

QUIVERS WITH RELATIONS FOR SYMMETRIZABLE CARTAN MATRICES AND ALGEBRAIC LIE THEORY

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Abstract

We give an overview of our effort to introduce (dual) semicanonical bases in the setting of symmetrizable Cartan matrices.

1 Introduction

One of the original motivations of Fomin and Zelevinsky for introducing cluster algebras was “to understand, in a concrete and combinatorial way, G. Lusztig’s theory of total positivity and canonical bases” [Fomin \[2010\]](#). This raised the question of finding a cluster algebra structure on the coordinate ring of a unipotent cell, and to study its relation with Lusztig’s bases. In a series of works culminating with [Geiß, Leclerc, and Schröer \[2011\]](#) and [Geiß, Leclerc, and Schröer \[2012\]](#), we showed that the coordinate ring of a unipotent cell of a symmetric Kac-Moody group has indeed a cluster algebra structure, whose cluster monomials belong to the dual of Lusztig’s semicanonical basis of the enveloping algebra of the attached Kac-Moody algebra. Since the semicanonical basis is built in terms of constructible functions on the complex varieties of nilpotent representations of the preprojective algebra of a quiver, it is not straightforward to extend those results to the setting of symmetrizable Cartan matrices, which appears more natural from the Lie theoretic point of view. The purpose of these notes is to give an overview of [Geiß, Leclerc, and Schröer \[2017a\]](#) - [Geiß, Leclerc, and Schröer \[2017d\]](#), where we are trying to make progress into this direction.

The starting point of our project was [Hernandez and Leclerc \[2016\]](#), where they observed that certain quivers with potential allowed to encode the q -characters of the Kirillov-Reshetikhin modules of the quantum loop algebra $U_q(L\mathfrak{g})$, where \mathfrak{g} is a complex simple Lie algebra of arbitrary Dynkin type. This quiver with potential served as model for the definition of our generalized preprojective algebras $\Pi = \Pi_K(C, D)$ associated to a symmetrizable Cartan matrix C with symmetrizer D over an arbitrary field K , which extends

the classical construction of [Gelfand and Ponomarev \[1979\]](#). After the completion of a preliminary version of [Geiß, Leclerc, and Schröer \[2017a\]](#) we learned that [Cecotti and Del Zotto \[2012\]](#) and [Yamakawa \[2010\]](#) had introduced similar constructions for quite different reasons. In comparison to the classical constructions of [Dlab and Ringel \[1974\]](#), [Dlab and Ringel \[1980\]](#) for a symmetrizable Cartan matrix C , we replace field extensions by truncated polynomial rings. Many of the core results of representations of species carry over to this setting if we restrict our attention to the so-called locally free modules, see [Geiß, Leclerc, and Schröer \[2017a\]](#). In particular, we have for each orientation Ω of C an algebra $H = H_K(C, D, \Omega)$ such that in many respects Π can be considered as the preprojective algebra of H . Our presentation of these results in [Section 3](#) is inspired by the thesis [Geuenich \[2016\]](#), he obtains similar results for a larger class of algebras.

Since our construction works in particular over algebraically closed fields, we can extend to our algebras H and Π several basic results about representation varieties of quivers and of varieties of nilpotent representations of the preprojective algebra of a quiver in our new context, again if we restrict our attention to locally free modules, see [Section 4](#). [Nandakumar and Tingley \[2016\]](#) obtained similar results by studying the set of K -rational points of the representation scheme of a species preprojective algebra, which is defined over certain infinite, non algebraically closed fields K .

In our setting we can take $K = \mathbb{C}$, and study algebras of constructible functions on those varieties of locally free modules and realize in this manner the universal enveloping algebra $U(\mathfrak{n})$ of the positive part \mathfrak{n} of a complex semisimple Lie algebra, together with a Ringel type PBW-basis in terms of the representations of H . For arbitrary symmetrizable Cartan matrices we can realize $U(\mathfrak{n})$ together with a semicanonical basis, modulo our *support conjecture*, see [Section 5](#).

Conventions. We use basic concepts from representation theory of finite dimensional algebras, like Auslander-Reiten theory or tilting theory without further reference. A good source for this material is [Ringel \[1984\]](#). For us, a quiver is an oriented graph $Q = (Q_0, Q_1, s, t)$ with vertex set Q_0 , arrow set Q_1 and functions $s, t: Q_1 \rightarrow Q_0$ indicating the start and terminal point of each arrow. We also write $D = \text{Hom}_K(-, K)$. We say that an A -module M is *rigid* if $\text{Ext}_A^1(M, M) = 0$.

2 Combinatorics of symmetrizable Cartan matrices

2.1 Symmetrizable Cartan matrices and quivers. Let $I = \{1, 2, \dots, n\}$. A *symmetrizable Cartan matrix* is an integer matrix $C = (c_{ij}) \in \mathbb{Z}^{I \times I}$ such that the following holds:

- $c_{ii} = 2$ for all $i \in I$ and $c_{ij} \leq 0$ for all $i \neq j$,

- there exist $(c_i)_{i \in I} \in \mathbb{N}_+^I$ such that $\text{diag}(c_1, \dots, c_n) \cdot C$ is a symmetric.

In this situation $D := \text{diag}(c_1, \dots, c_n) \in \mathbb{Z}^{I \times I}$ is called the *symmetrizer* of C . Note that the symmetrizer is not unique. In particular, for all $k \in \mathbb{N}_+$ also kD is a symmetrizer of C .

It is easy to see that the datum (C, D) of a symmetrizable Cartan matrix C and its symmetrizer D is equivalent to displaying a weighted graph (Γ, \underline{d}) with

- I the set of vertices of Γ ,
- $g_{ij} := \text{gcd}(c_{ij}, c_{ji})$ edges between i and j ,
- $\underline{d}: I \rightarrow \mathbb{N}_+, i \mapsto c_i$.

Here we agree that $\text{gcd}(0, 0) = 0$. We have then $c_{ij} = -\frac{\text{lcm}(c_i, c_j)}{c_i} g_{ij}$ for all $i \neq j$.

2.2 Bilinear forms, reflections and roots. We identify the root lattice of the Kac-Moody Lie algebra $\mathfrak{g}(C)$ associated to C with $\mathbb{Z}^I = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$, where the simple roots $(\alpha_i)_{i \in I}$ form the standard basis. We define on \mathbb{Z}^I by

$$(\alpha_i, \alpha_j)_{C, D} = c_i c_{ij},$$

a symmetric bilinear form. The *Weyl group* $W = W(C)$ is the subgroup of $\text{Aut}(\mathbb{Z}^I)$, which is generated by the simple reflections s_i for $i \in I$, where

$$s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i.$$

The *real roots* are the set

$$\Delta_{\text{re}}(C) := \cup_{i \in I} W(\alpha_i).$$

The *fundamental region* is

$$F := \{\alpha \in \mathbb{N}^I \mid \text{supp}(\alpha) \text{ is connected, and } (\alpha, \alpha_i)_{C, D} \leq 0 \text{ for all } i \in I\}.$$

Here, $\text{supp}(\alpha)$ is the full subgraph of $\Gamma(C)$ with vertex set $\{i \in I \mid \alpha(i) \neq 0\}$. Then the imaginary roots are by definition the set

$$\Delta_{\text{im}}(C) := W(F) \cup W(-F).$$

Finally the set of all roots is

$$\Delta(C) := \Delta_{\text{re}} \cup \Delta_{\text{im}}(C).$$

The positive roots are $\Delta^+(C) := \Delta(C) \cap \mathbb{N}^I$, and it is remarkable that $\Delta(C) = \Delta^+(C) \cup -\Delta^+(C)$.

A sequence $\mathbf{i} = (i_1, i_2, \dots, i_l) \in I^l$ is called a *reduced expression* for $w \in W$ if $w = s_{i_l} \cdots s_{i_2} s_{i_1}$ and w can't be expressed as a product of less than $l = l(w)$ reflections of the form s_i ($i \in I$). In this case we set

$$(2.1) \quad \beta_{\mathbf{i},k} := s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) \text{ and } \gamma_{\mathbf{i},k} := s_{i_l} s_{i_{l-1}} \cdots s_{i_{k+1}}(\alpha_{i_k})$$

for $k = 1, 2, \dots, l$, and understand $\beta_{\mathbf{i},1} = \alpha_{i_1}$ as well as $\gamma_{\mathbf{i},l} = \alpha_{i_l}$. It is a standard fact that $\beta_{\mathbf{i},k} \in \Delta^+$ for $k = 1, 2, \dots, l$, and that these roots are pairwise different. Obviously,

$$w(\beta_{\mathbf{i},k}) = -\gamma_{\mathbf{i},k} \text{ for } k = 1, 2, \dots, l.$$

The following result is well known.

Proposition 2.1. *For a connected, symmetrizable Cartan matrix C the following are equivalent:*

- C is of Dynkin type.
- The Weyl group $W(C)$ is finite.
- The root system $\Delta(C)$ is finite
- All roots are real: $\Delta(C) = \Delta_{\text{re}}(C)$.

Moreover, if in this situation \mathbf{i} is a reduced expression for w_0 , the longest element of W , then $\Delta^+ = \{\beta_{\mathbf{i},1}, \beta_{\mathbf{i},2}, \dots, \beta_{\mathbf{i},l}\}$.

2.3 Orientation and Coxeter elements. An *orientation* of C is a set $\Omega \subset I \times I$ such that

- $|\Omega \cap \{(i, j), (j, i)\}| > 0 \iff c_{ij} < 0$,
- for each sequence i_1, i_2, \dots, i_{k+1} with $(i_j, i_{j+1}) \in \Omega$ for $j = 1, 2, \dots, k$ we have $i_1 \neq i_{k+1}$.

The orientation Ω can be interpreted as upgrading the weighted graph (Γ, \underline{d}) of (C, D) to a weighted quiver (Q°, \underline{d}) with g_{ij} arrows $\alpha_{ij}^{(1)}, \dots, \alpha_{ij}^{(g_{ij})}$ from j to i if $(i, j) \in \Omega$, such that $Q^\circ = Q^\circ(C, \Omega)$ has no oriented cycles.

For an orientation Ω of the symmetrizable Cartan matrix $C \in \mathbb{Z}^{I \times I}$ and $i \in I$ we define

$$s_i(\Omega) := \{(r, s) \in \Omega \mid i \notin \{r, s\}\} \cap \{(s, r) \in I \times I \mid (r, s) \in \Omega \text{ and } i \in \{r, s\}\}.$$

Thus, in $Q^\circ(C, s_i(\Omega))$ the orientation of precisely the arrows in $Q^\circ(C, \Omega)$, which are incident with i , is changed. If i is a sink or a source of $Q^\circ(C, \Omega)$ then $s_i(\Omega)$ is also an orientation of C . It is convenient to define

$$\Omega(-, i) := \{j \in I \mid (j, i) \in \Omega\} \text{ and } \Omega(j, -) := \{i \in I \mid (j, i) \in \Omega\}.$$

We have on \mathbb{Z}^I the *non-symmetric bilinear form*

$$(2.2) \quad \langle -, - \rangle_{C, D, \Omega} : \mathbb{Z}^I \times \mathbb{Z}^I \rightarrow \mathbb{Z}, (\alpha_i, \alpha_j) \mapsto \begin{cases} c_i & \text{if } i = j, \\ c_i c_{ij} & \text{if } (j, i) \in \Omega, \\ 0 & \text{else.} \end{cases}$$

We leave it as an exercise to verify that

$$(2.3) \quad \langle \alpha, \beta \rangle_{C, D, \Omega} = \langle s_i(\alpha), s_i(\beta) \rangle_{C, D, s_i(\Omega)}$$

if i is a sink or a source for Ω .

We say that a reduced expression $\mathbf{i} = (i_1, i_2, \dots, i_l)$ of $w \in W$ is *+admissible* for Ω if i_1 is a sink of $Q^\circ(C, \Omega)$, and i_k is a sink of $Q^\circ(C, s_{i_{k-1}} \cdots s_{i_2} s_{i_1}(\Omega))$ for $k = 2, 3, \dots, l$. If moreover $l = n$ and $\{i_1, \dots, i_n\} = I$, we say that $c = s_{i_n} \cdots s_{i_2} s_{i_1}$ is the *Coxeter element* for (C, Ω) .

2.4 Kac-Moody Lie algebras. For a symmetrizable Cartan matrix $C \in \mathbb{Z}^{I \times I}$, the derived Kac-Moody Lie algebra $\mathfrak{g}' = \mathfrak{g}'(C)$ over the complex numbers has a presentation by $3n$ generators e_i, h_i, f_i ($i \in I$) subject to the following relations:

- (i) $[e_i, f_j] = \delta_{ij} h_i$;
- (ii) $[h_i, h_j] = 0$;
- (iii) $[h_i, e_j] = c_{ij} e_j, \quad [h_i, f_j] = -c_{ij} f_j$;
- (iv) $(\text{ad } e_i)^{1-c_{ij}}(e_j) = 0, \quad (\text{ad } f_i)^{1-c_{ij}}(f_j) = 0 \quad (i \neq j)$.

Note that for C of Dynkin type this is the Serre presentation of the corresponding semisimple Lie algebra. In case $\text{rank } C < |I|$ we have $\mathfrak{g}'(C) \neq \mathfrak{g}(C)$ and the latter has in this case a slightly larger Cartan subalgebra, which makes for a more complicated definition, see for example [Geiß, Leclerc, and Schröer \[2017c, Sec. 5.1\]](#) for a few more details. Of course, the main reference is [Kac \[1990\]](#).

Let $\mathfrak{n} = \mathfrak{n}(C)$ be the Lie subalgebra generated by the e_i ($i \in I$). Then $U(\mathfrak{n})$ is the associative \mathbb{C} -algebra with generators e_i ($1 \leq i \leq n$) subject to the relations

$$(2.4) \quad (\text{ad } e_i)^{1-c_{ij}}(e_j) = 0, \quad (i, j \in I, i \neq j).$$

$U(\mathfrak{n})$ is \mathbb{N}^I graded with $\deg(e_i) = \alpha_i$ ($i \in I$). With

$$\mathfrak{n}_\alpha := \mathfrak{n} \cap U(\mathfrak{n})_\alpha \text{ for } \alpha \in \Delta^+(C)$$

we recover the usual root space decomposition of \mathfrak{n} .

3 Quivers with relations for symmetrizable Cartan matrices

We keep the notations from the previous section, in particular $C \in \mathbb{Z}^{I \times I}$ is a symmetrizable Cartan matrix with symmetrizer D and Ω is an orientation for C .

3.1 A class of 1-Iwanaga-Gorenstein algebras. Let K be a field and $Q = Q(C, D, \Omega)$ the quiver obtained from $Q^\circ(C, D, \Omega)$, see [Section 2.3](#), by adding a loop ϵ_i at each vertex $i \in I$. Then $H = H_K(C, D, \Omega)$ is the path algebra KQ modulo the ideal which is generated by the following relations:

- $\epsilon_i^{c_i}$ for all $i \in I$
- $\epsilon_i^{-c_{ji}/g_{ji}} \alpha_{ij}^{(k)} - \alpha_{ij}^{(k)} \epsilon_j^{-c_{ij}/g_{ij}}$ for all $(i, j) \in \Omega$ and $k = 1, 2, \dots, g_{ij}$.

Recall that $g_{ij} = g_{ji} = \gcd(c_{ij}, c_{ji})$, thus $-c_{ij}/g_{ij} = \text{lcm}(c_i, c_j)/c_i$.

For $(i, j) \in \Omega$ let $c'_{ij} = c_{ij}/g_{ij}$ and $c'_{ji} = c_{ji}/g_{ij}$. We may consider the following symmetrizable Cartan matrix, symmetrizer and orientation:

$$C^{(i,j)} = \begin{pmatrix} 2 & c'_{ij} \\ c'_{ji} & 2 \end{pmatrix}, \quad D^{(i,j)} = \begin{pmatrix} c_i & 0 \\ 0 & c_j \end{pmatrix} \quad \text{and} \quad \Omega^{(i,j)} = \{(i, j)\}.$$

Thus,

$$Q^{(i,j)} := Q(C^{(i,j)}, \Omega^{(i,j)}) = \begin{array}{c} \epsilon_i \curvearrowright i \xleftarrow{\alpha_{ij}} j \curvearrowleft \epsilon_j \end{array}$$

and

$$H^{(i,j)} := H_K(C^{(i,j)}, D^{(i,j)}, \Omega^{(i,j)}) = KQ^{(i,j)} / \langle \epsilon_i^{c_i}, \epsilon_j^{c_j}, \epsilon_i^{-c'_{ji}} \alpha_{ij} - \alpha_{ij} \epsilon_j^{-c'_{ij}} \rangle.$$

Note, that with

$${}_i H'_j := e_i H^{(i,j)} e_j \quad \text{and} \quad H_i = e_i H_i e_i = K[\epsilon_i] / \langle \epsilon_i^{c_i} \rangle$$

it is easy to see that ${}_i H_j := {}_i H'_j \oplus g_{ij}$ is a H_i - H_j -bimodule, which is free of rank $-c_{ij}$ as a H_i -module, and free of rank $-c_{ji}$ as H_j -(right)-module. If we define similarly $H^{(j,i)} :=$

$H_K(C^{(i,j)}, D^{(i,j)}, \{(j,i)\})$ and ${}_j H'_i := e_j H^{(j,i)} e_j$, then ${}_j H_i = {}_j H'_i{}^{\oplus g_{ij}}$ is a H_j - H_i -bimodule, which is free of rank $-c_{ji}$ as H_j -module and free of rank $-c_{ij}$ as H_i -(left)-module. It is easy to see that we get an isomorphism of H_i - H_j -bimodules

$${}_i H_j \cong \text{Hom}_K({}_j H_i, K).$$

The adjunction yields for H_k -modules M_k , for $k \in \{i, j\}$, a natural isomorphism of vector spaces

$$(3.1) \quad \text{Hom}_{H_i}({}_i H_j \otimes_{H_j} M_j, M_i) \rightarrow \text{Hom}_{H_j}(M_j, {}_j H_i \otimes_{H_i} M_i), f \mapsto f^\vee.$$

Quite similarly to the representation theory of modulated graphs, in the sense of [Dlab and Ringel \[1974\]](#), we have the following basic results from [Geiß, Leclerc, and Schröer \[2017a, Prop. 6.4\]](#) and [Geiß, Leclerc, and Schröer \[ibid., Prop. 7.1\]](#).

Proposition 3.1. *Set $H := H_K(C, D, \Omega)$. With $S := \times_{i \in I} H_i$ we can consider $B := \bigoplus_{(i,j) \in \Omega} {}_i H_j$ as an S - S -bimodule and find:*

$$(a) \quad H \cong T_S(B) := \bigoplus_{j \in \mathbb{N}} B^{\otimes_S j}, \text{ i.e. } H \text{ is a tensor algebra.}$$

(b) *There is a canonical short exact sequence of H - H -bimodules*

$$0 \rightarrow H \otimes_S B \otimes_S H \xrightarrow{\delta} H \otimes_S H \xrightarrow{\text{mult}} H \rightarrow 0,$$

$$\text{where } \delta(h_l \otimes b \otimes h_r) = h_l b \otimes h_r - h_l \otimes b h_r.$$

Note that the H - H -bimodules $H \otimes_S B \otimes_S H$ and $H \otimes_S H$ are in general only projective as H -left- or right-modules, but not as bimodules. Anyway, the above sequence yields a functorial projective resolution for certain modules which we are going to define now. We say that a H -module M is *locally free* if $e_i M$ is a free H_i -module for all $i \in I$. In this case we define

$$\underline{\text{rank}}(M) := (\text{rank}_{H_i}(e_i M))_{i \in I}.$$

For example, there is a unique (indecomposable) locally free H -module E_i with $\underline{\text{rank}}(E_i) = \alpha_i$ for each $i \in I$. For later use we define for all $\mathbf{r} \in \mathbb{N}^I$ the module $\mathbf{E}^{\mathbf{r}} := \bigoplus_{i \in I} E_i^{\mathbf{r}(i)}$, and observe that $\underline{\text{rank}}(\mathbf{E}^{\mathbf{r}}) = \mathbf{r}$. Let us write down the following consequences of [Proposition 3.1](#), see [Geiß, Leclerc, and Schröer \[ibid., Sec. 3.1\]](#) and [Geiß, Leclerc, and Schröer \[ibid., Cor.7.1\]](#).

Corollary 3.2. *For H as above we have:*

(a) *The projective and injective H -modules are locally free. More precisely we have*

$$\underline{\text{rank}}(He_{i_k}) = \beta_{\mathbf{i},k} \quad \text{and} \quad \underline{\text{rank}}(De_{i_k}H) = \gamma_{\mathbf{i},k} \quad \text{for } k \in I,$$

where \mathbf{i} is a reduced expression for the Coxeter element of (C, Ω) .

(b) *Each locally free H -module M has a functorial projective resolution*

$$0 \rightarrow H \otimes_S B \otimes_S M \xrightarrow{\delta \otimes M} H \otimes_S M \xrightarrow{\text{mult}} M \rightarrow 0.$$

Moreover, if M is not locally free, then $\text{proj. dim } M = \infty$.

(c) *H is 1-Iwanaga-Gorenstein, i.e. $\text{proj. dim}_H(DH) \leq 1$ and $\text{inj. dim}_H(H) \leq 1$. Moreover an H -module M is locally free if and only if $\text{proj. dim}(M) \leq 1$.*

It follows that the Ringel (homological) bilinear form descends as the non-symmetric bilinear form (2.2) to the Grothendieck group of locally free modules, where we identify the classes of the generalized simples E_i with the coordinate vector α_i ($i \in I$), see also Geiß, Leclerc, and Schröer [2017a, Prop. 4.1].

Corollary 3.3. *If M and N are locally free H -modules, we have*

$$\dim \text{Hom}_H(M, N) - \dim \text{Ext}_H^1(M, N) = \langle \underline{\text{rank}}(M), \underline{\text{rank}}(N) \rangle_{C, D, \Omega}.$$

By combining Corollary 3.2 with standard results from Auslander-Reiten theory we obtain now the following result.

Corollary 3.4. *Let M be an indecomposable, non projective, locally free H -module such that the Auslander-Reiten translate $\tau_H M$ is locally free. Then*

$$\underline{\text{rank}}(\tau_H M) = c \cdot (\underline{\text{rank}}(M)),$$

where $c = s_{i_n} \cdots s_{i_1}$ is the Coxeter element for (C, Ω) . Moreover, if we take $R \in \mathbb{Z}^{I \times I}$, such that $D \cdot R$ is the matrix of $\langle -, - \rangle_{C, D, \Omega}$ with respect to the standard basis, we get $c = -R^{-1}(C - R)$.

This is the K -theoretic shadow of a deeper connection between the Auslander-Reiten translate and reflection functors, which we will discuss in the next subsection.

3.2 Auslander-Reiten theory and Coxeter functors. By Proposition 3.1 we may view $H = H_K(C, D, \Omega)$ as a tensor algebra. Thus, we identify a H -module M naturally with a S -module $\mathbf{M} = \bigoplus_{i \in I} M_i$ together with an element $(M_{ij})_{(i,j) \in \Omega}$ of

$$(3.2) \quad H(\mathbf{M}) := \bigoplus_{(i,j) \in \Omega} \in \text{Hom}_{H_i}(iH_j \otimes_{H_j} M_j, H_i).$$

Write $s_i(H) := H_k(C, D, s_i(\Omega))$ for any $i \in I$. If k is a sink of $Q^\circ(C, \Omega)$, we have for each H -module M a canonical exact sequence

$$(3.3) \quad 0 \rightarrow \text{Ker}(M_{k,\text{in}}) \rightarrow \bigoplus_{j \in \Omega(k, -)} {}_k H_j \otimes_{H_j} M_j \xrightarrow{M_{k,\text{in}}} M_k, \text{ where } M_{k,\text{in}} = \bigoplus_{j \in \Omega(k, -)} M_{kj}.$$

We can define now the BGP-reflection functor

$$F_k^+ : \text{rep}(H) \rightarrow \text{rep}(s_i(H)), \quad (F_k^+ M)_i = \begin{cases} M_i & \text{if } i \neq k, \\ \text{Ker}(M_{k,\text{in}}) & \text{if } i = k. \end{cases}$$

We can moreover define in this situation dually the left adjoint $F_k^- : \text{rep}(s_k(H)) \rightarrow \text{rep}(H)$. Note that k is a source of $Q^\circ(C, s_k \Omega)$. See [Geiß, Leclerc, and Schröer \[ibid., Sec. 9.2\]](#) for more details. We observe that the definitions imply easily the following:

Lemma 3.5. *If k is a sink for Ω and M is a locally free H -module which has no direct summand isomorphic to E_k and $F_k^+(M)$ is locally free, then $\underline{\text{rank}}(F_k^+ M) = s_k(\underline{\text{rank}}(M))$.*

The proof of [Geiß, Leclerc, and Schröer \[ibid., Prop. 9.6\]](#) implies the following, less obvious result:

Lemma 3.6. *Suppose that k is a sink for Ω and M a locally free rigid H -module, with no direct summand isomorphic to E_k , then $\text{Hom}_H(M, E_k) = 0$.*

We can interpret F_k^+ as a kind of APR-tilting functor [Auslander, Platzeck, and Reiten \[1979\]](#). See [Geiß, Leclerc, and Schröer \[2017a, Sec. 9.3\]](#) for a proof of this non-trivial result.

Theorem 3.7. *Let k be a sink of $Q^\circ(C, \Omega)$. Then $X := {}_H H / He_k \oplus \tau^- He_k$ is a classical tilting module for H . With $B := \text{End}_H(X)^{\text{op}}$ we have an equivalence $S : \text{rep}(s_k(H)) \rightarrow \text{rep}(B)$ such that the functors $S \circ F_k^+$ and $\text{Hom}_H(X, -)$ are isomorphic.*

Standard tilting theory arguments and Auslander–Reiten theory, together with [Lemma 3.5](#) and [Lemma 3.6](#) yield the following important consequence:

Corollary 3.8. *Let $k \in I$ be a sink for Ω and M a locally free rigid H -module, then $F_k^+(M)$ is a rigid, locally free $s_k(H)$ -module.*

Consider the algebra automorphism of H , which is defined by multiplying the non-loop arrows of $Q(C, \Omega)$ by -1 . It induces the so called *twist* automorphism $T : \text{rep}(H) \rightarrow \text{rep}(H)$. Moreover, let $s_{i_n} \cdots s_{i_2} s_{i_1}$ be the Coxeter element for (C, Ω) , corresponding to the $+$ -admissible sequence i_1, i_2, \dots, i_n , see [Section 2.3](#). Now we can define the *Coxeter functor*

$$C^+ := F_{i_n}^+ \circ \cdots \circ F_{i_2}^+ \circ F_{i_1}^+ : \text{rep}(H) \rightarrow \text{rep}(H).$$

Following ideas of [Gabriel \[1980, Sec. 5\]](#), by a careful comparison of the definitions of the reflection functors and Auslander–Reiten translate, we obtain the following result. See [Geiß, Leclerc, and Schröer \[2017a, Sec. 10\]](#) for the lengthy proof.

Theorem 3.9. *With the H - H -bimodule $Y := \text{Ext}_H^1(DH, H)$ we have an isomorphism of endofunctors of $\text{rep}(H)$:*

$$\text{Hom}_H(Y, -) \cong T \circ C^+$$

If M is locally free, we have functorial isomorphisms

$$\tau_H(M) \cong \text{Hom}_H(Y, M) \quad \text{and} \quad \tau_H^- M \cong Y \otimes_H M.$$

In particular, in this case the Coxeter functor C^+ and the Auslander–Reiten translate τ may be identified up to the twist T .

It is not true in general that the Auslander–Reiten translate of a locally free H -module is again locally free. In [Geiß, Leclerc, and Schröer \[ibid., pp. 13.6–13.8\]](#) several examples of this behavior are documented. This motivates the following definition. A H -module M is τ -locally free if $\tau^k M$ is locally free for all $k \in \mathbb{Z}$. In particular, rigid locally free modules are τ -locally free. We call an indecomposable H -module *preprojective*, resp. *preinjective*, if it is of the form $\tau^{-k}(He_i)$ resp. $\tau^k(De_i H)$ for some $k \in \mathbb{N}_0$ and $i \in I$. Thus, these modules are particular cases of rigid τ -locally free modules.

3.3 Dynkin type. By combining the findings of previous section with standard Auslander–Reiten theory and the characterization of Dynkin diagrams in [Proposition 2.1](#), we obtain the following analog of Gabriel’s theorem, see [Geiß, Leclerc, and Schröer \[ibid., Thm. 11.10\]](#).

Theorem 3.10. *Let $H = H_K(C, D, \Omega)$ be as above. There are only finitely many isomorphism classes of indecomposable, τ -locally free H -modules if and only if C is of Dynkin type. In this case the map $M \mapsto \underline{\text{rank}}(M)$ induces a bijection between the isomorphism classes of indecomposable, τ -locally free modules and the positive roots $\Delta^+(C)$. Moreover, all these modules are preprojective and preinjective.*

Note however, that even for C of Dynkin type, the algebra $H(C, D, \Omega)$ is in most cases not of finite representation type, see [Geiß, Leclerc, and Schröer \[ibid., Prop. 13.1\]](#) for details.

Let C be a symmetrizable Cartan matrix of Dynkin type and $\mathbf{i} = (i_1, i_2, \dots, i_r)$ a reduced expression for the longest element w_0 of the Weyl group W , which is $+$ -admissible for the orientation Ω . With the notation of [\(2.1\)](#) we abbreviate $\beta_j = \beta_{\mathbf{i}, j}$ for $j = 1, \dots, r$, and recall that this gives a complete list of the positive roots. By [Theorem 3.10](#) we have for each j a unique, locally free, indecomposable and rigid representation $M(\beta_j)$ with $\underline{\text{rank}}(M(\beta_j)) = \beta_j$.

Proposition 3.11. *With the above notations we have*

$$\langle \beta_i, \beta_j \rangle_{C,D,\Omega} = \begin{cases} \dim \operatorname{Hom}_H(M(\beta_i), M(\beta_j)) & \text{if } i \leq j, \\ -\dim \operatorname{Ext}_H^1(M(\beta_i), M(\beta_j)) & \text{if } i > j. \end{cases}$$

In particular, $\operatorname{Hom}_H(M(\beta_i), M(\beta_j)) = 0$ if $i > j$ and $\operatorname{Ext}_H^1(M(\beta_i), M(\beta_j)) = 0$ if $i \leq j$.

In fact, by [Theorem 3.7](#) and equation (2.3) we may assume that either $i = 1$ or $j = 1$. In any case $M(\beta_1) = E_{i_1}$ is projective. In the first case we have $\operatorname{Ext}_H^1(E_1, M(\beta_j)) = 0$. In the second case we have $\operatorname{Hom}_H(M(\beta_i), E_{i_1}) = 0$ by [Lemma 3.6](#). Now our claim follows by [Corollary 3.3](#).

The next result is an easy adaptation of similar results by [Dlab and Ringel \[1979\]](#) for species. The proof uses heavily [Proposition 3.11](#) and reflection functors. This version was worked out in Omlor's Masters thesis [Omlor \[2016\]](#), see also [Geiß, Leclerc, and Schröer \[2017d, Sec. 5\]](#).

Proposition 3.12. *With the same setup as above let $k \in \{1, 2, \dots, r\}$ and $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$ such that $\beta_k = \sum_{j=1}^r m_j \beta_j$ and $m_k = 0$. Then $M(\beta_k)$ admits a non-trivial filtration by locally free submodules*

$$0 = M_{(0)} \subset M_{(1)} \subset \dots \subset M_{(r)} = M(\beta_k)$$

such that $M_{(j)}/M_{(j-1)} \cong M(\beta_j)^{m_j}$ for $j = 1, 2, \dots, r$. It follows, that $M(\beta_k)$ has no filtration by locally free submodules

$$0 = M^{(r)} \subset M^{(r-1)} \subset \dots \subset M^{(0)} = M(\beta_k),$$

such that $\operatorname{rank}(M^{(j-1)}/M^{(j)}) = m_j \beta_j$ for $j = 1, 2, \dots, r$.

3.4 Generalized preprojective algebras. Let $\bar{Q} = \bar{Q}(C)$ be the quiver which is obtained from $Q(C, \Omega)$ by inserting for each $(i, j) \in \Omega$ additional g_{ij} arrows $\alpha_{ji}^{(1)}, \dots, \alpha_{ji}^{(g_{ij})}$ from i to j , and consider the potential

$$W = \sum_{(i,j) \in \Omega} \sum_{k=1}^{g_{ij}} (\alpha_{ji}^{(k)} \alpha_{ij}^{(k)} \epsilon_j^{-c_{ij}/g_{ij}} - \alpha_{ij}^{(k)} \alpha_{ji}^{(k)} \epsilon_i^{-c_{ji}/g_{ij}}).$$

The choice of Ω only affects the signs of the summands of W . Recall that for a cyclic path $\alpha_1 \alpha_2 \dots \alpha_l$ in \bar{Q} by definition

$$\partial_\alpha^{\text{cyc}}(\alpha_1 \alpha_2 \dots \alpha_l) := \sum_{i \in \{j \in [1, l] \mid \alpha_j = \alpha\}} \alpha_{i+1} \alpha_{i+2} \dots \alpha_l \alpha_1 \alpha_2 \dots \alpha_{i-1}.$$

The generalized preprojective algebra of H is

$$\Pi = \Pi(Q, D) := K\bar{Q}/\langle \partial_{\alpha}^{\text{cyc}}(W) \mid_{\alpha \in \bar{Q}_1}, \epsilon_i^{c_i} \mid_{i \in I} \rangle.$$

It is easy to see that Π does not depend on the choice of Ω , up to isomorphism. Notice that for $(i, j) \in \Omega$ we have

$$\partial_{\alpha_{ji}^{(k)}}^{\text{cyc}}(W) = \alpha_{ij}^{(k)} \epsilon_j^{-c_{ij}/g_{ij}} - \epsilon_i^{-c_{ji}/g_{ij}} \alpha_{ij}^{(k)}.$$

It follows, that for any orientation Ω of C we can equip $\Pi_K(C, D)$ with a \mathbb{N}_0 -grading by assigning each arrow $\alpha_{ji}^{(k)}$ with $(i, j) \in \Omega$ degree 1 and the remaining arrows get degree 0. We write then

$$\Pi_K(C, D) = \bigoplus_{i=0}^{\infty} \Pi(C, D, \Omega)_i,$$

and observe that $\Pi_K(C, D, \Omega)_0 = H_K(C, D, \Omega)$. We obtain from [Theorem 3.9](#) the following alternative description of our generalized preprojective algebra, which justifies its name:

Proposition 3.13. *Let C be a symmetrizable Cartan matrix with symmetrizer D , and Ω an orientation for C . Then, with $H = H_K(C, D, \Omega)$ we have*

$$\Pi(C, D, \Omega)_1 \cong \text{Ext}_H^1(DH, H)$$

as an H - H -bimodule, moreover

$$\Pi(C, D) \cong T_H(\text{Ext}_H^1(DH, H)) \quad \text{and} \quad {}_H\Pi(C, D) \cong \bigoplus_{i \in I, k \in \mathbb{N}_0} \tau_H^{-k} H e_i.$$

Here the first isomorphism is an isomorphism of K -algebras, and the second one of H -modules.

Similarly to [Proposition 3.1](#) we have the following straightforward description of our generalized preprojective algebra as a tensor algebra modulo canonical relations [Geiß, Leclerc, and Schröer \[2017a, Prop. 6.1\]](#), which yields a standard bimodule resolution. See [Geiß, Leclerc, and Schröer \[ibid., Sec. 12.1\]](#) for the proof, where we closely follow [Crawley-Boevey and Shaw \[2006, Lem. 3.1\]](#). See also [Brenner, Butler, and King \[2002, Sec. 4\]](#).

Proposition 3.14. *Let C be a symmetrizable, connected Cartan matrix and $\Pi := \Pi_K(C, D)$. With $\bar{B} := \bigoplus_{(i,j) \in \Omega} (i H_j \oplus_j H_i)$ we have $\Pi \cong T_S(\bar{B})/\langle \partial_{\epsilon_i}^{\text{cyc}}(W) \mid_{i \in I} \rangle$, where we interpret $\partial_{\epsilon_i}^{\text{cyc}}(W) \in \bar{B} \otimes_S \bar{B}$ in the obvious way. We obtain an exact sequence of Π - Π -bimodules*

$$(3.4) \quad \Pi \otimes_S \Pi \xrightarrow{f} \Pi \otimes_S \bar{B} \otimes_S \Pi \xrightarrow{g} \Pi \otimes_S \Pi \xrightarrow{h} \Pi \rightarrow 0,$$

where

$$f(e_i \otimes e_i) = \partial_{\epsilon_i}^{\text{cyc}}(W) \otimes e_i + e_i \otimes \partial_{\epsilon_i}^{\text{cyc}}(W), \quad g(e_i \otimes b \otimes e_j) = e_i b \otimes e_j - e_i \otimes b e_j$$

and h is the multiplication map. Moreover $\text{Ker}(f) \cong \text{Hom}_{\Pi}(D\Pi, \Pi)$ if C is of Dynkin type, otherwise f is injective.

We collect below several consequences, which can be found with detailed proofs in [Geiß, Leclerc, and Schröer \[2017a, Sec. 12.2\]](#). They illustrate that locally free Π -modules behave in many aspects like modules over classical preprojective algebras. Note that part (b) is an extension of Crawley-Boevey's remarkable formula [Crawley-Boevey \[2000, Lem. 1\]](#)

Corollary 3.15. *Let C be a connected, symmetrizable Cartan matrix, and $\Pi = \Pi_K(C, D)$ as above. Moreover, let M and N be locally free Π -modules.*

(a) *If N finite-dimensional, we have a functorial isomorphism*

$$\text{Ext}_{\Pi}^1(M, N) \cong D \text{Ext}_{\Pi}^1(N, M).$$

(b) *If M and N are finite-dimensional, we have*

$$\dim \text{Ext}_{\Pi}^1(M, N) = \dim \text{Hom}_{\Pi}(M, N) + \dim \text{Hom}_{\Pi}(N, M) - (\underline{\text{rank}}(M), \underline{\text{rank}}(N))_{C, D}.$$

(c) *If C is not of Dynkin type, $\text{proj. dim}(M) \leq 2$.*

(d) *If C is of Dynkin type, Π is a finite-dimensional, self-injective algebra and $\text{rep}_{1.f.}(\Pi)$ is a 2-Calabi-Yau Frobenius category.*

Similar to [Corollary 3.2 \(b\)](#) the complex (3.4) yields (the beginning of) a functorial projective resolution for all locally free Π -modules. Thus (a), (b) and (c) follow by exploring the symmetry of the above complex. For (d) we note that in this case Π is finite-dimensional and ${}_{\Pi}\Pi$ is a locally free module by [Theorem 3.10](#) and [Proposition 3.13](#).

4 Representation varieties

4.1 Notation. Let K be now an algebraically closed field. For Q a quiver and $\rho_j \in e_{t_j}(KQ_{\geq 2})e_{s_j}$ for $j = 1, 2, \dots, l$ we set $A = KQ/\langle \rho_1, \dots, \rho_l \rangle$. Note, that every finite dimensional basic K -algebra is of this form. We abbreviate $Q_0 = I$ and set for $\mathbf{d} \in \mathbb{N}_0^I$:

$$\text{Rep}(KQ, \mathbf{d}) := \times_{a \in Q_1} \text{Hom}_K(K^{\mathbf{d}(sa)}, K^{\mathbf{d}(ta)}) \quad \text{and} \quad \text{GL}_{\mathbf{d}} := \times_{i \in I} \text{GL}_{\mathbf{d}(i)}(K).$$

The reductive algebraic group $\text{GL}_{\mathbf{d}}$ acts on $\text{Rep}(KQ, \mathbf{d})$ by conjugation, and the $\text{GL}_{\mathbf{d}}$ -orbits correspond bijectively to the isoclasses of K -representations of Q . For $M \in$

$\text{Rep}(KQ, \mathbf{d})$ and $\rho \in e_i KQ e_j$ we can define $M(\rho) \in \text{Hom}_K(K^{\mathbf{d}(j)}, K^{\mathbf{d}(i)})$ in a natural way. We have then the $\text{GL}_{\mathbf{d}}$ -stable, Zariski closed subset

$$\text{Rep}(A, \mathbf{d}) := \{M \in \text{Rep}(KQ, \mathbf{d}) \mid M(\rho_i) = 0 \text{ for } j = 1, 2, \dots, l\}.$$

The $\text{GL}_{\mathbf{d}}$ -orbits on $\text{Rep}(A, \mathbf{d})$ correspond now to the isoclasses of representations of A with dimension vector \mathbf{d} . It is in general a hopeless task to describe the irreducible components of the affine variety $\text{Rep}(A, \mathbf{d})$.

4.2 Varieties of locally free modules for H . The set of locally free representations of $H = H_K(C, D, \Omega)$ is relatively easy to describe. Clearly, for each locally free $M \in \text{rep}(H)$ we have $\underline{\dim}(M) = D \cdot \underline{\text{rank}}(M)$.

Proposition 4.1. *For $\mathbf{r} \in \mathbb{N}^I$ we have the open subset*

$$\text{Rep}_{\text{l.f.}}(H, \mathbf{r}) := \{M \in \text{rep}(H, D \cdot \mathbf{r}) \mid M \text{ is locally free}\} \subset \text{Rep}(H, D \cdot \mathbf{r}),$$

which is irreducible and smooth with $\dim \text{rep}_{\text{l.f.}}(H, \mathbf{r}) = \dim \text{GL}_{D \cdot \mathbf{r} - \frac{1}{2}(\mathbf{r}, \mathbf{r})C, D}$.

In fact, it is well known that the modules of projective dimension at most 1 form always an open subset of $\text{rep}(A, \mathbf{d})$. One verifies next that $\text{Rep}_{\text{l.f.}}(H, \mathbf{r})$ is a vector bundle over the $\text{GL}_{D \cdot \mathbf{r}}$ -orbit $\mathcal{O}(\bigoplus_{i \in I} E_i^{\mathbf{r}(i)})$, with the fibers isomorphic to the vector space $H(\mathbf{r}) := H(\mathbf{E}^{\mathbf{r}})$, see (3.2).

This yields the remaining claims. Note that the (usually) non-reductive algebraic group

$$G_{\mathbf{r}} := \times_{i \in I} \text{GL}_{\mathbf{r}(i)}(H_i) = \text{Aut}_S(\bigoplus_{i \in I} E_i^{\mathbf{r}(i)})$$

acts on the affine space $H(\mathbf{r})$ naturally by conjugation, and the orbits are in bijection with isoclasses of locally free H -modules with rank vector \mathbf{r} .

As a consequence, if M and N are rigid, locally free modules with $\underline{\text{rank}}(M) = \underline{\text{rank}}(N)$, then already $M \cong N$, since the orbits of rigid modules are open.

4.3 Varieties of E-filtered modules for Π . Recall the description of $\Pi_K(C, D)$ in Proposition 3.14. A $\overline{H} := T_S(\overline{B})$ -module M is given by a S -module $\mathbf{M} = \bigoplus_{i \in I} M_i$ such that M_i is a H_i -module for $i \in I$, together with an element $(M_{ij})_{(i,j) \in \overline{\Omega}}$ of

$$\overline{H}(\mathbf{M}) := \bigoplus_{(i,j) \in \overline{\Omega}} \text{Hom}_{H_i}({}_i H_j \otimes_{H_j} M_j, M_i),$$

where $\bar{\Omega} = \Omega \cap \Omega^{\text{op}}$. Extending somewhat (3.3) we set

$$M_{i,\text{in}} := \left(\bigoplus_{j \in \bar{\Omega}(i,-)} \text{sgn}(i,j) M_{ij} \right) : \bigoplus_{j \in \bar{\Omega}(i,-)} {}_i H_j \otimes_{H_j} M_j \rightarrow M_i \quad \text{and}$$

$$M_{i,\text{out}} := \left(\prod_{j \in \bar{\Omega}(-,i)} M_{ji}^\vee \right) : M_i \rightarrow \bigoplus_{j \in \bar{\Omega}(-,i)} {}_i M_j \otimes_{H_j} M_j.$$

We define now for any S -module \mathbf{M} , as above, the affine variety

$$\text{Rep}^{\text{fib}}(\Pi, \mathbf{M}) := \{(M_{ij})_{(i,j) \in \bar{\Omega}} \in \bar{H}(\mathbf{M}) \mid M_{k,\text{in}} \circ M_{k,\text{out}} = 0 \text{ for all } k \in I\},$$

and observe that the orbits of the, usually non-reductive, group $\text{Aut}_S(\mathbf{M})$ on $\text{Rep}^{\text{fib}}(\Pi, \mathbf{M})$ correspond to the isoclasses of possible structures of representations of Π on \mathbf{M} , since the condition $M_{k,\text{in}} \circ M_{k,\text{out}}$ corresponds to the relation $\partial_{\epsilon_k}^{\text{cyc}}(W)$.

Similarly to the previous section we can define the open subset

$$\text{Rep}_{\text{l.f.}}(\Pi, \mathbf{r}) := \{M \in \text{Rep}(\Pi, D \cdot \mathbf{r}) \mid M \text{ locally free}\} \subset \text{Rep}(\Pi, D \cdot \mathbf{r}),$$

and observe that $\text{Rep}_{\text{l.f.}}(\Pi, \mathbf{r})$ is a fiber bundle over the $\text{GL}_{D \cdot \mathbf{r}}$ -orbit $\mathcal{O}(E^{\mathbf{r}})$, with typical fiber $\text{Rep}^{\text{fib}}(\Pi, \mathbf{E}^{\mathbf{r}})$. Finally we define for any projective S -module \mathbf{M} the constructible subset

$$\Pi(\mathbf{M}) = \{(M_{ij})_{(i,j) \in \bar{\Omega}} \in \text{Rep}^{\text{fib}}(\Pi, \mathbf{M}) \mid ((M_{ij})_{ij}, \mathbf{M}) \text{ is } \mathbf{E}\text{-filtered}\}.$$

Here, a Π -module X is \mathbf{E} -filtered if it admits a flag of submodules $0 = X_{(0)} \subset X_{(1)} \subset \cdots \subset X_{(l)} = X$, such that for all k we have $X_{(k)}/X_{(k-1)} \cong E_{i_k}$ for some $i_1, i_2, \dots, i_l \in I$. Note that for C symmetric and D trivial this specializes to Lusztig's notion of a nilpotent representation for the preprojective algebra of a quiver. However, if C is not symmetric even in the Dynkin case there exist finite-dimensional, locally free Π -modules which are not \mathbf{E} -filtered, see [Geiß, Leclerc, and Schröer \[2017c, Sec. 8.2.2\]](#) for an example.

We consider $\Pi(\mathbf{r})$ with the Zariski topology and call it by a slight abuse of notation a variety. In any case, it makes sense to speak of the dimension of $\Pi(\mathbf{r})$ and we can consider the set

$$\text{Irr}(\Pi(\mathbf{r}))^{\max}$$

of top-dimensional irreducible components of $\Pi(\mathbf{r})$.

Theorem 4.2. *Let C be a symmetrizable generalized Cartan matrix with symmetrizer D and $H = H_K(C, D, \Omega)$, $\Pi = \Pi_K(C, D)$ for an algebraically closed field K . For the spaces $\Pi(\mathbf{r})$ of \mathbf{E} -filtered representations of Π we have*

$$(a) \dim \Pi(\mathbf{r}) = \dim H(\mathbf{r}) = \sum_{i \in I} c_i \mathbf{r}(i)^2 - \frac{1}{2}(\mathbf{r}, \mathbf{r})_{C,D} \text{ for all } \mathbf{r} \in \mathbb{N}^I.$$

(b) The set $\mathfrak{B} := \coprod_{\mathbf{r} \in \mathbb{N}^I} \text{Irr}(\Pi(\mathbf{r}))^{\max}$ has a natural structure of a crystal of type $B_C(-\infty)$ in the sense of Kashiwara. In particular, we have

$$|\text{Irr}(\Pi(\mathbf{r}))^{\max}| = \dim U(\mathfrak{n})_{\mathbf{r}},$$

where $U(\mathfrak{n})$ the universal enveloping algebra of the positive part \mathfrak{n} of the Kac-Moody Lie algebra $\mathfrak{g}(C)$.

We will sketch in the next two sections a proof of these two statements, which are the main result of [Geiß, Leclerc, and Schröer \[2017c\]](#).

4.4 Bundle constructions. The bundle construction in this section is crucial. It is our version [Geiß, Leclerc, and Schröer \[ibid., Sec. 3\]](#) of Lusztig's construction [Lusztig \[1991, Sec. 12\]](#).

For $m \in \mathbb{N}$ we denote by \mathcal{P}_m the set of sequences of integers $\mathbf{p} = (p_1, p_2, \dots, p_t)$ with $m \geq p_1 \geq p_2 \geq \dots \geq p_t \geq 0$. Obviously \mathcal{P}_{c_k} parametrizes the isoclasses of H_k -modules, and we define $H_k^{\mathbf{p}} = \bigoplus_{j=1}^t H_k / (\epsilon_k^{p_j})$. For $k \in I$ and $M \in \text{rep}(\Pi)$ we set

$$\text{fac}_k(M) := M_k / \text{Im}(M_{k,\text{in}}) \quad \text{and} \quad \text{sub}_k(M) := \text{Ker}(M_{k,\text{out}}).$$

With this we can define

$$\begin{aligned} \Pi(\mathbf{M})^{k,\mathbf{p}} &= \{M \in \Pi(\mathbf{M}) \mid \text{fac}_k(M) \cong H_k^{\mathbf{p}}\} \text{ and} \\ \Pi(\mathbf{M})_{k,\mathbf{p}} &= \{M \in \Pi(\mathbf{M}) \mid \text{sub}_k(M) \cong H_k^{\mathbf{p}}\} \end{aligned}$$

for $\mathbf{p} \in \mathcal{P}_{c_k}$. We abbreviate $\Pi(\mathbf{M})^{k,m} = \Pi(\mathbf{M})^{k,c_k^m}$. In what follows, we will focus our exposition on the varieties of the form $\Pi(\mathbf{M})^{k,\mathbf{p}}$, however one should be aware that similar statements and constructions hold for the dual versions $\Pi(\mathbf{M})_{k,\mathbf{p}}$.

For an \mathbf{E} -filtered representation $M \in \text{rep}(\Pi)$ there exists always a $k \in I$ such that $\text{fac}_k(M)$, viewed as an H_k -module, has a non-trivial free summand. It is also important to observe that $\Pi(\mathbf{M})^{k,0}$ is an open subset of $\Pi(\mathbf{M})$.

Fix now $k \in I$, let \mathbf{M} be a projective S -module and \mathbf{U} be a proper, projective S -submodule of \mathbf{M} with $U_j = M_j$ for all $j \neq k$. Thus, $\mathbf{M}/\mathbf{U} \cong E_k^r$ for some $r \in \mathbb{N}_+$, and can choose a (free) complement T_k , such that $M_k = U_k \oplus T_k$. For partitions $\mathbf{p} = (c_k^r, q_1, q_2, \dots, q_t)$ and $\mathbf{q} = (q_1, \dots, q_t)$ in \mathcal{P}_{c_k} we set moreover $\text{Hom}_S^{\text{inj}}(\mathbf{U}, \mathbf{M}) := \{f \in \text{Hom}_S(\mathbf{U}, \mathbf{M}) \mid f \text{ injective}\}$, and define

$$Y^{k,\mathbf{p},\mathbf{q}} := \{(U, M, f) \in \Pi(\mathbf{U})^{k,\mathbf{q}} \times \text{Rep}^{\text{fib}}(\Pi, \mathbf{M}) \times \text{Hom}_S^{\text{inj}}(\mathbf{U}, \mathbf{M}) \mid f \in \text{Hom}_{\Pi}(U, M)\}.$$

Note that for $(U, M, f) \in Y^{k, \mathbf{p}, \mathbf{q}}$ we have in fact $M \in \Pi(\mathbf{M})^{k, \mathbf{p}}$, and that the group $\text{Aut}_{\mathcal{S}}(\mathbf{U})$ acts freely on $Y^{k, \mathbf{p}, \mathbf{q}}$ via

$$g \cdot (U, M, f) := ((g_i U_{ij} (\text{id} \otimes g_j^{-1}))_{(i,j) \in \bar{\Omega}}, M, g \cdot f^{-1}).$$

Lemma 4.3. *Consider in the above situation the diagram*

$$\begin{array}{ccc} & Y^{k, \mathbf{p}, \mathbf{q}} & \\ p' \swarrow & & \searrow p'' \\ \Pi(\mathbf{U})^{k, \mathbf{q}} \times \text{Hom}_{\mathcal{S}}^{\text{inj}}(\mathbf{U}, \mathbf{M}) & & \Pi(\mathbf{M})^{k, \mathbf{p}} \end{array}$$

with $p'(U, M, f) = (U, f)$ and $p''(U, M, f) = M$. Then the following holds:

(a) p' is a vector bundle of rank m , where

$$m = \sum_{j \in \bar{\Omega}(-, k)} \dim_K \text{Hom}_K(T_{k, k} M_j \otimes_{H_j} M_j) - \dim_K \text{Hom}_{H_k}(T_k, \text{Im}(U_{k, \text{in}})).$$

(b) p'' is a fiber bundle with smooth irreducible fibers isomorphic to

$$\text{Aut}_{\mathcal{S}}(\mathbf{U}) \times \text{Gr}_{H_k}^{T_k}(H_k^{\mathbf{p}}),$$

$$\text{where } \text{Gr}_{H_k}^{T_k}(H_k^{\mathbf{p}}) := \text{Hom}_{H_k}^{\text{surj}}(H_k^{\mathbf{p}}, T_k) / \text{Aut}_{H_k}(T_k).$$

Corollary 4.4. *In the situation of [Lemma 4.3](#), the correspondence*

$$Z' \mapsto p''(p'^{-1}(Z' \times \text{Hom}_{\mathcal{S}}^{\text{inj}}(\mathbf{U}, \mathbf{M}))) := Z''$$

induces a bijection between the sets of irreducible components $\text{Irr}(\Pi(\mathbf{U})^{k, \mathbf{q}})$ and $\text{Irr}(\Pi(\mathbf{M})^{k, \mathbf{p}})$. Moreover we have then

$$\dim Z'' - \dim Z' = \dim H(\mathbf{M}) - \dim(\mathbf{U}).$$

Note, that this implies already part (a) of [Theorem 4.2](#). In fact the Corollary allows us to conclude by induction that $\dim \Pi(\mathbf{r}) \leq \dim \text{Rep}^{\text{fib}}(H, \mathbf{r})$. On the other hand, we can identify $H(\mathbf{r})$ with an irreducible component of $\Pi(\mathbf{r})$.

4.5 Crystals. For $M \in \text{rep}(\Pi)$ and $j \in I$ there are two canonical short exact sequences

$$0 \rightarrow K_j(M) \rightarrow M \rightarrow \text{fac}_j(M) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{sub}_j(M) \rightarrow M \rightarrow C_j(M) \rightarrow 0.$$

We define recursively that M is a *crystal module* if $\text{fac}_j(M)$ and $\text{sub}_j(M)$ are locally free for all $j \in I$, and $K_j(M)$ as well as $C_j(M)$ are crystal modules for all $j \in I$. Clearly, if M is a crystal module, for all $j \in I$ there exist $\varphi_j(M), \varphi_j^*(M) \in \mathbb{N}$ such that

$$(4.1) \quad \text{sub}_j(M) \cong E_j^{\varphi_j(M)} \text{ and } \text{fac}_j(M) \cong E_j^{\varphi_j^*(M)}.$$

Note moreover, that crystal modules are by construction \mathbf{E} -filtered. It is now easy to see that for all projective S -modules \mathbf{M} the set

$$\Pi(\mathbf{M})^{\text{cr}} := \{M \in \Pi(\mathbf{M}) \mid M \text{ is a crystal representation}\}$$

is a constructible subset of $\Pi(\mathbf{M})$. The following result from [Geiß, Leclerc, and Schröer \[2017c, Sec. 4\]](#) is crucial for the proof of [Theorem 4.2 \(b\)](#). It has no counterpart for the case of trivial symmetrizers.

Proposition 4.5. *For each projective S -module \mathbf{M} the set $\Pi(\mathbf{M})^{\text{cr}}$ is a dense and equidimensional subset of the union of all top dimensional irreducible components of $\Pi(\mathbf{M})$.*

This allows us in particular to define for all $Z \in \text{Irr}(\Pi(\mathbf{M}))^{\text{max}}$ and $i \in I$ the value $\varphi_i(Z)$, see (4.1), such that for a dense open subset $U \subset Z$ we have $\varphi_i(M) = \varphi_i(Z)$ for all $M \in U$. Similarly, we can define $\varphi_i^*(Z)$.

Next we set

$$\text{Irr}(\Pi(\mathbf{r})^{i,p})^{\text{max}} := \{Z \in \text{Irr}(\Pi(\mathbf{r})^{i,p}) \mid \dim Z = \dim H(\mathbf{r})\}$$

for $i \in I$ and $p \in \mathbb{N}_0$, and similarly $\text{Irr}(\Pi(\mathbf{r})_{i,p})$. By [Lemma 4.3](#) we get a bijection

$$e_i^*(\mathbf{r}, p): \text{Irr}(\Pi(\mathbf{r})^{i,p})^{\text{max}} \rightarrow \text{Irr}(\Pi(\mathbf{r} + \alpha_i)^{i,p+1})^{\text{max}}, Z \mapsto p''(p'^{-1}(Z \times J_0))$$

Similarly we obtain a bijection

$$e_i(\mathbf{r}, p): \text{Irr}(\Pi(\mathbf{r})_{i,p})^{\text{max}} \rightarrow \text{Irr}(\Pi(\mathbf{r} + \alpha_i)_{i,p+1})^{\text{max}}.$$

This allows us to define for all $\mathbf{r} \in \mathbb{N}^I$ the operators

$$\tilde{e}_i: \text{Irr}(\Pi(\mathbf{r}))^{\text{max}} \rightarrow \text{Irr}(\Pi(\mathbf{r} + \alpha_i)), Z \mapsto \overline{e_i(\mathbf{r}, \varphi_i(Z))}(Z^\circ),$$

where $Z^\circ \in \text{Irr}(\Pi(\mathbf{r})_{i, \varphi_i(Z)})^{\text{max}}$ is the unique irreducible component with $\overline{Z^\circ} = Z$. Similarly, we can define the operators \tilde{e}_i^* in terms of the bijections $e_i^*(\mathbf{r}, p)$. We define now

$$(4.2) \quad \mathfrak{B} := \coprod_{\mathbf{r} \in \mathbb{N}_0} \text{Irr}(\Pi(\mathbf{r}))^{\text{max}} \text{ and } \text{wt}: \mathfrak{B} \rightarrow \mathbb{Z}^I, Z \mapsto \underline{\text{rank}}(Z).$$

It is easy to see that $(\mathfrak{B}, \text{wt}, (\tilde{e}_i, \varphi_i)_{i \in I})$ is special case of a lowest weight crystal in the sense of [Kashiwara \[1995, Sec. 7.2\]](#), namely we have

- $\varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$, $\text{wt}(\tilde{e}_i(b)) = \text{wt}(b) + \alpha_i$,
- with $\{b_-\} := \text{Irr}(\Pi(0))^{\max}$, for each $b \in \mathfrak{B}$ there exists a sequence i_1, \dots, i_l of elements of I with $\tilde{e}_{i_1} \tilde{e}_{i_2} \cdots \tilde{e}_{i_l}(b_-) = b$,
- $\varphi_i(b) = 0$ implies $b \notin \text{Im}(\tilde{e}_i)$.

Similarly $(\mathfrak{B}, \text{wt}, (\tilde{e}_i^*, \varphi_i^*)_{i \in I})$ is a lowest weight crystal with the same lowest weight element b_- .

Lemma 4.6. *The above defined operators and functions on \mathfrak{B} fulfill additionally the following conditions:*

- (a) *If $i \neq j$, then $\tilde{e}_i^* \tilde{e}_j(b) = \tilde{e}_j \tilde{e}_i^*(b)$.*
- (b) *For all $b \in \mathfrak{B}$ we have $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle \geq 0$.*
- (c) *If $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle = 0$, then $\tilde{e}_i(b) = \tilde{e}_i^*(b)$.*
- (d) *If $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle \geq 1$, then $\varphi_i(\tilde{e}_i^*(b)) = \varphi_i(b)$ and $\varphi_i^*(\tilde{e}_i(b)) = \varphi_i^*(b)$.*
- (e) *If $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle \geq 2$, then $\tilde{e}_i \tilde{e}_i^*(b) = \tilde{e}_i^* \tilde{e}_i(b)$.*

The proof of this Lemma in [Geiß, Leclerc, and Schröer \[2017c, Sec. 5.6\]](#) uses the homological features of locally free Π -modules from [Corollary 3.15](#) in an essential way. Note that here, by definition, $\langle \mathbf{r}, \alpha_i \rangle = (C \cdot \mathbf{r})_i$.

Altogether this means, by a criterion of [Kashiwara and Saito \[1997, Prop. 3.2.3\]](#), which we use here in a reformulation due to [Tingley and Webster \[2016, Prop. 1.4\]](#), that $(\mathfrak{B}, \text{wt}, (\tilde{e}_i, \varphi_i)_{i \in I})$ $(\mathfrak{B}, \text{wt}, (\tilde{e}_i^*, \varphi_i^*)_{i \in I}) \cong B_C(-\infty)$. Here, $B_C(-\infty)$ is the crystal graph of the quantum group $U_q(\mathfrak{n}(C))$. This implies part (b) of [Theorem 4.2](#).

Remark 4.7. We did not give here Kashiwara's general definition of a crystal graph, or that of a lowest weight crystal associated to a dominant integral weight. The reason is that, due to limitations of space, we can not to set up the, somehow unwieldy, notations for the integral weights of a Kac-Moody Lie algebra. The interested reader can look up the relevant definitions, in a form which is compatible with these notes, in [Geiß, Leclerc, and Schröer \[2017c, Sec. 5.1, 5.2\]](#).

5 Algebras of constructible functions

5.1 Constructible functions and Euler characteristic. Recall that the topological Euler characteristic, defined in terms of singular cohomology with compact support and rational coefficients, defines a ring homomorphism from the Grothendieck ring of complex varieties to the integers.

By definition, a constructible function $f : X \rightarrow \mathbb{C}$ on a complex algebraic variety X has finite image, and $f^{-1}(c) \subset X$ is a constructible set for all $c \in \mathbb{C}$. By the above remark it makes sense to define

$$\int_{x \in X} f d\chi := \sum_{c \in \mathbb{C}} c \chi(f^{-1}(c)).$$

If $\varphi : X \rightarrow Y$ is a morphism of varieties, we can define the *push forward* of constructible functions via $(\varphi_*(f))(y) := \int_{x \in \varphi^{-1}(y)} f d\chi$. This is functorial in the sense that $(\psi \circ \varphi)_*(f) = \psi_*(\varphi_*(f))$ for $\psi : Y \rightarrow Z$ another morphism, by result of MacPherson [1974, Prop. 1]. See also Joyce [2006, Sec. 3] for a careful discussion.

5.2 Convolution algebras as enveloping algebras. Let $A = \mathbb{C}Q$ as in Section 4.1. We consider for a dimension vector $\mathbf{d} \in \mathbb{N}^I$ the vector space $\mathfrak{F}(A)_{\mathbf{d}}$ of constructible functions $f : \text{Rep}(A, \mathbf{d}) \rightarrow \mathbb{C}$ which are constant on $\text{GL}_{\mathbf{d}}(\mathbb{C})$ -orbits and set

$$\mathfrak{F}(A) := \bigoplus_{\mathbf{d} \in \mathbb{N}^I} \mathfrak{F}(A)_{\mathbf{d}}.$$

Following Lusztig [1991] $\mathfrak{F}(A)$ has the structure of a unitary, graded associative algebra. The multiplication is defined by

$$(f * g)(X) = \int_{U \in \text{Gr}_{\mathbf{d}}^A(X)} f(U)g(X/U)d\chi,$$

where $f \in \mathfrak{F}(A)_{\mathbf{d}}, g \in \mathfrak{F}(A)_{\mathbf{e}}, X \in \text{Rep}(A, \mathbf{d} + \mathbf{e})$, and $\text{Gr}_{\mathbf{d}}^A(X)$ denotes the quiver Grassmannian of \mathbf{d} -dimensional subrepresentations of X . The associativity of the multiplication follows easily from the functoriality of the push-forward of constructible functions. We have an algebra homomorphism

$$(5.1) \quad c : \mathfrak{F}(A) \rightarrow \mathfrak{F}(A \times A), \text{ with } (c(f))(X, Y) = f(X \oplus Y),$$

see for example Geiß, Leclerc, and Schröer [2016, Sec. 4.3]. The proof depends crucially on the Białynicki-Birula result about the fixpoints of algebraic torus actions Białynicki-Birula [1973, Cor. 2]. This fails for example over the real numbers.

Remark 5.1. If $\mathbf{X} = (X_j)_{j \in J}$ is a family of indecomposable representations of A , we define the characteristic functions $\theta_j \in \mathfrak{F}_{\dim X_j}(A)$ of the $\text{GL}_{\dim X_j}$ -orbit $\mathcal{O}(X_j) \subset \text{Rep}(A, \underline{\dim} X_j)$ and consider the graded subalgebra $\mathfrak{M}(A) = \mathfrak{M}_{\mathbf{X}}(A)$ of $\mathfrak{F}(A)$, which is generated by the θ_j . Clearly, the homogeneous components of \mathfrak{M} are finite dimensional. If $\mathbf{j} =$

(j_1, j_2, \dots, j_l) is a sequence of elements of J we have by the definition of the multiplication

$$\theta_{j_1} * \theta_{j_2} * \dots * \theta_{j_l}(X) = \chi(\text{Fl}_{\mathbf{X}, \mathbf{j}}^A(M)),$$

where $\text{Fl}_{\mathbf{X}, \mathbf{j}}^A(M)$ denotes the variety of all flags of submodules

$$0 = M^{(0)} \subset M^{(1)} \subset \dots \subset M^{(l)} = M$$

with $M^{(k)}/M^{(k-1)} \cong X_{j_k}$ for $k = 1, 2, \dots, l$. In particular, if M has no filtration with all factors isomorphic to some X_j , we have $f(M) = 0$ for all $f \in \mathfrak{M}(A)_{\dim M}$. See [Geiß, Leclerc, and Schröer \[2016, Lemma 4.2\]](#).

Lemma 5.2. *The morphism c from (5.1) induces a comultiplication $\Delta: \mathfrak{M}(A) \rightarrow \mathfrak{M}(A) \otimes \mathfrak{M}(A)$ with $\Delta(\theta_j) = \theta_j \otimes 1 + 1 \otimes \theta_j$ for all j . With this structure \mathfrak{M} is a cocommutative Hopf algebra, which is isomorphic to the enveloping algebra $U(\mathcal{P}(\mathfrak{M}))$ of the Lie algebra of its primitive elements $\mathcal{P}(\mathfrak{M})$.*

See [Geiß, Leclerc, and Schröer \[ibid., Prop. 4.5\]](#) for a proof. Recall, that an element x of a Hopf algebra is called *primitive* iff $\Delta(x) = x \otimes 1 + 1 \otimes x$. It is straightforward to check that the primitive elements of a Hopf algebra form a Lie algebra under the usual commutator $[x, y] = xy - yx$.

Remark 5.3. It is important to observe that, by the definition of the comultiplication, the support of any primitive element of \mathfrak{M} consists of indecomposable, \mathbf{X} -filtered modules. In fact, for $f \in \mathcal{P}(\mathfrak{M})$ and $M, N \in \text{rep}(A)$ we have $f(M \oplus N) = cf(M, N) = (f \otimes 1 + 1 \otimes f)(M, N)$. See [Geiß, Leclerc, and Schröer \[ibid., Lem. 4.6\]](#).

We are here interested in the two special cases when $A = H_{\mathbf{C}}(C, D, \Omega)$ or $A = \Pi_{\mathbf{C}}(C, D)$ and $\mathbf{X} = \mathbf{E} = (E_i)_{i \in I}$. Note that by [Remark 5.1](#) only locally free modules can appear in the support of any $f \in \mathfrak{M}_{\mathbf{E}}(H)$. Similarly, the support of any $f \in \mathfrak{M}_{\mathbf{E}}(\Pi)$ consists only of \mathbf{E} -filtered modules. For this reason we will consider in what follows, both $\mathfrak{M}_{\mathbf{E}}(H)$ and $\mathfrak{M}_{\mathbf{E}}(\Pi)$ as graded by *rank vectors*. In other words, from now on

$$\mathfrak{M}(H) := \mathfrak{M}_{\mathbf{E}}(H) = \bigoplus_{\mathbf{r} \in \mathbb{N}^I} \mathfrak{M}_{\mathbf{r}}(H) \quad \text{and} \quad \mathfrak{M}(\Pi) = \mathfrak{M}_{\mathbf{E}}(\Pi) = \bigoplus_{\mathbf{r} \in \mathbb{N}^I} \mathfrak{M}_{\mathbf{r}}(\Pi),$$

where we may consider the elements of $\mathfrak{M}_{\mathbf{r}}(H) := \mathfrak{M}_{\mathbf{E}}(H)_{D, \mathbf{r}}$ as constructible functions on $H(\mathbf{r})$. Similarly we may consider the elements of $\mathfrak{M}_{\mathbf{r}}(\Pi) := \mathfrak{M}_{\mathbf{E}}(\Pi)_{D, \mathbf{r}}$ as constructible functions on $\Pi(\mathbf{r})$.

5.3 $\mathfrak{M}_{\mathbf{E}}(H)$ and a dual PBW-basis in the Dynkin case. We have the following basic result from [Geiß, Leclerc, and Schröer \[ibid., Cor. 4.10\]](#).

Proposition 5.4. *Let C be a symmetrizable Cartan matrix, D a symmetrizer and Ω an orientation for C . With $H = H_{\mathbb{C}}(C, D, \Omega)$ we have an surjective Hopf algebra homomorphism*

$$\eta_H : U(\mathfrak{n}(C)) \rightarrow \mathfrak{M}_{\mathbb{E}}(H) \text{ defined by } e_i \mapsto \theta_i (i \in I).$$

The main point is to show that for the θ_i ($i \in I$) fulfill the Serre relations (2.4). More precisely we need that the primitive elements

$$\theta_{ij} := (\text{ad } \theta_i)^{1-c_{ij}}(\theta_j) \in \mathcal{P}(\mathfrak{M}(H))_{(1-c_{ij})\alpha_i + \alpha_j} \quad (i \neq j)$$

actually vanish. For this it is enough, by Remark 5.3, to show that there exists no indecomposable, locally free H -module M with $\text{rank}(M) = (1 - c_{ij})\alpha_i + \alpha_j$. This is carried out in the proof of Geiß, Leclerc, and Schröer [2016, Prop. 4.9].

The proof of the following result, which is Geiß, Leclerc, and Schröer [ibid., Thm. 6.1], occupies the major part of that paper.

Proposition 5.5. *Let C be a symmetrizable Cartan matrix of Dynkin type, D a symmetrizer and Ω an orientation for C and $H = H_{\mathbb{C}}(C, D, \Omega)$. Then for each positive root $\beta \in \Delta^+$ there exists a primitive element $\theta_{\beta} \in \mathcal{P}(\mathfrak{M}(H))_{\beta}$ with $\theta_{\beta}(M(\beta)) = 1$.*

The idea of the proof is as follows: By Geiß, Leclerc, and Schröer [2017b, Cor. 1.3] for any $\beta \in \Delta^+(C)$ and any sequence \mathbf{i} in I , the Euler characteristic $\chi(\text{Fl}_{\mathbb{E}, \mathbf{i}}^H(M(\beta)))$ is independent of the choice of the symmetrizer D . So, we may assume that C is connected and D minimal. In the symmetric (quiver) case, our claim follows now by Schofield's result Schofield [n.d.], who showed that in this case $\mathcal{P}(\mathfrak{M}(H))$ can be identified with $\mathfrak{n}(C)$. By Gabriel's theorem in this case the θ_{β} are the characteristic function of the GL_{β} -orbit of $M(\beta)$.

In the remaining cases, we construct the θ_{β} by induction on the height of β in terms of (iterated) commutators of “smaller” θ_{γ} . Note however that in this case this construction is delicate since the support of the θ_{β} may contain several indecomposable, locally free modules. See for example Geiß, Leclerc, and Schröer [2017d, Sec. 13.2(d)].

Since in the Dynkin case all weight spaces of $\mathfrak{n}(C)$ are one-dimensional, the main result of Geiß, Leclerc, and Schröer [2016], Theorem 1.1 (ii), follows easily:

Theorem 5.6. *If C is of Dynkin type, the Hopf algebra homomorphism η_H is an isomorphism.*

Recall the notation used in Proposition 3.11. In particular, \mathbf{i} is a reduced expression for the longest element $w_0 \in W(C)$, which is $+$ -adapted to Ω , and $\beta_k = \beta_{\mathbf{i}, k}$ for $k = 1, 2, \dots, r$. Let us abbreviate

$$\theta_{\mathbf{m}} := \frac{1}{m_r! \cdots m_1!} \theta_{\beta_r}^{m_r} * \cdots * \theta_{\beta_1}^{m_1} \quad \text{and} \quad M(\mathbf{m}) := \bigoplus_{k=1}^r M(\beta_k)^{m_k}$$

for $\mathbf{m} = (m_1, m_2, \dots, m_r) \in \mathbb{N}^r$. By the above results $(\theta_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^r}$ is a normalized PBW-basis of $\mathfrak{M}(H) \cong U(\mathfrak{n}(C))$ in the Dynkin case.

Moreover we consider the graded dual $\mathfrak{M}(H)^*$ of $\mathfrak{M}(H)$, and the evaluation form $\delta_{M(\mathbf{m})} \in \mathfrak{M}(H)^*$ with $\delta_{\mathbf{m}}(f) := f(M(\mathbf{m}))$. By the definition of the comultiplication in $\mathfrak{M}(H)$, the graded dual is a commutative Hopf algebra, and $\delta_{M(\mathbf{m})} \cdot \delta_{M(\mathbf{n})} = \delta_{M(\mathbf{m}+\mathbf{n})}$. Our next result is essentially [Geiß, Leclerc, and Schröer \[2017d, Thm. 1.3\]](#).

Proposition 5.7. *With the above notation we have*

$$\delta_{M(\mathbf{m})}(\theta_{\mathbf{n}}) = \delta_{\mathbf{m}, \mathbf{n}} \text{ for all } \mathbf{m}, \mathbf{n} \in \mathbb{N}^r.$$

Thus $(\delta_{M(\mathbf{m})})_{\mathbf{m} \in \mathbb{N}^r}$ is a basis of $\mathfrak{M}(H)^*$ which is dual to the PBW-basis $(\theta_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^r}$, and $\mathfrak{M}(H)^* = \mathbb{C}[\delta_{M(\beta_1)}, \dots, \delta_{M(\beta_r)}]$.

In the quiver case (with trivial symmetrizer) this result is easy to prove, since with Gabriel's theorem and [Proposition 3.11](#) follows quickly that θ_r is the characteristic function of the orbit of $M(\mathbf{r})$. However, in our more general setting, already the θ_{β_k} are usually not the characteristic function of $M(\beta_k)$, as we observed above. The more sophisticated [Proposition 3.12](#) implies, by the definition of the multiplication in $\mathfrak{M}_{\mathbf{E}}(H)$, that $\theta_{\mathbf{m}}(M(\beta_k)) = 0$ if $\mathbf{m} \neq \mathbf{e}_k$, the k -th unit vector. The remaining claims follow now by formal arguments, see the proof of [Geiß, Leclerc, and Schröer \[ibid., Thm. 6.1\]](#).

Remark 5.8. For $M \in \text{rep}_{\text{l.f.}}(H)$ and $\mathbf{e} \in \mathbb{N}^I$ we introduce the quasi-projective variety

$$\text{Grf}_{\mathbf{e}}^H(M) := \{U \subset M \mid U \text{ locally free submodule and } \underline{\text{rank}}(U) = \mathbf{e}\},$$

which is an open subset of the usual quiver Grassmannian $\text{Gr}_{D, \mathbf{e}}^H(M)$. With this notation we can define

$$F_M := \sum_{\mathbf{e} \in \mathbb{N}^I} \chi(\text{Grf}_{\mathbf{e}}^H(M)) Y^{\mathbf{e}} \in \mathbb{Z}[Y_1, \dots, Y_n] \text{ and } \mathbf{g}_M := -R \cdot \underline{\text{rank}}(M),$$

where R is the matrix introduced in [Corollary 3.4](#). By the main result of [Geiß, Leclerc, and Schröer \[ibid.\]](#) this yields for $M = M(\beta)$ with $\beta \in \Delta^+(C)$ the F -polynomial and g -vector, in the sense of [Fomin and Zelevinsky \[2007\]](#), for all cluster variables of a finite type cluster algebra [Fomin and Zelevinsky \[2003\]](#) of type C with respect to an acyclic seed defined by Ω . The proof is based on [Proposition 5.7](#), and on the description by [Yang and Zelevinsky \[2008\]](#) of the F -polynomial of a cluster variable in terms of generalized minors.

5.4 Semicanonical functions and the support conjecture for $\mathfrak{M}_{\mathbf{E}}(\Pi)$. Recall, that we abbreviate $\Pi = \Pi_{\mathbb{C}}(C, D)$ for a symmetrizable Cartan matrix C with symmetrizer D . By definition $\mathfrak{M}(\Pi) = \mathfrak{M}_{\mathbf{E}}(\Pi) \subset \mathfrak{F}(\Pi)$ is generated by the functions $\tilde{\theta}_i \in \mathfrak{M}_{\alpha_i}(\Pi)$ for $i \in I$, where $\tilde{\theta}_i$ is the characteristic function of the orbit of E_i , viewed as a Π -module. We use here the notation $\tilde{\theta}_i$ rather than θ_i to remind us that the multiplication is now defined in terms of constructible functions on a larger space. More precisely, we have for each $\mathbf{r} \in \mathbb{N}^I$ an injective $\text{Aut}_{\mathbb{S}}(\mathbf{E}^{\mathbf{r}})$ -equivariant, injective morphism of varieties

$$\iota_{\mathbf{r}}: H(\mathbf{r}) \rightarrow \Pi(\mathbf{d}).$$

These morphisms induce, via restriction, a surjective morphism of graded Hopf algebras

$$\iota_{\Omega}^*: \mathfrak{M}(\Pi) \rightarrow \mathfrak{M}(H), \quad \tilde{\theta}_i \mapsto \theta_i \text{ for } i \in I.$$

The proof of the following result is, almost verbatim, the same induction argument as the one used by Lusztig [2000], see Geiß, Leclerc, and Schröer [2017c, Lem. 7.1].

Lemma 5.9. *Let $\mathbf{r} \in \mathbb{N}^I$. For each $Z \in \text{Irr}(\Pi(\mathbf{r}))^{\max}$ there exists an open dense subset $U_Z \subset Z$ and a function $f_Z \in \mathfrak{M}_{\mathbf{r}}(\Pi)$ such that for $Z, Z' \in \text{Irr}(\Pi(\mathbf{r}))^{\max}$ and any $u' \in U_{Z'}$ we have*

$$f_Z(u') = \delta_{Z, Z'}.$$

In particular, the functions $(f_Z)_{Z \in \text{Irr}(\Pi(\mathbf{r}))^{\max}}$ are linearly independent in $\mathfrak{M}_{\mathbf{r}}(\Pi)$.

Note however, that the result is not trivial since we claim that the $f_Z \in \mathfrak{M}_{\mathbf{e}}(\Pi)$ and not in the much bigger space $\mathfrak{F}(\Pi)_{\mathbb{C}, \mathbf{r}}$. On the other hand, it is important to observe that the inductive construction of the *semicanonical functions* f_Z involves some choices.

As in Section 5.3, we define now for each $i \neq j$ in I the primitive element

$$\tilde{\theta}_{ij} = (\text{ad } \tilde{\theta}_i)^{1-c_{ij}}(\tilde{\theta}_j) \in \mathcal{P}(\mathfrak{M}(\Pi)).$$

Unfortunately, we have the following result, which is a combination of Lemma 6.1, Proposition 6.2 and Lemma 6.3 from Geiß, Leclerc, and Schröer [ibid.].

Lemma 5.10. *Suppose with the above notations that $c_{ij} < 0$.*

- (a) *If $c_i \geq 2$ then there exists an indecomposable, $\Pi = \Pi(C, D)$ -module $X = X_{(ij)}$ with $\underline{\text{rank}}(X_{(ij)}) = (1 - c_{ij})\alpha_i + \alpha_j$ and $\tilde{\theta}_{ij}(X_{(ij)}) \neq 0$.*
- (b) *If M is crystal module with $\underline{\text{rank}}(M) = (1 - c_{ij})\alpha_i + \alpha_j$ we have $\tilde{\theta}_{ij}(M) = 0$.*

This leads us to define in $\mathfrak{M}(\Pi)$ the ideal \mathfrak{I} , which is generated by the homogeneous elements $\tilde{\theta}_{ij}$ for $i, j \in I$ with $i \neq j$. We set moreover

$$\overline{\mathfrak{M}}(\Pi) = \mathfrak{M}(\Pi)/\mathfrak{I} \text{ and } \bar{f} := f + \mathfrak{I} \quad (f \in \mathfrak{M}(\Pi)).$$

Thus, by [Proposition 5.4](#), the morphism ι_Ω^* induces a surjective algebra homomorphism $\bar{\iota}_\Omega^*: \overline{\mathfrak{M}}(\Pi) \rightarrow \mathfrak{M}(H)$. On the other hand, we can define for each $\mathbf{r} \in \mathbb{N}^I$ the space of functions with non maximal support

$$\mathfrak{S}_\mathbf{r} := \{f \in \mathfrak{M}_\mathbf{r}(\Pi) \mid \dim \text{supp}(f) < \dim H(\mathbf{r})\} \text{ and } \mathfrak{S} := \bigoplus_{\mathbf{r} \in \mathbb{N}^I} \mathfrak{S}_\mathbf{r}.$$

Recall that $\dim \Pi(\mathbf{r}) = \dim H(\mathbf{r})$. [Proposition 4.5](#) and [Lemma 5.10](#) imply at least that $\tilde{\theta}_{ij} \in \mathfrak{S}$. In view of [Lemma 5.9](#) and [Theorem 4.2](#) it is easy to show the following result:

Proposition 5.11. *The following three conditions are equivalent:*

$$(1) \mathfrak{I} \subset \mathfrak{S}, \quad (2) \mathfrak{I} = \mathfrak{S}, \quad (3) \mathfrak{S} \text{ is an ideal.}$$

In this case the surjective algebra homomorphism

$$\eta: U(\mathfrak{n}) \rightarrow \overline{\mathfrak{M}}(\Pi), e_i \mapsto \tilde{\theta}_i + \mathfrak{I}$$

would be an isomorphism, and the $(\eta^{-1}(\bar{f}_Z))_{\mathfrak{B}}$ would form a basis of $U(\mathfrak{n})$ which is independent of the possible choices for the $(f_Z)_{Z \in \mathfrak{B}}$.

Thus we call the equivalent conditions of the above proposition our *Support conjecture*.

Remark 5.12. Our semicanonical basis would yield, similarly to [Lusztig \[2000, Sec. 3\]](#), in a natural way a basis for each integrable highest weight representation $L(\lambda)$ of $\mathfrak{g}(C)$, if the support conjecture is true. See [Geiß, Leclerc, and Schröer \[2017c, Se. 7.3\]](#) for more details.

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