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PLURIPOTENTIAL THEORY AND COMPLEX DYNAMICS IN HIGHER DIMENSION

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Abstract

Positive closed currents, the analytic counterpart of effective cycles in algebraic geometry, are central objects in pluripotential theory. They were introduced in complex dynamics in the 1990s and become now a powerful tool in the field. Challenging dynamical problems involve currents of any dimension. We will report recent developments on positive closed currents of arbitrary dimension, including the solutions to the regularization problem, the theory of super-potentials and the theory of densities. Applications to dynamics such as properties of dynamical invariants (e.g. dynamical degrees, entropies, currents, measures), solutions to equidistribution problems, and properties of periodic points will be discussed.

Contents

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1 Introduction

Let X be a compact Kähler manifold of dimension k. Let $f : X \to X$ be a dynamical system associated with a dominant holomorphic map, or more generally, a meromorphic map or correspondence, i.e. multivalued map. As a basic example, one can consider the complex affine space $\mathbb{C}^k$ as the complement of a projective hyperplane in the complex projective space $\mathbb{P}^k$. Then, any polynomial map from $\mathbb{C}^k$ to $\mathbb{C}^k$ extends to a meromorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$.

Denote by $f^n := f \circ \cdots \circ f$ (n times) the iterate of order n of f. The aim of the theory of complex dynamics is to study the longtime asymptotic behaviour of the sequence $(f^n)_{n\geq 0}$. This includes not only the study of the orbits of points, sets, currents, measures, under the action of f, but also the dynamical invariants such as dynamical degrees, entropies, Green currents, equilibrium measures, and the distribution of periodic points, etc.

Complex dynamics in dimension 1 has a long history, going back to the works by Fatou and Julia in 1920s, see e.g. Berteloot and Mayer [2001] and Carleson and Gamelin [1993]. In 1965, Brolin considered the harmonic measure of the Julia set of a polynomial in one complex variable which turns out to be a fundamental dynamical object, see Brolin [1965]. In 1981, Sibony considered the Green functions associated with Brolin's measures of polynomials of fixed degree. They can be obtained as the rate of escaping to infinity of the orbits of points in $\mathbb{C}$ under the action of the polynomials, see Sibony [1984, 1999]. Sibony also considered these Green functions in a family which constitute the Green function for some dynamical systems in higher dimension. Hubbard extended this notion of Green function to complex Hénon maps on $\mathbb{C}^2$, see Hubbard [1986]. In 1990, Sibony considered positive closed currents associated to these Green functions and their intersection, see Bedford, Lyubich, and Smillie [1993b, p.78] and also Sibony [1999].

Green currents and their intersections turn out to be fundamental objects in dynamics and pluripotential theory becomes a powerful tool in the field. The theory of complex dynamics of several variables has been developed quickly, see for example, the works by Bedford, Lyubich, and Smillie [1993a,b] and Bedford and Smillie [1991, 1992] and Fornæss and Sibony [1992, 1994a,b,c, 1995a] among others. One can observe that many works only involve currents of bi-degree $(1,1)$ and their intersections because pluripotential theory has been developed first in this setting. However, some very basic questions already show the necessity of using positive closed currents of arbitrary bi-degree.

We will see in this survey different applications of such currents. Let's illustrate here their important role in the following basic picture. The periodic points of period n of f are the solutions of the equation $f^n(z) = z$. They can be identified with the intersection of the graph Γ_n of f^n with the diagonal Δ of $X \times X$. When n goes to infinity, for interesting dynamical systems, the volume of Γ_n tends to infinity. So in order to study the distribution of periodic points when n tends to infinity, it is necessary to consider the positive closed

(k,k)-current $[\Gamma_n]$ associated with Γ_n. Indeed, in this way, one can normalize $[\Gamma_n]$ to have mass 1 and consider the limit as n tends to infinity.

It is worth noting that in general Γ_n is not a complete intersection of hypersurfaces in $X \times X$: we may need more than k hypersurfaces in order to get Γ_n as their intersection. For example, the diagonal of $\mathbb{P}^k \times \mathbb{P}^k$, with $k \geq 2$, which is the graph of the identity map on $\mathbb{P}^k$, is not a complete intersection. More generally, the current associated with Γ_n is rarely the intersection of positive closed $(1,1)$-currents. So it is not enough to use $(1,1)$-currents to study Γ_n. Furthermore, computing the limit of a sequence of intersections of $(1,1)$-currents requires strong conditions on these currents which are not always available in the dynamical setting.

We will focus our discussion on the recent developments of pluripotential theory for currents of arbitrary bi-degree and their applications to dynamics. We refer the reader to the non-exhaustive list of references at the end of the paper, in particular the surveys Dinh and Sibony [2010a, 2017], Fornæss [1996], and Sibony [1999], for a more complete panorama of the theory of complex dynamics in higher dimension.

In Section 2, we will recall basic facts on currents and discuss the problem of approximating positive closed currents by appropriate smooth differential forms. As consequences, we will give some calculus on positive closed currents. Dynamical degrees, topological and algebraic entropies will be introduced together with the famous Gromov's inequality saying that the topological entropy is bounded from above by the algebraic one. The regularization theorem is a key point in the proofs.

In Section 3, we will introduce the notion of super-potentials which are canonical functions associated with positive closed (p,p)-currents. They play the role of quasi-plurisubharmonic functions which are used as quasi-potentials for positive closed $(1,1)$-currents. An intersection theory for positive closed currents of arbitrary bi-degrees will be presented. We then state some theorems in dynamics on the equidistribution of orbits of points and varieties. Unique ergodicity property and rigidity for dynamical currents will be discussed.

In Section 4, we will introduce the theory of densities for positive closed currents. A basic example of the theory is the case of two analytic subsets whose intersection is larger than expected, in terms of dimension. The densities are introduced in order to measure the dimension excess for the intersection of positive closed currents, see Fulton [1998] for an algebraic counterpart. Applications to dynamics concerning the distribution or the counting of periodic points will be considered.

Finally, in Section 5, some open problems in dynamics will be stated. They are related to the discussions in the previous sections and will require new ideas from pluripotential theory or from complex geometry. We expect that the solutions to these questions will provide new tools for complex dynamics.

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2 Regularization of currents, dynamical degrees and entropies

In this section, we will discuss a regularization theorem for positive closed currents and its applications. We refer the reader to Demailly [2012], Hörmander [1990], Siu [1974] for basic notions and results of pluripotential theory and to Voisin [2002] for Hodge theory on compact Kähler manifolds.

Let X be a compact Kähler manifold of dimension k and let ω be a Kähler form on X. Let T be a positive closed (p,p)-current on X. The pairing $\langle T, \omega^{k-p}\rangle$, i.e. the value of T at the test form ω^{k-p}, depends only on the (Hodge or de Rham) cohomology classes of T and ω. Moreover, this quantity is comparable with the mass of T which is, by definition, the norm of T as a linear operator on the space of continuous test $(k-p,k-p)$-forms. Therefore, a large part of the computations with positive closed currents reduces to a computation with cohomology classes which is often simpler.

Positive closed currents can be seen as positive closed differential forms with distribution coefficients. In general, they are singular and calculus with them requires suitable regularization processes. The following result gives us a regularization with a control of the positivity loss, see Demailly [1992] and Dinh and Sibony [2004]. The loss of positivity is unavoidable in general. For simplicity, we also call $\|T\| := \langle T, \omega^{k-p}\rangle$ *the mass* of T.

Theorem 2.1 (Demailly for $p=1$, Dinh–Sibony for $p \geq 1$)**.** *Let (X,ω) be a compact Kähler manifold. There is a constant $c>0$ depending only on X and ω satisfying the following property. If T is a positive closed (p,p)-current on X, there are positive closed (p,p)-currents T^+ and T^- which can be approximated by smooth positive closed (p,p)-forms and such that*

$$T = T^+ - T^- \qquad \textit{and} \qquad \|T^\pm\| \leq c\|T\|.$$

This result still holds for larger classes of currents, e.g. positive dd^c-closed currents. It is the analytic counterpart of the known fact in algebraic geometry that any cycle can be represented as the difference of movable effective cycles. The regularization process used in the proof preserves good properties of T when they exist. We will give now two consequences of the regularization theorem. They are used to prove the properties of dynamical degrees and entropies that we will discuss later.

Corollary 2.2. *Let X, ω be as above and let U be an open subset of X. Let $T_1, \dots, T_n$ be positive closed currents on X of mass at most equal to 1 whose total bi-degree is at most (k, k). Assume that $T_1, \dots, T_{n-1}$ are given by smooth positive closed forms on U; so the intersection (wedge-product) $T_1 \wedge \dots \wedge T_n$ is a well-defined positive closed current on U. Then, there is a positive closed current S on X such that $T_1 \wedge \dots \wedge T_n \leq S$ on U and the mass of S on X is bounded by a constant depending only on X, ω.*

In the dynamical setting, we need to work with positive closed forms which are smooth outside an analytic subset of X. This corollary allows us to show that the integrals involving such singular forms do not explode near the set of singularities.

Recall that a meromorphic map from X to X is a holomorphic map f from a dense Zariski open set Ω of X to X whose graph in $\Omega \times X$ is a Zariski open set of an irreducible analytic subset Γ of dimension k in $X \times X$. For simplicity, we call Γ the graph of the meromorphic map $f : X \to X$. We assume that f is *dominant*, that is, the image of f contains a non-empty open subset of X, see Oguiso [2016b,a, 2017] and Oguiso and Truong [2015] for some recent examples.

Denote by π_1 and π_2 the two canonical projections from $X \times X$ to X. So the map π_1 restricted to Γ is generically 1:1. Let $I(f)$ be the set of points $x \in X$ such that $\Gamma \cap \pi_1^{-1}(x)$ is not a single point, or equivalently, of positive dimension. This is the *indeterminacy set* of f which is an analytic subset of codimension at least 2 in X. It is non-empty when f is not holomorphic on X.

Consider two dominant meromorphic maps f and f' from X to X. We can define the composition $f' \circ f$ as a holomorphic map on a suitable Zariski open set of X and then extend it to a meromorphic map from X to X. By composing f with itself, we obtain the iterates of f.

Let S be a (p, q)-current on X. Define formally the *pull-back* of S by f by

$$f^*(S) := (\pi_1)_*(\pi_2^*(S) \wedge [\Gamma]),$$

when the last expression makes sense. Since the operators π_i^* and $(\pi_i)_*$ are well-defined on all currents, the last definition is meaningful when the wedge-product $\pi_2^*(S) \wedge [\Gamma]$ is meaningful. Similarly, the *push-forward* operator f_* is defined by

$$f_*(S) := (\pi_2)_*(\pi_1^*(S) \wedge [\Gamma]),$$

when the last expression makes sense.

Consider the particular case of a smooth differential (p, q)-form ϕ on X. The wedge-product $\pi_2^*(\phi) \wedge [\Gamma]$ is well-defined because $\pi_2^*(\phi)$ is smooth. So $f^*(\phi)$ is well-defined in the sense of currents. Moreover, the value of $f^*(\phi)$ at a point x is roughly the sum of the values of $\pi_2^*(\phi)$ on the fiber $\pi_1^{-1}(x) \cap \Gamma$. We can check that $f^*(\phi)$ is in general an

L^1 form and it may be singular at the indeterminacy set $I(f)$. So we cannot iterate the operator f^* on smooth forms.

Recall that the Hodge cohomology group $H^{p,q}(X,\mathbb{C})$ of X can be defined using either smooth forms or singular currents. When ϕ is closed or exact then $f^*(\phi)$ is also closed or exact. Therefore, the above operator f^* induces a linear map from $H^{p,q}(X,\mathbb{C})$ to itself, that we still denote by f^*. The operator f_* on $H^{p,q}(X,\mathbb{C})$ is defined similarly. We can iterate those operators as for every linear operator on a vector space but in general we don't have $(f^n)^* = (f^*)^n$ on $H^{p,q}(X,\mathbb{C})$.

Consider an arbitrary positive closed (p,p)-current T on X. The pull-back $f^*(T)$ and the push-forward $f_*(T)$ of T are not always well-defined. We can however define a *strict transform* of T by f in the following way. Choose a Zariski open set Ω of X such that π_2 restricted to $\Gamma\cap\pi_2^{-1}(\Omega)$ defines a unramified covering over Ω. Then the pull-back of T by π_2 is well-defined on $\Gamma\cap\pi_2^{-1}(\Omega)$. We can show using Theorem 2.1 that it has finite mass and then its extension by 0 is a positive closed current on $X\times X$, according to a theorem of Skoda [1982]. The push-forward of the last current by π_1 is a positive closed (p,p)-current of X that we denote by $f^\bullet(T)$. We define $f_\bullet(T)$ in a similar way.

In the following result, the norms of the operators f^* and f_* are considered using a fixed norm on the vector space $H^{p,p}(X,\mathbb{C})$.

Corollary 2.3. *There is a constant $c > 0$ depending only on X,ω and the norm on $H^{p,p}(X,\mathbb{C})$ such that*

$$\|f^\bullet(T)\| \le c\|T\|\|f^* : H^{p,p}(X,\mathbb{C}) \to H^{p,p}(X,\mathbb{C})\|$$

and

$$\|f_\bullet(T)\| \le c\|T\|\|f_* : H^{p,p}(X,\mathbb{C}) \to H^{p,p}(X,\mathbb{C})\|.$$

This result is clear when T is a smooth form. We then deduce the general case using Theorem 2.1. Note that the operators $f^\bullet$ and $f_\bullet$ depend on the choice of a Zariski open set. However, when we work with L^1 forms for example, this choice is not important. Note also that the constants involved in the above results do not depend on T nor on f. In the proofs of the results below, they will intervene under the form $c^{1/n}$ and their role will be negligible when n goes to infinity.

As mentioned above, we don't have in general $(f^n)^* = (f^*)^n$ on $H^{p,q}(X,\mathbb{C})$. However, we can show that the following quantities are always well-defined.

Definition 2.4. We call *dynamical degree* of order p of f the following limit

$$d_p(f) := \lim_{n\to\infty} \|(f^n)^* : H^{p,p}(X,\mathbb{C}) \to H^{p,p}(X,\mathbb{C})\|^{1/n},$$

and *algebraic entropy* of f the following quantity

$$h_a(f) := \max_{0\le p\le k} \log d_p(f).$$

The last dynamical degree $d_k(f)$ is also called *topological degree* because it is equal to the number of points in $f^{-1}(a)$ for a generic point a in X.

Note that by Poincaré duality, we also have

$$d_p(f) := \lim_{n\to\infty} \|(f^n)_* : H^{k-p,k-p}(X,\mathbb{C}) \to H^{k-p,k-p}(X,\mathbb{C})\|^{1/n}.$$

Theorem 2.5 (Dinh–Sibony)**.** *The limit in the above definition of $d_p(f)$ always exists. It is finite and doesn't depend on the choice of the norm on $H^{p,p}(X,\mathbb{C})$. Moreover, the dynamical degrees and the algebraic entropy are bi-meromorphic invariants of the dynamical system: if $\pi : X' \to X$ is a bi-meromorphic map between compact Kähler manifolds, then*

$$d_p(\pi^{-1}\circ f\circ \pi) = d_p(f) \quad \textit{and} \quad h_a(\pi^{-1}\circ f\circ \pi) = h_a(f).$$

We also have for $n \geq 1$ that $d_p(f^n) = d_p(f)^n$ and $h_a(f^n) = nh_a(f)$.

When X is a projective space, the first statement was used by Fornæss–Sibony for $p = 1$ in order to construct the Green dynamical $(1,1)$-current Fornæss and Sibony [1994c]. Also for projective spaces, it was extended by Russakovskii–Shiffman for higher degrees Russakovskii and Shiffman [1997]. In this case, the group $H^{p,p}(X,\mathbb{C})$ is of dimension 1 and the action of $(f^n)^*$ is just the multiplication by an integer $d_{p,n}$. Therefore, we easily get $d_{p,n+m} \leq d_{p,n}d_{p,m}$ which implies the existence of the limit of $(d_{p,n})^{1/n}$ as n tends to infinity.

The proof of the above theorem in the general case uses in an essential way a computation with positive closed currents and Theorem 2.1 plays a crucial role. We refer to Dinh and Sibony [2004, 2005] for details and Dinh, Nguyên, and Truong [2012], Esnault and Srinivas [2013], and Truong [2016] for related results. We also obtained in these works the following result, which is due to Gromov for holomorphic maps Gromov [2003].

Theorem 2.6 (Gromov, Dinh–Sibony)**.** *Let X and f be as above. Then the topological entropy $h_t(f)$ of f is bounded from above by its algebraic entropy $h_a(f)$. In particular, the topological entropy of f is finite.*

The topological entropy is an important dynamical invariant. It measures the rate of divergence of the orbits of points. The formal definition for meromorphic maps is the same as the Bowen's definition for continuous maps, except that we don't consider orbits which reach the indeterminacy set. Therefore, it is not obvious that the entropy of a meromorphic map is finite. Note also that when f is a holomorphic map, the above result combined with a theorem by Yomdin [1987] implies that the topological entropy is indeed equal to the algebraic one. This property still holds for large families of meromorphic maps. We don't know if in general, there is always a map $\hat{f}$ bi-meromorphically conjugate to f such that $h_t(\hat{f}) = h_a(\hat{f})$, see Problem 5.1 below.

Observe that the action of f^n on $H^{p,q}(X,\mathbb{C})$ is not explicitly used in the above property of entropies when $p \neq q$. This can be explained by the following inequality from Dinh [2005]

$$\limsup_{n\to\infty} \|(f^n)^* : H^{p,q}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C})\|^{1/n} \leq \sqrt{d_p(f)d_q(f)}.$$

Let T be a positive closed (p,p)-current on X, for example, the current of integration on a complex subvariety of codimension p. Applying Corollary 2.3 to f^n instead of f, we obtain that the mass of $(f^n)^\bullet(T)$ is bounded by a constant times $(d_p(f)+\epsilon)^n$ for every $\epsilon > 0$. Similarly, the mass of $(f^n)_\bullet(T)$ is bounded by a constant times $(d_{k-p}(f)+\epsilon)^n$. We see that dynamical degrees measures the growth of the degree and volume of varieties under the action of f or its inverse f^{-1}. So dynamical degrees are fundamental invariants in the study of the orbits of varieties. They play, with some variants, an important role in the problem of classification of meromorphic dynamical systems using invariant meromorphic fibrations, see Amerik and Campana [2008], Dinh, Nguyên, and Truong [2012], Nakayama and Zhang [2009], Oguiso [2016a], and Zhang [2009a,b] for details

Finally, recall that a direct consequence of the mixed Hodge–Riemann theorem applied to (resolutions of singularities of) the graphs of f^n, see e.g. Dinh and Nguyên [2006] and Gromov [1990], implies that, the function $p \mapsto \log d_p(f)$ is concave. Equivalently, we have

$$d_p(f)^2 \geq d_{p-1}(f)d_{p+1}(f) \quad \text{for} \quad 1 \leq p \leq k-1.$$

In particular, we have $1 \leq d_p(f) \leq d_1(f)^p$, $h_a(f) > 0$ if and only if $d_1(f) > 1$, and there are two numbers r and s with $0 \leq r \leq s \leq k$ such that

$$1 = d_0(f) < \cdots < d_r(f) = \cdots = d_s(f) > \cdots > d_k(f).$$

The maximal dynamical degree $d_r(f)$ is also called *the main dynamical degree*. The algebraic entropy of f is then equal to $\log d_r(f)$.

3 Super-potentiel theory and equidistribution problems

Super-potentials have been introduced in order to deal with positive closed currents of arbitrary bi-degree. Let T be a positive closed (p,p)-current on a compact Kähler manifold X as above. Any analytic set of pure codimension p in X defines by integration a positive closed (p,p)-current. So the current T can be seen as a generalization of analytic sets of codimension p.

When $p = 1$, the current T can be seen as a generalization of hypersurfaces. Locally, we can write $T = dd^c u$, where u is a plurisubharmonic (p.s.h. for short) function. This function is unique up to an additive pluriharmonic function which is real analytic. Globally, if α is a smooth closed real $(1,1)$-form on X, in the cohomology class of T, by the classical $\partial\overline{\partial}$-lemma, one can write $T = \alpha + dd^c u$. Here, u is a quasi-p.s.h. function on X, that is, u is locally the sum of a p.s.h. function and a smooth function. It is uniquely determined by T and α, up to an additive constant. In particular, there is a unique function u such that $\max u = 0$. Recall that $d^c := \frac{1}{2\pi i}(\partial - \overline{\partial})$ and $dd^c = \frac{i}{\pi}\partial\overline{\partial}$.

For the case of higher bi-degree, the current T corresponds to a generalized algebraic cycle of higher co-dimension. We still can write T in a similar way but u will be a current of bi-degree $(p-1, p-1)$. It doesn't satisfy a similar uniqueness property and there is no intrinsic choice for u. Super-potentials are canonical functions defined on some infinite dimensional spaces. They play the role of quasi-potentials as quasi-p.s.h. functions do for bi-degree $(1,1)$. For simplicity, we will not introduce this notion in full generality and refer the reader to Dinh and Sibony [2009, 2010c] for details.

Let $\mathfrak{D}_q(X)$ denote the real vector space spanned by positive closed (q,q)-currents on X. Define the *$*$-norm* on this space by $\|R\|_* := \min(\|R^+\| + \|R^-\|)$, where $R^\pm$ are positive closed (q,q)-currents satisfying $R = R^+ - R^-$. We consider this space of currents with the following topology : a sequence $(R_n)_{n\geq 0}$ in $\mathfrak{D}_q(X)$ converges in this space to R if $R_n \to R$ weakly and if $\|R_n\|_*$ is bounded independently of n. On any $*$-bounded subset of $\mathfrak{D}_q(X)$, this topology coincides with the classical weak topology for currents. By Theorem 2.1, the subspace $\widetilde{\mathfrak{D}}_q(X)$ of real closed smooth (q,q)-forms is dense in $\mathfrak{D}_q(X)$ for the considered topology.

Let $\mathfrak{D}_q^0(X)$ and $\widetilde{\mathfrak{D}}_q^0(X)$ denote the linear subspaces in $\mathfrak{D}_q(X)$ and $\widetilde{\mathfrak{D}}_q(X)$ respectively of currents whose cohomology classes in $H^{q,q}(X,\mathbb{R})$ vanish. Their co-dimensions are equal to the dimension of $H^{q,q}(X,\mathbb{R})$ which is finite. Fix a real smooth and closed (p,p)-form α in the cohomology class of T in $H^{p,p}(X,\mathbb{R})$. We will consider the super-potential of T which is the real function $\mathcal{U}_T$ on $\widetilde{\mathfrak{D}}^0_{k-p+1}(X)$ defined by

$$\mathcal{U}_T(R) := \langle T-\alpha, U_R\rangle \qquad \text{for} \qquad R \in \widetilde{\mathfrak{D}}^0_{k-p+1}(X),$$

where U_R is any smooth form of bi-degree $(k-p, k-p)$ such that $dd^c U_R = R$. This form always exists because the cohomology class of R vanishes. Note that since the cohomology class of $T-\alpha$ vanishes, we can write $T-\alpha = dd^c U_T$ for some current U_T. By Stokes theorem, we have

$$\mathcal{U}_T(R) = \langle dd^c U_T, U_R\rangle = \langle U_T, dd^c U_R\rangle = \langle U_T, R\rangle.$$

We deduce from these identities that $\mathcal{U}_T(R)$ doesn't depend on the choice of U_R and U_T. However, $\mathcal{U}_T$ depends on the reference form α. Note also that if T is smooth, it is not necessary to take R and U_R smooth.

For simplicity, we will not consider other super-potentials of T. They are some affine extensions of $\mathcal{U}_T$ to some subspaces of $\mathfrak{D}_{k-p+1}(X)$. The following notions do not depend on the choice of super-potential nor on the reference form α. We say that T has a *bounded super-potential* if $\mathcal{U}_T$ is bounded on each $*$-bounded subset of $\widetilde{\mathfrak{D}}^0_{k-p+1}(X)$. We say that T has a *continuous super-potential* if $\mathcal{U}_T$ can be extended to a continuous function on $\mathfrak{D}^0_{k-p+1}(X)$ with respect to the topology previously introduced.

As the definition of super-potentials introduces a new space $\mathfrak{D}_{k-p+1}(X)$, their calculus is not immediate. Recently, with Nguyen and Vu, we proved that if a positive closed current is bounded by another one with bounded or continuous super-potentials, then it satisfies the same property Dinh, Nguyên, and Truong [2017b]. The result plays a role in some constructions of dynamical Green currents and the study of periodic points. Super-potentials also permit to build an intersection theory, see Bedford and Taylor [1982], Demailly [2012], and Fornæss and Sibony [1995b] for the case of bi-degree $(1,1)$. In the dynamical setting, they allow us to define invariant measures as intersections of dynamical Green currents.

Consider two positive closed currents T and S on X of bi-degree (p,p) and (q,q) respectively. Assume that $p+q\leq k$ and that T has a continuous super-potential. So $\mathcal{U}_T$ is defined on whole $\mathfrak{D}^0_{k-p+1}(X)$. We can define the wedge-product $T\wedge S$ by

$$\langle T\wedge S,\phi\rangle := \langle \alpha\wedge S,\phi\rangle + \mathcal{U}_T(S\wedge dd^c\phi)$$

for every smooth real test form ϕ of bi-degree $(k-p-q,k-p-q)$. Note that $S\wedge dd^c\phi$ belongs to $\mathfrak{D}^0_{k-p+1}(X)$ because it is equal to $dd^c(S\wedge\phi)$. It is not difficult to check that $T\wedge S$ is equal to the usual wedge-product of T and S when one of them is smooth. The current $T\wedge S$ is positive and closed, see Dinh, Nguyên, and Truong [2017b], Dinh and Sibony [2009, 2010c], and Vu [2016b] for details.

In this short survey, we will not be able to discuss all properties of super-potentials. Let us focus our discussion in a key property which is crucial in the solution of equidistribution problems. It also illustrates how one can use super-potentials in a similar way that one can do with quasi-p.s.h. functions.

It is not difficult to show that quasi-p.s.h. functions are integrable with respect to the Lebesgue measure on X. However, we have the following much stronger property, see e.g. Dinh, Nguyên, and Sibony [2010], Kaufmann [2017], and Vu [2016a]. It implies that quasi-p.s.h. functions are L^p for all $1\leq p<\infty$.

Theorem 3.1 (Skoda). *Let X and ω be as above. Let α be a smooth real closed $(1,1)$-form on X. There are constants $\lambda>0$ and $c>0$ such that if T is any positive closed $(1,1)$-current in the cohomology class of α and u is the quasi-p.s.h. function satisfying $dd^cu=T-\alpha$ and $\max u=0$, then we have*

$$\int_X e^{\lambda|u|}\omega^k\leq c.$$

It is not clear how to generalize this result to super-potentials because there is no natural measure on the domain of definition of super-potentials. The following result of Dinh and Sibony [2009, 2010c] gives us an answer to this question.

Theorem 3.2 (Dinh–Sibony). *Let X and ω be as above. Let α be a smooth real closed (p,p)-form on X. There is a constant $c>0$ such that if T is any positive closed (p,p)-current in the cohomology class of α and $\mathcal{U}_T$ is its super-potential defined above, then*

$$|\mathcal{U}_T(R)| \leq c(1+\log^+ \|R\|_{\mathscr{C}^1}),$$

for $R \in \mathfrak{D}^0_{k-p+1}(X)$ with $\|R\|_* = 1$, where $\log^+ := \max(\log, 0)$.

If we remove $\log^+$ from the statement, the obtained estimate is much weaker and easy to prove. So the contribution of $\log^+$ here is similar to the role of the exponential in Theorem 3.1. Several applications of super-potentials in dynamics have been obtained. We will only present here two results and refer the reader to Ahn [2016], De Thélin and Vigny [2010], and Dinh and Sibony [2009, 2010c] for some other applications, in particular, for dynamics of automorphisms of compact Kähler manifolds.

Let $\mathcal{H}_d$ denote the family of all holomorphic self-maps of $\mathbb{P}^k$ such that the first dynamical degree is an integer $d \geq 2$. This can be identified to a Zariski open subset of a projective space. A generic map from $\mathbb{C}^k$ to $\mathbb{C}^k$ whose components are polynomials of degree d can be extended to a holomorphic self-map of $\mathbb{P}^k$. The following result was obtained in Dinh and Sibony [2009], see also Ahn [2016] for some extension.

Theorem 3.3 (Dinh–Sibony). *There is an explicit dense Zariski open subset $\mathcal{H}'_d$ of $\mathcal{H}_d$ such that for every f in $\mathcal{H}'_d$ and every analytic subset V of pure codimension p and of degree* $\deg(V)$ *of $\mathbb{P}^k$ we have*

$$\lim_{n\to\infty} \frac{1}{d^{pn}\deg(V)}(f^n)^*[V] = T^p,$$

where T^p is the p-th power of the dynamical Green $(1,1)$-current T of f. Moreover, the convergence is uniform on V and exponentially fast with respect to some natural distances on the space of positive currents.

We have not yet introduced the Green current T. This is a positive closed $(1,1)$-current on $\mathbb{P}^k$, invariant by $d^{-1}f^*$, with unit mass and continuous potentials. The power T^p is well-defined and is called *Green (p,p)-current* of f. The last theorem gives us a construction of T by pulling back a hypersurface V by f^n (case $p=1$). However, T was originally constructed for every $f \in \mathcal{H}_d$ by pulling back smooth positive closed $(1,1)$-forms, see Dinh and Sibony [2010a] and Sibony [1999] for details.

Note that Theorem 3.3 still holds if we replace $[V]$ by any positive closed (p, p)-current. So f satisfies the unique ergodicity for currents. The result is however not true for every map f in $\mathcal{H}_d$. In general, there may exist exceptional analytic sets V for which the convergence in the theorem doesn't hold, see Conjecture 5.2 below. There are satisfactory equidistribution results for general maps f only when $p = k$ and $p = 1$, that is, when V is a point or a hypersurface. The full result for $p = k$ was obtained by Dinh and Sibony [2003, 2010b] generalizing results obtained by Fornæss and Sibony [1994c] and Briend and Duval [2001]. The case $p = 1$ was obtained in Dinh and Sibony [2008] and Taflin [2011] generalizing results obtained earlier by Fornæss and Sibony [1994b], Russakovskii and Shiffman [1997] and Favre and Jonsson [2003] (for $p = 1, k = 2$). The same property for dynamics in one variable, except the rate of convergence, has been proved by Brolin [1965], Freire, Lopes, and Mañé [1983] and Ljubich [1983].

The convergence of currents in Theorem 3.3 is equivalent to the convergence of their super-potentials. The rate of the convergence of super-potentials implies the rate of convergence of currents with respect to some natural distances for positive currents. These distances are analogous to the classical Kantorovich–Wasserstein distance for measures.

We discuss now the second result where, as for the last result, super-potentials and Theorem 3.2 play crucial roles in the proof. Let f be a polynomial automorphism of $\mathbb{C}^k$. We extend it to a birational map on the projective space $\mathbb{P}^k$. Denote by $I(f)$ and $I(f^{-1})$ the indeterminacy sets of f and f^{-1} respectively. They are analytic subsets of the hyperplane at infinity $\mathbb{P}^k \setminus \mathbb{C}^k$. The following notion was introduced under the name of regular automorphisms in Sibony [1999].

Definition 3.4 (Sibony)**.** We say that f is a *Hénon-type automorphism* if f is not an automorphism of $\mathbb{P}^k$ and $I(f) \cap I(f^{-1}) = \varnothing$.

This is a large family of maps. In dimension 2, all polynomial automorphisms of $\mathbb{C}^2$ are conjugated either to Hénon-type maps as in Definition 3.4 or to elementary maps whose dynamics is simple to study, see Friedland and Milnor [1989]. Consider a Hénon-type map f as above. It is known that there is an integer p such that $\dim I(f) = k - p - 1$ and $\dim I(f^{-1}) = p - 1$. The action of f on cohomology is simple and $d_p(f)$ is the main dynamical degree.

It is also known that the set $I(f^{-1})$ is attractive for f. Let $\mathcal{U}(f)$ denote the basin of $I(f^{-1})$ which is an open neighbourhood of $I(f^{-1})$ in $\mathbb{P}^k$. The set $\mathcal{K}(f) := \mathbb{C}^k \setminus \mathcal{U}(f)$ is the set of all points $z \in \mathbb{C}^k$ whose orbits by f are bounded in $\mathbb{C}^k$. The closure $\overline{\mathcal{K}(f)}$ of $\mathcal{K}(f)$ in $\mathbb{P}^k$ is known to be the union of $\mathcal{K}(f)$ with $I(f)$. The following result was obtained in Dinh and Sibony [2009] generalizing results by Bedford, Lyubich, and Smillie [1993b] and Fornæss and Sibony [1994c], where the case of dimension 2, except the rate of convergence, was considered.

Theorem 3.5 (Dinh–Sibony)**.** *Let V be an analytic subset of pure dimension $k-p$ and of degree* $\deg(V)$ *in $\mathbb{P}^k$ such that $V \cap I(f^{-1}) = \varnothing$. Then* $\deg(V)^{-1} d_p(f)^{-n} (f^n)^*[V]$ *converges exponentially fast to a positive closed (p,p)-current $T(f)$ with support in $\overline{\mathcal{K}(f)}$. Moreover, the set $\overline{\mathcal{K}(f)}$ is rigid in the sense that $T(f)$ is the unique positive closed (p,p)-current of mass* 1 *with support in $\overline{\mathcal{K}(f)}$.*

Note that the rigidity of $\overline{\mathcal{K}(f)}$ implies the convergence of $\deg(V)^{-1} d_p(f)^{-n} (f^n)^*[V]$ to $T(f)$ because $f^{-n}(V)$ converges to $\overline{\mathcal{K}(f)}$. However, it doesn't imply the rate of convergence. The current $T(f)$ is the dynamical Green (p,p)-current of f. It was constructed by Sibony as a power of the dynamical Green $(1,1)$-current. The later has been obtained by pulling back smooth positive closed $(1,1)$-forms, see Sibony [1999] and Taflin [2011] for details. Observe that Theorem 3.5 still holds if we replace $[V]$ by any positive closed (p,p)-current whose support is disjoint from $I(f^{-1})$. The result can be applied for f^{-1} instead of f since f^{-1} is also a Hénon-type automorphism of $\mathbb{C}^k$. We refer the reader to the survey Dinh and Sibony [2014] for a more complete panorama on rigidity property in dynamics.

4 Theory of densities of currents and periodic points

The theory of densities has been introduced in order to study the intersection between positive closed currents of arbitrary dimension. These currents may not admit an intersection in the classical sense. In particular, the theory permits to measure the dimension excess of the intersection and to understand what happens for the limit of such intersections. Such situations appear in several dynamical problems. We will not report on the theory in full generality and refer the reader to Dinh and Sibony [2012, 2018] for details. Some applications in dynamics will be discussed at the end of this section.

Consider the case of two positive closed currents : the first one is a general positive closed (p,p)-current T and the second one is the current of integration on a submanifold V of X. We want to understand the densities of T, i.e. the repartition of mass in various directions, along V via a notion of tangent current. The case where V is a point corresponds to the classical theory of Lelong number for positive closed currents. The rough idea is to dilate the manifold X in the normal directions to V. When the dilation factor tends to infinity, the image of T by the dilation admits limits that we will call tangent currents of T along V. They may not be unique but belong to the same cohomology class. However, in general, there is no natural dilations in the normal directions to V and tangent currents are defined in a more sophisticated way.

Let E denote the normal vector bundle to V in X and $\overline{E}$ its canonical compactification. Denote by $A_\lambda : \overline{E} \to \overline{E}$ the map induced by the multiplication by λ on fibers of E with $\lambda \in \mathbb{C}^*$. We also identify V with the zero section of E. The tangent currents to T along

V will be positive closed (p,p)-currents on $\overline{E}$ which are V*-conic*, i.e. invariant under the action of A_λ.

Let τ be a diffeomorphism between a neighbourhood of V in X and a neighbourhood of V in E whose restriction to V is identity. Assume that τ is *admissible* in the sense that the endomorphism of E induced by the differential of τ is the identity map from E to E. Using exponential maps associated with a Kähler metric on X, it is not difficult to show that such maps exist. Here is a main result of the theory of densities.

Theorem 4.1. *Let $X, V, T, E, \overline{E}, A_\lambda$ and τ be as above. Then the family of currents $T_\lambda := (A_\lambda)_*\tau_*(T)$ is relatively compact and any limit current, for $\lambda \to \infty$, is a positive closed (p,p)-current on E whose trivial extension is a positive closed (p,p)-current on $\overline{E}$. Moreover, if S is such a current, it is V-conic, i.e. invariant under $(A_\lambda)_*$, and its cohomology class in $H^{p,p}(\overline{E},\mathbb{R})$ does not depend on the choice of τ and S.*

Note that T_λ is not of bi-degree (p,p) in general and one cannot talk about its positivity. The above theorem not only states the existence of a unique cohomology class, but it claims that it can be computed using any admissible τ. The result still holds and we obtain the same family of limit currents using local admissible diffeomorphisms. This flexibity is very useful in the analytic calculus with tangent currents and densities while the use of global admissible diffeomorphisms is convenient for calculus on cohomology.

We say that S is a *tangent current* to T along V. Its cohomology class is called the *total tangent class* of T along V. Note that this notion generalizes a notion of tangent cone in the algebraic setting where T is also given by a manifold. It measures the densities of T along V. The cohomology ring of $\overline{E}$ is generated by the cohomology ring of V and the tautological $(1,1)$-class on $\overline{E}$. Therefore, we can decompose the cohomology class of S and associate to it cohomology classes of different degrees on V. These classes represent different parts of the tangent class of T along V.

Note also that for the general case of two arbitrary positive closed currents T and T' on X (the manifold V is replaced by a general current T'), the densities between T and T' are determined by the densities between the tensor product $T \otimes T'$ on $X \times X$ and the diagonal of $X \times X$. As already mentioned above, we will not develop the general case in this report.

It is important to estimate or compute the densities. The following particular case of Dinh and Sibony [2012, Th.4.11] is used in the proofs of the dynamical properties presented below. It is analogous to a result by Siu for Lelong numbers.

Theorem 4.2. *Let T_n be a sequence of positive closed (p,p)-current converging to a positive closed (p,p)-current T on X. Let V be a submanifold of X and denote by κ_n, κ the total tangent classes of T_n, T along V. Let c be the cohomology class of a projective subspace of a fiber of $\overline{E}$. Assume that $\kappa = \lambda c$ for some non-negative constant λ. Then* *any cluster value of κ_n has the form $\lambda' c$ with a constant $0 \leq \lambda' \leq \lambda$. In particular, if $\lambda = 0$ then κ_n tends to 0.*

We will give now some applications in dynamics where the last two statements play a crucial role in the proof. For the following result, see Bedford, Lyubich, and Smillie [1993a] and Dinh and Sibony [2016].

Theorem 4.3 (Bedford–Lyubich–Smillie for $k = 2$, Dinh–Sibony for $k \geq 2$)**.** *Under the hypotheses of Theorem 3.5, let P_n denote the set of periodic points of period n of f in $\mathbb{C}^k$. Then the points in $P_n(f)$ are asymptotically equidistributed with respect to the equilibrium measure μ of f. More precisely, if δ_a denotes the Dirac mass at a point a, then*

$$\lim_{n\to\infty} d_p(f)^{-n} \sum_{a\in P_n(f)} \delta_a = \mu.$$

The result still holds if we replace $P_n(f)$ by the set of saddle periodic points of period n.

The measure μ was constructed by Sibony [1999]. With the notations of Theorem 3.5, it is equal to the intersection of the Green current $T(f)$ of f and the Green current $T(f^{-1})$ of f^{-1}. It has support in the compact set $\mathcal{K}(f) \cap \mathcal{K}(f^{-1})$ in $\mathbb{C}^k$. If Δ denotes the diagonal of $\mathbb{P}^k \times \mathbb{P}^k$, then μ can be identified with the intersection between the current $[\Delta]$ and the tensor product $T(f) \otimes T(f^{-1})$.

Let Γ_n denote the graph of f^n in $\mathbb{P}^k \times \mathbb{P}^k$. The set $P_n(f)$ can be identified with the intersection of Γ_n and Δ in $\mathbb{C}^k \times \mathbb{C}^k$. Denote for simplicity $d := d_p(f)$. We can show that the positive closed (k,k)-currents $d^{-n}[\Gamma_n]$ converge to the current $T(f) \otimes T(f^{-1})$. Therefore, Theorem 4.3 is equivalent to the identity

$$\lim_{n\to\infty} \big([\Delta] \wedge d^{-n}[\Gamma_n]\big) = [\Delta] \wedge \big(\lim_{n\to\infty} d^{-n}[\Gamma_n]\big)$$

on $\mathbb{C}^k \times \mathbb{C}^k$.

In the general setting of the theory of currents, the two operations of intersection and of taking the limit, even when they are well-defined, may not commute. In our setting, the last identity requires a transversality property described below for the intersection between Γ_n and Δ which is, in some sense, uniform in n. To establish this property requires a delicate analysis using in particular a result by de Thélin [2008].

Let $\mathrm{Gr}(\mathbb{P}^k \times \mathbb{P}^k, k)$ denote the Grassmannian bundle over $\mathbb{P}^k \times \mathbb{P}^k$ where each point corresponds to a pair $(x, [v])$ of a point $x \in \mathbb{P}^k \times \mathbb{P}^k$ and the direction $[v]$ of a simple tangent k-vector v of $\mathbb{P}^k \times \mathbb{P}^k$ at x. Let $\widetilde{\Gamma}_n$ denote the set of points $(x,[v])$ in $\mathrm{Gr}(\mathbb{P}^k \times \mathbb{P}^k, k)$ with $x \in \Gamma_n$ and v a k-vector not transverse to Γ_n at x. Let $\widehat{\Delta}$ denote the lift of Δ to $\mathrm{Gr}(\mathbb{P}^k \times \mathbb{P}^k, k)$, i.e. the set of points $(x,[v])$ with $x \in \Delta$ and v tangent to Δ. The intersection $\widetilde{\Gamma}_n \cap \widehat{\Delta}$ corresponds to the non-transverse points of intersection between Γ_n

and Δ. Note that $\dim \widetilde{\Gamma}_n + \dim \widehat{\Delta}$ is smaller than the dimension of $\mathrm{Gr}(\mathbb{P}^k \times \mathbb{P}^k, k)$ and the intersection of subvarieties of such dimensions are generically empty.

We show that the currents $d^{-n}[\widetilde{\Gamma}_n]$ cluster towards a positive closed current whose tangent currents along $\widehat{\Delta}$ vanish. This together with Theorem 4.2 implies that the intersection between Γ_n and Δ is asymptotically transverse as n goes to infinity. As mentioned above, this is the key point in the proof of Theorem 4.3.

We will end this section with another application of the theory of densities. Let f be a general dominant meromorphic map from X to X. When the periodic points of period n of f are isolated, then their number, counting with multiplicity, can be obtained using Lefschetz fixed point formula. In general, this set may have components of positive dimension, but we still want to study the distribution of isolated periodic points, in particular, to count them. The following result was recently obtained in Dinh, Nguyên, and Truong [2017a] as a consequence of Theorem 4.2 and some properties of the sequence Γ_n.

Theorem 4.4 (Dinh–Nguyen–Truong)**.** *Let f be a dominant meromorphic self-map on a compact Kähler manifold X. Let $h_a(f)$ be its algebraic entropy and $P_n(f)$ its number of isolated periodic points of period n counted with multiplicity. Then we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(f) \leq h_a(f).$$

In particular, f is an Artin–Mazur map, i.e., its number of isolated periodic points of period n grows at most exponentially fast with n.

Note that there are smooth real maps on compact manifolds which are not Artin–Mazur maps, see e.g. Artin and Mazur [1965] and Kaloshin [2000]. For large families of meromorphic maps or correspondences, we can obtain a sharp upper bound for the cardinality of $P_n(f)$ which is equal to $1 + o(1)$ times the number given by the Lefschetz fixed point formula Dinh, Nguyên, and Truong [2015, 2017a,b]. This is a crucial step in the study of the equidistribution property for these points. Lower bounds for the cardinality of $P_n(f)$ were also obtained in some cases using other ideas from dynamics. We refer to Cantat [2001], Diller, Dujardin, and Guedj [2010], Dujardin [2006], Favre [1998], Iwasaki and Uehara [2010], Jonsson and Reschke [2015], Saito [1987], and Xie [2015] for lower bounds and related results.

5 Some open problems

In this section, we will state three open problems which are related to our discussion in the previous three sections. We think that they are important problems in complex dynamics. They require new ideas and may provide new techniques that can be used to solve other questions.

The following problem is related to Theorem 2.6. It may require some ideas from complex analysis together with the techniques used by Yomdin [1987]. It was already briefly mentioned in Section 2.

Problem 5.1. Let $f : X \to X$ be a dominant meromorphic map. Does there always exist a bi-meromorphic map $\pi : X' \to X$ between compact Kähler manifolds such that the algebraic and topological entropies of $\pi^{-1} \circ f \circ \pi$ are equal ?

Some trivial examples show that we don't have this equality without modifying the manifold X, see Guedj [2005].

The following conjecture was stated in Dinh and Sibony [2008]. It may requires a deep understanding on the space of positive closed currents which is of infinite dimension. Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. A proper analytic subset of $\mathbb{P}^k$ is said to be *totally invariant* if it is invariant by both f and f^{-1}. They appear as exceptional sets, where the multiplicity of f is large. Recall that the family of all these analytic sets is either empty or finite, see Dinh and Sibony [2010b].

Conjecture 5.2. *Let T be the Green $(1,1)$-current of f and let p be an integer with $2 \leq p \leq k-1$. Then $(\deg V)^{-1} d^{-pn} (f^n)^*[V]$ converge to T^p for every analytic subset V of $\mathbb{P}^k$ of pure codimension p which is generic. Here, V is generic if either $V \cap E = \varnothing$ or $\operatorname{codim} V \cap E = p + \operatorname{codim} E$ for any irreducible component E of a totally invariant proper analytic subset of $\mathbb{P}^k$.*

Finally, the following problem seems to be very challenging. The current approach to get the equidistribution of periodic points in Theorem 4.3 contains different steps. Several of them are quantifiable but some of them need to be substituted by new ideas from pluripotential theory.

Problem 5.3. Study the rate of convergence of periodic points of Hénon-type maps toward the equilibrium measure.

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