W)}[\sigma, \sigma']$ for σ any finite

length representation of $SO(V)$, and σ' any finite length representation of $SO(W)$, where V and W are quadratic spaces over F with $V = X + D + W + Y$ with W a quadratic subspace of V of codimension $2k + 1$ with X and Y totally isotropic subspaces of V in duality with each other under the underlying bilinear form, and D an anisotropic line in V. Let $Z = X + Y$.

Let $\underline{\mathfrak{T}}$ denote the set of elliptic tori T in $SO(W)$ such that there exist quadratic subspaces W_T, W'_T of W such that:

1. $W = W_T \oplus W'_T$, and $V = W_T \oplus W'_T \oplus D \oplus Z$.

2. $\dim(W_T)$ is even, and $SO(W'_T)$ and $SO(W'_T \oplus D \oplus Z)$ are quasi-split.

3. T is a maximal (elliptic) torus in $SO(W_T)$.

Clearly the group $SO(W)$ operates on $\underline{\mathfrak{T}}$. Let $\mathfrak{T}$ denote a set of orbits for this action of $SO(W)$ on $\underline{\mathfrak{T}}$. For our purposes we note the most important elliptic torus $T = \langle e \rangle$ corresponding to $W_T = 0$.

For σ an admissible representation of $SO(V)$ of finite length, define a function $c_\sigma(t)$ for regular elements of a torus T belonging to $\underline{\mathfrak{T}}$ by the germ expansion of the character $\theta_\sigma(t)$ of σ on the centralizer of t in the Lie algebra of $SO(V)$, and picking out 'the' leading term. The semi-simple part of the centralizer of t in the Lie algebra of V is the Lie algebra of $SO(W'_T \oplus D \oplus Z)$ which, if $W'_T \oplus D \oplus Z$ has odd dimension, has a unique conjugacy class of regular nilpotent element, but if $W'_T \oplus D \oplus Z$ has even dimension, then although there are several regular nilpotent conjugacy classes, there is one which is 'relevant', and is what is used to define $c_\sigma(t)$. Similarly, for σ' an admissible representation of $SO(W)$ of finite length, one defines a function $c_{\sigma'}(t)$ for regular elements of a torus T belonging to $\underline{\mathfrak{T}}$ by the germ expansion of the character $\theta_{\sigma'}(t)$ of σ'.

Define a function Δ_T on an elliptic torus T belonging to $\underline{\mathfrak{T}}$ with $W = W_T \oplus W'_T$, by $\Delta(t) = |\det(1 - t)|_{W_T}|_F$, and let D^H denote the discriminant function on $H(F)$. For a torus T in H, define the Weyl group $W(H, T)$ by the usual normalizer divided by the centralizer: $W(H, T) = N_{H(F)}(T)/Z_{H(F)}(T)$.

The following theorem is proved by Waldspurger in J.-L.Waldspurger [2010] and J.-L.Waldspurger [2012b].

Theorem 7.1. *Let $V = X + D + W + Y$ be a quadratic space over the non-archimedean local field F with W a quadratic subspace of codimension $2k + 1$ as above. Then for any irreducible admissible representation σ of $SO(V)$ and irreducible admissible representation σ' of $SO(W)$,*

$$\sum_{T \in \mathfrak{T}} |W(H, T)|^{-1} \int_{T(F)} c_\sigma(t) c_{\sigma'}(t) D^H(t) \Delta^k(t) dt,$$

is a finite sum of absolutely convergent integrals. (The Haar measure on $T(F)$ is normalized to have volume 1.) If either σ is a supercuspidal representation of $\mathrm{SO}(V)$, and σ' is arbitrary irreducible admissible representation of $\mathrm{SO}(W)$, or both σ and σ' are tempered representations, then

$$\dim \mathrm{Hom}_{\mathrm{Bes}(V,W)}[\sigma, \sigma'] = \sum_{T \in \mathfrak{T}} |W(H,T)|^{-1} \int_{T(F)} c_\sigma(t) c_{\sigma'}(t) D^H(t) \Delta^k(t) dt.$$

Given this theorem of Waldspurger, it is most natural to propose the following conjecture on Euler-Poincaré pairing.

Conjecture 7.1. *Let $V = X + D + W + Y$ be a quadratic space over the non-archimedean local field F with W a quadratic subspace of V of codimension $2k + 1$ as before. Then for any irreducible admissible representation σ of $\mathrm{SO}(V)$ and irreducible admissible representation σ' of $\mathrm{SO}(W)$,*

1.

$$\mathrm{EP}_{\mathrm{Bes}(V,W)}[\sigma, \sigma'] = \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathrm{Bes}(V,W)}[\sigma, \sigma']$$

$$= \sum_{T \in \mathfrak{T}} |W(H,T)|^{-1} \int_{T(F)} c_\sigma(t) c_{\sigma'}(t) D^H(t) \Delta^k(t) dt.$$

2. If σ and σ' are irreducible tempered representations, then $\mathrm{Ext}^i_{\mathrm{Bes}(V,W)}[\sigma, \sigma'] = 0$ for $i > 0$.

Remark 7.1. Note that a supercuspidal representation of $\mathrm{SO}(V)$ is a projective object in the category of smooth representations of $\mathrm{SO}(V)$, and hence by Proposition 2.4, it remains a projective object in the category of smooth representations of $\mathrm{SO}(W) \cdot U$. Therefore if σ or σ' is supercuspidal, $\mathrm{Ext}^i_{\mathrm{Bes}(V,W)}[\sigma, \sigma'] = 0$ for $i > 0$. (We note that for a supercuspidal representation σ' of $\mathrm{SO}(W)$, the representation $\sigma' \otimes \psi$ is an injective module in the category of smooth representations of $\mathrm{SO}(W) \cdot U$.) Thus Waldspurger's theorem is equivalent to the conjectural statement on Euler-Poincaré characteristic if σ or σ' is supercuspidal (except that it is not proved if σ' is supercuspidal, but σ is arbitrary). Part 2 of the conjecture is there as the simplest possible explanation of Waldspurger's theorem for tempered representations!

Example 7.1. Assume that either $G = \mathrm{SO}_{n+1}(F)$ is a split group, and σ is induced from a character of a Borel subgroup of G, or $H = \mathrm{SO}_n(F)$ is a split group and σ' is induced from a character of a Borel subgroup of H. Then the conjectural formula on Euler-Poincaré becomes $\mathrm{EP}[\sigma, \sigma'] = 1$ which is a consequence of Corollary 6.3.

Remark 7.2. We consider the Waldspurger integral formula as some kind of Riemann-Roch theorem. Recall that for X a smooth projective variety with Todd class T_X, and for any coherent sheaf $\mathfrak{F}$ on X with Chern class $c(\mathfrak{F})$, one has,

$$EP(X, \mathfrak{F}) = \sum_i (-1)^i \dim H^i(X, \mathfrak{F}) = \sum_i (-1)^i \dim \text{Ext}^i(O_X, \mathfrak{F}) = \int_X (T_X \cdot c(\mathfrak{F})).$$

In our case, $EP[\pi_1, \pi_2] = \sum_i (-1)^i \dim \text{Ext}^i_H[\pi_1, \pi_2]$ is conjecturally expressed as

$$EP[\pi_1, \pi_2] = \int_X T_X \cdot c(\pi_1, \pi_2),$$

where X is a certain set of elliptic tori in H, T_X is a function on this set of elliptic tori, and $c(\pi_1, \pi_2)$ is a function on these elliptic tori defined in terms of the germ expansion of π_1 and π_2.

8   The Schneider-Stuhler duality theorem

The following theorem is a mild generalization of a duality theorem of Schneider and Stuhler due to M.Nori and Prasad [2017]; it turns questions on $\text{Ext}^i[\pi_1, \pi_2]$ to $\text{Ext}^j[\pi_2, \pi_1]$, and therefore is of central importance to our theme in this paper.

Theorem 8.1. *Let G be a reductive p-adic group, and π an irreducible, admissible representation of G. Let $d(\pi)$ be the largest integer $i \geq 0$ such that there is an irreducible, admissible representation π' of G with $\text{Ext}^i_G[\pi, \pi']$ nonzero. Then,*

1. *There is a unique irreducible representation π' of G with $\text{Ext}^{d(\pi)}_G[\pi, \pi'] \neq 0$.*

2. *The representation π' in (1) is nothing but $D(\pi)$ where $D(\pi)$ is the Aubert-Zelevinsky involution of π, and $d(\pi)$ is the split rank of the Levi subgroup M of G which carries the cuspidal support of π.*

3. *$\text{Ext}^{d(\pi)}_G[\pi, D(\pi)] \cong \mathbb{C}$.*

4. *For any smooth representation π' of G, the bilinear pairing*

$$(*) \quad \text{Ext}^i_G[\pi, \pi'] \times \text{Ext}^j_G[\pi', D(\pi)] \to \text{Ext}^{i+j=d(\pi)}_G[\pi, D(\pi)] \cong \mathbb{C},$$

   is nondegenerate in the sense that if $\pi' = \varinjlim \pi'_n$ of finitely generated G-submodules π'_n, then $\text{Ext}^i_G[\pi, \pi'] = \varinjlim \text{Ext}^i_G[\pi, \pi'_n]$, a direct limit of finite dimensional vector spaces over $\mathbb{C}$, and $\text{Ext}^j_G[\pi', D(\pi)] = \varprojlim \text{Ext}^j_G[\pi'_n, D(\pi)]$, an inverse limit

of finite dimensional vector spaces over $\mathbb{C}$, *and the pairing in* $(*)$ *is the direct limit of perfect pairings on these finite dimensional spaces:*

$$\mathrm{Ext}^i_G[\pi, \pi'_n] \times \mathrm{Ext}^j_G[\pi'_n, D(\pi)] \to \mathrm{Ext}^{i+j=d(\pi)}_G[\pi, D(\pi)] \cong \mathbb{C}.$$

(Observe that a compatible family of perfect pairings on finite dimensional vector spaces $B_n : V_n \times W_n \to \mathbb{C}$ *with* V_n *part of an inductive system, and* W_n *part of a projective system, gives rise to a natural pairing* $B : \varinjlim V_n \times \varprojlim W_n \to \mathbb{C}$ *such that the associated homomorphism from* $(\varinjlim V_n)^\star$ *to* $\varprojlim W_n$ *is an isomorphism.*

As an example, the following proposition giving complete classification of irreducible submodules π of the tensor product $\pi_1 \otimes \pi_2$ of two (irreducible, infinite dimensional) representations π_1, π_2 of $\mathrm{GL}_2(F)$ with the product of their central characters trivial, is essentially a translation of vanishing of $\mathrm{Ext}^1_{PGL_2(F)}[\pi_1 \otimes \pi_2, \pi_3]$ by this duality theorem. The vanishing itself follows because $\mathrm{Ext}^2_{PGL_2(F)}[\pi_1 \otimes \pi_2, \pi_3] = 0$ by Proposition 2.9, and both EP and Hom spaces have the same dimension.

Proposition 8.1. *Let* π_1, π_2 *be two irreducible admissible infinite dimensional representations of* $\mathrm{GL}_2(F)$ *with product of their central characters trivial. Then the following is the complete list of irreducible sub-representations* π *of* $\pi_1 \otimes \pi_2$ *as* $\mathrm{PGL}_2(F)$-*modules.*

1. π *is a supercuspidal representation of* $\mathrm{PGL}_2(F)$, *and appears as a quotient of* $\pi_1 \otimes \pi_2$.

2. π *is a twist of the Steinberg representation, which we assume by absorbing the twist in* π_1 *or* π_2 *to be the Steinberg representation* St *of* $\mathrm{PGL}_2(F)$. *Then* St *is a submodule of* $\pi_1 \otimes \pi_2$ *if and only if* π_1, π_2 *are both irreducible principal series representations, and* $\pi_1 \cong \pi_2^\vee$.

9   Geometrization of Ext groups

A natural way to construct exact sequences in representation theory is via the Bernstein-Zelevinsky exact sequence arising from the inclusion of an open set $X - Y$ in a topological space X equipped with an ℓ-sheaf $\mathfrak{F}$, with Y a closed subspace of X, giving rise to

$$0 \to \mathcal{S}(X - Y, \mathfrak{F}) \to \mathcal{S}(X, \mathfrak{F}) \to \mathcal{S}(Y, \mathfrak{F}) \to 0.$$

Observe that in this exact sequence, the larger space $\mathcal{S}(X - Y, \mathfrak{F})$ arises as a subspace, whereas the smaller space $\mathcal{S}(Y, \mathfrak{F})$ arises as a quotient of $\mathcal{S}(X, \mathfrak{F})$. Assuming that a group G operates on the space X, preserving the closed subspace Y, as well as the sheaf $\mathfrak{F}$, then this exact sequence gives rise to an element of $\mathrm{Hom}_G[\mathcal{S}(X, \mathfrak{F}), \mathcal{S}(Y, \mathfrak{F})]$, as well as

an element of $\mathrm{Ext}^1_G[\mathcal{S}(Y, \mathcal{F}), \mathcal{S}(X - Y, \mathcal{F})]$. Note that the Hom is from a larger space $\mathcal{S}(X, \mathcal{F})$ to a smaller space $\mathcal{S}(Y, \mathcal{F})$, whereas the Ext is between a smaller space and a larger space.

Similarly, if X_2, X_1 are closed subsets of an ℓ-space X with $X_2 \subset X_1 \subset X$, and endowed with an ℓ-sheaf $\mathcal{F}$, we have exact sequences,

$$0 \to \mathcal{S}(X - X_1, \mathcal{F}) \to \mathcal{S}(X - X_2, \mathcal{F}) \to \mathcal{S}(X_1 - X_2, \mathcal{F}) \to 0,$$

$$0 \to \mathcal{S}(X_1 - X_2, \mathcal{F}) \to \quad \mathcal{S}(X_1, \mathcal{F}) \quad \to \quad \mathcal{S}(X_2, \mathcal{F}) \quad \to 0,$$

which can be spliced together to give rise to the exact sequence,

$$0 \to \mathcal{S}(X - X_1, \mathcal{F}) \to \mathcal{S}(X - X_2, \mathcal{F}) \to \mathcal{S}(X_1, \mathcal{F}) \to \mathcal{S}(X_2, \mathcal{F}) \to 0,$$

which gives an element of $\mathrm{Ext}^2_G[\mathcal{S}(X_2, \mathcal{F}), \mathcal{S}(X - X_1, \mathcal{F})]$; so as the representation $\pi_2 = \mathcal{S}(X_2, \mathcal{F})$ becomes smaller and smaller compared to $\pi_1 = \mathcal{S}(X - X_1, \mathcal{F})$ (as the space X_2 is 'two step smaller' than X), it may be expected to contribute to higher and higher Ext groups $\mathrm{Ext}^i_G[\pi_2, \pi_1]$.

Various examples around the present work suggest that homomorphisms between representations, or extensions between them correspond to some geometric spaces (and ℓ-sheaves on them) as above, in particular, a typical homomorphism is from a larger space to smaller ones, whereas a typical Ext is the other way around!

Although most geometric spaces have algebraic geometric origin, that is not necessarily the case when thinking about the Bernstein-Zelevinsky exact sequence. For instance, one can use the action of G on its Bruhat-Tits building and its various compactifications. If X is the tree associated to $\mathrm{PGL}_2(F)$, then one knows that there is a compactification $\overline{X}$ of X on which $\mathrm{PGL}_2(F)$ continues to act with $\overline{X} - X = \mathbb{P}^1(F)$, a closed subset of $\overline{X}$. The zero-skeleton X^0 of X together with $\overline{X} - X$ is a compact topological space, call it $\overline{X}_0$, with an action of $\mathrm{PGL}_2(F)$ with two orbits: X^0 which is the open orbit, and $\mathbb{P}^1(F)$ which is the closed orbit.

An unramified character χ of B gives rise to a sheaf, say $\mathbb{C}_\chi$, on $\mathbb{P}^1(F)$, which can be extended to a $\mathrm{PGL}_2(F)$-equivariant sheaf on $\overline{X}_0$ by making it $\mathrm{ind}_{\mathrm{PGL}_2(\mathcal{O}_F)}^{\mathrm{PGL}_2(F)} \mathbb{C}$ on X^0. Call the extended sheaf on $\overline{X}_0$ also as $\mathbb{C}_\chi$; note that the restriction of $\mathbb{C}_\chi$ to X^0 is the constant sheaf $\mathbb{C}$. Thus we have an exact sequence,

$$0 \to \mathcal{S}(X^0, \mathbb{C}) \to \mathcal{S}(\overline{X}_0, \mathbb{C}_\chi) \to \mathcal{S}(\mathbb{P}^1(F), \mathbb{C}_\chi) \to 0.$$

Since $\mathrm{PGL}_2(F)$ acts transitively on the zero-skeleton X^0 with stabilizer $\mathrm{PGL}_2(\mathcal{O}_F)$, we have $\mathcal{S}(X^0, \mathbb{C}) \cong \mathrm{ind}_{\mathrm{PGL}_2(\mathcal{O}_F)}^{\mathrm{PGL}_2(F)} \mathbb{C}$. We have thus constructed an element of

$$\mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\mathrm{Ind}_B^{\mathrm{PGL}_2(F)} \chi, \mathrm{ind}_{\mathrm{PGL}_2(\mathcal{O}_F)}^{\mathrm{PGL}_2(F)} \mathbb{C}]$$

It may be hoped that many extensions which representation theory offers will be matched by geometric action of G on topological spaces with finitely many G-orbits coming either from algebraic geometric spaces, or from the Bruhat-Tits building and its compactifications; there is then also the issue of proving that geometric actions do give non-trivial extensions!

We end with a precise question but before that we need to make a definition. In what follows, groups and spaces are what are called ℓ-groups and ℓ-spaces in Bernstein and Zelevinsky [1976] and Bernstein [1992]; we will also use the notion of an ℓ-sheaf from these references which we recall is a sheaf, say $\mathcal{F}$, over X of vector spaces over $\mathbb{C}$ with the space of compactly supported global sections $\mathcal{S}(X, \mathcal{F}) = \mathcal{F}_c(X)$, a module over $\mathcal{S}(X)$, which is nondegenerate in the sense that $\mathcal{S}(X) \cdot \mathcal{F}_c(X) = \mathcal{F}_c(X)$; the functor $\mathcal{F} \to \mathcal{F}_c(X)$ gives an equivalence between ℓ-sheaves on X and nondegenerate modules over $\mathcal{S}(X)$.

Definition: A complex representation V of $G = G(F)$ is said to be of geometric origin if

1. there is a G-space X_V with finitely many G-orbits,

2. a G-equivariant sheaf $\mathcal{F}_V$ on X_V,

such that on each G-orbit $Y \subset X_V$ of the form G/H_Y, $\mathcal{F}_V|_Y$ is the equivariant sheaf associated to a finite dimensional representation W_Y of H_Y, and $V \cong \mathcal{S}(X_V, \mathcal{F}_V)$ (cf. §1.16 of Bernstein and Zelevinsky [1976] for the definition of the restriction of an ℓ-sheaf to a locally closed subset such as Y).

Example 9.1. Parabolic induction and Jacquet functor take representations of geometric origin to representations of geometric origin. It is expected that all supercuspidal representations are of geometric origin (proved for $\mathrm{GL}_n(F)$ and classical groups in odd residue characteristic). In understanding the class of representations of geometric origin given by the action of a group G on a space X, one difficulty seems to be to glue vector bundles on various orbits to an ℓ-sheaf on X.

Question 9.1. Suppose that we have two complex smooth representations V_1 and V_2 of $G = G(F)$ of geometric origin with $\mathrm{Ext}^1_G[V_1, V_2] \neq 0$ with $V \in \mathrm{Ext}^1_G[V_1, V_2]$ represented by the extension

$$0 \to V_2 \to V \to V_1 \to 0.$$

Then is V of geometric origin?

Remark 9.1. A representation of *geometric origin* comes equipped with considerable additional data as in the Bernstein–Zelevinsky exact sequence, which may be important to refine the question above.

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