DIFFUSIVE AND SUPER-DIFFUSIVE LIMITS FOR RANDOM WALKS AND DIFFUSIONS WITH LONG MEMORY

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Abstract

We survey recent results of normal and anomalous diffusion of two types of random motions with long memory in $\mathbb{R}^d$ or $\mathbb{Z}^d$. The first class consists of random walks on $\mathbb{Z}^d$ in divergence-free random drift field, modelling the motion of a particle suspended in time-stationary incompressible turbulent flow. The second class consists of self-repelling random diffusions, where the diffusing particle is pushed by the negative gradient of its own occupation time measure towards regions less visited in the past. We establish normal diffusion (with square-root-of-time scaling and Gaussian limiting distribution) in three and more dimensions and typically anomalously fast diffusion in low dimensions (typically, one and two). Results are quoted from various papers published between 2012-2017, with some hints to the main ideas of the proofs. No technical details are presented here.

1 Random walks in divergence-free random drift field

1.1 Set-up and notation. Let $(\Omega, \mathcal{F}, \pi, \tau_z : z \in \mathbb{Z}^d)$ be a probability space with an ergodic $\mathbb{Z}^d$-action. Denote by $\mathcal{E} := \{k \in \mathbb{Z}^d : |k| = 1\}$ the set of possible steps of a nearest-neighbour walk on $\mathbb{Z}^d$, and let $p_k : \Omega \to [0, s^*]$, $k \in \mathcal{E}$, be bounded measurable functions. These will be the jump rates of the RWRE considered (see (2) below) and assume they are *doubly stochastic*,

$$(1) \qquad \sum_{k \in \mathcal{E}} p_k(\omega) = \sum_{k \in \mathcal{E}} p_{-k}(\tau_k \omega).$$

Given these, define the continuous time nearest neighbour random walk $t \mapsto X(t) \in \mathbb{Z}^d$ as a Markov process on $\mathbb{Z}^d$, with $X(0) = 0$ and conditional jump rates

$$(2) \qquad \mathbf{P}_\omega \left(X(t + dt) = x + k \mid X(t) = x \right) = p_k(\tau_x \omega) dt,$$

where the subscript ω denotes that the random walk $X(t)$ is a Markov process on $\mathbb{Z}^d$ *conditionally*, with fixed $\omega \in \Omega$, sampled according to π. The continuous time setup is for convenience only. Since the jump rates are bounded this is fully equivalent with a discrete time walk.

We will use the notation $\mathbf{P}_\omega(\cdot)$ and $\mathbf{E}_\omega(\cdot)$ for *quenched* probability and expectation. That is: probability and expectation with respect to the distribution of the random walk $X(t)$, *conditionally, with given fixed environment* ω. The notation $\mathbf{P}(\cdot) := \int_\Omega \mathbf{P}_\omega(\cdot) \, d\pi(\omega)$ and $\mathbf{E}(\cdot) := \int_\Omega \mathbf{E}_\omega(\cdot) \, d\pi(\omega)$ will be used for *annealed* probability and expectation. That is: probability and expectation with respect to the random walk trajectory $X(t)$ *and* the environment ω, averaged out with the distribution π.

It is well known (and easy to check, see e.g. Kozlov [1985a]) that due to double stochasticity (1) the annealed set-up is stationary and ergodic in time: the process of the environment as seen from the position of the random walker

$$(3) \qquad \eta(t) := \tau_{X(t)} \omega$$

is a stationary and ergodic Markov process on (Ω, π) and, consequently, the random walk $t \mapsto X(t)$ will have stationary and ergodic annealed increments.

The local *quenched* drift of the random walk is

$$\mathbf{E}_\omega \left(dX(t) \mid X(t) = x \right) = \sum_{k \in \mathcal{E}} k p_k(\tau_x \omega) dt =: \varphi(\tau_x \omega) dt.$$

It is convenient to separate the symmetric and skew-symmetric part of the jump rates: for $k \in \mathcal{E}$, let $s_k : \Omega \to [0, s^*]$, $v_k : \Omega \to [-s^*, s^*]$,

$$(4) \qquad s_k(\omega) := \frac{p_k(\omega) + p_{-k}(\tau_k \omega)}{2}, \qquad v_k(\omega) := \frac{p_k(\omega) - p_{-k}(\tau_k \omega)}{2}.$$

Note that from the definitions (4) it follows that

$$(5) \qquad s_k(\omega) - s_{-k}(\tau_k \omega) = 0, \qquad v_k(\omega) + v_{-k}(\tau_k \omega) = 0.$$

In addition, the bi-stochasticity condition (1) is equivalent to

$$(6) \qquad \sum_{k \in \mathcal{E}} v_k(\omega) \equiv 0, \qquad \pi\text{-a.s.}$$

The second identity in (5) and (6) jointly mean that $(v_k(\tau_x\omega))_{k\in\mathcal{E}, x\in\mathbb{Z}^d}$ is a stationary *sourceless flow* (or, a *divergence-free lattice vector field*) on $\mathbb{Z}^d$. The physical interpretation of the divergence-free condition (6) is that the walk (2) models the motion of a particle suspended in stationary, *incompressible flow*, with thermal noise.

In order that the walk $t \mapsto X(t)$ have *zero annealed mean drift* we assume that for all $k \in \mathcal{E}$

$$(7) \qquad \int_\Omega v_k(\omega)\, d\pi(\omega) = 0.$$

Our next assumption is an *ellipticity* condition for the symmetric part of the jump rates: there exists another positive constant $s_* \in (0, s^*]$ such that for π-almost all $\omega \in \Omega$ and all $k \in \mathcal{E}$

$$(8) \qquad s_k(\omega) \geq s_*, \qquad \pi\text{-a.s.}$$

Note that the ellipticity condition is imposed only on the symmetric part s_k of the jump rates and not on the jump rates p_k. It may happen that $\pi(\{\omega : \min_{k\in\mathcal{E}} p_k(\omega) = 0\}) > 0$, as it is the case in some of the examples given in Section 1.4.

Finally, we formulate the notorious $\mathcal{H}_{-1}$-*condition* which plays a key role in diffusive scaling limits. Denote for $i, j = 1, \ldots, d$, $x \in \mathbb{Z}^d$, $p \in [-\pi, \pi)^d$,

$$(9) \quad C_{ij}(x) := \int_\Omega \varphi_i(\omega)\varphi_j(\tau_x\omega)\, d\pi(\omega), \qquad \widehat{C}_{ij}(p) := \sum_{x\in\mathbb{Z}^d} e^{\sqrt{-1}x\cdot p} C_{ij}(x).$$

That is: $C_{ij}(x)$ is the covariance matrix of the drift field, and $\widehat{C}_{ij}(p)$ is its Fourier-transform.

By Bochner's theorem, the Fourier transform $\widehat{C}$ is positive definite $d\times d$-matrix-valued-measure on $[-\pi, \pi)^d$. The no-drift condition (7) is equivalent to $\widehat{C}_{ij}(\{0\}) = 0$, for all $i, j = 1, \ldots, d$. With slight abuse of notation we denote this measure formally as $\widehat{C}_{ij}(p)dp$ even though it could be not absolutely continuous with respect to Lebesgue.

The $\mathcal{H}_{-1}$-*condition* is the following:

$$(10) \qquad \int_{[-\pi,\pi)^d} \left(\sum_{j=1}^d (1 - \cos p_j)\right)^{-1} \sum_{i=1}^d \widehat{C}_{ii}(p)\, dp < \infty.$$

This is an *infrared bound* on the correlations of the drift field, $x \mapsto \varphi(\tau_x\omega) \in \mathbb{R}^d$. It implies diffusive upper bound on the annealed variance of the walk and turns out to be a natural sufficient condition for the diffusive scaling limit (that is, CLT for the annealed walk). We'll see further below some other equivalent formulations of the $\mathcal{H}_{-1}$-condition (10). Note that the $\mathcal{H}_{-1}$-condition (10) formally implies the no-drift condition (7).

For later reference we state here the closely analogous problem of diffusion in divergence-free random drift field. Let $(\Omega, \mathcal{F}, \pi, \tau_z : z \in \mathbb{R}^d)$ be now a probability space with an ergodic $\mathbb{R}^d$-action, and $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ a stationary vector field which is π-almost-surely C^1 and divergence-free:

$$(11) \qquad\qquad \operatorname{div} F \equiv 0, \qquad \pi\text{-a.s.}$$

The diffusion considered is

$$(12) \qquad\qquad dX(t) = dB(t) + F(X(t))dt.$$

The SDE (12) has unique strong solution, π-almost surely. The main question is the same as in the case of the random walk (2): What is the asymptotic scaling behaviour and scaling limit of $X(t)$, as $t \to \infty$? Under what conditions does the central limit theorem with diffusive scaling and Gaussian limit distribution hold? Although the physical phenomena described by (2)-(1) and (12)-(11) are very similar, the technical details of various proofs are not always the same. In particular, PDE methods and techniques used for the diffusion problem (12)-(11) are not always easily implementable for the lattice problem (2)-(1). On the other hand, often restrictive local regularity conditions must be imposed on the diffusion problem (12)-(11).

The results reported in this section refer mainly to the random walk problem (2)-(1). The diffusion problem (12)-(11) will be tangentially mentioned in an example in Section 1.5.3 and in the historical notes of Section 1.6.

1.2 The infinitesimal generator of the environment process.

All forthcoming analysis will be done in the Hilbert space $\mathcal{H} := \{f \in \mathcal{L}^2(\Omega, \pi) : \int_\Omega f(\omega)d\pi(\omega) = 0\}$. The $\mathcal{L}^2(\Omega, \pi)$-gradients and Laplacian are bounded operators on $\mathcal{H}$:

$$\nabla_k f(\omega) := f(\tau_k \omega) - f(\omega) \qquad\qquad \Delta := 2\sum_{k \in \mathcal{E}} \nabla_k = -\sum_{k \in \mathcal{E}} \nabla_{-k}\nabla_k.$$

Note that Δ is self-adjoint and negative. Thus, the operators $|\Delta|^{1/2}$ and $|\Delta|^{-1/2}$ are defined in terms of the spectral theorem. The domain of the unbounded operator $|\Delta|^{-1/2}$ is

$$\mathcal{H}_{-1} := \{\phi \in \mathcal{H} : \lim_{\lambda \searrow 0}(\phi, (\lambda I - \Delta)^{-1}\phi)_{\mathcal{H}} < \infty\}.$$

The $\mathcal{H}_{-1}$-condition gets its name from the fact that (10) is equivalent to requesting that for $k \in \mathcal{E}$,

$$(13) \qquad\qquad v_k \in \mathcal{H}_{-1}.$$

We will also use the multiplication operators $M_k, N_k : \mathcal{L}^2(\Omega, \pi) \to \mathcal{L}^2(\Omega, \pi), k \in \mathcal{E}$,

$$M_k f(\omega) := v_k(\omega) f(\omega), \qquad N_k f(\omega) := (s_k(\omega) - s_*) f(\omega).$$

The following commutation relations are direct consequences of (in fact, equivalent with) (5) and (6)

(14) $$\sum_{k \in \mathcal{E}} M_k \nabla_k = -\sum_{k \in \mathcal{E}} \nabla_{-k} M_k, \qquad \sum_{k \in \mathcal{E}} N_k \nabla_k = \sum_{k \in \mathcal{E}} \nabla_{-k} N_k$$

We will denote

$$S := -\frac{s_*}{2} \Delta + \sum_{k \in \mathcal{E}} N_k \nabla_k, = S^* \qquad A := \sum_{k \in \mathcal{E}} M_k \nabla_k, = -A^*.$$

The infinitesimal generator L of the Markovian semigroup $P_t : \mathcal{L}^2(\Omega, \pi) \to \mathcal{L}^2(\Omega, \pi)$ of the environment process (3) is

$$L = -S + A.$$

Note that due to ellipticity (8) and boundedness of the jump rates the (absolute value of the) Laplacian minorizes and majorizes the self-adjoint part of the infinitesimal generator: $s_* |\Delta| \leq 2S \leq s^* |\Delta|$. The inequalities are meant in operator sense.

1.3 Helmholtz's theorem and the stream tensor.

In its most classical form Helmholtz's theorem states that in $\mathbb{R}^3$ (under suitable conditions of moderate increase at infinity) a divergence-free vector field can be realised as the curl (or rotation) of another vector field, called the vector potential. Helmholtz's theorem in our context is the following:

Proposition 1. *Let $v : \Omega \to \mathbb{R}^{\mathcal{E}}$ be such that $v_k \in \mathcal{H}$, and assume that (5) and (6) hold.*

(i) There exists a zero mean, square integrable, antisymmetric tensor cocycle $H : \Omega \times \mathbb{Z}^d \to \mathbb{R}^{\mathcal{E} \times \mathcal{E}}, H_{k,l}(\cdot, x) \in \mathcal{H}$:

(15) $$H_{k,l}(\omega, y) - H_{k,l}(\omega, x) = H_{k,l}(\tau_x \omega, y - x) - H_{k,l}(\tau_x \omega, 0),$$

(16) $$H_{l,k}(\omega, x) = H_{-k,l}(\omega, x + k) = H_{k,-l}(\omega, x + l) = -H_{k,l}(\omega, x),$$

such that

(17) $$v_k(\tau_x \omega) = \sum_{l \in \mathcal{E}} H_{k,l}(\omega, x).$$

The realization of the tensor field H is unique up to an additive term $H_{k,l}^0(\omega)$, not depending on $x \in \mathbb{Z}^d$ (but obeying the symmetries (16)).

(ii) *The $\mathcal{H}_{-1}$-condition* (10)/(13) *holds if and only if the cocycle H in (i) is stationary. That is, there exists $h : \Omega \to \mathbb{R}^{\mathcal{E} \times \mathcal{E}}$, with $h_{k,l} \in \mathcal{H}$, such that*

$$(18) \qquad h_{l,k}(\omega) = h_{-k,l}(\tau_k \omega) = h_{k,-l}(\tau_l \omega) = -h_{k,l}(\omega),$$

and

$$(19) \qquad v_k(\omega) = \sum_{l \in \mathcal{E}} h_{k,l}(\omega).$$

The tensor field H is realized as the stationary lifting of h: $H_{k,l}(\omega, x) = h_{k,l}(\tau_x \omega)$.

The fact that v is expressed in (17) as the curl of the tensor field H having the symmetries (16), is essentially the lattice-version of Helmholtz's theorem. Note that (16) means that the stream tensor field $x \mapsto H(\omega, x)$ is in fact a function of the *oriented plaquettes* of $\mathbb{Z}^d$. In particular, in two-dimensions $x \mapsto H(\omega, x)$ defines a *height function* with stationary increments, on the dual lattice $\mathbb{Z}^2 + (1/2, 1/2)$, in three-dimensions $x \mapsto H(\omega, x)$ defines an *oriented flow* (that is: a lattice vector field) with stationary increments on the dual lattice $\mathbb{Z}^3 + (1/2, 1/2, 1/2)$. In Helmholtz's theorem, if $d > 2$, there is much freedom in the choice of the *gauge* of H. The cocycle condition (15) is met by the *Coulomb gauge*, which makes the construction essentially uniquely determined.

In Kozma and Tóth [2017] it was shown that for a RWRE (2) whose environment satisfies conditions (1), (8) and (10) the central limit theorem holds, under diffusive scaling and Gaussian limit with finite and nondegenerate asymptotic covariance, *in probability with respect to the environment*. See Theorem 1 below.

In order to obtain the quenched version, that is central limit theorem for the displacement $X(t)$ at late times, with frozen environment, π-almost surely, we impose a slightly stronger integrability condition on the stream-tensor-field,

$$(20) \qquad h \in \mathcal{L}^{2+\varepsilon}(\Omega, \pi),$$

for some $\varepsilon > 0$, rather than being merely square integrable. This stronger integrability condition is needed in the proof of quenched tightness of the diffusively scaled displacement $t^{-1/2} X(t)$. We will refer to the $\mathcal{H}_{-1}$-condition complemented with the stronger integrability assumption (20) as the *turbo-$\mathcal{H}_{-1}$-condition*. In Tóth [n.d.] the quenched version of the central limit theorem for the displacement of the random walker was proved under the conditions (1), (8) and (10) and (20). See Theorem 3 below.

1.4 Examples. Bounded stream tensor: Let $((\chi_{ij}(x))_{1 \le i < j \le d})_{x \in \mathbb{Z}^d}$, be a stationary and ergodic (with respect to $x \in \mathbb{Z}^d$) sequence of bounded random variables (say, $|\chi_{ij}(x)| \le$

1), and extend them to $i, j \in \{1, \ldots, d\}$ as $\chi_{ji} = -\chi_{ij}$, $\chi_{ii} = 0$. Define for $k, l \in \mathcal{E}$, $x \in \mathbb{Z}^d$,

$$H_{k,l}(x) := (k \cdot e_i)(l \cdot e_j)\chi_{ij}(x + (k \cdot e_i - 1)e_i/2 + (l \cdot e_j - 1)e_j/2).$$

(This formula extends the random variables χ to a tensor field, consistent with the symmetries (16)). Define $v_k(\omega)$ as in (17) and let $s_k(\omega) \equiv s_* \geq 2(d-1)$. This is the most general construction of the case when $h \in \mathfrak{L}^\infty(\Omega, \pi)$. In particular, it covers those cases when the random environment of jump probabilities admits a *bounded cycle representation*, cf. Kozlov [1985a], Komorowski and Olla [2003], Deuschel and Kösters [2008], Komorowski, Landim, and Olla [2012] (chapter 3.3). Due to Proposition 1 the $\mathcal{H}_{-1}$-condition (10)/(13) holds.

Randomly oriented Manhattan lattice: Let $u_i(y)$, $i \in \{1, \ldots, d\}$, $y \in \mathbb{Z}^{d-1}$, be i.i.d. random variables with the common distribution $\mathbf{P}(u = \pm 1) = 1/2$, and define for $x \in \mathbb{Z}^d$ and $k \in \mathcal{E}$

$$v_k(\tau_x\omega) := \sum_{i=1}^{d}(k \cdot e_i)u_i(x_1, \ldots, x_{i-1}, \not{x_i}, x_{i+1}, \ldots, x_d),$$

One can easily compute the covariances (9): $C_{ij}(x) = \delta_{i,j} \prod_{i' \neq i} \delta_{x_{i'},0}$, and their Fourier transforms $\widehat{C}_{ij}(p) = \delta_{i,j}\delta(p_i)$. From here it follows that in this particular model the $\mathcal{H}_{-1}$-condition fails robustly (with power law divergence in (10)) in $d = 2$, fails marginally (with logarithmic divergence in (10)) in $d = 3$, and holds if $d \geq 4$.

$\binom{2d}{d}$**-vertex models on** $\mathbb{Z}^d$**:** Let Ω be the set of all possible orientations of the edges of $\mathbb{Z}^d$ with the constraint that at all vertices the number of edges oriented towards the site is equal to the number of edges oriented away, d out of $2d$. In this way, locally at every vertex $\binom{2d}{d}$ configurations of orientations are possible and there is a very rigid constraint on the configurations. Ω is a compact metric space and the group of translations $\tau_z : \Omega \to \Omega, z \in \mathbb{Z}^d$, acts naturally on it. Let, for $k \in \mathcal{E}$, $v_k(\omega) = \pm 1$ be the orientation of the edge $\overline{0k}$ in the configuration $\omega \in \Omega$, and $p_k(\omega) = 1 + v_k(\omega)$. Any translation invariant ergodic measure π on Ω realizes a model of our RWRE. The most natural choice is the one when π is the unique weak limit of the uniform distribution of the allowed $\binom{2d}{d}$-vertex configurations on the discrete torus $(-L, L] \times \cdots \times (-L, L]$, with periodic boundary conditions, as $L \to \infty$. In 2-dimensions this is the notorious (uniform) six-vertex model. In this case - in 2-dimensions - the $\mathcal{H}_{-1}$-condition fails: the integral in (10) is logarithmically divergent.

1.5 Scaling limits.

1.5.1 Central limit theorem in probability w.r.t. the environment under the $\mathfrak{H}_{-1}$ condition.

Theorem 1. (Source: Kozma and Tóth [2017]) *Conditions* (1), (8), (10) *are assumed. The asymptotic annealed covariance matrix*

$$(21) \qquad (\sigma^2)_{ij} := \lim_{t \to \infty} t^{-1} \mathbf{E}\left(X_i(t) X_j(t)\right)$$

exists, and it is finite and non-degenerate. For any bounded and continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$(22) \qquad \lim_{T \to \infty} \int_{\Omega} \left| \mathbf{E}_{\omega}\left(f\left(T^{-1/2} X(T)\right)\right) - \int_{\mathbb{R}^d} \frac{e^{-\frac{|y|^2}{2}}}{(2\pi)^{\frac{d}{2}}} f(\sigma^{-1} y) dy \right| d\pi(\omega) = 0.$$

Theorem 1 is proved in Kozma and Tóth [ibid.], and, weak convergence in the sense of (22) of all finite dimensional marginal distributions of $t \mapsto T^{-\frac{1}{2}} X(Tt)$, as $T \to \infty$, to those of a d-dimensional Brownian motion with covariance σ^2 is established. We sketch the main points. Start with a most natural martingale decomposition

$$(23) \qquad X(t) = \left\{ X(t) - \int_0^t \varphi(\eta(s)) ds \right\} + \int_0^t \varphi(\eta(s)) ds =: M(t) + I(t).$$

In this decomposition $M(t)$ is clearly a square integrable martingale with stationary and ergodic annealed increments. The main issue is an efficient martingale approximation of the term $I(t)$, à la Kipnis-Varadhan.

We rely on the following general result on Kipnis-Varadhan type of martingale approximation. Let $\eta(t)$ be a stationary and ergodic Markov process on the probability space (Ω, π), and L be the infinitesimal generator of its Markovian semigroup acting on $\mathcal{L}^2(\Omega, \pi)$. Denote $S := -(L + L^*)/2$, $A := (L - L^*)/2$ and assume that the symmetric part S is minorised and majorised by a postitive operator $D \geq 0$: $s_* D \leq S \leq s^* D$, with $0 < s_* \leq s^* < \infty$. Further, denote

$$B_\lambda := (\lambda I + D)^{-1/2} A (\lambda I + D)^{-1/2}.$$

Theorem 2. (Source: Horváth, Tóth, and Vető [2012b], Kozma and Tóth [2017]) *Assume that there exist a dense subspace* $\mathfrak{B} \subseteq \mathcal{L}^2(\Omega, \pi)$ *and a linear operator* $B : \mathfrak{B} \to \mathcal{L}^2(\Omega, \pi)$ *which is essentially skew-self-adjoint on the core* $\mathfrak{B}$ *and such that for any* $\varphi \in \mathfrak{B}$ *there exists a sequence* $\varphi_\lambda \in \mathcal{L}^2(\Omega, \pi)$ *such that*

$$\lim_{\lambda \to 0} \|\varphi_\lambda - \varphi\| = 0. \qquad and \qquad \lim_{\lambda \to 0} \|B_\lambda \varphi_\lambda - B\varphi\| = 0.$$

Then for any $f \in \mathrm{Dom}(D^{-1/2})$ there exists a martingale $M_f(t)$ (on the probability space and with respect to the filtration of the Markov process $t \mapsto \eta_t$) such that

$$\lim_{t \to \infty} t^{-1} \int_{\Omega} \mathbf{E}_{\omega} \left(\left| \int_0^t f(\eta(s))ds - M_f(t) \right|^2 \right) = 0.$$

In plain and informal words: if the operator $B = D^{-1/2}AD^{-1/2}$ makes sense as a densely defined unbounded *skew-self-adjoint* operator then integrals along the Markov process trajectory of functions in $\mathcal{H}_{-1} \subset \mathcal{L}^2(\Omega, \pi)$ defined in terms of the positive operator D are efficiently approximated with martingales, à la Kipnis-Varadhan. As shown in Horváth, Tóth, and Vető [2012b], the condition of Theorem 2 is weaker than the graded sector condition of Sethuraman, Varadhan, and H.-T. Yau [2000] (which in turn is weaker than the strong sector condition of Varadhan [1995]).

In our particular case define $B : \mathcal{H}_{-1} \to \mathcal{H}$ as

$$(24) \qquad B := -\sum_{l \in \mathcal{E}} |\Delta|^{-1/2} \nabla_{-l} M_l |\Delta|^{-1/2}.$$

(Note that the operators $|\Delta|^{-1/2} \nabla_{-l}, l \in \mathcal{E}$ are bounded.) From the commutation relations (14) it follows that the operator B is *skew-symmetric* on the dense subspace $\mathcal{B} := \mathcal{H}_{-1}$. It is not difficult to show that if $h \in \mathcal{L}^{\infty}$ then B is a bounded operator and thus, its skew-self-adjointness drops out for free. (This is essentially the same as Varadhan's strong sector condition, cf. Varadhan [ibid.]) On the other hand, if $h \notin \mathcal{L}^{\infty}$ then B is genuinely unbounded and proving its (essential) skew-self-adjointness is far from trivial.

The main technical result in Kozma and Tóth [2017] is the proof of the fact that B is in fact *essentially skew-self-adjoint* on $\mathcal{H}_{-1}$. By applying von Neumann's criterion for (skew-)self-adjointness this boils down to proving that the lattice PDE

$$(25) \qquad \Delta \Psi(\cdot, \omega) + V(\cdot, \omega) \cdot \nabla \Psi(\cdot, \omega) = 0,$$

does not have a non-trivial cocycle solution $\Psi(x, \omega)$. Here now ∇ and Δ denote the lattice gradient, respectively, the lattice Laplacian, $V(x, \omega) = v(\tau_x \omega)$ and $\Psi(x, \omega)$ is a zero mean cocycle to be determined.

1.5.2   Quenched central limit theorem under the turbo-$\mathcal{H}_{-1}$-condition.

Theorem 3. (Source: Tóth [n.d.]) *Conditions* (1), (8), (10), *and* (20) *are assumed. For any bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$,*

$$\lim_{T \to \infty} \mathbf{E}_{\omega} \left(f(T^{-1/2} X(T)) \right) = \int_{\mathbb{R}^d} \frac{e^{-\frac{|y|^2}{2}}}{(2\pi)^{\frac{d}{2}}} f(\sigma^{-1} y) dy, \qquad \pi\text{-a.s.}$$

with the non-degenerate covariance matrix σ^2 given in (21).

Theorem 3 is proved in Tóth [n.d.], and as in the case of Theorem 1, the weak convergence of all finite dimensional distributions follows. The proof consists of three major steps: (1) Proof of *quenched tightness* of the scaled displacement $t^{-1/2}X(t)$, as $t \to \infty$. (2) Construction of the *harmonic coordinates* for the walk. (3) Efficient estimate of the discrepancy between the actual position of the walker and the approximating harmonic coordinates.

Quenched tightness of $t^{-1/2}X(t)$:   The proof relies on adapting Nash's moment bound on reversible diffusions with strictly elliptic and bounded dispersion coefficients, cf. Nash [1958], to this type of non-reversible setup. The extension in the case of $h \in \mathcal{L}^\infty(\Omega, \pi)$ is essentially straightforward, following Osada [1983]. (Though, adaptation to the lattice walk case needs some attention.) The extension to $h \in \mathcal{L}^{2+\varepsilon}(\Omega, \pi)$ is tricky. An integration over time and the Chacon-Ornstein ergodic theorem help. Full details can be found in Tóth [n.d.]. This is the only part of the proof where the stronger integrability condition (20) is used.

Harmonic coordinates: The idea of harmonic coordinates for random walks in random environments originates in the classical works Kozlov [1979], G. C. Papanicolaou and Varadhan [1981], Osada [1983], Kozlov [1985a]. Since then it had been widely used in proving quenched central limit theorems, mostly for random walks among random conductances. That is: in time reversible cases. See, however, Deuschel and Kösters [2008] for a non-reversible application. The idea is very natural: find an $\mathbb{R}^d$-valued $\mathcal{L}^2(\Omega, \pi)$, zero mean random cocycle $\Theta(x, \omega)$, such that

$$(26) \qquad \sum_{k \in \mathcal{E}} p_k(\tau_x \omega)(k + \Theta(\omega, x + k) - \Theta(\omega, x)) = 0, \quad \pi\text{-a.s.}$$

If there exists a solution Θ to the equation (26) then the process $t \mapsto Y(t) := X(t) + \Theta(\omega, X(t))$ is a quenched martingale (that is, a martingale in its own filtration, with the environment $\omega \in \Omega$ frozen). It turns out that equation (26) is equivalent with the following equation in $\mathcal{H}$:

$$(27) \qquad (I + B^*)\chi = |\Delta|^{-1/2}\varphi,$$

where $\varphi \in \mathcal{H}_{-1}$ is given and χ is to be determined. The operator B^* on the left hand side is exactly the adjoint of B from (24). Since it was proved that the operator B is skew-self-adjoint it follows that $I + B^*$ is invertible and thus equation (27) has a unique solution. As a consequence, equation (26) also has a unique solution Θ which is an $\mathbb{R}^d$-valued cocycle, as required.

Once the harmonic coordinates are constructed the quenched central limit theorem for $t^{-1/2}Y(t)$ drops out via the martingale CLT, using ergodicity of the environment process (3).

Error bound: In order to obtain the quenched CLT for the scaled displacement $t^{-1/2}X(t)$ it remains to prove that for all $\delta > 0$ and π-almost all $\omega \in \Omega$,

$$\lim_{t\to\infty} \mathbf{P}_\omega \left(|\Theta(X(t))| > \delta\sqrt{t} \right) = 0.$$

The key ingredients of this are the a priori quenched tightness of $t^{-1/2}X(t)$ proved in the first main step, and a soft but nevertheless useful ergodic theorem for cocycles: Let $\Omega \times \mathbb{Z}^d \ni x \mapsto \Psi(\omega, x) \in \mathbb{R}$ be a zero-mean $\mathcal{L}^2$-cocycle. Then

$$\lim_{N\to\infty} N^{-(d+1)} \sum_{|x|\leq N} |\Psi(x)| = 0, \qquad \pi\text{-a.s.}$$

In 1-dimension this is a direct consequence of Birkhoff's ergodic theorem. For $d > 1$, however, the multidimensional *unconditional ergodic theorem* is invoked. See Tóth [n.d.] for the proof.

1.5.3 Superdiffusive bounds when the $\mathcal{H}_{-1}$-condition fails.

If the $\mathcal{H}_{-1}$-condition (10)/(13) fails, or, equivalently, the conditions of part (ii) of Proposition 1 don't hold, then there is no a priori upper bound on $t^{-1}\mathbf{E}\left(|X(t)|^2\right)$ and superdiffusive behaviour is expected. There is no general statement like this, but there are particular interesting cases studied. The fully worked out cases are, however, continuous-space diffusions on $\mathbb{R}^d$ rather than random walks on $\mathbb{Z}^d$. Since $d = 2$ is the most interesting we only mention the following two-dimensional example.

Let $x \mapsto F(x)$ be a stationary Gaussian random vector field with covariances $K_{ij}(x) := \mathbf{E}(F_i(0)F_j(x))$ as follows

$$(28) \quad K_{ij}(x) = \left(\partial_{ij}^2 - \delta_{i,j}(\partial_{11}^2 + \partial_{22}^2)\right) V * G(x), \quad \widehat{K}_{ij}(p) = \left(\delta_{i,j} - \frac{p_i p_j}{|p|^2}\right) \widehat{V}(p),$$

where $G(x) = \log|x|$ is the two-dimensional (Laplacian) Green function and $V : \mathbb{R}^2 \to \mathbb{R}_+$ is a C^∞ approximate identity with fast decay and positive Fourier transform, $\widehat{V}(p) > 0$. A good concrete choice could be $V(x) = (2\pi\sigma^2)^{-d/2}\exp\{-|x|^2/(2\sigma^2)\}$, with some $\sigma \in (0, \infty)$. In plain words: F is the rotation (curl) of the two-dimensional Gaussian free field, locally mollified by convolving with the convolution-square-root of V. As a rotation, the vector field F is divergence-free, cf. (11). Define the diffusion in this random drift field: $t \mapsto X(t) \in \mathbb{R}^2$ as the unique strong solution of the SDE (12).

From (28) it appears that the $\mathcal{H}_{-1}$-condition fails marginally: the integral on the right hand side of (10) diverges logarithmically. In Tóth and Valkó [2012] superdiffusive bounds are proved for this diffusion in the rotation field of the two-dimensional Gaussian free field, which look formally very similar to (38) in Section 2.4 below. This extends earlier results (with power-law divergences) of Komorowski and Olla [2002] to the

marginal case of the two-dimensional Gaussian free field (with logarithmic divergences). The random walk on the six-vertex model (see the third example in Section 1.4) behaves phenomenologically similarly, but its superdiffusivity is not yet proved. Applying the same methods as in Tóth and Valkó [2012] we obtain, however, superdiffusive bounds for the variance of $X(t)$ for the random walk on the randomly oriented Manhattan lattice (second example in Section 1.4) in dimensions $d = 2$ (robust, power law) and $d = 3$ (marginal, logarithmic), cf. Ledger, Tóth, and Valkó [n.d.].

1.6 Historical notes. The problems of scaling limit of diffusions in divergence-free random drift field (12)-(11) and that of the random walks in doubly stochastic random environment (2)-(1) are closely related. Although the physical phenomena modelled are very similar (tracer motion along the drift lines of incompressible turbulent flow), the technical details of various proofs are not always the same. In particular, PDE methods and techniques used for the diffusion problem (12)-(11) are not always easily implementable for the lattice problem (2)-(1). In the following list we give a summary of the main stations in the probability literature along the almost forty years history of the subject. The list is far from complete and contains only the probability results. See also the bibliographical notes of chapters 3 and 11 of Komorowski, Landim, and Olla [2012].

1979: Kozlov, respectively, Papanicolaou and Varadhan, independently and in parallel formulate the problem of scaling limits of diffusions in stationary random environment and prove the first CLT for the self-adjoint case under strong ellipticity condition, Kozlov [1979], G. C. Papanicolaou and Varadhan [1981].

1983: Osada proves quenched CLT for the diffusion (12) in divergence-free drift field (11), when the the stream tensor is bounded, Osada [1983].

1985: Kozlov formulates the problem of random walk in doubly stochasic random environment (1)-(2). An annealed CLT is stated for the case when the jump probabilities $((p_k(\tau_x \omega))_{k \in \varepsilon})_{x \in \mathbb{Z}^d}$ are *finitely dependent*, Kozlov [1985a]. Double stochasticity (1) and finite dependence of $(p(\tau_x \omega))_{x \in \mathbb{Z}^d}$, jointly are rather restrictive conditions. Natural examples are provided by a Bernoulli soup of bounded cycles.

1988: Oelschläger proves annealed invariance principle for the diffusion problem (12)-(11), under the optimal $\mathcal{H}_{-1}$-condition, and local regularity condition imposed on the drift field.

1996: Fannjiang and Papanicolaou consider the homogenisation for the parabolic problem corresponding to (12)-(11), under $\mathcal{H}_{-1}$-condition, Fannjiang and G. Papanicolaou [1996].

1997: Fannjiang and Komorowski prove quenched invariance principle for the diffusion (12)-(11), under the condition that $h \in \mathcal{L}^p$, with $p > d$, Fannjiang and Komorowski [1997]

2003: Komorowski and Olla prove annealed CLT for the random walk (2)-(1) when $h \in \mathcal{L}^\infty$, by applying Varadhan's strong sector condition, Komorowski and Olla [2003].

2008: Deuschel and Kösters prove quenched CLT for the random walk (2)-(1) when the jump probabilities admit a bounded cyclic representation, Deuschel and Kösters [2008]. This condition implies $h \in \mathcal{L}^\infty$ and thus the strong sector condition.

2012: Komorowski, Landim and Olla publish the proof of the CLT for the random walk problem (2)-(1) in the case when $h \in \mathcal{L}^p$, $p > d$, Komorowski, Landim, and Olla [2012].

2017: Annealed CLT for the doubly stochastic RWRE problem (2)-(1) under the optimal $\mathcal{H}_{-1}$-condition is proved in Kozma and Tóth [2017]. Quenched CLT under the additional integrability condition (20) is proved in Tóth [n.d.].

2   Self-repelling Brownian polymers

2.1   Set-up and notation.   We consider a self-repelling random process $t \mapsto X(t) \in \mathbb{R}^d$ which is pushed by the negative gradient of its own occupation time measure, towards regions less visited in the past. In this order, fix an approximate identity $V : \mathbb{R}^d \to \mathbb{R}_+$, which is infinitely differentiable, decays exponentially fast at infinity, and is of *positive type*:

$$(29) \qquad \widehat{V}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip \cdot x} V(x) dx \geq 0.$$

As an example, take $V(x) = (2\pi\sigma^2)^{-d/2} e^{-|x|^2/(2\sigma^2)}$, $\widehat{V}(p) = e^{-\sigma^2 |p|^2/2}$.

   Let $X(t)$ be the unique strong solution of the stochastic differential equation

$$(30) \qquad dX(t) = dB(t) - \left(\int_0^t \operatorname{grad} V(X(t) - X(u)) du \right) dt.$$

Denoting by $\ell(t, \cdot)$ the *occupation time measure* of the process X,

$$(31) \qquad \ell(t, A) := |\{0 < s \leq t : X(s) \in A\}|,$$

where $A \subset \mathbb{R}^d$ is any measurable set, the SDE (30) is written in the alternative form

$$(32) \qquad dX(t) = dB(t) - \operatorname{grad}(V * \ell(t, \cdot))(X(t)) dt,$$

which is more suggestive regarding the nature of the process $t \mapsto X(t)$: it is indeed driven by the negative gradient of an appropriate local regularization of its occupation time measure (local time). Following the terminology of the related probability literature we will refer to the process $X(t)$ defined in (30)/(32) as the *self-repelling Brownian polymer*. The

main question is: What is the long time asymptotic behaviour or X? How does the self-repulsion of the trajectory influence the long time scaling? The problem arose essentially in parallel, but unrelated, in the physics (random walk versions) and probability (diffusion versions) literature, cf. Amit, Parisi, and Peliti [1983], Obukhov and Peliti [1983], Peliti and Pietronero [1987], Norris, Rogers, and Williams [1987], Durrett and Rogers [1992], Cranston and Mountford [1996]. Mathematically non-rigorous, nevertheless strong and compelling scaling arguments appearing in physics papers Amit, Parisi, and Peliti [1983], Obukhov and Peliti [1983], Peliti and Pietronero [1987] convincingly suggest the following dimension dependent asymptotic scaling behaviour, as $t \to \infty$:

$$(33) \qquad X(t) \sim \begin{cases} t^{2/3} & \text{in } d = 1, \text{ with non-Gaussian scaling limit,} \\ t^{1/2}(\log t)^\gamma & \text{in } d = 2, \text{ with Gaussian scaling limit,} \\ t^{1/2} & \text{in } d \geq 3, \text{ with Gaussian scaling limit.} \end{cases}$$

In $d = 2$, the value of the exponent $\gamma \in (0, 1/2]$ in the logarithmic correction is disputed. However, there is good scaling reason to expect $\gamma = \frac{1}{4}$. In the following sections we are going to present the mathematically rigorous results related to the conjectures of (33).

In one-dimension, for some particular nearest-neighbour lattice walk versions of the self-repelling motion, the conjecture (in the first line of) (33) is fully settled. In Tóth [1995] a limit theorem is proved for $t^{-2/3}X(t)$, with an intricate non-Gaussian limit distribution, believed to be universally valid for the 1-d cases. In Tóth and Werner [1998] the presumed scaling limit process $t \mapsto \mathbb{X}(t)$ is constructed and fully analysed. In this note we are not going to cover those older results.

2.2   The environment process.   Let

$$\Omega := \left\{ \omega \in C^\infty(\mathbb{R}^d \to \mathbb{R}^d) \ : \ \partial_k \omega_l = \partial_l \omega_k, \ \ \|\omega\|_{k,m,r} < \infty, \right\},$$

where the seminorms are, for $k \in \{1, \ldots, d\}$, $m \in \mathbb{N}^d$, $r \in \mathbb{N}$,

$$\|\omega\|_{k,m,r} := \sup_{x \in \mathbb{R}^d} (1 + |x|)^{-1/r} \left| \partial^{m_1 + \cdots + m_d}_{m_1, \ldots, m_d} \omega_k(x) \right|.$$

I plain words: Ω is the space of gradient vector fields on $\mathbb{R}^d$, with all partial derivatives increasing slower than any power of $|x|$.

It turns out that the process $t \mapsto \eta(t, \cdot) \in \Omega$,

$$\eta(t, x) := \text{grad}(V * \ell(t, \cdot))(X(t) + x)$$

is a Markov process on the state space Ω with almost surely continuous sample paths. We allow for an initial profile $\eta(0, \cdot) \in \Omega$. (This means an initial signed measure $\ell(t, \cdot)$ in

(31).) The finite dimensional non-Markovian process $t \mapsto X(t) \in \mathbb{R}^d$ is traded for the infinite dimensional Markov process $t \mapsto \eta(t) \in \Omega$.

Next we define a Gaussian probability measure on Ω: the distribution of the gradient of the Gaussian free field on $\mathbb{R}^d$, locally regularised by convolving with the convolution-square-root of V. This is the point where positive definiteness (29) of the self-interaction potential V is essential. Let π be the Gaussian measure on Ω with zero mean and covariances $K_{kl}(y - x) := \int_\Omega \omega_k(x)\omega_l(y)d\pi(\omega)$,

$$(34) \qquad K_{kl}(x), = -\partial_{kl} V * G(x), \qquad \widehat{K}_{kl}(p) = \frac{p_k p_l}{|p|^2} \widehat{V}(p).$$

where $G : \mathbb{R}^d \setminus \{0\} \mapsto \mathbb{R}$ is the (Laplacian) Green function.

The group of translations $\tau_z : \Omega \to \Omega$, $z \in \mathbb{R}^d$, acts naturally as $\tau_z \omega(x) := \omega(z + x)$, and $(\Omega, \mathcal{F}, \pi, \tau_z : z \in \mathbb{R}^d)$ is ergodic.

Proposition 2. (Source: Tarrès, Tóth, and Valkó [2012], Horváth, Tóth, and Vető [2012a]) *The Gaussian probability measure π, with zero mean and covariances (34) is stationary and ergodic for the Markov process $\eta(t)$.*

This fact is consequence of the harmony between the two mechanisms driving the process η: diffusion pushed by $\eta(t, 0)$ and building up (gradient of) local time. Proposition 2 has two different proofs. In Tarrès, Tóth, and Valkó [2012] it is proved through careful Itô-calculus. In Horváth, Tóth, and Vető [2012a] a functional analytic proof is presented.

Tilted Gaussian measures with nonzero constant expectation and the same covariances as in (34) are also stationary and ergodic. We are not considering them because they result in ballistic behaviour (that is: nonzero overall speed) of the motion $X(t)$. We think (though, don't prove) that in $d = 1, 2$ these are the only stationary and ergodic probability measures for the Markov process $\eta(t)$. In $d \geq 3$, however, other stationary distributions of totally different character do exist.

The forthcoming results are all valid in this stationary regime. That is: the initial $\eta(0, \cdot)$ is sampled according to the distribution π. As in (23), the displacement of the random walker $X(t)$ will be decomposed as sum of a martingale with stationary and ergodic increments and its compensator

$$(35) \qquad X(t) = \left\{ X(t) - \int_0^t \varphi(\eta(s))ds \right\} + \int_0^t \varphi(\eta(s))ds =: M(t) + I(t),$$

where now $\varphi : \Omega \to \mathbb{R}^d$, $\varphi_l(\omega) = \omega_l(0)$. The first term, $M(t)$, in (35) is a square integrable martingale with stationary and ergodic increments (on the probability space and with respect to the natural filtration of the Markov process $t \mapsto \eta(t)$). So, that part is well understood from start: it is diffusive and the martingale central limit theorem applies to it.

In one and two dimensions *superdiffusive lower bounds* have been proved for the second term, $I(t)$, on the right hand side of (35), cf. Tarrès, Tóth, and Valkó [2012], Tóth and Valkó [2012] and Theorem 4 below. On the other hand, in three and more dimensions an efficient martingale approximation à la Kipnis-Varadhan holds for the compensator term, $I(t)$, on the right hand side of (35), cf. Horváth, Tóth, and Vető [2012a] and Theorem 5.

2.3 The infinitesimal generator of the environment process. All computations will be performed in the Hilbert space $\mathcal{H} := \{f \in \mathcal{L}^2(\Omega, \pi) : \int_\Omega f \, d\pi = 0\}$. Scalar product in the Hilbert space $\mathcal{H}$ will be denoted $\langle \cdot, \cdot \rangle$. This is a *Gaussian Hilbert space* with its natural grading:

$$(36) \qquad \mathcal{H} = \overline{\bigoplus_{n=1}^{\infty} \mathcal{H}_n,}$$

where $\mathcal{H}_n$ is the subspace spanned by the n-fold Wick products $:\omega_{k_1}(x_1), \ldots, \omega_{k_n}(x_n):, k_j \in \{1, \ldots, d\}, x_j \in \mathbb{R}^d$.

The shift operators $U_z : \mathcal{H} \to \mathcal{H}, U_z f(\omega) := f(\tau_z \omega), z \in \mathbb{R}^d$ form a unitary representation of $\mathbb{R}^d$. Denote by $\nabla_k, k \in \{1, \ldots, d\}$ the infinitesimal generators: $U_z = e^{\sum_{k=1}^d z_k \nabla_k}$ and $\Delta := \sum_{k=1}^d \nabla_k^2 = -\sum_{k=1}^d \nabla_k^* \nabla_k$. Note that the shift operators and all operators derived from them (e.g. ∇_k, Δ) preserve the grading (36).

We will also use the *creation and annihilation operators* $a_l^* : \mathcal{H}_n \to \mathcal{H}_{n+1}, a_l : \mathcal{H}_n \to \mathcal{H}_{n-1}$ defined on Wick monomials as follows and extended by linearity.

$$a_l^* :\omega_{k_1}(x_1), ..., \omega_{k_n}(x_n): = :\omega_l(0), \omega_{k_1}(x_1), ..., \omega_{k_n}(x_n):$$

$$a_l :\omega_{k_1}(x_1), ..., \omega_{k_n}(x_n): = \sum_{m=1}^n K_{lk_m}(x_m) :\omega_{k_1}(x_1), ..., \widehat{\omega_{k_m}(x_m)}, ..., \omega_{k_n}(x_n):$$

As suggested by notation the operators a_l and a_l^* are adjoints of each other and restricted to any finite grade they are bounded:

$$\|a_l\|_{\mathcal{H}_n \to \mathcal{H}_{n-1}} = \left(\int_{\mathbb{R}^d} |p|^{-2} p_l^2 \widehat{V}(p) dp \right)^{1/2} \sqrt{n}.$$

The infinitesimal generator L of the semigroup of the Markov process $\eta(t)$, acting on $\mathcal{H}$, $P_t f(\omega) := \mathbf{E} \left(f(\eta(t)) \mid \eta(0) = \omega \right)$ is expressed in terms of the operators introduced above, as

$$(37) \qquad L = -\frac{1}{2}\Delta + \sum_{l=1}^d a_l^* \nabla_l + \sum_{l=1}^d \nabla_l a_l = -S + A_+ + A_-.$$

The proof of this form of the infinitesimal generator relies – beside usual manipulations (integration by parts, etc.) – on *directional derivative* identities in the Gaussian Hilbert space $\mathcal{H}$ (that is: elements of Malliavin calculus). For details see Tarrès, Tóth, and Valkó [2012], Horváth, Tóth, and Vető [2012a]. The notation indicates that $A_+ : \mathcal{H}_n \to \mathcal{H}_{n+1}$ while $A_- : \mathcal{H}_n \to \mathcal{H}_{n+1}$, and clearly, $S = S^* \geq 0$, $A_+ = -A_-^*$.

2.4 Scaling limits.

2.4.1 Superdiffusive bounds in $d = 1$ and $d = 2$. Let, for $\lambda > 0$

$$\widehat{E}(\lambda) := \int_0^\infty e^{-\lambda t} \, \mathbf{E}\left(|X(t)|^2\right) dt.$$

Theorem 4. (Source: Tarrès, Tóth, and Valkó [2012], Tóth and Valkó [2012]) *In $d \leq 2$, the following bounds hold, with some constants $0 < c < C < \infty$, as $\lambda \to 0$:*

$$\text{in } d = 1 : \qquad c\lambda^{-\frac{9}{4}} < \widehat{E}(\lambda) < C\lambda^{-\frac{5}{2}},$$

$$\text{in } d = 2 : \qquad c\lambda^{-2}\log|\log\lambda| < \widehat{E}(\lambda) < C\lambda^{-2}\log|\lambda|.$$

With some minimal regularity assumption on the asymptotic behaviour of $t \mapsto \mathbf{E}\left(|X(t)|^2\right)$, as $t \to \infty$, the bounds in Theorem 4 imply bounds on their *Cesáro means*,

$$\text{in } d = 1 : \qquad ct^{\frac{5}{4}} < \frac{1}{t}\int_0^t \mathbf{E}\left(|X(s)|^2\right) ds < Ct^{\frac{3}{2}},$$

$$(38) \qquad \text{in } d = 2 : \qquad ct\log\log t < \frac{1}{t}\int_0^t \mathbf{E}\left(|X(s)|^2\right) ds < Ct\log t.$$

With some extra work the upper bounds can be improved to hold without Cesáro averaging, see e.g. Landim, Olla, and H. T. Yau [1998]. However, the loweer bounds are the more interesting here. The bounds are consistent with but don't quite match the asymptotic behaviour conjectured in (33). Nevertheless, robust superdiffusivity in $d = 1$ and marginal superdiffusivity in $d = 2$ is at least established. Note also, that in $d = 1$, in particular cases (of self-repelling lattice walks) the scaling $t^{2/3}X(t)$ has been rigorously established, cf. Tóth [n.d.], Tóth and Werner [1998], Tóth and Vető [2008].

The proof of Theorem 4 follows the *resolvent method* of Landim, Olla, and H. T. Yau [1998], Komorowski and Olla [2002], Landim, Quastel, Salmhofer, and H.-T. Yau [2004], with new input in the variational computations for the 2-dimensional case.

Due to the martingale decomposition (35) and stationarity of the process $\eta(s)$, applying a straightforward Schwarz inequality we obtain

$$\int_0^t (t-s)\langle \varphi, P_t\varphi\rangle ds - 2\alpha^2 t \leq \mathbf{E}\left(|X(t)|^2\right) \leq 4\int_0^t (t-s)\langle \varphi, P_t\varphi\rangle ds + 2\alpha^2 t,$$

where $\alpha^2 := t^{-1}\mathbf{E}\left(M(t)^2\right)$ is the variance rate of the first term, $M(t)$, in the decomposition (35), which is a square integrable martingale with stationary increments. (The value of α^2 is explicitly computable but does not matter.) Hence,

$$\lambda^{-2}\langle \varphi, R_\lambda\varphi\rangle - 2\alpha^2\lambda^{-2} \leq \widehat{E}(\lambda) \leq 4\lambda^{-2}\langle \varphi, R_\lambda\varphi\rangle + 2\alpha^2\lambda^{-2},$$

where R_λ is the resolvent of the semigroup P_t. Thus, lower and upper bounds on $\widehat{E}(\lambda)$ reduce to lower and upper bounds on $\langle \varphi, R_\lambda\varphi\rangle$. The following variational formula, proved in Landim, Olla, and H. T. Yau [1998], is valid in the widest generality for any contraction semigroup $P_t = e^{tL}$, with infinitesimal generator $L = -S + A$:

$$(39) \qquad \langle \varphi, R_\lambda\varphi\rangle = \sup_{\psi \in \mathcal{H}} \left\{2\langle \varphi, \psi\rangle - \langle \psi, (\lambda I + S)\psi\rangle - \langle A\psi, (\lambda I + S)^{-1}A\psi\rangle\right\}.$$

The upper bounds in Theorem 4 are obtained simply by dropping the third (negative!) term on the right hand side of (39). This is essentially for free. The lower bounds are obtained by bounding from below the variational expression on the right hand side of (39), *in the subspace* $\mathcal{H}_1$. This leads to a nontrivial variational problem in $u : \mathbb{R}^d \mapsto \mathbb{R}^d$ ($d = 1, 2$). In $d = 2$ the solution is tricky. For details see Tóth and Valkó [2012].

2.4.2   Diffusive limit in $d \geq 3$.

Theorem 5. (Source: Horváth, Tóth, and Vető [2012a]) *In $d \geq 3$, the asymptotic covariance matrix*

$$(\sigma^2)_{ij} := \lim_{t\to\infty} t^{-1}\mathbf{E}\left(X_i(t)X_j(t)\right)$$

exists, it is bounded and non-degenerate. For any bounded and continuous function f : $\mathbb{R}^d \to \mathbb{R}$,

$$(40) \qquad \lim_{T\to\infty}\int_\Omega \left|\mathbf{E}_\omega\left(f(T^{-1/2}X(T))\right) - \int_{\mathbb{R}^d} \frac{e^{-\frac{|y|^2}{2}}}{(2\pi)^{\frac{d}{2}}}f(\sigma^{-1}y)dy\right| d\pi(\omega) = 0.$$

Theorem 5 is proved in Horváth, Tóth, and Vető [ibid.], and weak convergence in the sense of (40) of all finite dimensional marginal distributions of the diffusively scaled process $t \mapsto T^{-\frac{1}{2}}X(Tt)$, as $T \to \infty$, to those of a d-dimensional Brownian motion is established. The proof relies on the efficient martingale approximation à la Kipnis-Varadhan of

the integral term $I(t)$ on the right hand side of (35). This is done by verifying the *graded sector condition* of Sethuraman, Varadhan, and H.-T. Yau [2000]. The graded structure of the Hilbert space $\mathcal{H}$ and of the infinitesimal generator L, cf. (37). Technical details to be found in Horváth, Tóth, and Vető [2012a].

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