Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 3 (2485–2512)

DE GIORGI–NASH–MOSER AND HÖRMANDER THEORIES: NEW INTERPLAYS

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Abstract

We report on recent results and a new line of research at the crossroad of two major theories in the analysis of partial differential equations. The celebrated De Giorgi–Nash–Moser theorem provides Hölder estimates and the Harnack inequality for uniformly elliptic or parabolic equations with rough coefficients in divergence form. The theory of hypoellipticity of Hörmander provides general "bracket" conditions for regularity of solutions to partial differential equations combining first and second order derivative operators when ellipticity fails in some directions. We discuss recent extensions of the De Giorgi–Nash–Moser theory to hypoelliptic equations of Kolmogorov (kinetic) type with rough coefficients. These equations combine a first-order skew-symmetric operator with a second-order elliptic operator involving derivatives in only part of the variables, and with rough coefficients. We then discuss applications to the Boltzmann and Landau equations in kinetic theory and present a program of research with some open questions.

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MSC2010: primary 35B65; secondary 82C40, 35R05, 35H10, 35Q20, 82D10.

1 Introduction

1.1 Kinetic theory. Modern physics goes back to Newton and classical mechanics, and was later expanded into the understanding of electric and magnetic forces were (Ampère, Faraday, Maxwell), large velocities and large scales (Lorentz, Poincaré, Minkowski, Einstein), small-scale particle physics and quantum mechanics (Planck, Einstein, Bohr, Heisenberg, Born, Jordan, Pauli, Fermi, Schrödinger, Dirac, De Broglie, Bose, etc.). However, all these theories are classically devised to study one physical system (planet, ship, motor, battery, electron, spaceship, etc.) or a small number of systems (planets in the Solar system, electrons in a molecule, etc.) In many situations though, one needs to deal with an assembly made up of elements so numerous that their individual tracking is not possible: galaxies made of hundreds of billions of stars, fluids made of more than 10^{20} molecules, crowds made of thousands of individuals, etc. Taking such large numbers into account leads to new effective laws of physics, requiring different models and concepts. This passage from microscopic rules to macroscopic laws is the founding principle of statistical physics. All branches of physics (classical, quantum, relativistic, etc.) can be studied from the point of view of statistical physics, in both stationary and dynamical perspectives. It was first done with the laws of classical mechanics, which resulted in kinetic theory, discovered by Maxwell [1867] and Boltzmann [1872] in the 19th century after precursory works by D. Bernoulli, Herapath, Waterston, Joule, König and Clausius.

Kinetic theory replaces a huge number of objects, whose physical states are described by a certain phase space, and whose properties are otherwise identical, by a *statistical distribution* over that phase space. The fundamental role played by the velocity (kinetic) variables inacessible to observation was counter-intuitive, and accounts for the denomination of *kinetic* theory. The theory introduces a distinction between three scales: the macroscopic scale of phenomena which are accessible to observation; the microscopic scale of molecules and infinitesimal constituents; and an intermediate scale, loosely defined and often called *mesoscopic*. This is the scale of phenomena which are not accessible to macroscopic observation but already involve a large number of particles, so that statistical effects are significant.

1.2 Main equations of kinetic theory. Maxwell wrote the first (weak) form of the evolution equation known now as the **Boltzmann equation**: the unknown is a (non-negative) density function $f(t, x, v)$, standing for the density of particles at time t in the phase space (x, v) (equipped with the reference Liouville measure $\mathrm{d}x\,\mathrm{d}v$); the equation, in modern writing and assuming the absence of external forces, is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f). \tag{1.1}$$

The left-hand side describes the evolution of f under the action of transport with *free streaming operator*. The right-hand side describes elastic collisions with the nonlinear *Boltzmann collision operator*:

$$(1.2)\quad Q(f,f) = \\ = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\omega)\Big(f(t,x,v')\,f(t,x,v'_*) - f(t,x,v)\,f(t,x,v_*)\Big)\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Note that this operator is localized in t and x, quadratic, and has the structure of a tensor product with respect to $f(t,x,\cdot)$. The velocities v' and v'_* should be thought of as the velocities of a pair of particles before collision, while v and v_* are the velocities after that collision: the formulas are $v' = v - \langle v-v_*,\omega\rangle\,\omega$ and $v'_* = v_* + \langle v-v_*,\omega\rangle\,\omega$. When one computes (v,v_*) from (v',v'_*) (or the reverse), conservation laws of the mass, momentum and energy are not enough to yield the result, with only 4 scalar conservation laws for 6 degrees of freedom. The unit vector $\omega\in\mathbb{S}^2$ removes this ambiguity: in the case of colliding hard spheres, it can be thought of as the direction of the line joining the two centers of the particles. The kernel $B(v-v_*,\omega)$ describes the relative frequency of vectors ω, depending on the relative impact velocity $v-v_*$; it only depends on the modulus $|v-v_*|$ and the deflection angle θ between $v-v_*$ and $v'-v'_*$. Maxwell computed it for hard spheres ($B\sim|v-v_*|\sin\theta$) and for inverse power forces: in the latter case the kernel factorizes as the product of $|v-v_*|^\gamma$ with a function $b(\cos\theta)$; Maxwell showed that if the force is repulsive, proportional to $r^{-\alpha}$ (r the interparticle distance), then $\gamma = (\alpha-5)/(\alpha-1)$ and $b(\cos\theta)\simeq\theta^{-(1+2s)}$ as $\theta\to 0$, where $2s = 2/(\alpha-1)$. In particular, the kernel is usually *nonintegrable* as a function of the angular variable: this is a general feature of *long-range interactions*, nowadays sometimes called “noncutoff property”.

The case $\alpha=5$, $\gamma=0$ and $2s=1/2$ is called *Maxwell molecules* Maxwell [1867], the case $\alpha\in(5,+\infty)$, $\gamma>0$ and $2s\in(0,1/2)$ is called *hard potentials (without cutoff)*, the case $\alpha\in[3,5)$, $\gamma\in[-1,0)$, $2s\in(1/2,1]$ is called *moderately soft potentials (without cutoff)*, and finally the case $\alpha\in(2,3)$, $\gamma\in(-3,-1)$, $2s\in(1,2)$ is called *very soft potentials (without cutoff)*. The limits between hard and soft potentials ($\gamma=0$) and between moderately and very soft potentials ($\gamma+2s=0$) are commonly taken as defining the “hard” / “moderately soft” / “very soft” terminology in any dimension, for kernel of the form $B=|v-v_*|^\gamma b(\cos\theta)$ with $b(\cos\theta)\simeq\theta^{-(1+2s)}$.

In order to find the stationary solutions, that is, time-independent solutions of (1.2), the first step is to identify particular *hydrodynamic* density functions, which make the collision contribution vanish: these are *Gaussian distributions with a scalar covariance* $f(v) = \rho\,(2\pi T)^{-3/2}\,e^{-\frac{|v-u|^2}{2T}}$, where the parameters $\rho>0$, $u\in\mathbb{R}^3$ and $T>0$ are the local density, mean velocity, and temperature of the fluid. These parameters can be fixed throughout the whole domain (providing in this case an *equilibrium distribution*), or depend on the position x and time t; in both cases collisions will have no effect. As pointed out in Maxwell's seminar paper, and later proved rigorously at least in some settings Bardos, Golse, and Levermore [1991, 1993] and Golse and Saint-Raymond [2004, 2005, 2009], the Boltzmann equation is connected to classical fluid mechanical equations on ρ, u and T, and one leads to the other in certain regimes. This provides a rigorous connexion between the mesoscopic (kinetic) level and the macroscopic level. At the other end of the scales, a rigorous derivation of the Boltzmann equation from many-body Newtonian mechanics for short time and short-range interactions was obtained by Lanford [1975] for hard spheres; see also King [1975] for an extension to more general short-range interactions, and Gallagher, Saint-Raymond, and Texier [2013] and Pulvirenti, Saffirio, and Simonella [2014] for re-visitation and extension of the initial arguments of Lanford and King. Note however that at the moment the equivalent of Lanford theorem for the Boltzmann equation with long-range interactions is still missing, see Ayi [2017] for partial progresses.

To summarise the key mathematical points: the Boltzmann equation is an integro-(partial)-differential equation with non-local operator in the kinetic variable v. Moreover for long-range interactions with repulsive force $F(r) \sim r^{-\alpha}$, this non-local operator has a singular kernel and shows fractional ellipticity of order $2/(\alpha-1)$. The Boltzmann equation "contains" the hydrodynamic, and it is a *fundamental* equation in the sense that it is derived rigorously, at least in some settings, from microscopic first principles. From now on, we consider the position variable in $\mathbb{R}^3$ or in the periodic box $\mathbb{T}^3$.

In the limit case $s \to 1$ (the Coulomb interactions), the Boltzmann collision operator is ill-defined. Landau [1936] proposed an alternative operator for these Coulomb interactions that is now called the *Landau–Coulomb operator*

$$Q(f,f) = \\ = \nabla_v \cdot \left(\int_{\mathbb{R}^3} \mathbf{P}_{(v-v_*)^\perp} \Big(f(t,x,v_*) \nabla_v f(t,x,v) - f(t,x,v) \nabla_v f(t,x,v_*) \Big) |v-v_*|^{\gamma+2} \, \mathrm{d}v_* \right)$$

where $\mathbf{P}_{(v-v_*)^\perp}$ is the orthogonal projection along $(v-v_*)^\perp$ and $\gamma = -3$. It writes as

$$Q(f,f) = \nabla_v \cdot \big(A[f] \nabla_v f + B[f] f \big) \tag{1.3}$$

$$\text{with} \begin{cases} A[f](v) = \displaystyle\int_{\mathbb{R}^3} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{\gamma+2} f(t,x,v-w) \, \mathrm{d}w, \\ B[f](v) = -\displaystyle\int_{\mathbb{R}^3} |w|^{\gamma} \, w \, f(t,x,v-w) \, \mathrm{d}w. \end{cases}$$

This operator is a nonlinear drift-diffusion operator with coefficients given by convolution-like averages of the unknown. This is a non-local integro-differential operator, with second-order local ellipticity. The resulting *Landau equation* (1.1)-(1.3) again "contains" the hydrodynamic. It is also considered *fundamental* because of its closed link to the Boltzmann equation for Coulomb interactions (note however that the equivalent to Lanford theorem for the Landau equation is lacking, even at a formal level, see Bobylev, Pulvirenti, and Saffirio [2013] for partial progresses). Because of the difficulty to handle the very singular kernel of the Landau–Coulomb operator, it is common to introduce artificially a scale of models by letting γ vary in $[-3, 1]$ (or even $[-d, 1]$ in general dimension d). The terminology *hard potentials*, *Maxwell molecules*, *soft potentials* are used as for the Boltzmann collision operator when $\gamma > 0$, $\gamma = 0$, $\gamma < 0$ respectively. The terminology *moderately soft potentials* corresponds here (since $s = 1$) to $\gamma \in (-2, 0)$.

1.3 Open problems and conjectures.

1.3.1 The Cauchy problem. The first mathematical question when studying the previous fundamental kinetic equations (Boltzmann and Landau equations) is the Cauchy problem, i.e. existence, uniqueness and regularity of solutions. Short-time solutions have been constructed, as well as global solutions close to the trivial stationary solution or with space homogeneity: see Gualdani, Mischler, and Mouhot [2017] for some of the most recent results and the references therein for the Boltzmann equation with short-range interactions, see Alexandre, Morimoto, Ukai, Xu, and Yang [2012, 2011a] and Gressman and Strain [2011] for the Boltzmann equation with long-range interactions, and see Guo [2002] for the Landau equation. However the construction of solutions "in the large" remains a formidable open problem. Since weak "renormalised" solutions have been constructed by DiPerna and Lions [1989b] that play a similar role to the Leray [1934] solutions in fluid mechanics, this open problem can be compared with the millenium problem of the regularity of solutions to 3D incompressible Navier–Stokes equations.

1.3.2 Study of a priori solutions. Given that the Cauchy problem in the large seems out of reach at the moment, Truesdell and Muncaster [1980] remarked almost 40 years ago that: "*Much effort has been spent toward proof that place-dependent solutions exist for all time. […] The main problem is really to discover and specify the circumstances that give rise to solutions which persist forever. Only after having done that can we expect to construct proofs that such solutions exist, are unique, and are regular.*" Cercignani then formulated a precise conjecture on the entropy production along this idea in Cercignani [1982]; its resolution lead to precise new quantitative informations on a priori solutions of the Boltzmann and Landau equation (see Desvillettes and Villani [2005], Desvillettes, Mouhot, and Villani [2011], Mouhot [2006], Gualdani, Mischler, and Mouhot [2017], and Carrapatoso and Mischler [2017]). The proof of optimal relaxation rates in physical spaces, conditionnally to some regularity and moments conditions, is now fairly well understood for many interactions. The results obtained along this line of research can all be summarised into the following general form:

Conditional relaxation. Any solution to the Boltzmann (resp. Landau) equation in $L^\infty_x(\mathbb{T}^3; L^1_v(\mathbb{R}^3, (1+|v|)^k \,\mathrm{d}v))$ (or a closely related functional space as large as possible) converges to the thermodynamical equilibrium with the optimal rate dictated by the linearized equation.

Note however that an interesting remaining open question in this program is to obtain a result equivalent to Gualdani, Mischler, and Mouhot [2017] and Carrapatoso and Mischler [2017] in the case of the Boltzmann equation with long-range interactions (with fractional ellipticity in the velocity variable).

1.3.3 Regularity conjectures for long-range interactions. In the case of long-range interactions, the Boltzmann and Landau–Coulomb operators show local ellipticity provided the solution enjoys some pointwise bounds on the hydrodynamical fields

$$\rho(t,x) := \int_{\mathbb{R}^3} f \,\mathrm{d}v \qquad\qquad e(t,x) := \int_{\mathbb{R}^3} f|v|^2 \,\mathrm{d}v$$

and the local entropy $h(t,x) := \int_{\mathbb{R}^3} f \ln f \,\mathrm{d}v$. Whereas it is clear in the case of the Landau–Coulomb operator, it was understood almost two decades ago in the case of the Boltzmann collision operator Lions [1998], Villani [1999], and Alexandre, Desvillettes, Villani, and Wennberg [2000a]. This had lead colleagues working on non-local operators and fully nonlinear elliptic problems like L. Silvestre and N. Guillen and co-authors to attempt to use maximum principle techniques à la Krylov and Safonov [1980] in order to obtain pointwise bounds for solutions to these equations. These first attempts, while unsuccessful, later proved crucial in attracting the attention of a larger community on this problem. And these authors rapidly reformulated the initial goal into, again, *conditional* conjectures on the regularity of the form:

Conditional regularity. Consider any solution to the Boltzmann equation with long-range interactions (resp. Landau equation) on a time interval $[0,T]$ such that its hydrodynamical fields are bounded:

$$(1.4) \qquad \forall\, t \in [0,T],\ x \in \mathbb{T}^3, \quad m_0 \le \rho(t,x) \le m_1, \quad e(t,x) \le e_1, \quad h(t,x) \le h_1$$

where $m_0, m_1, e_1, h_1 > 0$. Then the solution is bounded and smooth on $(0,T]$.

Note that this conjecture can be strenghtened by removing the assumption that the mass is bounded from below and replacing it by a bound from below on the total mass

$\int_{\mathbb{T}^3} \rho(t,x)\,\mathrm{d}x \geq M_0 > 0$. Mixing in velocity through collisions combined with transport effects indeed generate lower bounds in many settings, see Mouhot [2005], Filbet and Mouhot [2011], Briant [2015a], and Briant [2015b]; moreover it was indeed proved for the Landau equation with moderately soft potentials in Henderson, Snelson, and Tarfulea [2017].

This conjecture is now been partially solved in the case of the Landau equation, when the interaction is “moderately soft” $\gamma \in (-2, 0)$. This result has been the joint efforts of several groups Golse, Imbert, Mouhot, and Vasseur [2017], Henderson and Snelson [2017a], Henderson, Snelson, and Tarfulea [2017], and Imbert and Mouhot [2018], and this is the object of the next section. It is currently an ongoing program of research in the case of the Boltzmann equation with hard and moderately soft potentials, and this is the object of the fourth and last section. The conjecture interestingly remains open in the case of very soft potentials for both equations, and making progress in this setting is likely to require new conceptual tools.

2 De Giorgi–Nash–Moser meet Hörmander

2.1 The resolution of Hilbert 19-th problem. The De Giorgi–Nash–Moser theory De Giorgi [1956, 1957], Nash [1958], Moser [1960], and Moser [1964] was born out of the attempts to answer Hilbert’s 19th problem. This problem is about proving the analytic regularity of the minimizers u of an energy functional $\displaystyle\int_U L(\nabla u)\,\mathrm{d}x$, with $u : \mathbb{R}^d \to \mathbb{R}$ and where the *Lagrangian* $L : \mathbb{R}^d \to \mathbb{R}$ satisfies growth, smoothness and convexity conditions and $U \subset \mathbb{R}^d$ is some compact domain. The Euler–Lagrange equations for the minimizers take the form

$$\nabla \cdot \Big[\nabla L(\nabla u)\Big] = 0 \quad \text{i.e.} \quad \sum_{i,j=1}^{d} \underbrace{\big[(\partial_{ij} L)(\nabla u)\big]}_{b_{ij}} \partial_{ij} u = \sum_{i,j=1}^{d} b_{ij} \partial_{ij} u = 0.$$

For instance L can be the Dirichlet energy $L(p) = |p|^2$, or be nonlinear as for instance $L(p) = \sqrt{1+|p|^2}$ for minimal surfaces. With suitable assumptions on L and the domain, the pointwise control of ∇u was known. However the existence, uniqueness and regularity requires more: if $u \in C^{1,\alpha}$ with $\alpha > 0$ then $b_{ij} \in C^{\alpha}$ and Schauder estimates Schauder [1934] imply $u \in C^{2,\alpha}$. Then a bootstrap argument yields higher regularity, and analyticity follows from this regularity Bernstein [1904] and Petrowsky [1939].

Hence, apart from specific result in two dimensions Morrey [1938], the missing piece in solving Hilbert 19th problem, in the 1950s, was the proof of the Hölder regularity of ∇u.

The equation satisfied by a derivative $f := \partial_k u$ is the divergence form elliptic equation:

$$\sum_{i,j=1}^{d} \partial_i \Big[\underbrace{(\partial_{ij} L)(\nabla u)}_{a_{ij}} \partial_j f \Big] = \nabla \cdot (A \nabla f) = 0.$$

De Giorgi [1957] and Nash [1958] independently proved this Hölder regularity of f under the sole assumption that the symmetric matrix $A := (a_{ij})$ satisfies the controls $0 < \lambda \leq A \leq \Lambda$, and is measurable (no regularity is assumed). The proof of Nash uses what is now called the "Nash inequality", an $L \log L$ energy estimate, and refined estimates on the fundamental solution. The proof of De Giorgi uses an iterative argument to gain integrability, and an "isoperimetric argument" to control how oscillations decays when refining the scale of observation. Moser later gave an alternative proof Moser [1960] and Moser [1964] based on one hand on an iterative gain of integrability, formulated differently but similar to that of De Giorgi, and on the other hand on relating positive and negative Lebesgue norms through energy estimate on the equation satisfied by $g := \ln f$ and the use of a *Poincaré inequality*; the proof of Moser had and important further contribution in that it also proved the *Harnack inequality* for the equations considered, i.e. a universal control on the ratio between local maxima and local minima.

Let us mention that the De Giorgi–Nash–Moser (DGNM) theory only considers elliptic or parabolic equations in *divergence form*. An important counterpart result for non-divergence elliptic and parabolic equations was later discovered by Krylov and Safonov [1980]. The extension of the DGNM theory to hypoelliptic equations with rough coefficients that we present in this section requires the equation to be in divergent form. It is an open problem whether the Krylov–Safonov theory extends to hypoelliptic non-divergent equations of the form discussed below.

2.2 The theory of hypoellipticity. The DGNM theory has revolutionised the study of nonlinear elliptic and parabolic partial differential equations (PDEs). However it remained limited to PDEs where the diffusion acts in all directions of the phase space. In kinetic theory, as soon as the solution is non spatially homogeneous, the diffusion or fractional diffusion in velocity is combined to a conservative Hamiltonian dynamic in position and velocity. This structure is called *hypoelliptic*. It can be traced back to the short note of Kolmogoroff [1934]. The latter considered the combination of free transport with drift-diffusion in velocity: the law satisfies what is now sometimes called the *Kolmogorov equation*, that writes $\partial_t f + v \cdot \partial_x f = \Delta_v f$ on $x, v \in \mathbb{R}^d$ in the simpler case. It is the equation satisfied by the law of a Brownian motion integrated in time. Kolmogorov then wrote the fundamental solution associated with a Dirac distribution δ_{x_0,v_0} initial data:

$$G(t,x,v) = \left(\frac{\sqrt{3}}{2\pi t^2}\right)^d \exp\left\{-\frac{3|x-x_0-tv_0+t(v-v_0)/2|^2}{t^3} - \frac{|v|^2}{4t}\right\}.$$

The starting point of Hörmander's seminal paper Hörmander [1967] is the observation that this fundamental solution shows regularisation in all variables, even though the diffusion acts only in the velocity variable. The regularisation in (t,x) is produced by the interaction between the transport operator $v \cdot \nabla_x$ and the diffusion in v. Hörmander's paper then proposes precise geometric conditions for this regularisation, called *hypoelliptic*, to hold, based on commutator estimates. In short, given X_0, X_1, …, X_n a collection of smooth vector fields on $\mathbb{R}^N$ and the second-order differential operator $L = \frac{1}{2}\sum_{i=1}^n X_i^2 + X_0$, then the semigroup e^{tL^*} is regularising (hypoelliptic) as soon as the Lie algebra generated by X_0, …, X_n has dimension N throughout the domain of L.

Let us also mention the connexion with the *Malliavin calculus* in probability, which gives a probabilistic proof to the Hörmander theorem in many settings, see Malliavin [1978] as well as the many subsequent works, for instance Kusuoka and Stroock [1984], Kusuoka and Stroock [1985], Bismut [1981], and Norris [1986].

2.3 Extending the DGNM theory to hypoelliptic settings. The main question of interest here is the extension of the DGNM theory to hypoelliptic PDEs of divergent type. Hypoelliptic PDEs of second order naturally split into two classes: "type I" when the operator is a sum of squares of vector fields ($X_0 = 0$ in the Hörmander form described above), and the "type II" such as the Kolmogorov above, where the operator combines a first-order anti-symmetric operator with some partially diffusive second-order operator. Two main research groups had already been working on the question. On the one hand, Polidoro and collaborators Polidoro [1994], Manfredini and Polidoro [1998], Polidoro and Ragusa [2001], Pascucci and Polidoro [2004], and Di Francesco and Polidoro [2006] had generalised the DGNM theory to the "type I" equations and had obtained the improvement of integrability for the "type II" equations, as well as the Hölder regularity when assuming some continuity property on the coefficients. On the other hand, Wang and Zhang [2009, 2011] and Zhang [2011] had extended the proof of Moser for the "type II" equations to obtain Hölder regularity, with intricate technical calculations that did not seem easy to export. Note also that the use of the DGNM theory in kinetic theory had also been advocated almost a decade before in the premonitory lecture notes Villani [2003].

We present here the work Golse, Imbert, Mouhot, and Vasseur [2017] (see also the two previous related preprints Golse and Vasseur [2015] and Imbert and Mouhot [2015]) that (1) provides an elementary and robust proof of the gain of integrability and Hölder regularity in this "type II" hypoelliptic setting, (2) proves the stronger *Harnack inequality* for these equations (i.e. a quantitative version of the strong maximum principle).

Let us consider the following kinetic Fokker–Planck equation

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + s, \quad t \in (0,T),\ (x,v) \in \Omega, \tag{2.1}$$

where Ω is an open set of $\mathbb{R}^{2d}$, $f = f(t,x,v)$, B and s are bounded measurable coefficients depending in (t,x,v), and the $d \times d$ real matrices A, B and source term s are measurable and satisfy

$$0 < \lambda I \leq A \leq \Lambda I, \qquad |B| \leq \Lambda, \qquad s \text{ essentially bounded} \tag{2.2}$$

for two constants $\lambda, \Lambda > 0$. Given $z_0 = (x_0, v_0, t_0) \in \mathbb{R}^{2d+1}$, we define the "cylinder" $Q_r(z_0)$ centered at z_0 of radius r that respects the invariances of the equation:

$$Q_r(z_0) := \left\{(x,v,t) : |x - x_0 - (t-t_0)v_0| < r^3, |v - v_0| < r, t \in \left(t_0 - r^2, t_0\right]\right\}. \tag{2.3}$$

The *weak solutions* to equation (2.1) on $U_x \times U_v \times I$, $U_x \subset \mathbb{R}^d$ open, $U_v \subset \mathbb{R}^d$ open, $I = [a,b]$ with $-\infty < a < b \leq +\infty$, are defined as functions $f \in L^\infty_t(I, L^2_{x,v}(U_x \times U_v))) \cap L^2_{x,t}(U_x \times I, H^1_v(U_v))$ such that $\partial_t f + v \cdot \nabla_x f \in L^2_{x,t}(U_x \times I, H^{-1}_v(U_v))$ and f satisfies the equation (2.1) in the sense of distributions.

Theorem 1 (Hölder continuity Golse, Imbert, Mouhot, and Vasseur [2017]). *Let f be a weak solution of (2.1) in $\mathbb{Q}_0 := Q_{r_0}(z_0)$ and $\mathbb{Q}_1 := Q_{r_1}(z_0)$ with $r_1 < r_0$. Then f is α-Hölder continuous with respect to (x,v,t) in $\mathbb{Q}_1$ and*

$$\|f\|_{C^\alpha(\mathbb{Q}_1)} \leq C \left(\|f\|_{L^2(\mathbb{Q}_0)} + \|s\|_{L^\infty(\mathbb{Q}_0)}\right)$$

for some $\alpha \in (0,1)$ and $C > 0$ only depending on d, λ, Λ, r_0, r_1 (plus z_0 for C).

In order to prove such a result, we first prove that L^2 sub-solutions are locally bounded; we refer to such a result as an $L^2 - L^\infty$ estimate. We then prove that solutions are Hölder continuous by proving a lemma which is a hypoelliptic counterpart of De Giorgi's "isoperimetric lemma". We moreover prove the Harnack inequality:

Theorem 2 (Harnack inequality Golse, Imbert, Mouhot, and Vasseur [ibid.]). *If f is a non-negative weak solution of (2.1) in $Q_1(0,0,0)$, then*

$$\sup_{Q^-} f \leq C \left(\inf_{Q^+} f + \|s\|_{L^\infty(Q_1(0,0,0))}\right) \tag{2.4}$$

where $Q^+ := Q_R(0,0,0)$ and $Q^- := Q_R(0,0,-\Delta)$ and $C > 1$ and $R, \Delta \in (0,1)$ are small (in particular $Q^\pm \subset Q_1(0,0,0)$ and they are disjoint), and universal, i.e. only depend on dimension and ellipticity constants.

Remark 3. Using the transformation $\mathcal{T}_{z_0}(x, v, t) = (x_0 + x + t v_0, v_0 + v, t_0 + t)$, we get a Harnack inequality for cylinders centered at an arbitrary point $z_0 = (x_0, v_0, t_0)$.

Our proof combines the key ideas of De Giorgi and Moser and the *velocity averaging method*, which is a special type of smoothing effect for solutions of the free transport equation $(\partial_t + v \cdot \nabla_x) f = S$ observed for the first time in Agoshkov [1984] and Golse, Perthame, and Sentis [1985] independently, later improved and generalized in Golse, Lions, Perthame, and Sentis [1988] and DiPerna and Lions [1989a]. This smoothing effect concerns averages of f in the velocity variable v, i.e. expressions of the form $\int_{\mathbb{R}^d} f(x, v, t)\, \phi(v)\, \mathrm{d}v$ with, say, $\phi \in C_c^\infty$. Of course, no smoothing on f itself can be observed, since the transport operator is hyperbolic and propagates the singularities. However, when S is of the form $S = \nabla_v \cdot (A(x, v, t) \nabla_v f) + s$, where s is a given source term in L^2, the smoothing effect of velocity averaging can be combined with the H^1 regularity in the v variable implied by the energy inequality in order to obtain some amount of smoothing on the solution f itself. A first observation of this type (at the level of a compactness argument) can be found in Lions [1994]; Bouchut [2002] had then obtained quantitative Sobolev regularity estimates.

Our proof of the $L^2 - L^\infty$ gain of integrability follows the so-called “De Giorgi–Moser iteration”, see Golse, Imbert, Mouhot, and Vasseur [2017] where it is presented in both the equivalent presentations of De Giorgi and of Moser. We emphasize that, in both approaches, the main ingredient is a local gain of integrability of non-negative sub-solutions. This latter is obtained by combining a comparison principle and a Sobolev regularity estimate following from the velocity averaging method discussed above and energy estimates. We then prove the Hölder continuity through a De Giorgi type argument on the decrease of oscillation for solutions. We also derive the Harnack inequality by combining the decrease of oscillation with a result about how positive lower bounds on non-negative solutions deteriorate with time. It is worth mentioning here that our “hypoelliptic isoperimetric argument” is proved non-constructively, by a contradiction method, whereas the original isoperimetric argument of De Giorgi is obtained by a quantitative direct argument. It is an interesting open problem to obtain such quantitative estimates in the hypoelliptic case.

3 Conditional regularity of the Landau equation

3.1 Previous works and a conjecture. The infinite smoothing of solutions to the Landau equation has been investigated so far in two different settings. On the one hand, it has been investigated for weak spatially homogeneous solutions (non-negative in L^1 and with finite energy) see Desvillettes and Wennberg [2004] and the subsequent follow-up papers Alexandre and El Safadi [2005], Huo, Morimoto, Ukai, and Yang [2008], Alexandre, Morimoto, Ukai, Xu, and Yang [2008], Alexandre and Elsafadi [2009], Morimoto, Ukai, Xu, and Yang [2009], Arsen'ev and Buryak [1990], Desvillettes [2004], Villani [1998], and Desvillettes and Villani [2000a], and see also the related entropy dissipation estimates in Desvillettes and Villani [2000b] and Desvillettes [2015], and see the analytic regularisation of weak spatially homogeneous solutions for Maxwellian or hard potentials in H. Chen, Li, and Xu [2010]. Furthermore, Silvestre [2016b] derives an L^∞ bound (gain of integrability) for spatially homogeneous solutions in the case of moderately soft potentials without relying on energy methods. Let us also mention works studying modified Landau equations Krieger and Strain [2012] and Gressman, Krieger, and Strain [2012] and the work Gualdani and Guillen [2016] that shows that any weak radial solution to the Landau–Coulomb equation that belongs to $L^{3/2}$ is automatically bounded and C^2 using barrier arguments. On the other hand, the investigations of the regularity of classical spatially heterogeneous solutions had been more sparse, focusing on the regularisation of classical solutions see Y. Chen, Desvillettes, and He [2009] and Liu and Ma [2014].

The general question of conditional regularity hence suggests the following question in the context of the Landau equation:

Conjecture 1. Any solutions to the Landau equation (1.1)- (1.3) (with Coulomb interaction $\gamma = -3$) on $[0, T]$ satisfying (1.4) is bounded and smooth on $(0, T]$.

An important progress has been made by solving a weaker version of this conjecture when the exponent $\gamma \in (-2, 0)$, which corresponds to *moderately soft potentials*, i.e. $\gamma + 2s > 0$ since here $s = 1$. We describe in this section the different steps and combined efforts of different groups.

3.2 DGNM theory and local Hölder regularity.

The first step was the work Golse, Imbert, Mouhot, and Vasseur [2017] already mentioned. A corollary of the general regularity theorem, Theorem 1, is the following:

Theorem 4 (Local Hölder regularity for the LE Golse, Imbert, Mouhot, and Vasseur [ibid.]). *Given any $\gamma \in [-3, 1]$, there are universal constants $C > 0$, $\alpha \in (0, 1)$ such that any f essentially bounded weak solution of* (1.1)- (1.3) *in $B_1 \times B_1 \times (-1, 0]$ satisfying* (1.4) *is α-Hölder continuous with respect to $(x, v, t) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times (-\frac{1}{2}, 0]$ and*

$$\|f\|_{C^\alpha(B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times (-\frac{1}{2},0])} \le C \left(\|f\|_{L^2(B_1 \times B_1 \times (-1,0])} + \|f\|^2_{L^\infty(B_1 \times B_1 \times (-1,0])} \right).$$

Note that this theorem includes the physical case of Coulomb interactions $\gamma = -3$. The adjective "universal" for the constants refers to their independence from the solution.

3.3 Maximum principles and pointwise bounds.

This line of research originates in the work of L. Silvestre both on the spatially homogeneous Boltzmann (SHBE) and Landau (SHLE) equations Silvestre [2016a, 2017]. These papers build upon the ideas of "nonlinear maximal principles" introduced in Constantin and Vicol [2012] in the case of the Boltzmann collision operator, and upon the so-called "Aleksandrov–Bakelman–Pucci Maximum Principle" in the case of the Landau collision operator, see for instance Caffarelli and Cabré [1995] and Caffarelli and Silvestre [2009].

The main result of Silvestre [2017] is:

Theorem 5 (Pointwise bound for the SHLE). *Let $\gamma \in [-2, 0]$ (moderately soft potentials) and f a classical spatially homogeneous solution to the Landau equation* (1.1)- (1.3) *satisfying the assumptions* (1.4). *Then $f \lesssim 1 + t^{-3/2}$ with constant depending only on the bounds* (1.4).

As noted by the author, this estimate implies quite straightforwardly existence, uniqueness and infinite regularity for the spatially homogeneous solution. For the difficult case of very soft potentials $\gamma \in [-3, 2)$, this paper includes a weaker result where the L^∞ bound depends on a certain weighted Lebesgue norms; unfortunately it is not yet known how to control such norm along time. This conceptual barrier when crossing the "very soft potentials threshold" is reminiscent of the situation for the Cauchy theory in Lebesgue and Sobolev spaces of both the spatially homogeneous Boltzmann with long-range interactions Desvillettes and Mouhot [2009] and Landau equation Wu [2014].

The pointwise bounds estimates were then extended to the spatially inhomogeneous case in Cameron, Silvestre, and Snelson [2017]. The main result in this latter paper is:

Theorem 6 (Pointwise bound for the LE). *Let $\gamma \in (-2, 0]$ (moderately soft potentials without the limit case) and f a bounded weak solution to the Landau equation* (1.1)- (1.3) *satisfying the assumptions* (1.4). *Then $f \lesssim (1 + t^{-3/2})(1 + |v|)^{-1}$ with constant depending only on the bounds* (1.4) *(and* not *on the L^∞ norm of the solution). Moreover if $f_{in}(x, v) \le C_0 e^{-\alpha|v|^2}$, for some $C_0 > 0$ and a sufficiently small $\alpha > 0$ (depending on γ and* (1.4)*), then $f(t, x, v) \le C_1 e^{-\alpha|v|^2}$ with $C_1 > 0$ depending only on C_0, γ and the bounds* (1.4).

The proof relies on using locally the Harnack inequality in Theorem 2 adapted to the Landau equation and on devising a clever change of variable to track how this local estimate behaves at large velocities. The Gaussian bound is then obtained by combining existing maximum principle arguments at large velocities (using that well-constructed Gaussians provide supersolutions at large v) in the spirit of Gamba, Panferov, and Villani [2009], and the previous pointwise bound for not-so-large velocities. Finally the authors remarked that the Hölder regularity estimate of Theorem 4 can be made global using the Gaussian decay bound.

3.4 Schauder estimates and higher regularity. Once the L^∞ norm and the Hölder regularity is under control, the next step is to obtain higher-order regularity. The classical tool is the so-called *Schauder estimates* Schauder [1934]. The purpose of such estimates in general is to show that the solution to an elliptic or parabolic equation whose coefficients are Hölder continuous gains two derivatives with respect to the data (source term, initial data). The gain of the two derivatives is obtained in Hölder spaces: $C^\delta \to C^{2+\delta}$.

Two works have been obtained independently along this line of research. The first one Henderson and Snelson [2017b] focuses on the use of combination of Hölder estimates, maximum principles and Schauder estimates to obtain conditional infinite regularity for solutions to the Landau equation with moderately soft potentials $\gamma \in (-2, 0)$. The second one Imbert and Mouhot [2018] focuses on the use of these ingredients to "break the super-criticality" of the nonlinearity for a toy model of the Landau equation. Both these works develop, in different technical ways, Schauder estimates for this hypoelliptic equation.

The main result in Henderson and Snelson [2017b] is:

Theorem 7 (Conditional regularity for LE)**.** *Let $\gamma \in (-2, 0)$ (moderately soft potentials without the limit case) and f a bounded weak solution to the Landau equation* (1.1)- (1.3) *satisfying the assumptions* (1.4) *and $f_{in}(x, v) \le C_0 e^{-\alpha |v|^2}$, for some $C_0 > 0$ and a sufficiently small $\alpha > 0$ (depending on γ and* (1.4)*). Then f is smooth and its derivatives have some (possibly weaker) Gaussian decay.*

Note that (1) the regularity and decay bounds are uniform in time, as long as the bounds (1.4) remain uniformly bounded in time, (2) further conditional regularity are given in the paper for very soft potentials $\gamma \in [-3, -2]$ but they require higher $L^\infty_{t,x} L^1_v$ moments and the constants depend on time when $\gamma \in [-3, -5/2]$ in dimension 3, (3) a useful complementary result is provided by Henderson, Snelson, and Tarfulea [2017] where a local existence is proved in weighted locally uniform Sobolev spaces and the lower bound on the mass is relaxed by using the regularity to find a ball where the solution is uniformly positive: the combination of the two papers provide a conditional existence, uniqueness and regularity result for soft potentials, conditionally to upper bounds on the local mass, energy and entropy.

The work Imbert and Mouhot [2018] considers the toy model:

$$\partial f + v \cdot \nabla_x f = \rho[f] \nabla_v \left(\nabla_v f + v f \right), \quad \rho[f] := \int_{\mathbb{R}^d} f \, \mathrm{d}v, \tag{3.1}$$

in $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, $d \ge 1$. This model preserves the form of the steady state, the ellipticity in v, the non-locality, the bilinearity and the mass conservation of the LE. It however greatly simplifies the underlying hydrodynamic and the maximum principle structure. Here $H^k(\mathbb{T}^d \times \mathbb{R}^d)$ denotes the standard L^2-based Sobolev space. The main result states (note that solutions are constructed and not conditional here):

Theorem 8. *For all initial data f_{in} such that $f_{in}/\sqrt{\mu} \in H^k(\mathbb{T}^d \times \mathbb{R}^d)$ with $k > d/2$ and satisfying $C_1\mu \le f_{in} \le C_2\mu$, there exists a unique global-in-time solution f of (3.1) with initial data f_{in} satisfying for all time $t > 0$: $f(t)/\sqrt{\mu} \in H^k(\mathbb{T}^d \times \mathbb{R}^d)$ and $C_1\mu \le f \le C_2\mu$ and $f(t,\cdot,\cdot) \in C^\infty$.*

Note that the initial regularity could be relaxed with more work. A key step of the proof is the Schauder estimate. It gives the following additional information on this solution: the *hypoelliptic Hölder norm* $\mathcal{H}^\alpha$ (defined below) of $f/\sqrt{\mu}$ is uniformly bounded in terms of the L^2 norm of $f_{in}/\sqrt{\mu}$ for times away from 0. This norm is defined on a given open connected set $\mathcal{Q}$ by

$$\|g\|_{\mathcal{H}^\alpha(\mathcal{Q})} := \sup_{\mathcal{Q}} |g| + \sup_{\mathcal{Q}} |(\partial_t + v\cdot\nabla_x)g| + \sup_{\mathcal{Q}} |D_v^2 g| + [(\partial_t + v\cdot\nabla_x)g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} + [D_v^2 g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})}$$

where $[\cdot]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})}$ is a Hölder anisotropic semi-norm, i.e. the smallest $C > 0$ such that

$$\forall\, z_0 \in \mathcal{Q},\ r > 0 \ \text{ s.t. } \ Q_r(z_0) \subset \mathcal{Q}, \quad \|g - g(z_0)\|_{L^\infty(Q_r(z_0))} \le C r^\alpha$$

where

$$Q_r(z_0) := \left\{ z : \frac{1}{r}(z_0^{-1} \circ z) \in Q_1 \right\} = \left\{ (t,x,v) : t_0 - r^2 < t \le t_0,\ |x - x_0 - (t-t_0)v_0| < r^3,\ |v - v_0| < r \right\}$$

and $rz := (r^2 t, r^3 x, rv)$ and $z_1 \circ z_2 := (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2)$.

The specific contribution of this work is the study of the Cauchy problem: the maximum principle provides Gaussian upper and lower bounds on the solution, and we then provide energy estimates and a blow-up criterion *à la* Beale, Kato, and Majda [1984]. We then use the extensions of the DGNM and Schauder theories to control the quantity governing the blow-up. We prove Hölder regularity through the method of Golse, Imbert, Mouhot, and Vasseur [2017]. We then develop Schauder estimates following the method of Krylov [1996] (see also Polidoro [1994], Manfredini [1997], Di Francesco and Polidoro [2006], Bramanti and Brandolini [2007], Lunardi [1997], Radkevich [2008], and Henderson and Snelson [2017a]). New difficulties arise compared with the parabolic case treated in Krylov [1996] in relation with the hypoelliptic structure and we develop trajectorial hypoelliptic commutator estimates to solve them and also borrow some ideas from hypocoercivity Villani [2009] in the so-called gradient estimate.

Note that it would be interesting to give a proof of Schauder estimates for such hypoelliptic equations that is entirely based on scaling arguments in the spirit Simon [1997] (see also the proof and use of such estimates in Hairer [2014]). This might indeed prove useful for generalising such estimates to the integral Boltzmann collision operator, see the next section.

4 Conditional regularity of the Boltzmann equation

4.1 Previous works and a conjecture. Short time existence of solutions to (1.1)- (1.2) was obtained in Alexandre, Morimoto, Ukai, Xu, and Yang [2010a] for sufficiently regular initial data f_0. Global existence was obtained in Desvillettes and Mouhot [2009] for moderately soft potentials in the spatially homogeneous case. In the next subsections, we present the progresses made so far in the case of moderately soft potentials: the estimate in L^∞ for $t > 0$ was obtained in Silvestre [2016a], the local Hölder regularity in Imbert and Silvestre [2017], and finally the polynomial pointwise decay estimates in Imbert, Mouhot, and Silvestre [2018]. The bootstrap mechanism to obtain higher regularity through Schauder estimates remains however unsolved at now.

Let us briefly review the existing results about regularisation. In Alexandre, Morimoto, Ukai, Xu, and Yang [2010a], the authors prove that if the solution f has five derivatives in L^2, with respect to all variables t, x and v, weighted by $(1 + |v|)^q$ for arbitrarily large powers q, and in addition the mass density is bounded below, then the solution f is C^∞. It is not known however whether these hypotheses are implied by (1.4). Note also the previous partial result Desvillettes and Wennberg [2004] and the subsequent follow-up papers Alexandre and El Safadi [2005], Huo, Morimoto, Ukai, and Yang [2008], Alexandre, Morimoto, Ukai, Xu, and Yang [2008], Alexandre and Elsafadi [2009], and Morimoto, Ukai, Xu, and Yang [2009] in the spatially homogeneous case, with less assumptions on the initial data.

Note that, drawing inspiration from the case of the Landau equation, in order for the iterative gain of regularity in Henderson and Snelson [2017b] to work, it is necessary to start with a solution that decays, as $|v| \to \infty$, faster than any algebraic power rate $|v|^{-q}$. We expect the same general principle to apply to the Boltzmann equation, even the appropriate Schauder type estimates for kinetic integro-differential equations to carry out an iterative gain in regularity are not yet available.

Finally, we highlight the related results of regularisation for the Boltzmann equation with long-range interactions Desvillettes [1995] and Y. Chen and He [2011, 2012], and the related perturbative results for the Landau and (long-range interaction) Boltzmann equation Guo [2002], Gressman and Strain [2011], Alexandre, Morimoto, Ukai, Xu, and Yang [2010b, 2011b], Alexandre [2009], Wu [2014], and Alexandre, Liao, and Lin [2015].

The question of conditional regularity suggests the following conjecture in the context of the Boltzmann equation with long-range interactions:

Conjecture 2. Any solutions to the Boltzmann equation (1.1)- (1.2) with long-range interactions $\gamma \in (-3, 1]$, $s \in (0, 1)$, $\gamma + 2s \in (-1, 1)$) on $[0, T]$ satisfying (1.4) is bounded and smooth on $(0, T]$.

The rest of this section is devoted to describing the partial progresses made in the case of, again, moderately soft potentials $\gamma + 2s > 0$.

4.2 Maximum principle and pointwise L^∞ bound. This first breakthrough is due to Silvestre [2016a]. This article draws inspiration from his own previous works on non-local operators and the "nonlinear maximum principle" of Constantin and Vicol [2012]. It is based on a maximum principle argument for a barrier supersolution that is constant in x, v and blowing-up as $t \to 0^+$; it uses the decomposition of the collision and "cancellation lemma" going back to Alexandre, Desvillettes, Villani, and Wennberg [2000b], the identification of a cone of direction for $(v' - v)$ is order to obtain lower bounds on the f-dependent kernel of the elliptic part of the operator, and finally some Chebycheff inequality and nonlinear lower bound on the collision integral. The main result is:

Theorem 9 (Pointwise bound for the BE Silvestre [2016a])**.** *Let $\gamma \in [-2, 1]$, $s \in (0, 1)$ with $\gamma + 2s > 0$ (moderately soft potentials). Let f be a classical solution to the Boltzmann equation* (1.1)- (1.3) *satisfying the assumptions* (1.4)*. Then $f \le C(1 + t^{-\beta})$ with $C > 0$ and $\beta > 0$ and constant depending only on γ, s and the bounds* (1.4)*.*

Note that the paper also includes further results in the case of very soft potentials but conditionally to additional estimates of the form $L^\infty_{t,x} L^p_v (1 + |v|^q)$ for some $p > 1, q > 0$; it is not known at present how to deduce the latter estimates from the hydrodynamic bounds (1.4).

4.3 Weak Harnack inequality and local Hölder regularity. The second breakthrough is the paper Imbert and Silvestre [2017]. In comparison to the Landau equation, the Boltzmann equation has a more complicated integral structure, that shares similarity with "fully nonlinear" fractional elliptic operators. The main result proved is:

Theorem 10 (Local Hölder regularity for the BE Imbert and Silvestre [ibid.])**.** *Given any $\gamma \in (-3, 1]$ and $s \in (0, 1)$ with $\gamma + 2s > 0$, there are universal constants $C > 0$, $\alpha \in (0, 1)$ such that any f essentially bounded weak solution of* (1.1)- (1.2) *in $B_1 \times \mathbb{R}^3 \times (-1, 0]$ satisfying* (1.4) *is α-Hölder continuous with respect to $(x, v, t) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times (-\frac{1}{2}, 0]$, where $C > 0$ and $\alpha \in (0, 1)$ are constants depending on the L^∞ bound of f and the bounds* (1.4)*.*

The proof goes in two steps. The first step is a local $L^2 \to L^\infty$ gain of integrability, following the approach of De Giorgi and Moser as reformulated in a kinetic context in Pascucci and Polidoro [2004] and Golse, Imbert, Mouhot, and Vasseur [2017]. It requires further technical work to formulate the De Giorgi iteration for such integro-differential equations with degenerate kernels (see also the related works Kassmann [2009], Felsinger and Kassmann [2013], and Bux, Kassmann, and Schulze [2017]). The regularity mechanism at the core of the averaging velocity method is used, however under a different presentation, by relying on explicit calculations on the fundamental solution of the fractional Kolmogorov equation. In the second step of the proof, the authors establish a weak Harnack inequality, i.e. an the local control from above of local $L^{\epsilon}_{t,x,v}$ averages with $\epsilon > 0$ small by a local infimum multiplied by a universal constant. This inequality is sufficient to deduce the Hölder regularity. Two different strategies are used depending on whether $s \in (0, 1/2)$ or $s \in [1/2, 1)$. In the first case, they construct a barrier function to propagate lower bounds as in the method by Krylov and Safonov for nondivergence equations. In the second case, they use a variant of the isometric argument of De Giorgi proved by compactness as in Golse, Imbert, Mouhot, and Vasseur [2017]. Again the regularity of velocity averages plays a crucial role but is exploited by direct calculation on the fundamental solution of the fractional Kolmogorov equation.

4.4 Maximum principle and decay at large velocities. Finally in the paper Imbert, Mouhot, and Silvestre [2018], the nonlinear maximum principle argument of Silvestre [2016a] is refined to obtain "pointwise counterpart" of velocity moments.

Let us recall that in order for an iterative gain of regularity similar to Henderson and Snelson [2017b] and Imbert and Mouhot [2018] to work, it is necessary to start with a solution that decays, as $|v| \to \infty$, faster than any algebraic power rate $|v|^{-q}$, and we expect the same general to be true for the Boltzmann equation. The main result established in this paper is:

Theorem 11 (Decay at large velocities for the BE Imbert, Mouhot, and Silvestre [2018])**.** *Given any $\gamma \in (-3, 1]$ and $s \in (0, 1)$ with $\gamma + 2s > 0$, there are universal constants $C > 0$, $\alpha \in (0, 1)$ such that for any f classical solution of* (1.1)- (1.2) *in $\mathbb{T}^3 \times \mathbb{R}^3 \times [0, T]$ satisfying* (1.4) *it holds for any $q > 0$: (i) pointwise polynomial decay is propagated: if $f_{in} \lesssim (1 + |v|)^{-q}$ then for all $t > 0$, $x \in T^3$ then $f(x, v, t) \le C(1 + |v|)^{-q}$, (ii) if $\gamma > 0$ all the polynomial moments are generated: $f(x, v, t) \le C'(1 + t^{-\beta})(1 + |v|)^{-q}$. All the constants depend on γ, s, q and the bounds* (1.4).

The study of large velocity decay, known as the study of *moments*, is an old and important question in kinetic equations. The study of moments was initiated for Maxwellian potentials ($\gamma = 0$) in Ikenberry and Truesdell [1956]. In the case of hard potentials ($\gamma > 0$), Povzner identities Povzner [1962], Elmroth [1983], Wennberg [1996], and Bobylev [1997] play an important role. For instance, Elmroth [1983] used them to prove that if moments are initially bounded, then they remain bounded for all times. Desvillettes [1993] then proved that only one moment of order $s > 2$ is necessary for the same conclusion to hold true. It is explained in Wennberg [1996] and Mischler and Wennberg [1999] that even the condition on one moment of order $s > 2$ can be dispensed with, in both (homogeneous) cutoff and non-cutoff case. These moment estimates were used by Bobylev [1997] in order to derive (integral) Gaussian tail estimates. In the case of soft potentials, Desvillettes [1993] proved for $\gamma \in (-1, 0)$ that initially bounded moments grow at most linearly with time and it is explained in Villani [2002] that the method applies to $\gamma \in [-2, 0)$. The case of measure-valued solutions is considered in Lu and Mouhot [2012].

However the extension of these integral moments estimates to the spatially inhomogeneous case is a hard and unclear question at the moment. The only result available is Gualdani, Mischler, and Mouhot [2018, Lemma 5.9 & 5.11] which proves the propagation and appearance of certain exponential moments for the spatially inhomogeneous Boltzmann equation for hard spheres (or hard potentials with cutoff), however in a space of the form $W^{3,1}_x L^1(1 + |v|^q)$. Another line of research opened by Gamba, Panferov, and Villani [2009] consists in establishing exponential Gaussian pointwise decay by maximum principle arguments (see also Bobylev and Gamba [2017], Alonso, Gamba, and Tasković [2017], and Gamba, Pavlović, and Tasković [2017]). However these works rely on previously establishing exponential integral moments, therefore it is not clear how to use them in this context.

We finally recall that the last part of the research program, the Schauder estimates, is missing for the Boltzmann equation with moderately soft potentials, and is an interesting open question for future researches.

Acknowledgments. This review article greatly benefited from the many discussions of the author with Cyril Imbert and Luis Silvestre, in particular concerning the works on the conditional regularity initiated by Luis Silvestre that form a key part of this review, and concerning their important joint work on the weak Harnack inequality for the Boltzmann equation. Moreover the author gratefully acknowledges the many crucial discussions with Alexis Vasseur about his insights on the De Giorgi method, and with François Golse about averaging lemma and homogeneisation. The author acknowledges partial funding by the ERC grants MATKIT 2011-2016 and MAFRAN 2017-2022.

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Received 2018-15-01.

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