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SOME 20+ YEAR OLD PROBLEMS ABOUT BANACH SPACES AND OPERATORS ON THEM

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Abstract

In the last few years numerous 20+ year old problems in the geometry of Banach spaces were solved. Some are described herein.

1 Introduction

In this note I describe some problems in Banach space theory from the 1970s and 1980s that were solved after they had been opened for 20+ years. The problems are mostly not connected to one another, so each section is independent from the other sections. I use standard Banach space notation and terminolgy, as is contained e.g. in Lindenstrauss and Tzafriri [1977] or Albiac and N. J. Kalton [2006]. In this introduction I just recall some definitions that are used repeatedly. Other possibly unfamilar definitions are introduced in the sections in which they are used.

All spaces are Banach spaces and subspaces are closed linear subspaces. An operator is a bounded linear operator between Banach spaces. An isomorphism is a not necessarly surjective linear homeomorphism. $L(X, Y)$ denotes the space of operators from X to Y. This is abbreviated to $L(X)$ when $X = Y$. B_X denotes the closed unit ball of the space X. An operator T with domain X is compact if TB_X has compact closure and is weakly compact if TB_X has weakly compact closure. An operator T is strictly singular if the restriction of T to any infinite dimensional subspace of its domain is not an isomorphism. If Y is a Banach space and T is an operator, T is said to be Y-singular if the restriction of T to any subspace of its domain that is isomorphic to Y is not an isomorphism. So T is strictly singular if T is Y-singular for every infinite dimensional space Y. The isomorphism constant or Banach-Mazur distance between Banach spaces X_1 and X_2 is defined as

$$d(X_1, X_2) = \inf \|T\| \cdot \|T^{-1}\|$$

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where the infimum is taken over all isomorphisms from X_1 onto X_2. So $d(X_1, X_2) = \infty$ if X_1 is not isomorphic to X_2.

2 The diameter of the isomorphism class of a Banach space

Given a Banach space X, define

$$D(X) = \sup\{d(X_1, X_2) : X_1, X_2 \text{ are isomorphic to } X\}.$$

J. J. Schäffer considered the following question to be well-known when he included it in his 1976 book Schäffer [1976]:

Is $D(X) = \infty$ for all infinite dimensional X?

My first PhD student, E. Odell (who died much too young) and I gave an affirmative answer for separable X Johnson and E. Odell [2005]. A new definition helped: Call a Banach space X K-*elastic* provided every isomorph of X K-embeds into X. Call X *elastic* if X is K-elastic for some $K < \infty$.

Ted and I proved the following, which easily implies that $D(X) = \infty$ if X is separable and infinite dimensional.

Theorem 2.1. *If X is a separable Banach space so that for some K, every isomorph of X is K-elastic, then X is finite dimensional.*

The only "obvious" example of a separable elastic space is $C[0, 1]$. It is 1-elastic because Mazur proved that every separable Banach space is isometrically isomorphic to a subspace of $C[0, 1]$. Odell and I suspected that isomorphs of $C[0, 1]$ are the only elastic separable spaces and remarked that our proof of Theorem 2.1 could be streamlined a lot if this is true. We could not prove this but were able to use Bourgain's ℓ_∞ index theory Bourgain [1980] to prove that a separable elastic space contains a subspace that is isomorphic to c_0 and used that information in the proof of Theorem 2.1. Ten years later my third PhD student, D. Alspach, and B. Sari created a new index that they used to verify that our suspicion was correct. Their proof is rather complicated, but even more recently Beanland and Causey [n.d.] simplified the proof somewhat by using more descriptive set theory. It looks likely that the Alspach-Sari index will be used more down the road.

Schäffer's problem remains open for non separable spaces. In some models of set theory (GCH) there are spaces of every density character that are 1-elastic by virtue of being universal, but some tools that were used in the separable setting are not available when the spaces are non separable. Godefroy [2010] proved that under Martin's Maximum Axiom Schäffer's problem has an affirmative answer for subspaces of ℓ_∞.

3 Commutators

The commutator of two elements A and B in a Banach algebra is given by

$$[A, B] = AB - BA.$$

A natural problem that arises in the study of derivations on a Banach algebra $\mathfrak{A}$ is to classify the commutators in the algebra. Probably the most natural non commutative Banach algebras other than C^* algebras are the spaces $\mathfrak{L}(\mathfrak{X})$ of bounded linear operators on a Banach space $\mathfrak{X}$. When X is $n < \infty$ dimensional, $\mathfrak{L}(\mathfrak{X})$ can be identified with the n by n matrices of scalars, and it is classical that such a matrix is a commutator if and only if it has trace zero. There is generally no trace on $L(X)$ when X is infinite dimensional, and the only general obstruction to an operator being a commutator is due to Wintner [1947], who proved that the identity in a unital Banach algebra is not a commutator. It follows immediately by passing to the quotient algebra $\mathfrak{L}(\mathfrak{X})/\mathfrak{I}(\mathfrak{X})$ that no element of the form $\lambda I + K$, where K belongs to a proper norm closed ideal $\mathfrak{I}(\mathfrak{X})$ of $\mathfrak{L}(\mathfrak{X})$ and $\lambda \neq 0$, can be a commutator. With this in mind we call a Banach space $\mathfrak{X}$ a *Wintner space* provided the only non commutators in $\mathfrak{L}(\mathfrak{X})$ are elements of the form $\lambda I + K$ with $\lambda \neq 0$ and K in a proper closed ideal.
Here is Wielandt's elegant proof Wielandt [1949] of Wintner's theorem that I is not a commutator:
If $I = AB - BA$ then by induction

$$\forall n \quad A^n B - BA^n = nA^{n-1}.$$

So A cannot be nilpotent and

$$n\|A^{n-1}\| \leq 2\|A\| \cdot \|B\| \cdot \|A^{n-1}\|.$$

To determine whether a Banach space $\mathfrak{X}$ is a Wintner space, the first thing one most know is what elements in $\mathfrak{L}(\mathfrak{X})$ lie in a proper closed ideal, so one needs to know what are the maximal ideals in $\mathfrak{L}(\mathfrak{X})$ (maximal ideals in a unital Banach algebra are automatically closed because the invertible elements are open). In certain classical spaces, such as ℓ_p for $1 \leq p < \infty$, and c_0, there is only one proper closed ideal; namely, the ideal of compact operators on $\mathfrak{X}$, (Gohberg, Markus, and Feldman [1960], see also Whitley [1964, Theorem 6.2]), so it is not surprising that these spaces received the most attention early on. After a decade of so research on commutators by numerous people, in 1965 Brown and Pearcy [1965]) made a breakthrough by proving that ℓ_2 is a Wintner space. In 1972, Apostol [1972a] verified that ℓ_p for $1 < p <$ is a Wintner space and a year later Apostol [1973] he proved that c_0 is a Wintner space. Apostol obtained information about commutators on ℓ_1 and ℓ_∞, and there was also research done around the same time about commutators on L_p, but it was only 30 years later that another classification theorem was proved. In 2009, my student D. Dosev showed in his dissertation that ℓ_1 is a Wintner space. In D. Dosev and Johnson [2010], he and I codified what is needed for the technology developed by Brown–Pearcy, Apostol, and him in order to prove that an operator is a commutator in spaces that have a *Pełczyński decomposition*. (The space $\mathfrak{X}$ is said to have a Pełczyński decomposition if X is isomorphic to $\left(\sum \mathfrak{X}\right)_p$ with $1 \le p \le \infty$ or $p = 0$.) Notice that if $\mathfrak{X}$ has a Pełczyński decomposition then one can define right and left shifts of infinite multiplicity on $\mathfrak{X}$. Such shifts can be used to show that certain operators on $\mathfrak{X}$ are commutators. In D. Dosev and Johnson [ibid.] the following theorem was proved (but it was only stated in D. Dosev, Johnson, and Schechtman [2013]).

Theorem 3.1. *Let $\mathfrak{X}$ be a Banach space such that $\mathfrak{X}$ is isomorphic to $\left(\sum \mathfrak{X}\right)_p$, where $1 \le p \le \infty$ or $p = 0$. Let $T \in \mathcal{L}(\mathfrak{X})$ be such that there exists a subspace $X \subset \mathfrak{X}$ such that $\simeq \mathfrak{X}$, $T_{|X}$ is an isomorphism, $X + T(X)$ is complemented in $\mathfrak{X}$, and $d(X, T(X)) > 0$. Then T is a commutator.*

In practice, Theorem 3.1 allows one to avoid operator theoretic arguments when trying to check whether a space $\mathfrak{X}$ is a Wintner space and concentrate on the geometry of $\mathfrak{X}$. This is particularly important when $K(\mathfrak{X})$ is not the only closed ideal in $\mathcal{L}(\mathfrak{X})$, as is the case in all classical spaces other than ℓ_p, $1 \le p < \infty$ and c_0. In D. Dosev and Johnson [2010] Dosev and I used Theorem 3.1 to prove that ℓ_∞ is a Wintner space and in D. Dosev, Johnson, and Schechtman [2013] together with Schechtman we used it to prove that $L_p := L_p(0,1)$ is a Wintner space. In ℓ_∞ the unique maximal ideal is not too bad–it is the ideal of strictly singular operators. However, in L_p, the unique maximal ideal is horrendously large and hard to deal with–it is the ideal of L_p-singular operators. Theorem 3.1 also was used in Chen, Johnson, and Zheng [2011] and Zheng [2014].
Here is a wild conjecture that was made in D. Dosev and Johnson [2010]:
If $\mathfrak{X}$ has a Pełczyński decomposition then $\mathfrak{X}$ is a Wintner space.
The most interesting classical spaces not known to be Wintner spaces are the spaces $C(K)$ where K is an infinite compact metric space with $C(K)$ not isomorphic to c_0–all of these have a .Pełczyński decomposition. The best partial results on these spaces is contained in Dosev's paper D. T. Dosev [2015]. There are other recent papers that prove that some simpler spaces are Wintner spaces, including Zheng [2014] and Chen, Johnson, and Zheng [2011].
After D. Dosev and Johnson [2010] was written it was proved by Tarbard [2012] that not every infinite dimensional Banach space is a Wintner space. Building on the work of his advisor, R. Haydon, and S. Argyros that solved a famous 40+ year old problem that they will discuss at their 2018 ICM lecture, Tarbard constructed a Banach space $\mathfrak{X}$ such that every operator on $\mathfrak{X}$ has the form $\lambda I + \alpha S + K$ with λ and α scalars, K is compact, and S is special non compact operator whose square is compact. The strictly singular operators

form the unique maximal ideal in $L(\mathfrak{X})$ and it is clear that S is not a commutator, so $\mathfrak{X}$ is not a Wintner space.

Two other well-known open problems about commutators are worth mentioning.

Problem 1. If X is infinite dimensional, then is every compact operator on X a commutator?

I suspect that, to the contrary, there is an infinite dimensional space X such that every finite rank commutator on X has zero trace.

The following problem is open for *every* infinite dimensional space.

Problem 2. Is every compact operator the commutator of two compact operators?

4 Counting Ideals in $L(L_p)$

After C^* algebras, probably the most natural non commutative Banach algebras are the spaces of bounded linear operators on such classical Banach spaces as $L_p := L_p(0,1)$. In order to study any Banach algebra one must understand something about the closed ideals in the algebra. For $L(\ell_p)$, the space of bounded linear operators on ℓ_p, $1 \le p < \infty$, the situation is the same as for ℓ_2. The only non trivial closed ideal is the ideal of compact operators (see Gohberg, Markus, and Feldman [1960] and Whitley [1964]). The situation for $L(L_p)$, $1 \le p \ne 2 < \infty$, is much more complicated. Let's call an ideal $\mathfrak{I}$ *small* if $\mathfrak{I}$ is contained in the ideal of strictly singular operators. Call an ideal large if it is not small. The most natural way to construct a large ideal in $L(X)$ is to find a complemented subspace Y of X and consider the closed ideal $\mathfrak{I}_Y$ generated by a bounded linear projection from X onto Y. If, as is usually the case, Y is isomorphic to $Y \oplus Y$, this ideal is the closure of the collection of all operators on X that factor through Y. Then $\mathfrak{I}_Y$ is a proper ideal as long as X is not isomorphic to a complemented subspace of Y. Schechtman [1975] proved that $L(L_p)$, $1 < p \ne 2 < \infty$, has at least $\aleph_0$ ideals by constructing $\aleph_0$ isomorphically different complemented subspaces of L_p. With Bourgain and Rosenthal, he Bourgain, Rosenthal, and Schechtman [1981] improved this to $\aleph_1$ by constructing $\aleph_1$ isomorphically different complemented subspaces of L_p. It is still open whether in ZFC $L(L_p)$ has a continuum of large ideals.

Only recently was it proved that $L(L_p)$, $1 < p \ne 2 < \infty$, has infinitely many closed small ideals. In fact, building on some other recent work, Schlumprecht and Zsák [2018] show that $L(L_p)$ has a continuum of small closed ideals, solving in the process a problem in Pietsch's 1978 book Pietsch [1978]. It remains open whether $L(L_p)$, $1 < p \ne 2 < \infty$, has more than a continuum of closed ideals.

For $L(L_1)$, the situation was stagnant for an even longer time. In 1978 Pietsch [ibid.] recorded the well-known problem whether there are infinitely many closed ideals in $L(L_1)$.

At that time the only non trivial ideals in $L(L_1)$ known were the ideal of compact operators, the ideal of strictly singular operators, the ideal of operators that factor through ℓ_1, and the unique maximal ideal. It is easy to write down candidates for other ideals, but many turn out to be one of these four. For example, if $1 < p \le \infty$, the closure of the operators on L_1 that factor through L_p is the ideal of weakly compact operators, and on L_1 an operator is weakly compact if and only if it is strictly singular. Just in the past year, Johnson, Pisier, and Schechtman [n.d.] proved that there are other closed ideals. We constructed a continuum of closed small ideals in $L(L_1)$. For $2 < p < \infty$ we take a $\Lambda(p)$ sequence (x_n^p) of characters that has certain extra properties ("$\Lambda(p)$" means that the L_p and L_2 norms are equivalent on the linear span of the set of characters). Let J_p be the bounded linear operator from ℓ_1 into L_1 that maps the nth unit basis vector to x_n^p and let $\mathfrak{I}(p)$ be the closure of the operators on L_1 that factor through J_p. It turns out that $\mathfrak{I}(p) \neq \mathfrak{I}(q)$ when $p \neq q$.
It is open whether $L(L_1)$ has more than two large ideals. This is closely connected to the famous problem whether every infinite dimensional complemented subspace of L_1 is isomorphic either to ℓ_1 or to L_1.

5 Spaces that are uniformly homeomorphic to $\mathfrak{L}_1$ spaces

Banach spaces X and Y are said to be uniformly homeomorphic if there is an injective uniformly continuous function from X onto Y whose inverse is uniformly continuous. B. Maurey, G. Schechtman, and I gave an affirmative answer to the 1982 question of Heinrich and Mankiewicz [1982]:

Are the $\mathfrak{L}_1$ spaces are preserved under uniform homeomorphisms?

A Banach space X is said to be $\mathfrak{L}_1$ if its dual X^* is isomorphic to $C(K)$ for some compact Hausdorff space K. That is really a theorem Lindenstrauss and Rosenthal [1969]. The definition Lindenstrauss and Pełczyński [1968] is that X is the increasing union of finite dimensional subspaces that are uniformly isomorphic to finite dimensional L_1 spaces.
Subsequently N. J. Kalton [2012] proved that this theorem is optimal by constructing two separable $\mathfrak{L}_1$ spaces that are uniformly homeomorphic but not isomorphic.
At the heart of the question is a recurring problem:
Suppose a linear mapping $T : X \to Y$ admits a Lipschitz factorization through a Banach space Z; i.e., we have Lipschitz $F_1 : X \to Z$ and $F_2 : Z \to Y$ and $F_2 \circ F_1 = T$. What extra is needed to guarantee that T admits a linear factorization through Z?
Something extra is needed because the identity on $C[0,1]$ Lipschitz factors through c_0 Aharoni [1974], Lindenstrauss [1964].
The main result in Johnson, Maurey, and Schechtman [2009] is

Theorem 5.1. *Let X be a finite dimensional normed space, Y a Banach space with the Radon-Nikodym property (which means that every Lipschitz mapping from the real line into Y is differentiable almost everywhere) and $T : X \to Y$ a linear operator. Let Z be a separable Banach space and assume there are Lipschitz maps $F_1 : X \to Z$ and $F_2 : Z \to Y$ with $F_2 \circ F_1 = T$. Then for every $\lambda > 1$ there are linear maps $T_1 : X \to L_\infty(Z)$ and $T_2 : L_1(Z) \to Y$ with $T_2 \circ i_{\infty,1} \circ T_1 = T$ and $\|T_1\| \cdot \|T_2\| \leq \lambda \mathrm{Lip}\,(F_1)\mathrm{Lip}\,(F_2)$.*

If Z is $\mathfrak{L}_1$ then so is $L_1(Z)$ and hence T linearly factors through a $\mathfrak{L}_1$ space.
This and fairly standard tools in non linear geometric functional analysis give an affirmative answer to the Heinrich–Mankiewicz problem. The proof of Theorem 5.1 is based on a rather simple local-global linearization idea. For the application we need only the case where Y is finite dimensional.

6 Weakly null sequences in L_1

The first weakly null normalized sequences (WNNS) with no unconditional sub-sequence were constructed by Maurey and Rosenthal [1977]. Their technique was incorporated into the famous paper of Gowers and Maurey [1993] that contains an example of an infinite dimensional Banach space that contains NO unconditional sequence, but the examples in Maurey and Rosenthal [1977] are still interesting because the ambient spaces were $C(K)$ with K countable. These $C(K)$ spaces are hereditarily c_0 and so have unconditional sequences all over the place. Every subsequence of the WNNS they constructed reproduces the (conditional) summing basis on blocks.
In 1977 Maurey and Rosenthal [ibid.] asked whether every WNNS sequence in $L_1 := L_1(0,1)$ has an unconditional subsequence. Like the $C(K)$ spaces with K countable, every infinite dimensional subspace of L_1 contains an unconditional sequence. In Johnson, Maurey, and Schechtman [2007] we constructed a WNNS in L_1 such that every subsequence contains a block basis that is $1+\epsilon$–equivalent to the (conditional) Haar basis for L_1, which implies that the WNNS has no unconditional subsequence. In fact, the theorem stated this way extends to rearrangement invariant spaces which (in some appropriate sense) are not to the right of L_2 (e.g. L_p, $1 < p < 2$) and which are not too close to L_∞.

7 Subspaces of spaces that have an unconditional basis

A problem that goes back to the 1970s is to give an intrinsic characterization of Banach spaces that isomorphically embed into a space that has an unconditional basis. It was shown that every space with an unconditional expansion of the identity (in particular, every space with an unconditional finite dimensional decompostion) embeds into a space with unconditional basis Pełczyński and Wojtaszczyk [1971], Lindenstrauss and Tzafriri

[1977]. However for spaces that lack such a strong approximation property the only apparently useful invariant is that in a subspace of a space with unconditional basis, every weakly null normalized sequence (WNNS) has an unconditional subsequence. A quotient of a space with shrinking unconditional basis has this desirable property Johnson [1977], E. Odell [1992]. (A basis is shrinking provided the linear functionals biorthogonal to the basis vectors are a basis for the dual space. Every basis for a reflexive space is shrinking.) But this condition is not suffiencient even for reflexive spaces: in Johnson and Zheng [2008] my student B. Zheng and I used a variation of a construction of E. W. Odell and Schlumprecht [2006] to build a separable reflexive space that does not embed into a space with unconditional basis, yet every WNNS in the space has an unconditional subsequence. On the other hand, Feder [1980] proved that a reflexive quotient X of a space with shrinking unconditional basis embeds into a space with unconditional basis as long as X has the approximation property. Unfortunately, all classical reflexive spaces other than Hilbert spaces have quotients that fail the approximation property.

So there were two problems

1. Give an intrinsic characterization of Banach spaces that embed into a space that has an unconditional basis.

2. Does every quotient of a space with shrinking unconditional basis embed into a space with unconditional basis?

Much research centered around reflexive spaces. For example, in addition to the result of Feder mentioned above, it was proved that every reflexive subspace of a space with unconditional basis embeds into a reflexive space with unconditional basis Davis, Figiel, Johnson, and Pełczyński [1974], Figiel, Johnson, and Tzafriri [1975].

In Johnson and Zheng [2008] Zheng and I answered both problems in the affirmative for reflexive spaces. Our later paper Johnson and Zheng [2011] gives an affirmative answer to (2) in general and to (1) for spaces that have a separable dual. The answer to (1) for spaces with non separable dual must be completely different because of the space ℓ_1, which has an unconditional basis but also has the Schur property–every WNNS converges in norm to zero.

The answers for reflexive spaces follow from the following omnibus theorem, which basically says that every condition that *might* be equivalent to "the reflexive space X embeds into a space with an unconditional basis" actually *is* equivalent to it.

Theorem 7.1. *Let X be a separable reflexive Banach space. Then the following are equivalent.*

(a) *X has the UTP.*

(b) *X is isomorphic to a subspace of a Banach space with an unconditional basis.*

(c) *X is isomorphic to a subspace of a reflexive space with an unconditional basis.*

(d) X is isomorphic to a quotient of a Banach space with a shrinking unconditional basis.

(e) X is isomorphic to a quotient of a reflexive space with an unconditional basis.

(f) X is isomorphic to a subspace of a quotient of a reflexive space with an unconditional basis.

(g) X is isomorphic to a subspace of a reflexive quotient of a Banach space with a shrinking unconditional basis.

(h) X is isomorphic to a quotient of a subspace of a reflexive space with an unconditional basis.

(i) X is isomorphic to a quotient of a reflexive subspace of a Banach space with a shrinking unconditional basis.

(j) X^* has the UTP.

The UTP is a strengthening of the property "every WNNS has an unconditional subsequence". The weaker property for a reflexive space does NOT imply embeddability into a space with unconditional basis Johnson and Zheng [2008]. The definition of the UTP is due to E. W. Odell and Schlumprecht [2006]:

Definition 7.2. *A branch of a tree is a maximal linearly ordered subset of the tree under the tree order. We say X has the C-unconditional tree property (C-UTP) if every normalized weakly null infinitely branching tree in X has a C-unconditional branch. X has the UTP if X has the C-UTP for some $C > 0$.*

The proof of the theorem uses some new tricks, blocking methods developed in the 1970s Johnson and Zippin [1972], Johnson and Zippin [1974], Johnson and E. Odell [1974], Johnson and E. Odell [1981], Johnson [1977], and the Odell-Schlumprecht analysis E. W. Odell and Schlumprecht [2006] relating tree properties to embeddability into spaces that have a finite dimensional decomposition with the corresponding skipped blocking property.

For Banach spaces with a separable dual, there is a similar theorem Johnson and Zheng [2011], but the characterization involves the weak* UTP. A Banach space X is said to have the weak* UTP provided every normalized weak* null infinitely branching tree in X^* has a branch that is an unconditional basic sequence. The main new technical feature in Johnson and Zheng [ibid.] is that blocking and "killing the overlap" techniques originally developed for finite dimensional decompositions are adapted to work for blockings of shrinking M-bases (that is, biorthogonal sequences $\{x_n, x_n^*\}$ with span x_n dense in X and span x_n^* dense in X^*). Shrinking M-bases are known to exist in every Banach space that has a separable dual. These technical advances provide some simplifications of the argument in the reflexive case presented in Johnson and Zheng [2008] and likely will be used in the future to study the structure of Banach spaces that lack a good approximation property.

8 Operators on ℓ_∞ with dense range

In http://mathoverflow.net/questions/101253 A. B. Nasseri asked
"Can anyone give me an example of an *(sic)* bounded and linear operator $T : \ell_\infty \to \ell_\infty$ (the space of bounded sequences with the usual sup-norm), such that T has dense range, but is not surjective?"
This question quickly drew two close votes. Nevertheless it took a couple of years for Nasseri, G. Schechtman, T. Tkocz, and me to resolve it Johnson, Nasseri, Schechtman, and Tkocz [2015].
On separable infinite dimensional spaces, there are always dense range compact operators, but compact operators have separable ranges. On a non separable space, even on a dual to a separable space, it can happen that every dense range operator is surjective: Argyros, Arvanitakis, and Tolias [2006] constructed a separable space X so that X^* is non separable, hereditarily indecomposable (HI) in the sense of Gowers–Maurey, and every strictly singular operator on X^* is weakly compact. Since X^* is HI, every operator on X^* is of the form $\lambda I + S$ with S strictly singular Gowers and Maurey [1993]. If $\lambda \neq 0$, then $\lambda I + S$ is Fredholm of index zero by Kato's classical perturbation theory. On the other hand, since every weakly compact subset of the dual to a separable space is norm separable, every strictly singular operator on X^* has separable range.
It turns out that Nasseri's problem is related to Tauberian operators on $L_1 := L_1(0,1)$. An operator $T : X \to Y$ is called *Tauberian* if $T^{**-1}(Y) = X$ N. Kalton and Wilansky [1976]. The book of González and Martínez-Abejón [2010] on Tauberian operators contains:

Theorem 8.1. *Let $T : L_1(0,1) \to Y$. The following are equivalent.*

0. *T is Tauberian.*

1. *For all normalized disjoint sequences $\{x_i\}$, $\liminf_{i\to\infty} \|Tx_i\| > 0$.*

2. *If $\{x_i\}$ is equivalent to the unit vector basis of ℓ_1 then there is an N such that $T_{|[x_i]_{i=N}^\infty}$ is an isomorphism.*

3. *There are $\varepsilon, \delta > 0$ such that $\|Tf\| \geq \varepsilon\|f\|$ for all f with $|\mathrm{supp}(f)| < \delta$.*

What is the connection between Tauberian operators on L_1 and dense range, non surjective operators on ℓ_∞? If T is injective Tauberian, T^{**} is injective. Thus, if T is a Tauberian operator on L_1 that is injective but does not have closed range, then T^* is a dense range operator on L_∞ that is not surjective. Since L_∞ is isomorphic to ℓ_∞, having an injective Tauberian, non closed range operator on L_1 gives a positive answer to Nasseri's question. In fact, we checked that whether there is such an operator on L_1 is *a priori* equivalent to Nasseri's question.

One of the main open problems mentioned in González and Martínez-Abejón [ibid.], raised in 1984 by Weis and Wolff [1984], is whether there is a Tauberian operator T on L_1 whose kernel is infinite dimensional. If T satisfies this condition, then you can play around and get a perturbation S of T that is Tauberian, injective, and has dense, non closed range (so is not surjective). Taking the adjoint of S and replacing L_∞ by its isomorph ℓ_∞, you would have an injective, dense range, non surjective operator on ℓ_∞. (To get S from T, take an injective nuclear operator from the kernel on T that has dense range in L_1, extend it to a nuclear operator on L_1, and add it to T. This does not quite work, but some fiddling produces the desired S.) In fact, without knowing the solution to either problem, one can check that the Weis—Wolff question is equivalent to Nasseri's question. The bottom line is that the question whether there is a dense range non surjective operator on the non separable space ℓ_∞ is really a question about the existence of a Tauberian operator with infinite dimensional kernel on the separable space L_1.

It happened that T satisfying condition (3) in Theorem 8.1 and having an infinite dimensional kernel has a known finite dimensional analogue:

Theorem 8.2. *[CS result] For each n sufficiently large, putting $m = [3n/4]$, there is an operator $T : \ell_1^n \to \ell_1^m$ such that $\frac{1}{4}\|x\|_1 \le \|Tx\|_1 \le \|x\|_1$ for all x with $\sharp\mathrm{supp}(x) \le n/400$.*

This CS result (where "CS" can be interpreted either to mean "Computer Science" or "Compressed Sensing") is a very special case of a theorem due to Berinde, Gilbert, Indyk, Karloff, and Strauss [2008]. The kernel of T_n has dimension at least $n/4$, so if you take the ultraproduct $\tilde{T}$ of the T_n you get an operator with infinite dimensional kernel on some gigantic L_1 space. Let T be the restriction of $\tilde{T}$ to some separable $\tilde{T}$-invariant L_1 subspace that intersects the kernel of $\tilde{T}$ in an infinite dimensional subspace. As long as $\tilde{T}$ is Tauberian, the operator T will be a Tauberian operator with infinite dimensional kernel on L_1, and we will be done. It remains to isolate a condition implying Tauberianism that is possessed by all T_n and is preserved under ultraproducts.

Say an operator $T : X \to Y$ (X an L_1 space) is (r, N)-Tauberian provided whenever $(x_n)_{n=1}^N$ are disjoint unit vectors in X, then $\max_{1\le n\le N} \|Tx_n\| \ge r$.

Lemma 8.3. *$T : X \to Y$ is Tauberian iff $\exists\, r > 0$ and N such that T is (r, N)-Tauberian.*

Proof: T being (r, N)-Tauberian implies that if (x_n) is a disjoint sequence of unit vectors in X, then $\liminf_n \|Tx_n\| > 0$, so T is González and Martínez-Abejón [2010]. Conversely, suppose there are disjoint collections $(x_k^n)_{k=1}^n$, $n = 1, 2, \dots$ with $\max_{1 \le k \le n} \|Tx_k^n\| \to 0$ as $n \to \infty$. Then the closed sublattice generated by $\cup_{n=1}^\infty (x_k^n)_{k=1}^n$ is a separable L_1 space, hence is order isometric to $L_1(\mu)$ for some probability measure μ by Kakutani's theorem. Choose $1 \le k(n) \le n$ so that the support of $x_{k(n)}^n$ in $L_1(\mu)$ has measure at most $1/n$. Since T is Tauberian, necessarily $\liminf_n \|Tx_{k(n)}^n\| > 0$ González and Martínez-Abejón [ibid.], a contradiction.
It is not difficult to prove that the property of being (r, N)-Tauberian is stable under ultraproducts of uniformly bounded operators, so it is just a matter of observing that the operators T_n of Berinde, Gilbert, Indyk, Karloff, and Strauss [2008]. are all $(1/4, 400)$–Tauberian. □
Conclusion: There is a non surjective Tauberian operator on L_1 that has dense range. The operator can be chosen either to be injective or to have infinite dimensional kernel. Consequently, there is a dense range, non surjective, injective operator on ℓ_∞.
Conclusion from the proof: Computer science has applications to non separable Banach space theory!

9 Approximation properties

A Banach space X has the approximation property (AP) provided the identity operator is the limit of finite rank operators in the topology of uniform convergence on compact sets. If these operators can be taken to be uniformly bounded, we say that X has the bounded approximation property (BAP) or λ-BAP if the uniform bound can be λ. Grothendieck [1955] proved that a reflexive space that has the AP must have the 1-BAP, but there are non reflexive spaces that have the AP but fail the BAP Figiel and Johnson [1973]. Sometimes these properties come up when considering problems that, on the surface, have nothing to do with approximation. For example, given a family $\mathfrak{F}$ of operators between Banach spaces, it is natural to try to find a single (usually separable) Banach space Z such that all the operators in $\mathfrak{F}$ factor through Z. If $\mathfrak{F}$ is the collection of all operators between separable Banach spaces that have the BAP, there is such a separable Z; namely, the separable universal basis space of Pełczyński [1969], Pełczyński [1971], Kadec [1971]. This space, as well as smaller (even reflexive) spaces Johnson [1971] have the property that every operator that is uniformly approximable by finite rank operators factors through Z. A. Szankowski and I proved that there is not a separable space such that every operator between separable spaces (not even every operator between spaces that have the AP) factors through it Johnson and Szankowski [1976], but this paper left open the question: Is there a separable space such that every *compact* operator factors through it? 23 years later, in part 2 of Johnson and Szankowski [ibid.], we finally managed to proved that no such space exists Johnson and Szankowski [2009].

* * * * * * * *

A Banach space has the *hereditary approximation property* (HAP) provided every subspace has the approximation property. There are non Hilbertian spaces that have the HAP Johnson [1980], Pisier [1988]. All of these examples are *asymptotically Hilbertian*; i.e., for some K and every n, there is a finite codimensional subspace all of whose n-dimensional subspaces are K-isomorphic to ℓ_2^n. An asymptotically Hilbertian space must be superreflexive and cannot have a symmetric basis unless it is isomorphic to a Hilbert space. This led to two problems Johnson [1980]:
1. Can a non reflexive space have the HAP?
2. Does there exist a non Hilbertian space with a symmetric basis that has the HAP?
The HAP is very difficult to work with, partly because it does not have good permanence properties–there are spaces X and Y that have the HAP such that $X \oplus Y$ fails the HAP Casazza, García, and Johnson [2001].
The main result of Johnson and Szankowski [2012] gives an affirmative answer to problem 2 from Johnson [1980]:

Theorem 9.1. *There is a function $f(n) \uparrow \infty$ such that if for infinitely many n we have $D_n(X) \le f(n)$, then X has the HAP.*

Here $D_n(X) := \sup d(E, \ell_2^n)$, where the sup is over all n-dimensional subspaces of X. The proof combines the ideas in Johnson [ibid.] with the argument in Lindenstrauss and Tzafriri [1976].
You can build Banach spaces with a symmetric basis, even Orlicz sequence spaces, that are not isomorphic to a Hilbert space and yet $D_n(X)$ goes to infinity as slowly as is desired. Hence problem (2) has an affirmative answer.
It turns out that Theorem 9.1 can be used to give a footnote to the famous theorem of J. Lindenstrauss and L. Tzafriri Lindenstrauss and Tzafriri [1971] that Hilbert spaces are the only, up to isomorphism, Banach spaces in which every subspace is complemented. Timur Oikhberg asked us whether there is a non Hilbertian Banach space in which every subspace is isomorphic to a complemented subspace.

Theorem 9.2. *Johnson and Szankowski [2012] There is a separable, infinite dimensional Banach space not isomorphic to ℓ_2 that is complementably universal for all subspaces of all of its quotients.*

Let X be any non Hilbertian separable Banach space such that $D_{4^n}(X) \le f(n)$ for all n. Let (E_k) be a sequence of finite dimensional spaces that is dense (in the sense of the Banach-Mazur distance) in the collection of all finite dimensional spaces that are contained in some quotient of $\ell_2(X)$ and let Y be the ℓ_2-sum of the E_k. Then $D_n(Y) \le f(n)$ for all n. If you are old enough to know the right background, you can give a short argument to prove that Y is complementably universal for all subspaces of all of its quotients. Problem (1) remains open.

* * * * * * * * *

It was a privilege for Tadek Figiel and me to be co-authors on A. Pełczyński's last paper Figiel, Johnson, and Pełczyński [2011]. The solution to a (not especially important) problem that had eluded Tadek and me in the early 1970s Figiel and Johnson [1973] just dropped out, so I have an excuse to include a discussion of part of Figiel, Johnson, and Pełczyński [2011] in this note.

Let X be a Banach space, let $Y \subseteq X$ be a subspace, let $\lambda \geq 1$. The pair (X, Y) is said to have the λ-BAP if for each $\lambda' > \lambda$ and each subspace $F \subseteq X$ with $\dim F < \infty$, there is a finite rank operator $u : X \to X$ such that $||u|| < \lambda'$, $u(x) = x$ for $x \in F$ and $u(Y) \subseteq Y$. If (X, Y) has the λ-BAP then X/Y has the λ-BAP. Thus by a theorem due to Szankowski [2009], for $1 \leq p < 2$ there are subspaces Y of ℓ_p that have the BAP and yet (ℓ_p, Y) fails the BAP.

It is open whether (X, Y) has the BAP if X, Y, and X/Y all have the BAP, but I don't believe it.

If Y is a finite dimensional subspace of X and X has the λ-BAP then also (X, Y) has the λ-BAP and hence also X/Y has the λ-BAP. That is, the λ-BAP passes to quotients by finite dimensional subspaces. By duality you get that if X^* the λ-BAP then every finite codimensional subspace of X has the λ-BAP. In particular, every finite codimensional subspace of an L_1 space has the 1-BAP. Easy as this is, I don't think that anyone previously had noticed this.

In fact,

Proposition 9.3. *X^* has the λ-BAP iff (X, Y) has the λ-BAP for every finite codimensional subspace Y.*

The following proposition turned out to be useful.

Proposition 9.4. *Let X be a Banach space and let $Y \subseteq X$ be a closed subspace such that* $\dim X/Y = n < \infty$ *and Y has the $\lambda-BAP$. Then the pair (X, Y) has the $3\lambda-BAP$.*

This gives the corollary

Corollary 9.5. *If X is a Banach space and Y has the $\lambda-BAP$ for every finite codimensional subspace $Y \subseteq X$, then X^* has the $3\lambda-BAP$.*

Consequently, in contradistinction to the case of commutative L_1 spaces, for every λ there are finite codimensional subspaces Y of the non commutative L_1 space S_1 of trace class operators on ℓ_2 that fail the λ-BAP because Szankowski [1981] proved that $L(\ell_2)$ fails the AP and $L(\ell_2)$ is the dual to S_1.

The main result in my 1972 paper with Figiel and Johnson [1973] is that there is a subspace of c_0 that has the AP but fails the BAP. We could not prove the same result for ℓ_1.

Corollary 9.6. *Figiel, Johnson, and Pełczyński [2011] There is a subspace Y of ℓ_1 that has the AP but fails the BAP.*

Proof. Start with a subspace X of ℓ_1 that fails the approximation property Szankowski [1981]. From the existence of such a space it follows Johnson [1972] that if we let Z be the ℓ_1−sum of a dense sequence (X_n) of finite dimensional subspaces of X, then Z^* fails the BAP and yet Z has the BAP. Then Y can be the ℓ_1− sum of a suitable sequence of finite codimensional subspaces of Z because of Corollary 9.5.

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