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ANALYTIC TOOLS FOR THE STUDY OF FLOWS AND INVERSE PROBLEMS

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Abstract

In this survey, we review recent results in hyperbolic dynamical systems and in geometric inverse problems using analytic tools, based on spectral theory and microlocal methods.

1 Introduction

We describe recent results in dynamical systems and inverse problems using analytic tools based on microlocal analysis. These tools are designed to understand the long time dynamics of hyperbolic dynamical systems through spectral theory, and to solve transport equations in certain functional spaces, even when the flow is not dissipative. They allow for example to prove meromorphic extension of dynamical zeta functions in the smooth setting (while it was only known in the real analytic setting before).

These tools can also be applied to geometric inverse problems such as geodesic X-ray tomography and the boundary rigidity or lens rigidity problem, where one wants to determine a Riemannian metric from the length of its closed geodesics in the closed case, or the Riemannian distance between boundary point in the case with boundary.

In Section 2, we review some recent results concerning the study of hyperbolic flows and Ruelle resonances, while in Section 3 we discuss the boundary/lens rigidity problem and the analysis of X-ray tomography in the curved setting.

2 Microlocal analysis for Anosov and Axiom A flows

2.1 Anosov and Axiom A flows. Consider X a non-vanishing smooth vector field on a compact smooth manifold $\mathfrak{M}$ (with or without boundary), generating a flow $\varphi_t : \mathfrak{M} \to \mathfrak{M}$.

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We define the maps

$$\tau^{\pm} : \mathcal{M} \to \mathbb{R}^{\pm} \cup \{\pm\infty\}$$

by the condition that $(\tau^-(y), \tau^+(y))$ is the maximal interval of time where the flow $\varphi_t(y)$ is defined in $\mathcal{M}$ (we put $\tau^{\pm}(y) = 0$ if $\varphi_{\pm t}(y)$ is not defined for $t > 0$). We will call *trapped set* $\mathcal{K}$ the closed set of points where this interval is $\mathbb{R}$

$$\mathcal{K} := \{y \in \mathcal{M}; \tau^+(y) = +\infty, \tau^-(y) = -\infty\}$$

and we shall call *incoming tail* Γ_- and *outgoing tail* Γ_+ the sets

$$\Gamma_{\pm} := \{y \in \mathcal{M}; \tau^{\mp}(y) = \mp\infty\}.$$

Note that when $\partial\mathcal{M} = \varnothing$, we have $\Gamma_{\pm} = \mathcal{K} = \mathcal{M}$. We say that $\mathcal{K}$ is a hyperbolic set for the flow if there is a continuous flow-invariant splitting of $T\mathcal{M}$ over $\mathcal{K}$

$$T_{\mathcal{K}}\mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u$$

such that there are uniform constants $C > 0, \nu > 0$ satisfying

$$\text{(2-1)} \qquad \begin{aligned} \forall y \in \mathcal{K}, \forall \xi \in E_s(y), \forall t \geq 0, \quad & ||d\varphi_t(y)\xi|| \leq Ce^{-\nu t}||\xi|| \\ \forall y \in \mathcal{K}, \forall \xi \in E_u(y), \forall t \leq 0, \quad & ||d\varphi_t(y)\xi|| \leq Ce^{-\nu|t|}||\xi||. \end{aligned}$$

Here the norm is with respect to any fixed Riemannian metric on $\mathcal{M}$. When $\mathcal{K} = \mathcal{M}$ and $\mathcal{M}$ is a closed manifold, we say that the flow of X is *Anosov*. When $\mathcal{K}$ is a compact set in the interior $\mathcal{M}^{\circ}$ of $\mathcal{M}$, we shall say that the flow is *Axiom A*, following the terminology of Smale [1967]. By the spectral decomposition of hyperbolic flows Katok and Hasselblatt [1995, Theorem 18.3.1 and Exercise 18.3.7], the non-wandering set $\Omega \subset \mathcal{K}$ of φ_t decomposes into finitely many disjoint invariant topologically transitive sets $\Omega = \cup_{i=1}^{N}\Omega_i$ for φ_t. By Katok and Hasselblatt [ibid., Corollary 6.4.20], the periodic orbits of the flow are dense in Ω. Each Ω_i is called a basic set in the terminology of hyperbolic dynamical systems. General Axiom A flows are defined by Smale [1967] and essentially consist in a finite union of basic sets and fixed points for the flow (in that case X would have to vanish at some points). For example, gradient flows of Morse-Smale type have finitely many fixed hyperbolic points and are Axiom A as well.

2.2 Solving transport equations and continuation of the resolvent. The classical important objects for a flow as above are the periodic orbits and their length, the topological entropy of the flow, the invariant measures, the ergodicity and mixing properties, and solving cohomological equations. In some sense, all these quantities or properties are related to the transport equation

$$\text{(2-2)} \qquad (X - V)u = f$$

where $V \in C^0$ is a potential and u, f are functions or distributions.

For example, a periodic orbit γ gives rise to a Dirac distribution δ_γ given by $\langle \delta_\gamma, f\rangle = \int_\gamma f$ solving the equation

$$X\delta_\gamma = 0.$$

We say that δ_γ are invariant distributions for the flow (in fact they are invariant measures). The cohomological equation problem asks if $f \in C^\infty(\mathfrak{M})$ and $\langle w, f\rangle = 0$ for all $w \in \mathfrak{D}'(\mathfrak{M}) \cap \ker X$, then $f = Xu$ for some $u \in C^\infty$. The ergodicity and mixing of the flow with respect to a smooth measure can be read from the L^2 spectrum of X, and the entropy appears also as a leading eigenvalue of some operator $X - V$ for a well chosen potential V.

For an Anosov or Axiom A flow, we can then ask when the equation (2-2) can be solved, and in what spaces. A convenient way to analyse this is to view $P := -X + V$ as a first order differential operator and to define the resolvent

$$R_P(\lambda) := (P - \lambda)^{-1} : L^2(\mathfrak{M}) \to L^2(\mathfrak{M})$$

for $\mathrm{Re}(\lambda) \gg 1$. An explicit expression is given by the converging expression

$$R_P(\lambda)f(y) = -\int_{\tau^-(y)}^0 e^{\lambda t + \int_t^0 V(\varphi_s(y))ds} f(\varphi_t(y))dt$$

if $\mathrm{Re}(\lambda) \gg 1$ is large enough. However this operator can not be extended in $\lambda \in \mathbb{C}$ on $L^2(\mathfrak{M})$ when we reach its L^2-spectrum. This is for example a problem in the study of the cohomological equation (say when $V = 0$ and μ is a smooth invariant measure for X) since the equation $Xu = f$ corresponds to the spectral value $\lambda = 0$ and X has essential spectrum on $i\mathbb{R}$.

In the case of an Anosov flow, a major step was first made by Butterley and Liverani [2007]. They proved that the resolvent of P admits a meromorphic extension to $\mathbb{C}$ on certain functional spaces and that P has only discrete spectrum on those spaces. Another proof of microlocal nature appeared later, first in the work of Faure and Sjöstrand [2011] and then of Dyatlov and Zworski [n.d.]. Before we summarise these results, let us introduce the dual Anosov decomposition

$$T^*\mathfrak{M} = E_0^* \oplus E_s^* \oplus E_u^*, \text{ with}$$
$$E_0^*(E_u \oplus E_s) = 0, \quad E_u^*(E_u \oplus \mathbb{R}X) = 0, \quad E_s^*(E_s \oplus \mathbb{R}X) = 0.$$

and mention that we denote by $H^s(\mathfrak{M})$ the usual L^2-based Sobolev space when $s \in \mathbb{R}$.

Theorem 1 (Butterley and Liverani [2007], Faure and Sjöstrand [2011], and Dyatlov and Zworski [n.d.]). *Let X be a smooth vector field generating an Anosov flow on a compact manifold $\mathcal{M}$, let $V \in C^\infty(\mathcal{M})$ and let $P = -X + V$ be the associated first-order differential operator.*
1) There exists $C_0 \geq 0$ such that the resolvent $R_P(\lambda) := (P-\lambda)^{-1} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ of P is defined for $\mathrm{Re}(\lambda) > C_0$ and extends meromorphically to $\lambda \in \mathbb{C}$ as a family of bounded operators $R_P(\lambda) : C^\infty(\mathcal{M}) \to \mathfrak{D}'(M)$. The poles are called Ruelle resonances, the operator $\Pi_{\lambda_0} := -\mathrm{Res}_{\lambda_0} R_P(\lambda)$ at a pole λ_0 is a finite rank projector and there exists $p \geq 1$ such that $(P-\lambda_0)^p \Pi_{\lambda_0} = 0$. The distributions in $\mathrm{Ran}\,\Pi_{\lambda_0}$ are called generalized resonant states and those in $\mathrm{Ran}\,\Pi_{\lambda_0} \cap \ker(P-\lambda_0)$ are called resonant states.
2) There is $C_1 > 0$ depending only on the constant ν in (2-1) such that for each $N \in [0,\infty)$, there exists a Hilbert space $\mathcal{H}^N$ so that $C^\infty(\mathcal{M}) \subset \mathcal{H}^N \subset H^{-N}(\mathcal{M})$ and such that $R_P(\lambda) : \mathcal{H}^N \to \mathcal{H}^N$ is a meromorphic family of bounded operators in $\mathrm{Re}(\lambda) > C_0 - C_1 N$, and $(P-\lambda) : \mathrm{Dom}(P) \cap \mathcal{H}^N \to \mathcal{H}^N$ is an analytic family of Fredholm operators in that region with inverse given by $R_P(\lambda)$.
3) For a resonance λ_0, the wave-front set of each generalized resonant state $u \in \mathrm{Ran}(\Pi_{\lambda_0})$ is contained in E_u^*.

In Butterley and Liverani [2007] the space $\mathcal{H}^N$ is actually a Banach space, but we will focus here rather on the works Faure and Sjöstrand [2011] and Dyatlov and Zworski [n.d.] where $\mathcal{H}^N$ is indeed a Hilbert space defined by $\mathcal{H}^N = A_N(L^2(\mathcal{M}))$ where A_N is a certain pseudo-differential operator in an exotic class. The operator A_N is constructed as $A_N = \mathrm{Op}(a_N)$ where Op denotes a standard quantization procedure (see e.g. Zworski [2012]) and $a_N \in C^\infty(T^*\mathcal{M})$ is a symbol of the form $a_N(y,\xi) = \exp(m(y,\xi)\log|\xi|)$ for $|\xi| \geq 1$, and $m(y,\xi)$ is a homogeneous function of degree 0 in the fibers of $T^*\mathcal{M}$, equal to -1 near E_s^* and $+1$ near E_u^*. Roughly speaking, a function in $\mathcal{H}^N$ is in the classical Sobolev space $H^N(\mathcal{M})$ (microlocally) near E_s^* and in the classical Sobolev space $H^{-N}(\mathcal{M})$ near E_u^*. The behaviour outside the characteristic set $\{(y,\xi) \in T^*\mathcal{M}; \xi(X) = 0\} = E_u^* \oplus E_s^*$ of X has less importance. These $\mathcal{H}^N$ spaces are called *anisotropic Sobolev spaces*. Theorem 1 tells us that we can solve the transport equation $(-X+V-\lambda)u = f$ in a well-posed fashion provided f, u are in a good anisotropic Sobolev space, except for a discrete set of λ where f needs to be in the finite codimension range. These types of spaces were first introduced (or some Hölder version) for the case of hyperbolic diffeomorphisms, in the work of Blank, Keller, and Liverani [2002], Liverani [2005], Gouëzel and Liverani [2006], Baladi and Tsujii [2007] and with a microlocal approach in Faure, Roy, and Sjöstrand [2008]. We mention that there were previous important works on spectral approaches of hyperbolic dynamical systems by Ruelle, Fried, Pollicott, Rugh, Kitaev, etc, mostly in the case of real analytic diffeomorphisms and flows, but we won't focus on these aspects.

In the Axiom A case, we proved in Dyatlov and Guillarmou [2016] a result in the same spirit as Theorem 1. Our setting is a manifold $\mathfrak{M}$ with boundary and a non-vanishing vector field X with hyperbolic trapped set $\mathfrak{K} \subset \mathfrak{M}^\circ$, with a convexity condition on $\partial\mathfrak{M}$ (the boundary is strictly convex with respect to the flow lines of X). We note that this convexity is not really necessary and can be removed by the argument of Guillarmou, Mazzucchelli, and Tzou [n.d., Section 2.2], for there is always a convex neighborhood of $\mathfrak{K}$, even if $\partial\mathfrak{M}$ is not convex. For such flows, the stable space E_s over K extends continuously over Γ_- in a subbundle E_- satisfying hyperbolic estimates similar to E_s (i.e. those in (2-1)), while E_u extends to Γ_+ in a subbundle E_+ satisfying hyperbolicity estimates similar to E_u. In fact E_- is simply the union of tangent spaces to the stable manifolds of $\mathfrak{K}$ while E_+ is those for the unstable manifolds of $\mathfrak{K}$. We will also use the dual spaces $E_\pm^*$ over $\Gamma_\pm$ defined by

$$E_\pm^*(E_\pm \oplus \mathbb{R}X) = 0.$$

We will write below $\mathrm{WF}(u) \subset T^*\mathfrak{M}$ for the wave-front set of a distribution.

Theorem 2. *Dyatlov and Guillarmou [2016] Let $\mathfrak{M}$ be a manifold with boundary and X a smooth non vanishing vector field so that its trapped set $\mathfrak{K}$ is a compact hyperbolic set in $\mathfrak{M}^\circ$. Then for each $V \in C^\infty(\mathfrak{M})$ the resolvent $R_P(\lambda) = (P-\lambda)^{-1}$ of $P := -X + V$ admits a meromorphic extension from $\mathrm{Re}(\lambda) \gg 1$ to $\lambda \in \mathbb{C}$ with poles of finite multiplicity as a map $C_c^\infty(\mathfrak{M}^\circ) \to \mathfrak{D}'(\mathfrak{M}^\circ)$. The poles are called resonances and the generalized eigenstates $u \in \mathrm{Ran}(\mathrm{Res}_{\lambda_0}(R_P(\lambda))$ satisfy the following properties*

$$\mathrm{supp}(u) \subset \Gamma_+, \quad \mathrm{WF}(u) \subset E_+^*.$$

Moreover, for $f \in C_c^\infty(\mathfrak{M}^\circ)$, we have $R_P(\lambda)f \in C^\infty(\mathfrak{M} \setminus \Gamma_+) \cap H^{-N}(\mathfrak{M})$ for some $N > 0$ depending only on $\mathrm{Re}(\lambda)$, and $\mathrm{WF}(R_P(\lambda)f) \subset E_+^*$.

Here again, the proof uses the construction of anisotropic Sobolev spaces, but new complications come from the fact that the hyperbolicity is only on a compact subset of $\mathfrak{M}$. In the proof, we extend the flow to a compact manifold with boundary and add some absorbing and elliptic operators outside $\mathfrak{M}$. We notice that geodesic flows on closed negatively curved manifolds are examples of Anosov flows. Similarly, examples of Axiom A flows are given by geodesic flows on negatively curved non-compact manifold (M, g) satisfying the following conditions: there exists a strictly convex region M_0 such that the map $\psi := \mathbb{R}^+ \times \partial M_0 \to M \setminus M_0$ given by $\psi(t,x) = \exp_x(t\nu_x)$ is a diffeomorphism if ν_x is the unit normal to ∂M_0 pointing outside M_0. Convex co-compact hyperbolic manifolds are such examples, but we can also consider asymptotically hyperbolic manifolds with hyperbolic trapped set that are not necessarily negatively curved.

For Morse-Smale gradient flows, Dang and Rivière [n.d.(b)] studied Ruelle resonances using also a Faure-Sjöstrand approach, this is another (simpler) case of Axiom A flows. In that case the Ruelle spectrum can be explicitly computed using a normal form for the flow near a hyperbolic fixed point. We finally mention a forthcoming work of Bonthonneau and Weich [n.d.] for cases where the flow is hyperbolic but the trapped set is not compact: they show meromorphic extension of the resolvent of the geodesic flow in the case of finite volume manifolds with hyperbolic cusps. They can therefore define Ruelle resonances also in that setting.

2.3 Localisation of the spectrum and decay of correlations. We now assume that $\mathfrak{M}$ is closed, X generates an Anosov flow and that there is a smooth invariant measure μ for the flow φ_t, that is $\mathcal{L}_X\mu = 0$. In that case we formally have $X^* = -X$ and, for $P = -X$, the resolvent $R_P(\lambda)$ is analytic in $\mathrm{Re}(\lambda) > 0$. The constant functions belong to $\ker X \cap C^\infty(\mathfrak{M})$, thus 0 is a resonance. It is easy to prove that there is no Jordan block at $\lambda = 0$. Ergodicity of μ with respect to φ_t is equivalent to the fact that the only resonant states with resonance $\lambda = 0$ are the constants. Mixing of φ_t is equivalent to the fact that 0 is the only resonance on the imaginary line $\mathrm{Re}(\lambda) = 0$ (corresponding to the L^2 spectrum of X). The *correlation functions* are defined for $f_1, f_2 \in C^\infty(\mathfrak{M})$ by

$$C(f_1, f_2, t) := \langle \varphi_t^* f_1, f_2\rangle_{L^2(\mathfrak{M},\mu)}.$$

Understanding the speed of mixing, when there is mixing, amounts to studying the behaviour of $C(f_1, f_2, t)$ as $t \to \pm\infty$ for each observables f_1, f_2. It is easy to check that the resolvent is related to the correlation functions via a Laplace transform:

$$\langle R_P(\lambda) f_1, f_2\rangle = -\int_{-\infty}^{0} e^{\lambda t} C(f_1, f_2, t)dt. \tag{2-3}$$

When the correlations have an asymptotic expansion of the form

$$C(f_1, f_2, t) = \langle f_1, f_2\rangle + \sum_{j=1}^{N}\sum_{k=0}^{k_j} e^{-\lambda_j t} t^k \alpha_{j,k}(f_1, f_2) + \mathcal{O}(e^{-\nu|t|}) \tag{2-4}$$

as $t \to -\infty$, for some $\lambda_j \in \mathbb{C}$ with $\mathrm{Re}(\lambda_j) \in (-\nu, 0)$ with $\nu > 0$ and $k_j \in \mathbb{N}$, one easily get from (2-3) that $\langle R_P(\lambda) f_1, f_2\rangle$ has only finitely many poles in $\mathrm{Re}(\lambda) > -\nu$ given by the λ_j (and λ_j is a pole of order $k_j + 1$). It is a bit more difficult but still true to prove that if $\langle R_P(\lambda) f_1, f_2\rangle$ has only finitely many poles in $\mathrm{Re}(\lambda) > -\nu$ for each $f_1, f_2 \in C^\infty(\mathfrak{M})$, with a polynomial bound

$$|\langle R_P(\lambda) f_1, f_2\rangle| \leq C_{f_1,f_2}|\lambda|^p$$

for some constant C_{f_1,f_2} (depending bilinearly on f_1, f_2) and some $p \in \mathbb{R}$ independent of f_i, then an expansion of the form (2-4) holds true. This can be proved by some contour deformation when writing the operator $e^{tX} = \varphi_t^*$ in terms of resolvents, see for example Nonnenmacher and Zworski [2015, Corollary 5]. We will call the constant ν in (2-4) the *size of the essential spectral gap*.

Using representation theory, an exponential decay of correlations of mixing for geodesic flow on hyperbolic surfaces was proved by Ratner [1987] and it was extended to higher dimensions by Moore [1987]. In variable negative curvature for surfaces and more generally for Anosov flows with stable/unstable jointly non-integrable foliations, exponential decay of correlations was first shown by Dolgopyat [1998] and then by Liverani for contact flows Liverani [2004]. In these work, an essential gap of size $\epsilon > 0$ is shown, but $\epsilon > 0$ is not explicit.

Theorem 3 (Liverani [ibid.]). *For each contact Anosov flow, there is an essential spectral gap of positive size and the correlations decay exponentially fast.*

Later, the work of Tsujii [2010] gave a quantitative value for the size ν of the essential gap, and another proof appeared later in work of Nonnenmacher and Zworski [2015] (where they extended this result to general normally hyperbolic trapped sets).

Theorem 4 (Tsujii [2010, 2012] and Nonnenmacher and Zworski [2015]). *For contact Anosov flows, there is an essential gap of size ν for all $\nu < \nu_0$, where*

$$\nu_0 = \frac{1}{2}\Big(\liminf_{t\to\infty} \frac{1}{t} \inf_{y\in\mathfrak{M}} \log\det(d\varphi_t|_{E_u(y)})\Big)$$

More recently Tsujii proved that the contact assumption can be removed in dim 3, at least generically.

Theorem 5 (Tsujii [n.d.]). *On a manifold of dimension 3 admitting an Anosov flow, for $r \geq 3$ there is an open dense set in C^r of volume preserving Anosov flows that have an essential gap, and thus are exponentially mixing.*

For the billiard flow associated with a two-dimensional finite horizon Lorentz Gas (the Sinai billiard flow with finite horizon), Baladi, Demers, and Liverani [2018] recently proved an exponential decay of correlations and the existence of a non-explicit essential gap. In the Axiom A case, we note the result of Naud [2005] for hyperbolic convex co-compact surfaces and Stoyanov [2011, 2013] for more general cases proving an essential spectral gap; both results use Dolgopyat method. The recent work of Bourgain and Dyatlov [n.d.] gives an essential gap of size $1/2 + \epsilon$ for some $\epsilon > 0$ on all convex co-compact hyperbolic surfaces.

For contact Anosov flows with pinching of the Lyapunov exponents (holding for example in pinched negative curvature), Faure and Tsujii [2013, 2017] established the striking fact that the Ruelle resonance spectrum has a band structure.

Theorem 6 (Faure and Tsujii [2013, 2017]). *Let X be a contact Anosov flow on manifold $\mathfrak{M}$. There is $C>0$ such that for each $\epsilon>0$ the resonance spectrum in $|\mathrm{Im}(\lambda)|>C$ is contained in the union over $k\in\mathbb{N}_0$ of the bands*

$$B_k := \{\lambda\in\mathbb{C}; \mathrm{Re}(\lambda)\in[\gamma_k^- -\epsilon, \gamma_k^+ +\epsilon]\}$$

where

$$\gamma_k^+ := \lim_{t\to\infty}\sup_{y\in\mathfrak{M}}\frac{1}{t}\Big(-\frac{1}{2}\int_0^t \mathrm{div}(X|_{E_u})(\varphi_s(y))ds - k\log\big|\big|(d\varphi_t|_{E_u(y)})^{-1}\big|\big|^{-1}\Big),$$

$$\gamma_k^- := \lim_{t\to\infty}\inf_{y\in\mathfrak{M}}\frac{1}{t}\Big(-\frac{1}{2}\int_0^t \mathrm{div}(X|_{E_u})(\varphi_s(y))ds - k\log\big|\big|d\varphi_t|_{E_u(y)}\big|\big|\Big)$$

and $\mathrm{div}(X|_{E_u}) := (\partial_t \log\det d\varphi_t|_{E_u})|_{t=0}>0$.

This band structure was first observed in related (but different) settings in the works by Faure [2007], Faure and Tsujii [2015] and by Dyatlov [2015]. We note that only finitely many bands B_k do not intersect except for geodesic flows in constant negative curvature where the Lyapunov exponents are constants: in curvature -1, $\gamma_k^-=\gamma_k^+=-\frac{n}{2}-k$ if the dimension of the Riemannian manifold is $n+1$. Actually, in that setting, the manifold M is a quotient of hyperbolic space $\mathbb{H}^{n+1}$ by a co-compact group Γ, and the Ruelle resonance spectrum for the flow on SM has been (almost) completely characterised by Dyatlov, Faure, and Guillarmou [2015]: there is a one-to-one correspondence between the Ruelle resonances/resonant eigenstates with the spectrum/eigenfunctions of some Bochner Laplacians on certain bundles over M.

Theorem 7 (Dyatlov, Faure, and Guillarmou [ibid.]). *Let M be a compact hyperbolic manifold of dimension $n+1\geq 2$. Assume that $\lambda\in\mathbb{C}\setminus\big(-\frac{n}{2}-\frac{1}{2}\mathbb{N}_0\big)$. Denote by $m_X(\lambda)$ the multiplicity of $\lambda\in\mathbb{C}$ as a Ruelle resonance for the geodesic flow X on SM, and let $\Delta_k=\nabla^*\nabla$ be the rough Laplacian on the space of trace-free divergence-free symmetric tensors of order k. Then for $\lambda\notin -2\mathbb{N}$, we have*

$$m_X(\lambda)=\sum_{m\geq 0}\sum_{\ell=0}^{\lfloor m/2\rfloor}\dim\ker\Big(\Delta_{m-2\ell}+(\lambda+m+\frac{n}{2})^2-\frac{n^2}{4}-m+2\ell\Big)$$

and for $\lambda\in-2\mathbb{N}$, we have

$$m_X(\lambda)=\sum_{\substack{m\geq 0\\ m\neq-\lambda}}\sum_{\ell=0}^{\lfloor m/2\rfloor}\dim\ker\Big(\Delta_{m-2\ell}+(\lambda+m+\frac{n}{2})^2-\frac{n^2}{4}-m+2\ell\Big).$$

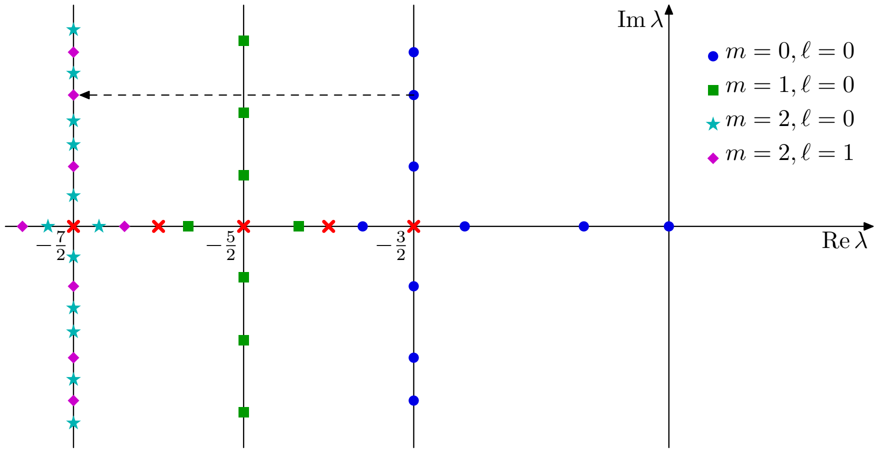

Figure 1: An illustration of Theorem 7 for $n = 3$. The red crosses mark exceptional points where the theorem does not apply. Note that the points with $m = 2, \ell = 1$ are simply the points with $m = 0, \ell = 0$ shifted by -2 (modulo exceptional points), as illustrated by the arrow.

The first band (actually line) of Ruelle resonances appear at $\mathrm{Re}(\lambda) = -\frac{n}{2} + i\mathbb{R}$, they correspond to the spectrum of the Laplacian on functions ($m = \ell = 0$ in Theorem 7). In dimension $n+1 = 2$, i.e. for surfaces, the statement is simpler since the space of trace-free divergence-free tensors is finite dimensional. Then the resonance spectrum is simply

$$\Big(-\mathbb{N}_0 + \bigcup_{1/4+r_j^2\in\sigma(\Delta)} (-\tfrac{1}{2} + ir_j)\Big) \bigcup (-\mathbb{N})$$

where $\sigma(\Delta_0)$ denotes the spectrum of the Laplacian Δ_0 acting on functions on $\Gamma\backslash\mathbb{H}^2$ (for the analysis of the special points $-\mathbb{N}/2$, see Guillarmou, Hilgert, and Weich [n.d.(a)]). In fact, in Dyatlov, Faure, and Guillarmou [2015], we show an explicit correspondence between the resonant states and the eigenfunctions of $\Delta_{m-2\ell}$ on M: for example, for the first band $m = 0, \ell = 0$, the correspondence is given by the pushforward map (integration in the fibers of SM)

$$\pi_{0*} : \ker_{\mathcal{H}^N}(-X - \lambda) \to \ker(\Delta_0 + \lambda(n + \lambda)), \quad \pi_{0*}u(x) := \int_{S_xM} u(x, v)dv.$$

where $\mathcal{H}^N$ is an anisotropic Sobolev space as in Theorem 1 with $N \gg 1$ large enough. We call this a *classical-quantum correspondence* between the eigenspaces. A partial generalisation to all compact rank-1 locally symmetric spaces is done by Guillarmou, Hilgert, and Weich [n.d.(b)]. The case of the flow acting on sections of certain bundles is worked out by Küster and Weich [n.d.].

In the case of convex co-compact hyperbolic manifolds, where the flow is Axiom A and the Laplacian has continuous spectrum, the description of the Ruelle resonance spectrum and the classical-quantum correspondence has been done completely in dimension 2 by Guillarmou, Hilgert, and Weich [n.d.(a)], and outside the special points $-\frac{n}{2}-\frac{1}{2}\mathbb{N}$ in higher dimension by Hadfield [n.d.].

In the analysis of the first band of resonances and resonant states for hyperbolic manifolds, we strongly use a differential operator $U_- : C^\infty(SM) \to C^\infty(SM; E_s^*)$ that is a covariant derivative in the direction of the unstable space:

$$\xi \in E_u(y), \quad U_- f(y)\xi := df(y)\xi,$$

recall that $E_s^*(E_s \oplus \mathbb{R}X) = 0$ thus E_s^* is a dual space to E_u. We prove that the resonant states associated to the first band are characterised as the solutions $u \in \mathfrak{D}'(SM)$, $Xu = -\lambda u$ with $U_- u = 0$ for the compact case, and with the additional condition $\mathrm{supp}(u) \subset \Gamma_+$ for the convex co-compact case. Such distributions u can be lifted to $S\mathbb{H}^{n+1}$ and are in one-to-one correspondence with distributions on $\partial\mathbb{H}^{n+1} = \mathbb{S}^n$ that have a particular conformal covariance with respect to the group Γ. The correspondence is done via pullback through the backward endpoint map $B_- : S\mathbb{H}^{n+1} \to \partial\mathbb{H}^{n+1}$ defined by $B_-(y) := \lim_{t\to+\infty} \exp_x(-tv)$. Applying the Poisson transform to those distributions we find eigenfunctions for Δ_0 on $\mathbb{H}^{n+1}$ that are Γ-equivariant, thus descend to $\Gamma\backslash\mathbb{H}^{n+1}$.

In variable curvature or more generally contact flows, a similar covariant derivative U_- can be defined, but the stable/unstable bundles are only Hölder continuous thus applying U_- to $\mathfrak{D}'(SM)$ does not make sense. In dimension $\dim(SM) = 3$, for contact flows, the operator U_- has $C^{2-\epsilon}(SM)$ coefficients for all $\epsilon > 0$ by Hurder and Katok [1990]. The first band of resonances has resonant states that are regular enough to apply U_- and we show with Faure that the rigidity $U_- u = 0$ of resonant states in constant curvature still holds in variable curvature.

Theorem 8 (Faure and Guillarmou [n.d.]). *Let $\mathfrak{M}$ be a smooth 3-dimensional compact oriented manifold and let X be a smooth vector field generating a contact Anosov flow. Assume that the unstable bundle is orientable. If λ_0 is a resonance of X with $\mathrm{Re}(\lambda_0) > -\mu_{\min}$ and if u is a generalised resonant state of P with resonance λ_0, then $U_- u = 0$. Here $\mu_{\min}$ is the minimal expansion rate given by*

$$\mu_{\min} := \lim_{t\to+\infty} \inf_{z\in\mathfrak{M}} -\frac{1}{t}\log\left|d\varphi_t(z)|_{E_s(z)}\right| = \lim_{t\to+\infty} \inf_{z\in\mathfrak{M}} -\frac{1}{t}\log\left|d\varphi_{-t}(z)|_{E_u(z)}\right|.$$

We also remark that for Morse-Smale flows, i.e. with finitely many hyperbolic fixed points and finitely many hyperbolic periodic orbits, the Ruelle spectrum has been computed explicitly by Dang and Rivière [n.d.(a)] (among other things dealt with in the article).

2.4 Dynamical zeta functions. Consider a smooth vector field X on $\mathcal{M}$ and $\mathbf{X} : C^\infty(\mathcal{M};\mathcal{E}) \to C^\infty(\mathcal{M};\mathcal{E})$ a first order differential operator on a bundle $\mathcal{E}$ satisfying

$$\forall f \in C^\infty(\mathcal{M}), \forall u \in C^\infty(\mathcal{M};\mathcal{E}),\ \mathbf{X}(fu) = (Xf)u + f(\mathbf{X}u).$$

Define the vector bundle $\mathcal{E}_0$ by

$$\mathcal{E}_0(x) = \{\eta \in T_x^*\mathcal{M} \mid \langle X(x), \eta\rangle = 0\}, \quad x \in \mathcal{M}$$

and the *linearized Poincaré map* by

$$\mathcal{P}_{x,t} : \mathcal{E}_0(x) \to \mathcal{E}_0(\varphi_t(x)), \quad \mathcal{P}_{x,t} = (d\varphi_t(x)^{-1})^T|_{\mathcal{E}_0(x)}.$$

Next, the parallel transport $\alpha_{x,t} : \mathcal{E}(x) \to \mathcal{E}(\varphi_t(x))$ is defined as follows: for each $\mathbf{u} \in C^\infty(\mathcal{M};\mathcal{E})$, we put $\alpha_{x,t}(\mathbf{u}(x)) = e^{-t\mathbf{X}}\mathbf{u}(\varphi_t(x))$. Now, assume that $\gamma(t) = \varphi^t(x_0)$ is a closed trajectory, that is $\gamma(T) = \gamma(0)$ for some $T > 0$. (We call T the period of γ, and regard the same γ with two different values of T as two different closed trajectories. The minimal positive $T^\sharp$ such that $\gamma(T^\sharp) = \gamma(0)$ is called the *primitive period*.) For such closed orbit γ, we define $\mathcal{P}_\gamma = \mathcal{P}_{x,T}$ where x is any point on the closed orbit γ, and similarly $\alpha_\gamma = \alpha_{x,T}$. We can note that $\mathrm{Tr}(\alpha_\gamma)$ and $|\det(1-\mathcal{P}_\gamma)|$ are independent of the choice of x on the closed orbit.

Giulietti, Liverani, and Pollicott [2013], and then Dyatlov and Zworski [n.d.], show the meromorphic extension of the zeta function for the flow and of the Ruelle zeta function for Anosov flows. In the Axiom A case, this is proved by Dyatlov and Guillarmou [2016].

Theorem 9 (Giulietti, Liverani, and Pollicott [2013], Dyatlov and Zworski [n.d.], and Dyatlov and Guillarmou [2016]). *1) Define for* $\mathrm{Re}\,\lambda \gg 1$*, the dynamical zeta function for* $\mathbf{X}$

$$Z_{\mathbf{X}}(\lambda) := \sum_\gamma \frac{e^{-\lambda T_\gamma} T_\gamma^\sharp \,\mathrm{Tr}(\alpha_\gamma)}{|\det(I-\mathcal{P}_\gamma)|} \tag{2-5}$$

where the sum is over all closed trajectories γ inside $\mathcal{M}$ (resp. inside $\mathcal{K}$) in the Anosov case (resp. in the Axiom A case), $T_\gamma > 0$ is the period of γ, and $T_\gamma^\sharp$ is the primitive period. Then $Z_{\mathbf{X}}(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$. The poles of $Z_{\mathbf{X}}(\lambda)$ are the

Ruelle resonances of $\mathbf{X}$ *and the residue at a pole* λ_0 *is equal to* $\mathrm{Rank}(\mathrm{Res}_{\lambda_0}(-\mathbf{X}-\lambda)^{-1})$.
2) The Ruelle zeta function defined by

$$\zeta(\lambda) := \prod_{\gamma^\sharp}\big(1-\exp(-T_{\gamma^\sharp}(\lambda))\big), \quad \mathrm{Re}\,\lambda \gg 1.$$

admits a meromorphic continuation to $\mathbb{C}$ *both in the Anosov and the Axiom A case.*

These results answer positively a conjecture of Smale. In the work Faure and Tsujii [2015], Faure and Tsujii define a Gutzwiller-Voros dynamical zeta function associated to the operator $-X+\frac{1}{2}\mathrm{div}(X|_{E_u})$ and show that its zeros in a band $\mathrm{Re}(\lambda) > -C$ for some $C>0$ are located asymptotically close to the imaginary line as $\mathrm{Im}(\lambda)\to\infty$, using their band structure results of Theorem 6 (with the suitable potential added). This zeta function is the natural generalisation of Selberg's zeta function in variable curvature.

3 Boundary rigidity and X-ray tomography problems

The boundary and lens rigidity problems are inverse problems consisting in determining a Riemannian manifold (M,g) with boundary from boundary measurements on the geodesic flow. As boundary data, we employ the *boundary distance function*

$$\beta_g := d_g|_{\partial M\times\partial M}, \tag{3-1}$$

where $d_g : M\times M\to[0,\infty)$ is the Riemannian distance, and the *lens data*

$$\tau_g^+ : \partial SM\to[0,\infty], \qquad \sigma_g : \partial SM\setminus\Gamma_-\to\partial SM.$$

Here, SM denotes the unit tangent bundle, $\Gamma_- := \{y\in\partial SM \mid \tau_g^+(y)=+\infty\}$, the *exit time* $\tau_g^+(x,v)$ is the maximal non-negative time of existence of the geodesic $\gamma_{x,v}(t)=\exp_x(tv)$, and the *scattering map* $\sigma_g(x,v) := (\gamma_{x,v}(\tau_g^+(x,v)),\dot\gamma_{x,v}(\tau_g^+(x,v)))$ gives the exit position and "angle" of $\gamma_{x,v}$. When τ_g^+ is everywhere finite, (M,g) is said to be *non-trapping*. The *boundary rigidity problem* asks whether the boundary distance β_g determine (M,g) up to diffeomorphisms fixing ∂M. Analogously, the *lens rigidity problem* asks whether the lens data (τ_g^+,σ_g) determine (M,g) up to diffeomorphisms fixing ∂M. For *simple* Riemannian manifolds, that is, compact Riemannian balls with strictly convex boundary and without conjugate points, these two rigidity problems are equivalent, since the boundary distance and the lens data can be easily recovered from each another. When the manifold has non-empty trapped set, non convex boundary or conjugate points, this equivalence is not in general true. There are easy counter examples to boundary rigidity when there are non-minimizing length geodesics (see C. B. Croke [1991]). We will say that a metric is *deformation lens rigid* if any one-parameter family of metrics with the same lens data are isometric.

In the closed setting, there is a corresponding problem that consists in determining a metric from the length of its closed geodesics, called the *length spectrum*. Here again there are counter examples due to Vignéras [1980] of non-isometric hyperbolic surfaces with same length spectrum. It is therefore more appropriate to ask wether the *marked length spectrum* determine the metric, where the marking of the geodesics is made using free homotopy classes (recall that in negative curvature there is a unique closed geodesic in each free homotopy class).

One way to attack these problems is the consider the linearised problem, which consists in analysing the kernel of the geodesic X-ray transform on symmetric tensors or order 2. More generally, the X-ray on the bundle $\otimes_S^m T^*M$ of symmetric tensors of order $m \in \mathbb{N}_0$ is defined in the case with boundary as

$$I_m : C^0(M; \otimes_S^m T^*M) \to L^\infty_{\rm loc}(\mathcal{G}), \quad I_m(f)(\gamma) := \int_0^{\ell_\gamma} f(\gamma(t))(\otimes^m \dot\gamma(t))dt$$

where $\mathcal{G}$ denotes the set of geodesics γ in SM with endpoints on the boundary ∂SM, thus having finite length $\ell(\gamma) < \infty$. In the closed case, a similar definition holds with $\mathcal{G}$ being the set of closed geodesics and $I_m(f)(\gamma)$ is defined as above but normalized by $1/\ell(\gamma)$ so that it is an $L^\infty(\mathcal{G})$ function. It is easy to check that $I_m(Df) = 0$ if $f \in C^1(M; \otimes_S^{m-1} T^*M)$ satisfies $f|_{\partial M} = 0$ and D is the symmetrised Levi-Civita covariant derivative (the condition $f|_{\partial M} = 0$ is obviously removed in the closed manifold setting). In general, the best one can get is injectivity of I_m on $\ker D^*$, i.e. divergence-free tensors, which is called *solenoidal injectivity*.

3.1 Simple metrics. Simple manifolds were introduced by Michel [1981/82] and can be defined as manifolds (M, g) that are a topological ball with strictly convex boundary and so that g has no conjugate points. Their exponential map is a diffeomorphism at each point $x \in M$. In particular, there is a unique geodesic between each pair of points $x, x' \in M$, and this geodesic has length $d_g(x, x')$. Michel made the conjecture that two simple manifolds (M, g_1) and (M, g_2) with same boundary distance $\beta_{g_1} = \beta_{g_2}$ verify that there is $\psi : M \to M$ such that $\psi^* g_2 = g_1$ and $\psi|_{\partial M} = \mathrm{Id}$. As mentionned above, the boundary rigidity and lens rigidity questions are equivalent in that setting.

For negatively curved and non-positively curved cases, it was shown by Otal [1990b] and C. B. Croke [1990] that the conjecture holds in dimension 2.

Theorem 10 (C. B. Croke [1990] and Otal [1990b]). *Two non-positively curved simple surfaces with the same boundary distance are isometric via an isometry fixing the boundary.*

In fact, in these works, we notice that the boundary is even allowed to be non-convex. The full conjecture in dimension 2 was later proved by L. Pestov and Uhlmann [2005], using earlier works of Muhometov [1981] which allow to recover a conformal factor from the boundary distance.

Theorem 11 (L. Pestov and Uhlmann [2005]). *Two simple surfaces with the same boundary distance are isometric via an isometry fixing the boundary. Moreover the scattering map σ_g determines the conformal class of a simple manifold.*

In higher dimension, little was known until recently. Burago and Ivanov [2010] proved that metrics close to flat simple metrics are boundary rigid and Stefanov and Uhlmann [2005] showed that generic simple metrics are boundary rigid. A more recent work due to Stefanov, Uhlmann, and Vasy [n.d.(b)] solves Michel's conjecture in the category of non-positively curved simple metrics, they even show a local result near boundary points.

Theorem 12 (Stefanov, Uhlmann, and Vasy [ibid.]). *Two simple manifolds in dimension $n \geq 3$ which are non-positively curved and with the same boundary distance are isometric via an isometry fixing the boundary. Moreover the boundary distance near a point $p \in \partial M$ determines the metric near p in M.*

In Stefanov, Uhlmann, and Vasy [ibid.], the condition for rigidity is weaker than non-positive curvature: it is asked that the manifolds are foliated by strictly convex hypersurfaces.

The analysis of X-ray transform is the main tool in the proof of Theorem L. Pestov and Uhlmann [2005] and Stefanov, Uhlmann, and Vasy [n.d.(b)]. These proofs are essentially of analytic nature, contrary to C. B. Croke [1990], Otal [1990b], and Burago and Ivanov [2010] where the method is more geometric. Let us quickly review some known results on this linearised problem, that is the injectivity of the X-ray transform. For simple metrics, I_0 and I_1 are known to be solenoidal injective, this was proved by Muhometov [1981] for I_0 and Anikonov and Romanov [1997] for I_1. In dimension 2, the injectivity on I_2 follows from L. Pestov and Uhlmann [2005] and the injectivity of I_m for $m > 2$ was only proved recently by Paternain, Salo, and Uhlmann [2013]. In dimension $n > 2$ and for $m \geq 2$, the injectivity of I_m in non-positive curvature was proved by L. N. Pestov and Sharafutdinov [1988]. The main tool that is used in these cases is an energy identity called *Mukhometov-Pestov identity*. We will review it quickly in the next section. We also notice that a local injectivity result (i.e. we consider the X-ray transform of a tensor only on an open subset of geodesics) has been recently proved by Uhlmann and Vasy [2016] for I_0 and Stefanov, Uhlmann, and Vasy [n.d.(a)] for I_1 and I_2 using new microlocal methods.

Theorem 13 (Uhlmann and Vasy [2016] and Stefanov, Uhlmann, and Vasy [n.d.(a)]). *Let (M,g) be a Riemannian manifold of dimension $n \geq 3$, and assume that $p \in \partial M$ is such that ∂M is strictly convex at p.*
1) Let $f \in C^\infty(M)$ and assume that $I_0(f)(\gamma) = 0$ for all γ passing through a small neighborhood of $T_p\partial M$, i.e. γ are short geodesics that are almost tangent to ∂M. Then $f = 0$ near p.
2) Let $f \in C^\infty(M; \otimes_S^m T^*M)$ with $m \in \{1,2\}$ such that $f = u + Dv$ with $v|_{\partial M} = 0$. If $I_m(f)(\gamma) = 0$ for all γ passing through a small neighborhood of $T_p\partial M$, then $u = 0$ near p.

This is the local result that allows Stefanov, Uhlmann, and Vasy [n.d.(b)] to prove Theorem 12 through a layer stripping method. The proof uses the scattering calculus of Melrose to analyse the normal operator $I_m^* I_m$. An artificial boundary is put near p in order to make the local analysis a global one on a new manifold, and the normal operator is somehow replaced by a localised one $I_m^* \chi I_m$ for some well chosen function $\chi(\gamma)$. For simple manifolds, the normal operator $I_m^* I_m$ is a pseudo-differential operator of order -1 that is elliptic on $\ker D^*$, this is a quite helpful fact to analyse the Fredholm properties and closed range properties of the operators of interest. The presence of conjugate points would ruin this property. In Uhlmann and Vasy [2016], the localisation using the χ in $I_m^* \chi I_m$ allows for example to avoid conjugate points since the geodesics almost tangent to ∂M are short and thus free of conjugate points, showing that $I_m^* \chi I_m$ is also pseudo-differential. In dimension 3 there are enough directions to get ellipticity of $I_m^* \chi I_m$, which is not the case in dimension $n = 2$, and in fact Uhlmann and Vasy [ibid.] show that this is a strong enough ellipticity to obtain injectivity (full ellipticity in the scattering calculus of Melrose).

3.2 Cases with trapped set, conjugate points or non-convex boundary. There are three different ways a manifold can be not simple: it has non-empty trapped set, it has non-convex boundary or pairs of conjugate points.

Trapped case. First, let us mention some recent results for the case with trapped set. In Guillarmou [2017b], we address the case where the trapped set is a hyperbolic set for the geodesic flow. For example, this condition is always satisfied in negative curvature. We consider a manifold (M,g) with strictly convex boundary, hyperbolic trapped set and no conjugate points. The simplest example is a hyperbolic cylinder with one closed geodesic. We are able to show injectivity of the X-ray transform on tensors.

Theorem 14 (Guillarmou [ibid.]). *Let (M,g) be a manifold with strictly convex boundary, hyperbolic trapped set and no conjugate points. The ray transforms I_0 and I_1 are* *solenoidal injective. If in addition the curvature of g is non-positive, I_m is solenoidal injective for all $m \geq 2$. Such a manifold is deformation lens rigid.*

This result shows in particular that all negatively curved manifold with strictly convex boundary have solenoidal injective ray transform for all tensors and are deformation lens rigid. The proof uses two steps, one is purely of dynamical system nature and is a Livsic type result (although Livsic theorem is usually for integration on closed orbits). Here $\mathfrak{M}$ will typically be the unit tangent bundle SM of M, where the geodesic flow lives.

Theorem 15 (Guillarmou [2017b]). *Let $\mathfrak{M}$ be a manifold with boundary and X a non-vanishing smooth vector field with trapped set $\mathcal{K} \subset \mathfrak{M}^\circ$ that is hyperbolic, and assume that $\partial\mathfrak{M}$ is strictly convex for the flow of X. If $f \in C^\infty(\mathfrak{M})$ vanishes to infinite order at $\partial\mathfrak{M}$ and satisfies $\int_\gamma f = 0$ for all integral curve γ with endpoints on $\partial\mathfrak{M}$, then there exists $u \in C^\infty(\mathfrak{M})$ such that $Xu = f$ and $u|_{\partial M} = 0$.*

If $I_0 f = 0$ we deduce from some classical argument using the short geodesics near ∂M that f vanishes at ∂M to infinite order, we can then apply Theorem 15 to get $u \in C^\infty(SM)$ such that $Xu = \pi_0^* f$ with $u|_{\partial SM} = 0$, where $\pi_0^* : C^\infty(M) \to C^\infty(SM)$ is the pull-back by the projection $\pi_0 : SM \to M$ on the base of the fibration. The Mukhometov-Pestov identity is the following identity: if $\dim(M) = n$, for each $w \in H^2(SM) \cap H^1_0(SM)$

$$||\nabla^v X w||^2_{L^2(SM)} = ||X\nabla^v w||^2_{L^2(sM)} + (n-1)||Xw||^2_{L^2(SM)} - \langle \mathcal{R}\nabla^v w, \nabla^v w\rangle_{L^2(SM)}.$$

Here $\nabla^v w = P^v \nabla w$ where ∇ is the gradient for the Sasaki metric and P^v is the orthogonal projection on the vertical space $\ker d\pi_0$ with respect to the same metric, $\mathcal{R}$ is a natural operator made from the Riemann curvature tensor. Applying to $w = u$, the left hand side is 0 since $\nabla^v \pi_0^* = 0$ and the quantity $||X\nabla^w||^2_{L^2} - \langle \mathcal{R}\nabla^v w, \nabla^v w\rangle_{L^2(SM)} \geq 0$ when there are no conjugate points, using the index theory for the energy functional of curves. This implies $Xu = 0$, thus $f = 0$. A similar argument works for I_1, and also for higher order tensors provided the curvature is non-positive.

We notice that a surface containing a flat cylinder is such that I_0 has infinite dimensional kernel (at least if I_0 maps to the space of geodesics γ with endpoints on the boundary), thus the hyperbolicity condition on the trapped set is somehow a condition that might be difficult to remove to get injectivity of X-ray in other trapped situations.

Using Theorem 14, we are able to show a "Pestov-Uhlmann" type result for surfaces.

Theorem 16 (Guillarmou [ibid.]). *Let (M_1, g_1) and (M_2, g_2) be two Riemannian surfaces with strictly convex boundary, hyperbolic trapped set and no conjugate points. Assume that $\partial M_1 = \partial M_2$ and that their scattering maps agree, i.e. $\sigma_{g_1} = \sigma_{g_2}$, then there is a diffeomorphism $\psi : M_1 \to M_2$ such that $\psi^* g_2 = e^\rho g_1$ for some $\rho \in C^\infty(M_1)$ vanishing at ∂M_1.*

So far we are not able to prove that the lens data allows to determine the remaining conformal factor ρ in Theorem 16, although we believe it does. However, in a work with Guillarmou and Mazzucchelli [2016], we show a marked lens rigidity result for the same class of Riemannian surface, that is the lens data in the universal cover (or equivalently the boundary distance in the universal cover) determine the metric.

The proof of Theorem 16 is quite complicated and uses the approach of L. Pestov and Uhlmann [2005], that mainly reduces the problem to showing solenoidal injectivity of I_1 and surjectviity of I_0^*, the dual transform to I_0 with respect to some natural measure on the set $\mathcal{G}$ of geodesics. We already know from Theorem 14 that I_0, I_1 are solenoidal injective. To prove surjectivity of I_0^*, the strategy is to prove that $I_0^* I_0$ is a Fredholm operator. We can check that

$$I_0^* I_0 = -2\pi_{0*} R_X(0) \pi_0^*$$

where $R_X(\lambda) = (-X - \lambda)^{-1}$ is the resolvent of the flow studied in Theorem 2, π_0^* is as above and π_{0*} is its adjoint consisting in integration in fibers. In the paper Dyatlov and Guillarmou [2016] with Dyatlov, we actually characterised the wave-front set of the Schwartz kernel of $R_X(\lambda)$ using propagation of singularities with radial points. Basically, writing $R_X(0) = -\int_{-\infty}^{0} e^{tX} dt$, and using that $e^{tX} = \varphi_t^*$ is a Fourier integral operator with well-known wave-front set, we already see that the conormal to the diagonal is in the wave-front set (the contribution of $t = 0$ in the integral) as well as the graph of the symplectic flow $\Phi_t = (d\varphi_t^{-1})^T$ on $T^*(SM)$. Another component appears from long time propagation, and that is where the propagation with radial point shows up, it is given by $E_+^* \times E_-^*$. Using standard rules for composition of wave-front sets, applying the push-forward $\pi_{0*} \otimes \pi_{0*}$ to the Schwartz kernel of $R_X(0)$, everything in the wave-front disappears except the conormal to the diagonal: this is a consequence of the no-conjugate points assumption and the fact that $E_\pm$ is transversal to the vertical space $\ker d\pi_0 \subset T(SM)$ in the characteristic set $\{\xi \in T^*(SM); \xi(X) = 0\}$.

There are a couple of other rigidity results in the trapped case, due to C. B. Croke and Herreros [2016] and C. Croke [2014]: in C. B. Croke and Herreros [2016] it is shown that a 2-dimensional negatively curved or flat cylinder with convex boundary is lens rigid, and C. Croke [2014] proved that the flat product metric on $B_n \times S^1$ is scattering rigid if B_n is the unit ball in $\mathbb{R}^n$.

Non-convex boundary. When the boundary is non-convex, there are also complications: the boundary distance is not a priori directly related to the lens data. In fact, it is shown to be the case for simply connected surfaces with no conjugate points by Guillarmou, Mazzucchelli, and Tzou [n.d.]. It is probably not true in higher dimension due to the fact that there are simply connected manifolds with boundary having geodesics with endpoints on ∂M and that are not length minimizing. The determination of the C^∞-jet in that case is also more complicated since there does not exist small geodesics near points in ∂M where ∂M is concave. This determination has however been proved by Stefanov and Uhlmann [2009] in the non-trapping with no-conjugate points case (and certain trapped cases). The injectivity of the X-ray transform for non-trapping manifolds with no-conjugate points has been proved by Dairbekov [2006] and extended by Guillarmou, Mazzucchelli, and Tzou [n.d.] to the case where the trapped set is a hyperbolic set not intersecting the boundary.

Theorem 17 (Dairbekov [2006] and Guillarmou, Mazzucchelli, and Tzou [n.d.])**.** *Assume that (M, g) has no conjugate points and that its trapped set $\mathcal{K}$ does not intersect ∂SM, then I_0 and I_1 are solenoidal injective. Moreover I_m is solenoidal injective if in addition the curvature is non-positive.*

Recall that the boundary rigidity results of Otal [1990b], C. B. Croke [1990] and Burago and Ivanov [2010] do not involve convexity of the boundary. In Guillarmou, Mazzucchelli, and Tzou [n.d.], we are recently able to extend L. Pestov and Uhlmann [2005], Otal [1990b], and C. B. Croke [1990] to non-trapping manifolds with no conjugate points, a class that is more general than simple manifolds.

Theorem 18 (Guillarmou, Mazzucchelli, and Tzou [n.d.])**.** *1) Let M be a simply connected compact surface with boundary. If g_1 and g_2 are two Riemannian metrics on M without conjugate points such that $\beta_{g_1} = \beta_{g_2}$, then there is a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{Id}$ and $\psi^* g_2 = g_1$.*
2) Let (M_1, g_1) and (M_2, g_2) be two non-trapping, oriented compact Riemannian surfaces with boundary, without conjugate points, and with the same lens data. Then there exists a diffeomorphism $\psi : M_1 \to M_2$ such that $\psi^* g_2 = g_1$.

This result uses the method of Pestov-Uhlmann and a careful analysis near the glancing trajectories to be able to show that I_0^* is a surjective operator. Working with the normal operator $I_0^* I_0$ in order to prove this property would not be a very good idea since this operator has problematic singularities due to glancing geodesics: it is not a pseudo-differential operator anymore as in the simple manifold case. We thus have to consider a modified normal operator that separates the glancing trajectories from the non-glancing ones. We then show that the scattering map σ_g determines (M, g) up to conformal diffeomorphism. The lens rigidity result in the non-simply connected case in Theorem 18 also uses some unpublished work of Zhou [2011] done in his PhD thesis under Croke's direction; this work is based on a result of C. Croke [2005] on lens rigidity for finite quotients. We finally conjecture that Theorem 18 should be true also in higher dimension; this would be a more general result than Michel's conjecture.

Conjugate points. Very little is known in cases with conjugate points. It is conjectured that non-trapping manifolds should have injective X-ray transform, even when there are conjugate points, but so far this conjecture remains open. There has been some recent analysis of the normal operator $I_0^* I_0$ by Stefanov and Uhlmann [2012], Monard, Stefanov, and Uhlmann [2015], Bao and Zhang [2014] and Holman and Uhlmann [n.d.] who prove that this is a Fourier integral operator under certain assumptions on the type of conjugate points. In certain cases, this implies in dimension $n \geq 3$ that the kernel of I_0 is finite dimensional. We also note that the results of Uhlmann and Vasy [2016] deal with certain cases with conjugate points in dimension $n \geq 3$, under the foliation by convex hypersurfaces condition.

3.3 Closed manifolds.

For closed Riemannian manifolds with Anosov geodesic flow, the main result is due to Otal [1990a] and C. B. Croke [1991] who proved the following:

Theorem 19 (Otal [1990a] and C. B. Croke [1991])**.** *Two closed Riemannian surface with negative curvature and with the same marked length spectrum are isometric.*

The method of proof is purely geometric: Otal proves first that the geodesic flows for the two metrics are conjugate, and that the conjugation preserves the Liouville measure. Then he uses a sequence of clever arguments based on Gauss-Bonnet formula for triangles to show that the conjugation of the flows comes from an isometry. An extension to certain manifolds with non-positive curvature has been obtained by C. Croke, Fathi, and Feldman [1992].

For manifolds conformal one to each other, the fact that the marked length spectrum determines the conformal factor has been proved by Katok [1988]; the proof is in dimension 2 but extends to higher dimension.

Maybe the first works on this topic were done by Guillemin and Kazhdan [1980a] and Guillemin and Kazhdan [1980b], where they proved deformation rigidity of the length spectrum in negative curvature for surfaces. This was extended by C. B. Croke and Sharafutdinov [1998] in higher dimension and in the Anosov setting for surfaces by Paternain, Salo, and Uhlmann [2014] and Guillarmou [2017a].

Theorem 20 (Guillemin and Kazhdan [1980a], C. B. Croke and Sharafutdinov [1998], Paternain, Salo, and Uhlmann [2014], and Guillarmou [2017a])**.** *1) Let g_s be a one-parameter family of negatively curved metrics on a closed manifold M. If g_s have the same length spectrum for all small $s \in (-\epsilon, \epsilon)$, then $g_s = \psi_s^* g_0$ for some smooth family of isometries ψ_s for s small.*
2) Let g_s be a one-parameter family of metrics with Anosov flows on a closed surface M. If g_s have the same length spectrum for all small $s \in (-\epsilon, \epsilon)$, then $g_s = \psi_s^* g_0$ for some smooth family of isometries ψ_s for s small.

These results are direct consequences of the solenoidal injectivity of the X-ray transform I_2, which is proved in negative curvature using Livsic theorem and a Mukhometov-Pestov energy identity in the same spirit as what we explained above for manifolds with boundary. For the Anosov case without negative (or non-positive) curvature assumption, the methods of Paternain, Salo, and Uhlmann [2014] and Guillarmou [2017a] use surjectivity of I_1^* and the complex structure of Riemann surfaces. The surjectivity of I_1^* consists in the construction of invariant distribution by the flow that have prescribed first Fourier coefficient in the Fourier decomposition in the fibers (that are circles). It is shown in Paternain, Salo, and Uhlmann [2014] that surjectivity of I_1^* follows from solenoidal injectivity of I_1, and that it implies surjectivity of I_2^* using Max Noether theorem, which in turn implies solenoidal injectivity of I_2. The argument is extended in Guillarmou [2017a] to prove solenoidal injectivity of I_m for all m for surfaces with Anosov geodesic flows.

Theorem 21 (Guillarmou [ibid.])**.** *If (M, g) is a closed surface with Anosov geodesic flow, I_m is solenoidal injective.*

The microlocal approach for hyperbolic flows of Faure and Sjöstrand [2011] and Dyatlov and Zworski [n.d.] and Theorem 1 is strongly used by Guillarmou [2017a] to show that the invariant distributions constructed in the surjectivity of I_1^* can be multiplied, through a careful analysis of their wave-front sets (shown to be contained in $E_s^* \cup E_u^*$).

We also mention that in the work Guillarmou [ibid.], we obtain new direct proofs of the regularity theory for the Livsic cohomological equation $Xu = f$ of Anosov flows, including in Sobolev spaces (which was not done), extending some results of de la Llave, Marco, and Moriyón [1986] and Journé [1986].

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