

BIRATIONAL GEOMETRY OF ALGEBRAIC VARIETIES

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Abstract

This is a report on some of the main developments in birational geometry in recent years focusing on the minimal model program, Fano varieties, singularities and related topics, in characteristic zero.

1 Introduction

It is not a comprehensive survey of all advances in birational geometry, e.g. we will not touch upon the positive characteristic case which is a very active area of research. We will work over an algebraically closed field k of characteristic zero. Varieties are all quasi-projective.

Birational geometry, with the so-called minimal model program at its core, aims to classify algebraic varieties up to birational isomorphism by identifying “nice” elements in each birational class and then classifying such elements, e.g. study their moduli spaces. Two varieties are birational if they contain isomorphic open subsets. In dimension one, a nice element in a birational class is simply a smooth and projective element. In higher dimension though there are infinitely many such elements in each class, so picking a representative is a very challenging problem. Before going any further let us introduce the canonical divisor.

1.1 Canonical divisor. To understand a variety X one studies subvarieties and sheaves on it. Subvarieties of codimension one and their linear combinations, that is, divisors play a crucial role. Of particular importance is the canonical divisor K_X . When X is smooth this is the divisor (class) whose associated sheaf $\mathcal{O}_X(K_X)$ is the canonical sheaf $\omega_X := \det \Omega_X$ where Ω_X is the sheaf of regular differential forms. When X is only normal, K_X is the closure of the canonical divisor of the smooth locus. In general, the canonical divisor is the only special non-trivial divisor attached to X . It plays an important

role in algebraic geometry, e.g. in duality theory and Riemann-Roch formula, and also in differential and arithmetic geometry. It is a central object in birational geometry.

Example. Assume $X = \mathbb{P}^d$. Then $K_X \sim -(d+1)H$ where $H \subset \mathbb{P}^d$ is a hyperplane.

Example. Assume $X \subset \mathbb{P}^d$ is a smooth hypersurface of degree r . Then we have $K_X \sim (-d-1+r)H|_X$ where $H \subset \mathbb{P}^d$ is a hyperplane not containing X .

Example. If X is a toric variety, then $K_X \sim -\Lambda$ where Λ is the sum of the torus-invariant divisors.

1.2 Varieties with special canonical divisor. Let X be a projective variety with “good” singularities (by this we mean klt or lc singularities defined below, see [Section 2.4](#)).

We say X is $\begin{cases} \textit{Fano} & \text{if } K_X \text{ is anti-ample} \\ \textit{Calabi-Yau} & \text{if } K_X \text{ is numerically trivial} \\ \textit{canonically polarised} & \text{if } K_X \text{ is ample} \end{cases}$

Note that here we consider Calabi-Yau varieties in a weak sense, that is, we do not require the vanishing $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$ which is usually assumed in other contexts. For example, abelian varieties are Calabi-Yau by our definition.

The special varieties just defined are of great importance in algebraic geometry (e.g. birational geometry, moduli theory, derived categories), differential geometry (e.g. Kähler-Einstein metrics, stability), arithmetic geometry (e.g. existence and density of rational points), and mathematical physics (e.g. string theory and mirror symmetry). They behave much better compared to a randomly chosen variety.

Example. Assume X is a smooth projective curve of genus g . If $g = 0$, then $X \simeq \mathbb{P}^1$ which is Fano. If $g = 1$, then X is an elliptic curve, hence a Calabi-Yau. If $g \geq 2$, then X is canonically polarised.

Example. Assume $X \subset \mathbb{P}^d$ is a smooth hypersurface of degree r . If $r \leq d$, then X is Fano. If $r = d + 1$, then X is Calabi-Yau. If $r > d + 1$, then X is canonically polarised.

1.3 Minimal model program. Now we give a brief description of the minimal model program (MMP). Pick a variety W . Using resolution of singularities we can modify W so that it is smooth and projective. However, being smooth and projective is not very special as in dimension at least two these properties are shared by infinitely many other varieties in the same birational class. It is then natural to look for a more special representative. One of the main aims of birational geometry is to show that we can dismantle W birationally and reconstruct it using canonically polarised, Calabi-Yau, and Fano varieties. To be more precise we want to establish the following conjecture formulated in its simplest form.

Conjecture 1.4 (Minimal model and abundance). *Each variety W is birational to a projective variety Y with “good” singularities such that either*

- Y is canonically polarised, or
- Y admits a Fano fibration, or
- Y admits a Calabi-Yau fibration.

In particular, even if W is smooth, Y may be singular. In fact singularity theory is an indispensable part of modern birational geometry.

As the name suggests the conjecture actually consists of two parts, the *minimal model conjecture* and the *abundance conjecture*. The minimal model conjecture essentially says that we can find Y such that K_Y is *nef* meaning K_Y intersects every curve non-negatively, or else there is a K_Y -negative fibration $Y \rightarrow Z$ which means we have a *Fano fibration*. The abundance conjecture essentially says that if Y is not canonically polarised and if it does not admit a Fano fibration, then it admits a K_Y -trivial fibration $Y \rightarrow Z$ which means we have a *Calabi-Yau fibration*. The minimal model conjecture holds in dimension ≤ 4 by Mori [1988], Shokurov [1993], Kawamata [1992c], and Shokurov [2003, 2009] in full generality, and in any dimension for varieties of general type by Birkar, Cascini, Hacon, and McKernan [2010] while the abundance conjecture is proved in dimension ≤ 3 by Miyaoka [1988] and Kawamata [1992a], and in any dimension for varieties of general type by Shokurov [1985] and Kawamata [1984] (also see Birkar [2012] and references therein for more results). We should also mention that the *non-vanishing conjecture* which is a special case of (a suitable reformulation of) the abundance conjecture implies the minimal model conjecture by Birkar [2010, 2011].

Given a smooth projective W , how can we get to Y ? This is achieved via *running the MMP* which is a step by step program making the canonical divisor K_W more positive by successively removing or replacing curves along which K_W is not positive. It gives a (conjecturally finite) sequence of birational transformations

$$W = W_1 \dashrightarrow W_2 \dashrightarrow \cdots \dashrightarrow W_n = Y$$

consisting of *divisorial contractions*, *flips*, and a last step canonically trivial contraction. The required contractions and flips exist by Shokurov [1985] and Kawamata [1984] and Birkar, Cascini, Hacon, and McKernan [2010] and Hacon and McKernan [2010]. An important ingredient is the *finite generation* of the k -algebra

$$R = \bigoplus_{m \geq 0} H^0(W, mK_W)$$

in its various forms; see Birkar, Cascini, Hacon, and McKernan [2010], Hacon and McKernan [2010], and Shokurov [2003].

A serious issue with the MMP is that we do not know whether it actually stops at some step W_n . What is not clear is if the MMP can produce an infinite sequence of flips. In other words, the minimal model conjecture is reduced to the following.

Conjecture 1.5 (Termination). *There is no infinite sequence of flips.*

The two-dimensional case of the MMP is classical developed in the early 20th century by Castelnuovo, Enriques, etc. The three-dimensional case (in characteristic zero) was developed in the 70's-90's through work of many people notably Iitaka, Iskovskikh, Kawamata, Kollár, Mori, Reid, Shokurov, Ueno, etc. The higher dimensional case is still conjectural but a large portion of it has been established since the turn of the century by many people including Birkar, Cascini, Hacon, McKernan, Shokurov, Xu, etc, involving many difficult problems of local and global nature.

1.6 Pluricanonical systems, Kodaira dimension and Iitaka fibration. Let W be a smooth projective variety. The space of sections $H^0(W, mK_W)$, for $m \in \mathbb{Z}$, and their associated linear systems $|mK_W|$ are of great importance. When W is one-dimensional the linear system $|K_W|$ determines its geometry to a large extent. Indeed the genus g of W is just $h^0(W, K_W)$ which is encoded in $|K_W|$. Moreover, if $g \geq 2$, then $|K_W|$ is base point free, and if in addition W is not hyperelliptic, then $|K_W|$ defines an embedding of X into a projective space of dimension $g - 1$. In higher dimension, however, $|K_W|$ often says little about W . One instead needs to study $|mK_W|$ for all $m \in \mathbb{Z}$ in order to investigate the geometry of W . This leads to the notion of *Kodaira dimension* $\kappa(W)$, an important birational invariant of W . This is defined to be the maximum of the dimension of the images of W under the maps defined by the linear systems $|mK_W|$ for $m > 0$. It takes values in $\{-\infty, 0, 1, \dots, \dim X\}$ where the case $-\infty$ corresponds to the situation when $h^0(W, mK_W) = 0$ for every $m > 0$.

Assume $\kappa(W) \geq 0$, that is, $h^0(W, mK_W) \neq 0$ for some $m > 0$. When $m > 0$ is sufficiently divisible, $|mK_W|$ defines a rational fibration $W \dashrightarrow X$ which is called the *Iitaka fibration* of W . This is usually defined up to birational equivalence. The dimension of X is simply the Kodaira dimension $\kappa(W)$. It is often possible to translate questions about W to corresponding questions about X . An old problem is the following:

Conjecture 1.7. *Assume $\kappa(W) \geq 0$. Then there exists $m \in \mathbb{N}$ depending only on $\dim W$ such that $|mK_W|$ defines the Iitaka fibration.*

If W is of *general type*, i.e. if $\kappa(W) = \dim W$, then the conjecture is already known by Hacon and McKernan [2006] and Takayama [2006] (also see Hacon, McKernan, and Xu [2013, 2014] for more recent and more general results). In this case we can take m such that $|mK_W|$ defines a birational embedding of W into some projective space. Note

that W is birational to its canonical model X , by [Birkar, Cascini, Hacon, and McKernan \[2010\]](#), which is a canonically polarised variety and understanding $|mK_W|$ is the same as understanding $|mK_X|$.

Now assume $0 \leq \kappa(W) < \dim W$. The most general known result is that the conjecture is true if we have bounds on certain invariants of the general fibres of the Iitaka fibration, by [Birkar and Zhang \[2016\]](#). This is done by using a canonical bundle formula for the Iitaka fibration and translating the conjecture into a question on the base of the fibration. Very roughly the main result of [Birkar and Zhang \[ibid.\]](#) says that the conjecture holds if one understands the case $\kappa(W) = 0$. Note that in this case, assuming the minimal model and abundance conjectures, W is birational to a Calabi-Yau variety, and understanding $|mK_W|$ is the same as understanding such systems on the Calabi-Yau variety.

Finally, assume $\kappa(W) = \infty$. Then all the linear systems $|mK_W|$, for $m > 0$, are empty. By the minimal model and abundance conjectures, W is birational to a variety Y admitting a Fano fibration $Y \rightarrow Z$. The general fibres of this fibration are Fano varieties. It is then natural to focus on Fano varieties F and study the linear systems $|-mK_F|$, for $m > 0$, in detail. There has been extensive studies of these systems, especially in low dimension, but general higher dimensional results are quite recent; see [Birkar \[2016a,b\]](#).

1.8 Fano varieties, and connection with families, singularities, and termination.

Let X be a Fano variety. A difficulty with investigating $|-mK_X|$ is that, unlike the case of varieties of general type, these systems can change dramatically if we change X birationally. On the other hand, a standard inductive technique to study $|-mK_X|$ is to use the elements of $|-mK_X|$ (usually with bad singularities) to create a particular kind of covering family of subvarieties of X and then use induction by restricting to members of this family. A difficulty in this approach is that a member of this family is not necessarily Fano, so it is hard to apply induction, again unlike the case of varieties of general type. Despite these difficulties there has been lots of progress in recent years.

In general there is $m \in \mathbb{N}$ depending only on $\dim X$ such that $|-mK_X|$ is non-empty. Moreover, there is an element of $|-mK_X|$ with good singularities by [Birkar \[2016a, Theorem 1.1\]](#): this is a special case of *boundedness of complements* (see [Theorem 3.3](#) below). In addition if we put a bound on the singularities of X , that is, if X is ϵ -lc where $\epsilon > 0$, then we can choose m so that $|-mK_X|$ defines a birational embedding of X into some projective space by [Birkar \[ibid.\]](#), [Theorem 1.2](#)) (see [Theorem 3.5](#) below). In fact one can go further in this case and show that we can choose m so that $-mK_X$ is very ample, hence $|-mK_X|$ defines an embedding of X into some projective space, and that the set of such X form a bounded family by [Birkar \[2016b, Theorem 1.1\]](#): this is the so-called *BAB conjecture* (see [Theorem 3.7](#) below). These results are proved along with various other

results and in conjunction with Shokurov's theory of complements. We will give ample explanations in subsequent sections.

So far we have only mentioned *global Fano* varieties but there are other (relative) Fano varieties. Assume X has good singularities, $f: X \rightarrow Z$ is a surjective projective morphism, and $-K_X$ is ample over Z . We call X *Fano over Z* . If Z is a point, then X is a usual Fano variety otherwise in general X is not projective. When $\dim X > \dim Z > 0$, then f is a Fano fibration. Such fibrations appear naturally in birational geometry, and in other contexts, e.g. families and moduli of Fano's.

Now assume f is birational. A special case is a *flipping contraction*, one of the corner stones of the MMP. Existence of flips basically means understanding the linear systems $| -mK_X |$ relatively over Z . Another important special case is when f is the identity morphism in which case we are just looking at the germ of a point on a variety, hence we are doing *singularity theory*. Another connection with singularity theory is that of singularities of \mathbb{R} -linear systems of divisors on varieties, in general, that is the variety may not be Fano and the divisors may not be related to canonical divisors (see [Theorem 4.5](#) below). This is necessary for the proof of BAB. Therefore, studying Fano varieties in the relative setting naturally overlaps with other important topics in birational and algebraic geometry.

There is also connection with the termination conjecture. It is understood that the termination conjecture is about understanding singularities (see [Section 6.5](#)). Moreover, understanding singularities is essentially about understanding Fano varieties in the relative birational case. On the other hand, problems about families of Fano varieties fits well in this theory (see [Sections 6.1](#) and [6.7](#)). It is then no surprise that recent advances on Fano varieties described above is expected to have a profound impact on further developments in birational geometry.

2 Preliminaries

In this section we recall some basic notions. We will try to keep technicalities to a minimum throughout the text. Most of what we need can be found in [Kollár and Mori \[1998\]](#) and [Birkar, Cascini, Hacon, and McKernan \[2010\]](#).

2.1 Contractions. A *contraction* is a projective morphism $f: X \rightarrow Z$ of varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$. In particular, f is surjective with connected fibres.

2.2 Hyperstandard sets. Let \mathfrak{R} be a subset of $[0, 1]$. We define

$$\Phi(\mathfrak{R}) = \left\{ 1 - \frac{r}{m} \mid r \in \mathfrak{R}, m \in \mathbb{N} \right\}$$

to be the set of *hyperstandard multiplicities* associated to \mathfrak{K} . We usually assume $0, 1 \in \mathfrak{K}$ without mention, so $\Phi(\mathfrak{K})$ includes $\Phi(\{0, 1\})$.

2.3 Divisors and resolutions. In algebraic geometry Weil divisors usually have integer coefficients. However, in birational geometry it is standard practice to consider \mathbb{R} -divisors. An \mathbb{R} -divisor on a normal variety X is of the form $M = \sum a_i M_i$ where M_i are distinct prime divisors and $a_i \in \mathbb{R}$. By $\mu_{M_i} M$ we mean the coefficient a_i . We say M is \mathbb{R} -Cartier if M can be written as an \mathbb{R} -linear combination of (not necessarily prime) Cartier divisors. For two \mathbb{R} -divisors M and N , $M \sim_{\mathbb{R}} N$ means $M - N$ is an \mathbb{R} -linear combination of principal Cartier divisors (a principal divisor is the divisor of zeros and poles of a rational function).

If X is equipped with a projective morphism $f : X \rightarrow Z$, an \mathbb{R} -Cartier divisor M is *nef* over Z if $M \cdot C \geq 0$ for every curve C contracted to a point by f . We say M is *ample* over Z if it is a positive \mathbb{R} -linear combination of ample Cartier divisors. We say M is *big* over Z if $M \sim_{\mathbb{R}} A + D$ where A is ample over Z and $D \geq 0$.

A *log resolution* $\phi : W \rightarrow X$ of (X, M) is a projective birational morphism where W is smooth, and the union of the exceptional locus of ϕ and the birational transform of $\text{Supp } M$ has simple normal crossing singularities.

2.4 Pairs. An important feature of modern birational geometry is that the main objects are pairs rather than varieties. Pairs are much better behaved when it comes to induction and passing from a variety to a birational model.

A *pair* (X, B) consists of a normal variety X and an \mathbb{R} -divisor $B \geq 0$ such that $K_X + B$ is \mathbb{R} -Cartier. If the coefficients of B are ≤ 1 , we say B is a *boundary*.

Let $\phi : W \rightarrow X$ be a log resolution of (X, B) . Let

$$K_W + B_W := \phi^*(K_X + B).$$

The *log discrepancy* of a prime divisor D on W with respect to (X, B) is defined as

$$a(D, X, B) := 1 - \mu_D B_W.$$

We say (X, B) is *lc* (resp. *klt*) (resp. ϵ -*lc*) if every coefficient of B_W is ≤ 1 (resp. < 1) (resp. $\leq 1 - \epsilon$). When $B = 0$ we just say X is lc, etc, instead of $(X, 0)$.

A *non-klt place* of (X, B) is a prime divisor D on birational models of X such that $a(D, X, B) \leq 0$. A *non-klt centre* is the image on X of a non-klt place. When (X, B) is lc, a non-klt centre is also called a *lc centre*.

If we remove the condition $B \geq 0$, the above definitions still make sense but we add *sub* to each notion defined, e.g. instead of lc we say sub-lc, etc.

Example. The simplest kind of pair is a *log smooth* one, that is, a pair (X, B) where X is smooth and $\text{Supp } B$ has simple normal crossing singularities. In this case (X, B) being lc (resp. klt) means every coefficient of B is ≤ 1 (resp. < 1).

Example. Let X be the cone over a rational curve of degree n (for a more precise definition see the example following [Theorem 3.7](#)). Then X is klt. But if X is the cone over an elliptic curve, then X is lc but not klt.

Example. Let X be a klt surface. Let $\phi: W \rightarrow X$ be the minimal resolution. The exceptional curves are all smooth rational curves and they intersect in a special way. There is a whose classification of the possible configurations (cf. [Kollár and Mori \[1998, Section 4\]](#)). Once we know the configuration and the self-intersections of the exceptional divisors it is a matter of an easy calculation to determine all the log discrepancies.

2.5 Generalised pairs. These pairs appear mainly when one considers the canonical bundle formula of a fibration, e.g. see case (2) of [Section 5.1](#). A generalised pair is roughly speaking a pair together with a birational polarisation, that is, a nef divisor on some birational model. They play an important role in relation with [Conjecture 1.7](#) by [Birkar and Zhang \[2016\]](#) and most of the results of [Birkar \[2016a,b\]](#). For the sake of simplicity we will try to avoid using these pairs and their subtle properties as much as possible but for convenience here we recall the definition in the projective case only. For detailed studies of generalised pairs see [Birkar and Zhang \[2016\]](#) and [Birkar \[2016a\]](#).

A projective *generalised (polarised) pair* consists of

- a normal projective variety X' ,
- an \mathbb{R} -divisor $B' \geq 0$ on X' ,
- a projective birational morphism $\phi: X \rightarrow X'$ from a normal variety, and
- a nef \mathbb{R} -Cartier divisor M on X ,

such that $K_{X'} + B' + M'$ is \mathbb{R} -Cartier, where $M' := \phi_* M$. We usually refer to the pair by saying $(X', B' + M')$ is a projective generalised pair with data $X \xrightarrow{\phi} X'$ and M . However, we want ϕ and M to be birational data, that is, if we replace X with a higher model, e.g. a resolution, and replace M with its pullback, then we assume the new data defines the same generalised pair.

Now we define generalised singularities. Replacing X we can assume ϕ is a log resolution of (X', B') . We can write

$$K_X + B + M = \phi^*(K_{X'} + B' + M')$$

for some uniquely determined B . We say $(X', B' + M')$ is *generalised lc* (resp. *generalised klt*) if every coefficient of B is ≤ 1 (resp. < 1).

Example. Assume $(X', B' + M')$ is a projective generalised pair with data $X \xrightarrow{\phi} X'$ and M , and assume $M = \phi^* M'$. Then $(X', B' + M')$ is generalised lc (resp. generalised klt) iff (X', B') is lc (resp. klt). In other words, in this case M' does not contribute to singularities.

Example. Let $X' = \mathbb{P}^2$, and $\phi: X \rightarrow X'$ be the blowup of a closed point $x' \in X'$. Assume $H \subset X'$ is a hyperplane. Let $M = 3\phi^* H - tE$ where E is the exceptional divisor of ϕ and $t \in [0, 3]$ is a real number. Then letting $B' = 0$, $(X', B' + M')$ is a projective generalised pair with data $X \xrightarrow{\phi} X'$ and M . Note that $M' = 0$. Now we can determine B in the formula above. Calculating intersection numbers we find $B = (t - 1)E$. Therefore, $(X', B' + M')$ is generalised lc (resp. generalised klt) iff $t \leq 2$ (resp. $t < 2$).

3 Fano varieties

3.1 Facets of Fano varieties. Grothendieck insisted on studying varieties (and schemes) in a relative setting. This philosophy has been very successfully implemented in birational geometry. This is particularly interesting in the case of Fano varieties, or we should say relative Fano varieties.

Let (X, B) be a klt pair and $f: X \rightarrow Z$ be a surjective projective morphism, and assume $-(K_X + B)$ is ample over Z . We then say (X, B) is *Fano over Z*. This relative notion unifies various classes of objects of central importance. There are three distinct cases.

- *Global case:* this is when Z is just a point, hence (X, B) is a *Fano pair* in the usual sense.
- *Fibration case:* this is when $\dim X > \dim Z > 0$, that is, f is a genuine fibration and its general fibres are global Fano pairs.
- *Birational case:* this is when f is birational. There are several important subcases here. If f is extremal and contracts one divisor, then f is a *divisorial contraction*. If f is extremal and contracts some subvariety but not a divisor, then f is a *flipping contraction*. If f is an isomorphism, then (X, B) is just the germ of a klt singularity.

3.2 Complements and anti-pluri-canonical systems. Assume (X, B) is an lc pair equipped with a projective morphism $X \rightarrow Z$. The *theory of complements* is essentially

the study of the systems $|-n(K_X + B)|$ where $n \in \mathbb{N}$, in a relative sense over Z . Obviously this is interesting only when some of these systems are not non-empty, e.g. Fano case. The theory was introduced by [Shokurov \[1993\]](#). The theory was further developed by [Shokurov \[2000\]](#), [Y. G. Prokhorov and Shokurov \[2001, 2009\]](#), and [Birkar \[2016a,b\]](#).

A *strong n -complement* of $K_X + B$ over a point $z \in Z$ is of the form $K_X + B^+$ where over some neighbourhood of z we have:

- (X, B^+) is lc,
- $n(K_X + B^+) \sim 0$, and
- $B^+ \geq B$.

From the definition we get

$$-n(K_X + B) \sim nB^+ - nB \geq 0$$

over some neighbourhood of z which in particular means the linear system $|-n(K_X + B)|$ is not empty over z , and that it contains a “nice” element. An n -complement (see [Birkar \[2016a\]](#)) is defined similarly but it is more complicated, so for simplicity we avoid using it. However, if $B = 0$, a complement and a strong complement are the same thing.

Theorem 3.3 ([Birkar \[ibid.\]](#), Theorems 1.7, 1.8, 1.9). *Let d be a natural number and $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and \mathfrak{R} satisfying the following. Assume (X, B) is a pair and $X \rightarrow Z$ a contraction such that*

- (X, B) is lc of dimension d ,
- the coefficients of B are in $\Phi(\mathfrak{R})$,
- X is Fano type over Z , and
- $-(K_X + B)$ is nef over Z .

Then for any point $z \in Z$, there is a strong n -complement $K_X + B^+$ of $K_X + B$ over z . Moreover, the complement is also an mn -complement for any $m \in \mathbb{N}$.

Here X of Fano type over Z means (X, G) is Fano over Z for some G . The theorem was conjectured by [Shokurov \[2000, Conjecture 1.3\]](#), it was proved in dimension 2 by [Shokurov \[ibid., Theorem 1.4\]](#), (see also [Y. G. Prokhorov and Shokurov \[2009, Corollary 1.8\]](#), and [Shokurov \[1993\]](#) for some cases). [Y. G. Prokhorov and Shokurov \[2001, 2009\]](#) proved various inductive statements regarding complements including some unconditional cases in dimension 3.

Example. When $X \rightarrow Z$ is toric morphism and $B = 0$ we can take $n = 1$ and B^+ to be the sum of the torus-invariant divisors.

Remark. Assume Z is a point. Assume for simplicity that $B = 0$ and that $-K_X$ is ample, that is, X is a usual Fano variety. When X is a smooth 3-fold, Šokurov [1979] proved that $|-K_X|$ contains a smooth K3 surface. In particular, K_X has a 1-complement. This is probably where the higher dimensional theory of complements originates.

Remark. Assume $X \rightarrow Z$ is birational. Assume again for simplicity that $B = 0$ and that $-K_X$ is ample over Z . When $X \rightarrow Z$ is a flipping contraction contracting one smooth rational curve only, Mori (cf. Kollár and Mori [1992, Theorem 1.7]) showed that there always exists a 1-complement over each $z \in Z$ but in the analytic sense, i.e. it exists over an analytic neighbourhood of z . This is used in Mori’s proof of existence of 3-fold flips, see Mori [1988].

Remark. Assume $X \rightarrow Z$ is an isomorphism, so we are looking at the germ of a klt singularity (X, B) around a point $x \in X$. For simplicity again assume $B = 0$. In general the Cartier index of K_X is not bounded even in dimension 2. The point of complement theory in this case is that the n -complement $K_X + B^+$ has Cartier index n which is bounded.

Remark. When X is a 3-fold with terminal singularities, $-K_X$ is ample over Z , and $B = 0$, the *general elephant conjecture* of Reid asks whether a general element of the linear system $|-K_X|$, relatively over Z , has canonical singularities. This is true in various cases, e.g. when X is Gorenstein and Z is a point by Reid [n.d.], or when $X \rightarrow Z$ is identity by Reid [1987].

Example. Lets look at the particular case of surfaces in the local case. Assume X is a surface, $X \rightarrow Z$ is the identity, and $B = 0$. If $x \in X$ is smooth, then K_X is a 1-complement of itself, that is, we can take $B^+ = 0$. In the singular case, from classification of the possible singularities one gets the following, by Shokurov [2000, p. 5.2.3]:

$$\text{if } x \in X \text{ is a type } \left\{ \begin{array}{l} A \text{ singularity, then } K_X \text{ has a 1-complement.} \\ D \text{ singularity, then } K_X \text{ has a 2-complement.} \\ E_6 \text{ singularity, then } K_X \text{ has a 3-complement.} \\ E_7 \text{ singularity, then } K_X \text{ has a 4-complement.} \\ E_8 \text{ singularity, then } K_X \text{ has a 6-complement.} \end{array} \right.$$

3.4 Effective birationality. Let X be a Fano variety. Theorem 3.3 says that $|-mK_X|$ is non-empty containing a nice element for some $m > 0$ depending only on $\dim X$. If we bound the singularities of X , we then have a much stronger statement.

Theorem 3.5 (Birkar [2016a, Theorem 1.2]). *Let d be a natural number and $\epsilon > 0$ a real number. Then there is a natural number m depending only on d and ϵ such that if X is any ϵ -lc Fano variety of dimension d , then $| -mK_X |$ defines a birational map.*

Note that m indeed depends on d as well as ϵ because the theorem implies the volume $\text{vol}(-K_X)$ is bounded from below by $\frac{1}{m^d}$. Without the ϵ -lc assumption, $\text{vol}(-K_X)$ can get arbitrarily small or large; see Hacon, McKernan, and Xu [2014, Example 2.1.1]. In dimension 2, the theorem is a consequence of BAB proved by V. Alexeev [1994], and in dimension 3, special cases are proved by Jiang [2015] using different methods. Cascini and McKernan have independently proved the theorem for canonical singularities, that is when $\epsilon = 1$, using quite different methods.

It is worth mentioning that the theorem also holds in the relative setting. It follows immediately from the global case stated above.

3.6 Boundedness of Fano varieties: BAB. It is possible to strengthen Theorem 3.5 so that $| -mK_X |$ defines an actual embedding. This follows from the next result.

Theorem 3.7 (Birkar [2016b, Theorem 1.1]). *Let d be a natural number and ϵ a positive real number. Then the projective varieties X such that*

- (X, B) is ϵ -lc of dimension d for some boundary B , and
- $-(K_X + B)$ is nef and big,

form a bounded family.

This was known as the Borisov-Alexeev-Borisov or BAB conjecture. Various special cases of it was considered by many people. It was known in the following cases (by taking $B = 0$): surfaces by V. Alexeev [1994], toric varieties by A. A. Borisov and L. A. Borisov [1992], Fano 3-folds with terminal singularities and Picard number one by Kawamata [1992b], Fano 3-folds with canonical singularities by Kollár, Miyaoka, Mori, and Takagi [2000], smooth Fano varieties by Kollár, Miyaoka, and Mori [1992], spherical Fano varieties by V. A. Alexeev and Brion [2004], Fano 3-folds with fixed Cartier index of K_X by A. Borisov [2001], and more generally, Fano varieties of given dimension with fixed Cartier index of K_X by Hacon, McKernan, and Xu [2014]; in a given dimension, the Fano varieties X equipped with a boundary Δ such that $K_X + \Delta \equiv 0$, (X, Δ) is ϵ -lc, and such that the coefficients of Δ belong to a DCC set, by Hacon, McKernan, and Xu [ibid.] (also see Hacon and Xu [2015] and Birkar [2016a, Theorem 1.4]).

Example. Now we look at an example of a non-bounded family of singular Fano surfaces. For $n \geq 2$ consider

$$\begin{array}{ccc} E & \subset & W_n \xrightarrow{f} X_n \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

where X_n is the cone over a rational curve of deg n , f is blowup of the vertex, and E is the exceptional curve. In other words, W_n is the projective bundle of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$, E is the section given by the summand $\mathcal{O}_{\mathbb{P}^1}(-n)$, and X_n is obtained from W_n by contracting E . Then an easy calculation, using $E^2 = -n$, shows that

$$K_{W_n} + \frac{n-2}{n}E = f^*K_{X_n},$$

hence X_n is a $\frac{2}{n}$ -lc Fano variety with one singular point (the larger is n , the deeper is the singularity). In particular, since the set of numbers $\{\frac{n-2}{n} \mid n \in \mathbb{N}\}$ is not finite, the set $\{X_n \mid n \in \mathbb{N}\}$ is not a bounded family. This example explains the role of the number ϵ in [Theorem 3.7](#).

Example. In this example we sketch the proof of [Theorem 3.7](#) in dimension two following [V. Alexeev and Mori \[2004\]](#). For simplicity assume $B = 0$ and that $-K_X$ is ample. There is $\Delta \geq 0$ such that (X, Δ) is ϵ -lc and $K_X + \Delta \sim_{\mathbb{R}} 0$. Let $\phi: W \rightarrow X$ be the minimal resolution and let $K_W + \Delta_W$ be the pullback of $K_X + \Delta$. Since (X, Δ) is klt, the exceptional divisors of ϕ are all smooth rational curves. Moreover, by basic properties of minimal resolutions, $\Delta_W \geq 0$. In particular, (W, Δ_W) is an ϵ -lc pair. Now a simple calculation of intersection numbers shows that $-E^2 \leq l$ for every exceptional curve of ϕ where $l \in \mathbb{N}$ depends only on ϵ . If the number of exceptional curves of ϕ is bounded, then the Cartier index of $-K_X$ is bounded which in turn implies $-nK_X$ is very ample for some bounded n . In particular, this holds if the Picard number of W is bounded from above. If in addition $\text{vol}(-K_X)$ is bounded, then X belongs to a bounded family. Note that $\text{vol}(-K_X) = \text{vol}(-K_W)$.

Running an MMP on K_W we get a morphism $W \rightarrow V$ where V is either \mathbb{P}^2 or a rational ruled surface (like W_n in the previous example), and the morphism is a sequence of blowups at smooth points. Let Δ_V be the pushdown of Δ_W . Then (V, Δ_V) is ϵ -lc and $K_V + \Delta_V \sim_{\mathbb{R}} 0$. It is easy to show that there are finitely many possibilities for V . In particular, from $\text{vol}(-K_W) \leq \text{vol}(-K_V)$, we deduce that $\text{vol}(-K_X) = \text{vol}(-K_W)$ is bounded from above. Thus it is enough to prove that the number of blowups in $W \rightarrow V$ is bounded. This number can be bounded by an elementary analysis of possible intersection numbers in the sequence (see [V. Alexeev and Mori \[ibid., Section 1\]](#) for more details).

3.8 Birational automorphism groups. An interesting consequence of [Theorem 3.7](#) concerns the Jordan property of birational automorphism groups of rationally connected varieties. [Y. Prokhorov and C. Shramov \[2016, Theorem 1.8\]](#) proved the next result assuming [Theorem 3.7](#).

Corollary 3.9 ([Birkar \[2016b, Corollary 1.3\]](#)). *Let d be a natural number. Then there is a natural number h depending only on d satisfying the following. Let X be a rationally connected variety of dimension d over k . Then for any finite subgroup G of the birational automorphism group $\text{Bir}(X)$, there is a normal abelian subgroup H of G of index at most h . In particular, $\text{Bir}(X)$ is Jordan.*

Here X *rationally connected* means that every two general closed points can be joined by a rational curve. If we take $X = \mathbb{P}^d$ in the corollary, then we deduce that the Cremona group $\text{Cr}_d := \text{Bir}(\mathbb{P}^d)$ is Jordan, answering a question of [Serre \[2009, p. 6.1\]](#).

4 Singularities of linear systems

4.1 Lc thresholds of \mathbb{R} -linear systems. Let (X, B) be a pair. The *log canonical threshold* (lc threshold for short) of an \mathbb{R} -Cartier \mathbb{R} -divisor $L \geq 0$ with respect to (X, B) is defined as

$$\text{lct}(X, B, L) := \sup\{t \mid (X, B + tL) \text{ is lc}\}.$$

It is a way of measuring the singularities of L taking into account the singularities of (X, B) as well.

Now let A be an \mathbb{R} -Cartier \mathbb{R} -divisor. The \mathbb{R} -linear system of A is

$$|A|_{\mathbb{R}} = \{L \geq 0 \mid L \sim_{\mathbb{R}} A\}.$$

We then define the *lc threshold* of $|A|_{\mathbb{R}}$ with respect to (X, B) (also called global lc threshold or α -invariant) as

$$\text{lct}(X, B, |A|_{\mathbb{R}}) := \inf\{\text{lct}(X, B, L) \mid L \in |A|_{\mathbb{R}}\}$$

which coincides with

$$\sup\{t \mid (X, B + tL) \text{ is lc for every } L \in |A|_{\mathbb{R}}\}.$$

This is an asymptotic invariant, so not surprisingly it is hard to compute in specific cases and study in general.

Due to connections with the notion of stability and existence of Kähler-Einstein metrics, lc thresholds of \mathbb{R} -linear systems have attracted a lot of attention, particularly, when A is

ample. An important special case is when X is Fano and $A = -K_X$ in which case many examples have been calculated, e.g. see [Cheltsov and K. A. Shramov \[2008\]](#).

Example. If $X = \mathbb{P}^d$, $B = 0$, and $A = -K_X$, then

$$\text{lct}(X, B, |A|_{\mathbb{R}}) = \frac{1}{d + 1}.$$

On the other hand, if $X \subset \mathbb{P}^d$ is a smooth hypersurface of degree $r \leq d$, $B = 0$, and $A = -K_X$, then

$$\text{lct}(X, B, |A|_{\mathbb{R}}) = \frac{1}{d + 1 - r}$$

by [Cheltsov and K. A. Shramov \[ibid., Example 1.3\]](#).

Another reason for studying the above threshold is connection with boundedness of Fano varieties. Indeed it plays a central role in the proof of [Theorem 3.7](#).

Theorem 4.2 ([Birkar \[2016b, Theorem 1.4\]](#)). *Let d be a natural number and ϵ a positive real number. Then there is a positive real number t depending only on d, ϵ satisfying the following. Assume*

- (X, B) is a projective ϵ -lc pair of dimension d , and
- $A := -(K_X + B)$ is nef and big.

Then

$$\text{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

This was conjectured by [Ambro \[2016\]](#) who proved it in the toric case. It can be derived from [Theorem 3.7](#) but in reality it is proved before [Theorem 3.7](#) (see next section). [Jiang \[2015, 2014\]](#) proved it in dimension two.

The lc threshold of an \mathbb{R} -linear system $|A|_{\mathbb{R}}$ is defined as an infimum of usual lc thresholds. [Tian \[1990, Question 1\]](#) asked whether the infimum is a minimum when $A = -K_X$ and X is Fano. The question was reformulated and generalised to Fano pairs in [Cheltsov and K. A. Shramov \[2008, Conjecture 1.12\]](#). The next result gives a positive answer when the lc threshold is at most 1.

Theorem 4.3 ([Birkar \[2016b, Theorem 1.5\]](#)). *Let (X, B) be a projective klt pair such that $A := -(K_X + B)$ is nef and big. Assume that $\text{lct}(X, B, |A|_{\mathbb{R}}) \leq 1$. Then there is $0 \leq D \sim_{\mathbb{R}} A$ such that*

$$\text{lct}(X, B, |A|_{\mathbb{R}}) = \text{lct}(X, B, D).$$

Moreover, if B is a \mathbb{Q} -boundary, then we can choose $D \sim_{\mathbb{Q}} A$, hence in particular, the lc threshold is a rational number.

Shokurov has an unpublished proof of the theorem in dimension two.

4.4 Lc thresholds of \mathbb{R} -linear systems with bounded degree. Next we treat lc thresholds associated with divisors on varieties, in a general setting. To obtain any useful result, one needs to impose certain boundedness conditions on the invariants of the divisor and the variety.

Theorem 4.5 (Birkar [2016b, Theorem 1.6]). *Let d, r be natural numbers and ϵ a positive real number. Then there is a positive real number t depending only on d, r, ϵ satisfying the following. Assume*

- (X, B) is a projective ϵ -lc pair of dimension d ,
- A is a very ample divisor on X with $A^d \leq r$, and
- $A - B$ is ample.

Then

$$\text{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

This is one of the main ingredients of the proof of [Theorem 4.2](#) but it is also interesting on its own. We explain briefly some of the assumptions of the theorem. The condition $A^d \leq r$ means that X belongs to a bounded family of varieties, actually, if we choose A general in its linear system, then (X, A) belongs to a bounded family of pairs. We can use the divisor A to measure how “large” other divisors are on X . Indeed, the ampleness of $A - B$ roughly speaking says that the “degree” of B is bounded from above, that is,

$$\deg_A B := A^{d-1} B < A^d \leq r.$$

Without such boundedness assumptions, one would not find a positive lower bound for the lc threshold as the next example shows.

Example. Assume $(X = \mathbb{P}^2, B)$ is ϵ -lc and $S \subset X$ is a line. Let $L = A = lS$ where $l \in \mathbb{N}$. Then the multiplicity of L at any closed point $x \in L$ is l , hence the lc threshold $\text{lct}(L, X, B) \leq \frac{1}{l}$. Thus the larger is l , the smaller is the threshold. Next we illustrate how the threshold depends on the degree of B . Let T be another line and x be the intersection point $S \cap T$. Let $X_1 \rightarrow X$ be the blowup at x , and let x_1 be the intersection of the exceptional divisor E_1 and the birational transform S^\sim . Let $X_2 \rightarrow X_1$ be the blowup at x_1 , and let x_2 be the intersection of the new exceptional divisor E_2 and S^\sim . At each step we blowup the intersection point of S^\sim and the newest exceptional divisor.

Put $W := X_r$. Then the exceptional locus of $\phi: W \rightarrow X$ consists of a chain of curves all of which are -2 -curves except one which is a -1 -curve. Then $-K_W$ is nef over X , in fact, it is semi-ample over X . Thus there is $0 \leq B_W \sim_{\mathbb{R}} \alpha \phi^* H - K_W$ for some $\alpha > 0$ such that (W, B_W) is $\frac{1}{2}$ -lc and $K_W + B_W \sim_{\mathbb{R}} 0/X$. Now let B be the pushdown of B_W .

Then (X, B) is $\frac{1}{2}$ -lc. Now let $L = S + T$. Then the coefficient of E_r in ϕ^*L is $r + 1$, hence

$$\text{lct}(L, X, B) = \text{lct}(\phi^*L, W, B_W) \leq \frac{1}{r + 1}.$$

Thus there is no lower bound on the lc threshold if r is arbitrarily large. This does not contradict [Theorem 4.5](#) because when $r \gg 0$, the degree $\deg_A B \gg 0$ and $A - B$ cannot be ample (here $A = lS$ with l fixed).

5 Brief sketch of proofs of main results

In this section we sketch some of the ideas of the proofs of [Theorem 3.3](#), [Theorem 3.5](#), [Theorem 4.5](#) and [Theorem 3.7](#). We try to remove technicalities as much as possible but this comes at the expense of being imprecise in various places and not elaborating on many of the new ideas.

5.1 Sketch of proof of boundedness of complements. ([Theorem 3.3](#)) For simplicity we look at the global case, that is, when Z is a point. Pick a sufficiently small $\epsilon \in (0, 1)$. Let $Y \rightarrow X$ be the birational morphism which extracts all the prime divisors with log discrepancy smaller than ϵ . Let $K_Y + B_Y$ be the pullback of $K_X + B$. Define Θ_Y to be the same as B_Y except that we replace each coefficient in $(1 - \epsilon, 1)$ with 1. Run an MMP on $-(K_Y + \Theta_Y)$ and let Y' be the resulting model and $\Theta_{Y'}$ be the pushdown of Θ_Y . We can run such MMP because Y turns out to be of Fano type, so we can run MMP on any divisor on Y .

As a consequence of local and global ACC of [Hacon, McKernan, and Xu \[2014, Theorems 1.1 and 1.5\]](#) (in practice we need their generalisations to generalised pairs, see [Birkar and Zhang \[2016, Theorems 1.5 and 1.6\]](#)), we can show that the MMP does not contract any component of $[\Theta_Y]$, $(Y', \Theta_{Y'})$ is lc, and $-(K_{Y'} + \Theta_{Y'})$ is nef. It is enough to construct a bounded complement for $K_{Y'} + \Theta_{Y'}$. Replacing (X, B) with $(Y', \Theta_{Y'})$ and applying further reductions, we can reduce the problem to one of the following cases:

1. B has a component S with coefficient 1 and $-(K_X + B)$ is nef and big, or
2. $K_X + B \equiv 0$ along a fibration $f : X \rightarrow T$, or
3. (X, B) is *exceptional*.

Here exceptional means that for any choice of $0 \leq P \sim_{\mathbb{R}} -(K_X + B)$ the pair $(X, B + P)$ is klt. These cases require very different inductive treatment.

Case (1): First apply *divisorial adjunction* to define $K_S + B_S = (K_X + B)|_S$. Further modification of the setting allows us to ensure that S is Fano type. Moreover, the coefficients of B_S happen to be in a set $\Phi(\mathfrak{S})$ for some fixed finite set \mathfrak{S} . By induction on

dimension $K_S + B_S$ has a strong n -complement for some bounded n . The idea then is to lift the complement to X using vanishing theorems. In the simplest case when (X, B) is log smooth and $B = S$, we look at the exact sequence

$$H^0(-n(K_X + B)) \rightarrow H^0(-n(K_X + B)|_S) \rightarrow H^1(-n(K_X + B) - S) = 0$$

where the vanishing follows from Kawamata-Viehweg vanishing theorem noting that

$$-n(K_X + B) - S = K_X - n(K_X + B) - (K_X + B) = K_X - (n + 1)(K_X + B)$$

Since $K_S + B_S$ has a strong n -complement, the middle space in the above sequence is non-trivial which implies the left hand side is also non-trivial by lifting the section corresponding to the complement. One then argues that the lifted section gives a strong n -complement for $K_X + B$.

Case (2): Apply the *canonical bundle formula* (also called adjunction for fibre spaces, derived from Kawamata [1998]) to write

$$K_X + B \sim_{\mathbb{R}} f^*(K_T + B_T + M_T)$$

where B_T is the *discriminant divisor* and M_T is the *moduli divisor*. It turns out that the coefficients of B_T happen to be in a set $\Phi(\mathfrak{S})$ for some fixed finite set \mathfrak{S} , and that pM_T is integral for some bounded number $p \in \mathbb{N}$. Now we want to find a complement for $K_T + B_T + M_T$ and pull it back to X . There is a serious issue here: $(T, B_T + M_T)$ is not a pair in the usual sense but it is a generalised pair. Thus we actually need to prove [Theorem 3.3](#) (at least in the global case) in the more general setting of generalised pairs. This makes life a lot more difficult but fortunately everything turns out to work. Once we have a bounded complement for $K_T + B_T + M_T$ it is straightforward to derive a bounded complement for $K_X + B$.

Case (3): In this case we use effective birationality. Perhaps after decreasing ϵ , the exceptionality condition implies that (X, B) is ϵ -lc. For simplicity assume $B = 0$ and that X is a Fano variety. Also assume we already have [Theorem 3.5](#). Then there is a bounded number $m \in \mathbb{N}$ such that $|-mK_X|$ defines a birational map. Pick $M \in |-mK_X|$ and let $B^+ = \frac{1}{m}M$. Since X is exceptional, (X, B^+) is automatically klt, hence $K_X + B^+$ is a strong m -complement. Although this gives some ideas of how one may get a bounded complement but in practice we cannot give a complete proof of [Theorem 3.5](#) before proving [Theorem 3.3](#). The two theorems are actually proved together. See [Birkar \[2016a, Sections 6 and 7\]](#) for more details.

5.2 Sketch of proof of effective birationality. ([Theorem 3.5](#)) Let $m \in \mathbb{N}$ be the smallest number such that $|-mK_X|$ defines a birational map, and let $n \in \mathbb{N}$ be a number such

that $\text{vol}(-nK_X) > (2d)^d$. Initially we take n to be the smallest such number but we will modify it during the proof. We want to show that m is bounded from above. The idea is first to show that $\frac{m}{n}$ is bounded from above, and then at the end show that m is bounded.

Applying a standard elementary technique we can create a covering family \mathcal{G} of sub-varieties of X such that if $x, y \in X$ are any pair of general closed points, then there is $0 \leq \Delta \sim_{\mathbb{Q}} -(n+1)K_X$ and $G \in \mathcal{G}$ such that (X, Δ) is lc at x with the unique non-plt centre G , and (X, Δ) is not plt at y .

Assume $\dim G = 0$ for all G . Then $G = \{x\}$ is an isolated non-plt centre. Using multiplier ideals and vanishing theorems we can lift sections from G and show that $|-nK_X|$ defines a birational map after replacing n with a bounded multiple, hence in particular $\frac{m}{n}$ is bounded from above in this case.

Now let's assume all G have positive dimension. If $\text{vol}(-mK_X|_G)$ is large, then again using some elementary arguments, we can create a new non-plt centre G' containing x but with $\dim G' < \dim G$. Thus we can replace G with G' and apply induction on dimension of G . We can then assume $\text{vol}(-mK_X|_G)$ is bounded from above.

Similar to the previous paragraph, we can cut G and decrease its dimension if $\text{vol}(-nK_X|_G)$ is bounded from below. Showing this lower boundedness is the hard part. A key point here is that although G is not necessarily a divisor and although the singularities of (X, Δ) away from x maybe quite bad but still there is a kind of adjunction formula, that is, if F is the normalisation of G , then we can write

$$(K_X + \Delta)|_F \sim_{\mathbb{R}} K_F + \Theta_F + P_F$$

where Θ_F is a boundary divisor with coefficients in a fixed DCC set Ψ depending only on d , and P_F is pseudo-effective. Replacing n with $2n$ and adding to Δ we can easily make P_F big and effective.

Now we would ideally want to apply induction on d but the difficulty is that F may not be Fano, in fact, it can be any type of variety. Another issue is that the singularities of $(F, \Theta_F + P_F)$ can be pretty bad. To overcome these difficulties we use the fact that $\text{vol}(-mK_X|_G)$ is bounded from above. From this boundedness one can deduce that there is a bounded projective log smooth pair $(\bar{F}, \Sigma_{\bar{F}})$ and a birational map $\bar{F} \dashrightarrow F$ such that $\Sigma_{\bar{F}}$ is reduced containing the exceptional divisor of $\bar{F} \dashrightarrow F$ and the support of the birational transform of Θ_F (and other relevant divisors).

Surprisingly, the worse the singularities of $(F, \Theta_F + P_F)$ the better because we can then produce divisors on \bar{F} with bounded "degree" but with arbitrarily small lc thresholds which would contradict a baby version of [Theorem 4.5](#). Indeed assume $(F, \Theta_F + P_F)$ is not plt. A careful study of the above adjunction formula allows to write $K_F + \Lambda_F := K_X|_F$ where $\Lambda_F \leq \Theta_F$ and (F, Λ_F) is sub- ϵ -lc. Put $I_F = \Theta_F + P_F - \Lambda_F$. Then

$$I_F = K_F + \Theta_F + P_F - K_F - \Lambda_F \sim_{\mathbb{R}} (K_X + \Delta)|_F - K_X|_F = \Delta|_F \sim_{\mathbb{R}} -(n+1)K_X|_F.$$

Moreover, $K_F + \Lambda_F + I_F$ is ample.

Let $\phi: F' \rightarrow F$ and $\psi: F' \rightarrow \overline{F}$ be a common resolution. Pull back $K_F + \Lambda_F + I_F$ to F' and then push it down to \overline{F} and write it as $K_{\overline{F}} + \Lambda_{\overline{F}} + I_{\overline{F}}$. Then the above ampleness gives

$$\phi^*(K_F + \Lambda_F + I_F) \leq \psi^*(K_{\overline{F}} + \Lambda_{\overline{F}} + I_{\overline{F}})$$

which implies that $(\overline{F}, \Lambda_{\overline{F}} + I_{\overline{F}})$ is not sub-plt. From this one deduces that $(\overline{F}, \Gamma_{\overline{F}} + I_{\overline{F}})$ is not plt where $\Gamma_{\overline{F}} = (1 - \epsilon)\Sigma_{\overline{F}}$. Finally, one argues that the degree of $I_{\overline{F}}$ gets arbitrarily small if $\text{vol}(-nK_X|_G)$ gets arbitrarily small, and this contradicts an easy case of [Theorem 4.5](#).

If singularities of $(F, \Theta_F + P_F)$ are good, then we again face some serious difficulties. Very roughly, in this case, we lift sections from F to X and use this section to modify Δ so that $(F, \Theta_F + P_F)$ has bad singularities, hence we reduce the problem to the above arguments. This shows $\frac{m}{n}$ is bounded.

Finally, to we still need to bound m . This can be done by arguing that $\text{vol}(-mK_X)$ is bounded from above and use this to show X is birationally bounded, and then work on the bounded model. See [Birkar \[2016a, Section 4\]](#) for more details.

5.3 Sketch of proof of boundedness of lc thresholds. ([Theorem 4.5](#)) Pick $0 \leq N \sim_{\mathbb{R}} A$. Let s be the largest number such that $(X, B + sN)$ is ϵ' -lc where $\epsilon' = \frac{\epsilon}{2}$. It is enough to show s is bounded from below. There is a prime divisor T on birational models of X with log discrepancy $a(T, X, \Delta) = \epsilon'$ where $\Delta := B + sN$. It is enough to show that the multiplicity of T in ϕ^*N is bounded on some resolution $\phi: V \rightarrow X$ on which T is a divisor. We can assume the image of T on X is a closed point x otherwise we can cut by hyperplane sections and apply induction on dimension.

There is a birational morphism $Y \rightarrow X$ from a normal projective variety which contracts exactly T . A key ingredient here is provided by the theory of complements: using the fact that $-(K_Y + T)$ is ample over X , we can find Λ_Y such that (Y, Λ_Y) is lc near T and $n(K_Y + \Lambda_Y) \sim 0/X$ for some bounded number $n \in \mathbb{N}$. One can think of $K_Y + \Lambda_Y$ as a local-global type of complement. The crucial point is that if Λ is the pushdown of Λ_Y , then we can make sure degree of Λ is bounded from above, that is, after replacing A we can assume $A - \Lambda$ is ample. By construction, the log discrepancy $a(T, X, \Lambda) = 0$ and $(X, \text{Supp } \Lambda)$ is bounded.

Next using resolution of singularities we can modify the setting and then assume that (X, Λ) is log smooth and Λ is reduced. The advantage of having Λ is that now T can be obtained by a sequence of blowups which is toroidal with respect to (X, Λ) . That is, in every step we blowup the centre of T which happens to be a stratum of (X, Λ) ; a stratum is just a component of the intersection of some of the components of Λ . The first step is just the blowup of x . One argues that it is enough to bound the number of these blowups.

By the previous paragraph, we can discard any component of Λ not passing through x , hence assume $\Lambda = S_1 + \dots + S_d$ where S_i are irreducible components. On the other hand, a careful analysis of $Y \rightarrow X$ allows us to further modify the situation so that $\text{Supp } \Delta$ does not contain any stratum of (X, Λ) apart from x . This is one of the difficult steps of the whole proof.

Since (X, Λ) is log smooth and bounded, we can find a surjective finite morphism $X \rightarrow \mathbb{P}^d$ which maps x to the origin $z = (0 : \dots : 0 : 1)$ and maps S_i on H_i where H_1, \dots, H_d are the coordinate hyperplanes passing through z . Since $\text{Supp } \Delta$ does not contain any stratum of (X, Λ) apart from x , it is not hard to reduce the problem to a similar problem on \mathbb{P}^d . From now on we assume $X = \mathbb{P}^d$ and that S_i are the coordinate hyperplanes. The point of this reduction is that now (X, Λ) is not only toroidal but actually toric, and $-(K_X + \Lambda)$ is very ample. In particular, replacing Δ with $t\Delta + (1-t)\Lambda$ for some sufficiently small $t > 0$ (and replacing ϵ' accordingly), we can make $K_X + \Delta$ anti-ample. Next by adding to Δ we can assume $K_X + \Delta$ is numerically trivial.

Let $W \rightarrow X$ be the sequence of blowups which obtains T as above. Since the blowups are toric, W is a toric variety. If $Y \rightarrow X$ is the birational morphism contracting T only, as before, then Y is also a toric variety. Moreover, if $K_Y + \Delta_Y$ is the pullback of $K_X + \Delta$, then (Y, Δ_Y) is ϵ' -lc and $K_Y + \Delta_Y$ is numerically trivial. Now running MMP on $-K_Y$ and using base point freeness gives another toric variety Y' which is Fano and ϵ' -lc. By the toric version of BAB proved by [A. A. Borisov and L. A. Borisov \[1992\]](#), Y' belongs to a bounded family. From this we can produce a klt strong m -complement $K_{Y'} + \Omega_{Y'}$ for some bounded $m \in \mathbb{N}$ which induces a klt strong m -complement $K_Y + \Omega_Y$ which in turn gives a klt strong m -complement $K_X + \Omega$.

Finally Ω belongs to a bounded family as its coefficients are in a fixed finite set and its degree is bounded. This implies that $(X, \Omega + u\Lambda)$ is klt for some $u > 0$ bounded from below. Now an easy calculation shows that the multiplicity of T in the pullback of Λ on W is bounded from above which in turn implies the number of blowups in $W \rightarrow X$ is bounded as required.

5.4 Sketch of proof of BAB. ([Theorem 3.7](#)) First applying [Hacon and Xu \[2015, Theorem 1, 3\]](#) it is enough to show that K_X has a klt strong m -complement for some bounded number $m \in \mathbb{N}$. Running an MMP on $-K_X$ and replacing X with the resulting model we can assume $B = 0$. By [Theorem 3.3](#), we know that we have an lc strong n -complement $K_X + B^+$. If X is exceptional, then the complement is klt, so we are done in this case. To treat the general case the idea is to modify the complement $K_X + B^+$ into a klt one. We will do this using birational boundedness.

We need to show $\text{vol}(-K_X)$ is bounded from above. This can be proved using arguments similar to the proof of the effective birationality theorem. Once we have this bound,

we can show that (X, B^+) is log birationally bounded, that is, there exist a bounded log smooth projective pair $(\bar{X}, \Sigma_{\bar{X}})$ and a birational map $\bar{X} \dashrightarrow X$ such that $\Sigma_{\bar{X}}$ contains the exceptional divisors of $\bar{X} \dashrightarrow X$ and the support of the birational transform of B^+ .

Next we pull back $K_X + B^+$ to a high resolution of X and push it down to \bar{X} and denote it by $K_{\bar{X}} + B_{\bar{X}}^+$. Then $(\bar{X}, B_{\bar{X}}^+)$ is sub-lc and $n(K_{\bar{X}} + B_{\bar{X}}^+) \sim 0$. Now support of $B_{\bar{X}}^+$ is contained in $\Sigma_{\bar{X}}$ so we can use the boundedness of $(\bar{X}, \Sigma_{\bar{X}})$ to perturb the coefficients of $B_{\bar{X}}^+$. More precisely, perhaps after replacing n , there is $\Delta_{\bar{X}} \sim_{\mathbb{Q}} B_{\bar{X}}^+$ such that $(\bar{X}, \Delta_{\bar{X}})$ is sub-klt and $n(K_{\bar{X}} + \Delta_{\bar{X}}) \sim 0$. Pulling $K_{\bar{X}} + \Delta_{\bar{X}}$ back to X and denoting it by $K_X + \Delta$ we get a sub-klt (X, Δ) with $n(K_X + \Delta) \sim 0$.

Now a serious issue here is that Δ is not necessarily effective. In fact it is by no means clear that its coefficients are even bounded from below. This is one of the difficult steps of the proof. However, this boundedness follows directly from [Theorem 4.2](#). The rest of the argument which modifies Δ to get a klt complement is an easy application of complement theory.

6 Some related problems and topics

6.1 Fano fibrations. One of the possible outcomes of the MMP is a Mori fibre space which is an extremal contraction $X \rightarrow Z$ where K_X is anti-ample over Z . This is a special kind of Fano fibration. Fano fibrations and more generally Fano type fibrations appear naturally in the course of applying induction on uniruled varieties, and in the context of moduli theory.

Suppose now that $f: X \rightarrow Z$ is a Mori fibre space where X is a 3-fold with \mathbb{Q} -factorial terminal singularities. Mori and Prokhorov proved that if Z is a surface, then Z has canonical singularities by [Mori and Y. Prokhorov \[2008\]](#), and if Z is a curve, then the coefficients of the fibres of f are bounded from above by 6 by [Mori and Y. G. Prokhorov \[2009\]](#).

M^cKernan proposed a generalisation of the first part to higher dimension:

Conjecture 6.2. *Assume $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$. Then there is $\delta \in \mathbb{R}^{>0}$ such that if $f: X \rightarrow Z$ is a Mori fibre space where X is ϵ -lc \mathbb{Q} -factorial of dimension d , then Z is δ -lc.*

On the other hand, independently, Shokurov proposed a more general problem which generalised both parts of Mori and Prokhorov result.

Conjecture 6.3. *Assume $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$. Then there is $\delta \in \mathbb{R}^{>0}$ such that if*

- (X, B) is an ϵ -lc pair of dimension d ,
- $f: X \rightarrow Z$ is a contraction with $\dim Z > 0$,

- $K_X + B \sim_{\mathbb{R}} 0/Z$, and $-K_X$ is big/ Z ,

then we can write

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

such that $(Z, B_Z + M_Z)$ is δ -lc where B_Z and M_Z are the discriminant and moduli parts of adjunction.

M^cKernan’s conjecture is known in the toric case by [V. Alexeev and A. Borisov \[2014\]](#). Shokurov’s conjecture is known when $\dim X - \dim Z \leq 1$ by [Birkar \[2016c\]](#), in particular for surfaces, and open in higher dimension but we have the following general result of [Birkar \[ibid.\]](#).

Theorem 6.4. *Shokurov conjecture holds for those f such that $(F, \text{Supp}B|_F)$ belongs to a bounded family where F is a general fibre of f .*

Note that by BAB (more precisely [Birkar \[2016b, Corollary 1.2\]](#)), F automatically belongs to a bounded family. However, one has little control over $\text{Supp}B|_F$ and this is the main difficulty. This issue is similar to the difficulties which appear in the proof of BAB and related results. It is expected that the methods developed to prove BAB also works to prove Shokurov’s conjecture but perhaps after some hard work.

6.5 Minimal log discrepancies and termination. The lc threshold plays an important role in birational geometry. This is clear from the proofs described in [Section 5](#). It is also related to the termination conjecture (1.5) by [Birkar \[2007\]](#). Another more subtle invariant of singularities is the *minimal log discrepancy* (mld) also defined by Shorkuov. Let (X, B) be a pair. The mld of (X, B) denoted $\text{mld}(X, B)$ is defined to be the minimum of log discrepancies $a(D, X, B)$ where D runs over all prime divisors on birational model of X . The mld is way harder to treat than the lc threshold. Shokurov proposed the following:

Conjecture 6.6 (ACC for mld’s). *Assume $d \in \mathbb{N}$ and $\Phi \subset [0, 1]$ is a set of numbers satisfying the descending chain condition (DCC). Then the set*

$$\{\text{mld}(X, B) \mid (X, B) \text{ is an lc pair and coefficients of } B \text{ are in } \Phi\}$$

satisfies the ascending chain condition (ACC).

This is known for surfaces by [V. Alexeev \[1993\]](#) but open in dimension ≥ 3 . Its importance is in relation with the termination conjecture and other topics of interest, see [Shokurov \[2004\]](#) and [Birkar and Shokurov \[2010\]](#). Shokurov showed that this ACC conjecture together with a semi-continuity conjecture about mld’s due to Ambro imply the termination conjecture by [Shokurov \[2004\]](#). The expectation is that the ACC conjecture can be tackled using the theory of complements and the methods described in this text but again after some hard work.

6.7 Stable Fano varieties. Existence of specific metrics, e.g. Kähler-Einstein metrics, on manifolds is a central topic in differential geometry. Unlike canonically polarised and Calabi-Yau manifolds (see [Yau \[1978\]](#) and references therein), Fano manifolds do not always admit such metrics. It is now an established fact that a Fano manifold admits a Kähler-Einstein metric iff it is so-called *K-polystable* (see [Chen, Donaldson, and Sun \[2015\]](#) and references therein).

On the other hand, it is well-known that Fano varieties do not behave as well as canonically polarised varieties in the context of moduli theory. For example, the moduli space would not be separated. A remedy is to consider only *stable* Fano's. The first step of constructing a moduli space is to prove a suitable boundedness result. In the smooth case this is not an issue by [Kollár, Miyaoka, and Mori \[1992\]](#) but in the singular case boundedness is a recent result. Using methods described in [Section 5, Jiang \[2017\]](#) proved such a result by showing that the set of *K*-semistable Fano varieties X of fixed dimension and $\text{vol}(-K_X)$ bounded from below forms a bounded family.

6.8 Other topics. There are connections between the advances described in this text and other topics of interest not discussed above. Here we only mention some works very briefly. [Lehmann, Tanimoto, and Tschinkel \[2014\]](#) and [Lehmann and Tanimoto \[2017\]](#) relate boundedness of Fano's and related invariants to the geometry underlying Manin's conjecture on distribution of rational points on Fano varieties. On the other hand, [Cerbo and Svaldi \[2016\]](#) studied boundedness of Calabi-Yau pairs where boundedness of Fano varieties appears naturally.

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Received 2017-12-05.

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