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BOUNDARY DYNAMICS FOR SURFACE HOMEOMORPHISMS

Andres Koropecki and Meysam Nassiri

Abstract

We discuss some aspects of the topological dynamics of surface homeomorphisms. In particular, we survey recent results about the dynamics on the boundary of invariant domains, its relationship with the induced dynamics in the prime ends compactification, and its applications in the area-preserving setting following our recent works with P. Le Calvez.

1 Introduction

The dynamics of homeomorphisms of the circle $\mathbb{S}^1$ has been completely understood and classified from the topological viewpoint since the early 20th century, with the classical Poincaré rotation number as a key concept. If $f: \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation-preserving homeomorphism, the rotation number $\rho(f) \in \mathbb{R}/\mathbb{Z}$ measures the asymptotic average rotation of its orbits, and one has:

(i) If $\rho(f) = p/q \in \mathbb{Q}/\mathbb{Z}$ then every nonwandering point is a fixed point of f^q, and all periodic points have the same least period. In particular, the ω-limit and α-limit sets of any point consist of fixed points of f^q.

(ii) If $\rho(f) \notin \mathbb{Q}/\mathbb{Z}$ then there are no periodic points. Moreover, f is topologically semi-conjugate to the rigid rotation by $\rho(f)$, and it is uniquely ergodic. In particular, there is a unique minimal set. If f is sufficiently smooth, this semi-conjugation is a conjugation.

On the other hand, the jump from dimension 1 to dimension 2 introduces new behavior and rich dynamical phenomena which suggest that one cannot expect a general classification. For instance, one may have coexistence of periodic orbits of arbitrary periods, positive topological entropy, different coexistent types of rotational behavior, mixing, among several other phenomena that are not present in dimension 1.

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Nevertheless, generalizations of the notion of rotation number have been useful in the study of two-dimensional dynamics. As in dimension one, the general idea is to measure the (asymptotic) average rotation of an orbit along a given homological direction of the surface, by means of a lift to the universal cover or an isotopy. It is observed that the co-existence of different types of rotational behavior often forces the existence of new types of orbits, periodic orbits, positive topological entropy, and other dynamical phenomena. The concept of rotation intervals for annulus homeomorphisms, motivated by the classical Poincaré-Birkhoff theorem, has been useful to obtain generalizations and refined versions of this result, in the context of twist maps Mather [1982a], area-preserving homeomorphisms, and even in the general setting Franks [1988] and Handel [1990]. The notion of rotation set of a toral homeomorphism had a similar success Franks [1989] and Llibre and MacKay [1991], and to a lesser extent its generalization to arbitrary surfaces of positive genus Pollicott [1992] and Koropecki and Tal [2017]. In late years, a great deal of progress has been made in understanding the dynamical consequences of rotation sets and intervals, e.g. Le Calvez and Tal [2017], Dávalos [2016], Jäger [2009], Koropecki and Tal [2014], Boyland, de Carvalho, and Hall [2016], Passeggi [2014], Addas-Zanata [2015], Kocsard [2016], and Koropecki, Passeggi, and Sambarino [2016].

A different approach consists in studying the dynamics of a surface homeomorphism at the boundary of an invariant domain. This is the focus of this article. To fix ideas, suppose U is a simply connected open set in an orientable surface S, invariant by an orientation-preserving homeomorphism f. Can one describe the dynamics of $f|_{\partial U}$ in simple terms, as in (i) and (ii) above? If ∂U is a simple curve, this is clearly the case. However, often ∂U is far from being a curve. An illustrative example is given by Wada-type continua, which are the common boundary of three or more disjoint topological disks. For instsnce the example in Figure 1 is nowhere locally connected and *indecomposable*, i.e. it is not the union of two proper nonempty subcontinua. This type of continuum appears frequently and robustly in smooth dynamics, for instance as a hyperbolic planar attractor Plykin [1974]. Another example, in a sense more drastic, is the *pseudo-circle* (see Figure 2), which is hereditarily indecomposable. This means every subcontinuum is indecomposable, and in particular it does not contain any arcs. The pseudo-circle may appear as an attractor Herman [1986] and Boroński and Oprocha [2015], as an invariant set of an area-preserving C^∞ diffeomorphism Handel [1982], or even as the boundary of a Siegel disk for a local holomorphic diffeomorphism Chéritat [2011]. Other relevant examples are the *hedgehogs* that appear in holomorphic dynamics Pérez-Marco [1997].

There is no hope for a classification such as (i)-(ii) in the general setting. Most of the rich dynamical dynamical phenomena that appear in dimension 2 can also appear simultaneously in the boundary of U. For instance, a connected hyperbolic attractor such as the Plykin attractor in the sphere includes dense periodic points of arbitrary periods, positive topological entropy, topological mixing, and yet it is the boundary of a simply connected

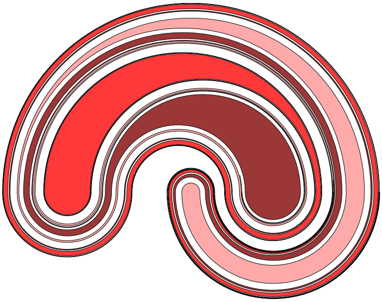

Figure 1: Wada-type continuum

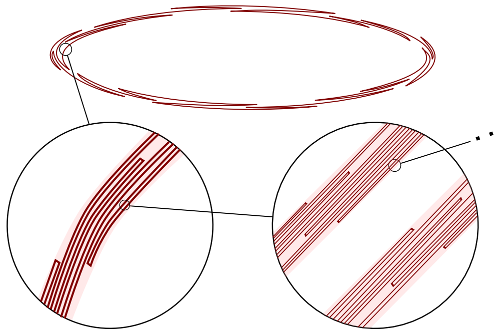

Figure 2: A pseudo-circle

domain. But it turns out that the presence of such rich dynamics in ∂U is related to the existence of attracting or repelling regions near ∂U, and if one introduces some mild form of recurrence near ∂U the situation changes completely. This is particularly the case in the area-preserving setting, which was the main motivation behind our joint works with P. Le Calvez described below.

The idea of studying the dynamics of an invariant continuum by means of its complementary components was already present in the work of Cartwright and Littlewood [1951] where they proved their celebrated fixed point theorem, and one of their key ideas was the use of Caratheodory's prime ends compactification and the prime ends rotation number. This was further explored by many authors (see for instance Mather [1981], Walker [1991], Barge and Gillette [1991], Alligood and Yorke [1992], Ortega and Ruiz del Portal [2011], and Hernández-Corbato, Ortega, and Ruiz del Portal [2012]). In the area-preserving setting, one of the motivations for this approach is the conjecture (dating back to Poincaré) that for a C^r-generic symplectic surface diffeomorphism the periodic points are dense. This is well known for $r = 1$ Pugh and Robinson [1983], but for $r > 1$ the usual local perturbation techniques do not work well, and a completely different approach is needed. Recent developments in symplectic topology led to a proof of the conjecture for any r in the case of Hamiltonian diffeomorphisms Asaoka and Irie [2016]. The general case remains open, but a relevant step is understanding the closures of invariant manifolds of hyperbolic periodic points. The following result, proved first in the sphere Franks and Le Calvez [2003] and later generalized to arbitrary surfaces Koropecki, Le Calvez, and Nassiri [2015] and Xia [2006] illustrates the usefulness of the topological study of boundaries of invariant domains (see Section 2.5):

Theorem 1.1. *For a C^r-generic area-preserving diffeomorphism of a closed surface, the set of all stable (or unstable) manifolds of hyperbolic periodic points is dense.*

Previous results by Mather [1981] showed that in the generic setting, the closures of any two stable or unstable branches of a hyperbolic periodic point coincide. This also relied on the use of prime ends.

If $U \subsetneq \mathbb{R}^2$ is an open simply connected set, the prime ends compactification $c_{\mathcal{E}}U$ is a way of compactifying U by adjoining a *circle of prime ends* $b_{\mathcal{E}}U \simeq \mathbb{S}^1$, in such a way that the resulting space is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$ (see Section 2). The most direct approach to define it is to consider a conformal Riemann uniformization $\phi\colon U \to \mathbb{D}$, and then define the compactification as $U \sqcup \mathbb{S}^1$ with the topology generated by sets of the form $\phi^{-1}(V) \cup (V \cap \mathbb{S}^1)$ where V is (relatively) open in $\overline{\mathbb{D}}$. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism such that $f(U) = U$, then $f|_U$ always extends to a homeomorphism $f_e\colon c_{\mathcal{E}}U \to c_{\mathcal{E}}U$. The restriction of f_e to the circle of of prime ends is an orientation-preserving circle homeomorphism if f is orientation-preserving, and this allows us to define a *prime ends rotation number* $\rho(f, U)$ by considering the Poincaré rotation number of f_e on the circle of prime ends. One may then try to describe the dynamics in ∂U in terms of the this rotation number, and hope to recover results such as (i) and (ii) above. This transition from the prime ends to the boundary dynamics is a subtle problem, and there are examples showing that one cannot hope to do it without any additional hypotheses (see Figures 3 and 4).

However, when U has finite area and f is area-preserving, or more generally when there is some mild form of recurrence near ∂U, this approach has been more successful: an argument due to Cartwright and Littlewood [1951] shows that if $\rho(f, U) = p/q$ then there is a fixed point of f^q in ∂U (see Theorem 2.1 ahead). On the other hand, Mather showed that under certain generic conditions for an area-preserving diffeomorphism, $\rho(f, U)$ is always irrational Mather [1981] (although he did not obtain any direct consequences about the dynamics of $f|_{\partial U}$). Finally, in recent joint works Koropecki, Le Calvez, and Nassiri [2015, 2017], we obtained a more complete picture, which is very similar to the situation for homeomorphisms of the circle. The results apply under a more general local condition near ∂U which we will call the *boundary condition*. The precise definition is given in Section 2.2. We only mention here that this condition holds whenever $f|_U$ is nonwandering (in particular if f is area-preserving and U has finite area). Summarizing some of the results from Koropecki, Le Calvez, and Nassiri [2015, 2017] we may state the following:

Theorem 1.2. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is orientation-preserving and leaves invariant an open simply connected set $U \subsetneq \mathbb{R}^2$. Assume further that f has the boundary condition in U. Then:*

(1) If $\rho(f, U) = p/q \in \mathbb{Q}$, then every nonwandering point of $f|_{\partial U}$ is a fixed point of f^q, and all periodic points have the same least period. In particular if U is bounded then the ω-limit and α-limit sets of any point in ∂U consist of fixed points of f^q.

(2) *If $\rho(f,U) \notin \mathbb{Q}$ then there are no periodic points in ∂U. Moreover, if U is unbounded, there are no periodic points in the complement of U.*

The theorem is stated in $\mathbb{R}^2$ for simplicity, but similar results hold in more general surfaces. Note that part (1) is identical to what happens for circle homeomorphisms. Part (2) is only partially so, since examples such as Handel [1982] show that one cannot hope to obtain a semiconjugation to an irrational rotation, except in some special cases (see Section 3.4).

Note that this may be seen as a result about dynamics, without invoking prime ends. For instance, we may state the following:

Corollary 1.3. *Under the same hypotheses, if there is a fixed point of f in ∂U then the nonwandering points of $f|_{\partial U}$ are all fixed points.*

We remark that the claim about the nonwandering points of $f|_{\partial U}$ is very strong. If U is bounded, it implies that every orbit goes from the fixed point set to the fixed point set. This leads to the following Koropecki, Le Calvez, and Nassiri [2017]:

Theorem 1.4. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an area-preserving diffeomorphism, and U is a bounded f-invariant open topological disk with a fixed point in its boundary. Then:*

- *The derivative of f at every fixed point in ∂U has positive eigenvalues. In particular, there are no elliptic points in ∂U.*
- *If there is no fixed point in ∂U with an eigenvalue 1, then ∂U is the union of (finitely many) hyperbolic saddles together with saddle connections. In particular, ∂U is locally connected.*

The same thing holds on any surface if f is isotopic to the identity, with a possible exceptional case on the sphere (see Figure 5). A consequence of this fact is that under the explicit C^r-generic condition that f has no saddle connections and all periodic points are either hyperbolic or elliptic, there are no periodic points in ∂U. A similar result, with a completely different proof (still unpublished), was also announced by Fernando Oliveira several years ago.

As part of the proof of Theorem 1.2 we also obtain results about the topology of ∂U. Let us mention a particularly striking one. To simplify, assume that U is bounded, and ∂U is also the boundary of the unbounded connected component of $\mathbb{R}^2 \setminus \overline{U}$. In that case, if $\rho(f,U) = p/q$ and f has the boundary condition in U, we show that there are two possibilities: either $f^q|_{\partial U} = \mathrm{Id}$, or ∂U is *compactly generated* in the annulus $A = \mathbb{R}^2 \setminus \{z_0\}$ where $z_0 \in U$ is any point. This means that there is a compact connected set in the universal cover of A which projects onto ∂U, and it is a restrictive condition. For example, the pseudo-circle is not compactly generated, so we can state the following:

Theorem 1.5. *Suppose that f is area-preserving and K is an invariant pseudo-circle. If there is a fixed point in K, then $f|_K = \mathrm{Id}$.*

This is another instance of a general property that is seen in invariant boundaries in the area-preserving setting: *if the topology of the boundary is too wild in a certain way, then its dynamics is forced to be trivial.* The results on homotopical boundedness described in Section 2.7 also reflect this observation.

2 Dynamics under a boundary condition

Instead of defining the prime ends compactification by means of a uniformizing conformal map Milnor [2011] and Pommerenke [1992], one may use a purely topological definition, which is more convenient and flexible in many situations. We summarize it here; more details can be found, for instance, in Mather [1982b] or Koropecki, Le Calvez, and Nassiri [2015]. Most definitions in the literature assume that U is relatively compact, but following Koropecki, Le Calvez, and Nassiri [ibid.] we do not make this assumption.

Let S be a surface (which is always assumed to be boundaryless, connected, orientable, of finite genus and endowed with a metric) and $U \subset S$ an open topological disk such that $S \setminus U$ has more than one point. A *cross-cut* of U in S is the image of a simple arc $\gamma\colon (0,1) \to U$ that extends to an arc $\overline{\gamma}\colon [0,1] \to \overline{U}$ joining two points[1] of ∂U, and such that each of the two components of[2] $U \setminus \gamma$ has some boundary point in $\partial U \setminus \overline{\gamma}$. A *cross-section* of U in S is any connected component of $U \setminus \gamma$ for some cross-cut γ of U in S. Each cross-cut corresponds to exactly two cross-sections, which are topological disks.

A *chain* for U in S is a sequence $\mathcal{C} = (D_n)_{n\in\mathbb{N}}$ of cross-sections such that $D_i \subset D_j$ for all $i \geq j \geq 1$ and $\overline{\partial_U D_i} \cap \overline{\partial_U D_j} = \varnothing$ for all $i \neq j$. If D is any cross-section of U, we say that the chain $\mathcal{C}$ *divides* D if $D_i \subset D$ for some $i \in \mathbb{N}$. If $\mathcal{C}' = (D'_n)_{n\in\mathbb{N}}$ is another chain, we say that $\mathcal{C}$ *divides* $\mathcal{C}'$ if $\mathcal{C}$ divides D'_n for each $n \in \mathbb{N}$. We say that $\mathcal{C}$ and $\mathcal{C}'$ are equivalent if $\mathcal{C}$ divides $\mathcal{C}'$ and $\mathcal{C}'$ divides $\mathcal{C}$. A chain $\mathcal{C}$ is called a *prime chain* if $\mathcal{C}$ divides $\mathcal{C}'$ whenever $\mathcal{C}'$ is a chain that divides $\mathcal{C}$. An equivalence class of prime chains is called a *prime end* of U.

For a cross-section D of U, we say that the prime end $\mathfrak{p}$ *divides* D if some (hence any) chain representing p divides D. We denote by $\mathcal{E}_U D$ the set of all prime ends that divide D, and by $\mathrm{b}_{\mathcal{E}} U$ the set of all prime ends of U. The *prime ends compactification* of U is the set $\mathrm{c}_{\mathcal{E}} U = U \sqcup \mathrm{b}_{\mathcal{E}} U$, with the topology which has as a basis of open sets the family of all open subsets of U together with all sets of the form $D \cup \mathcal{E}_U D$ for some cross-section D of U. With this topology, $\mathrm{c}_{\mathcal{E}} U$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$, and $\mathrm{b}_{\mathcal{E}} U$ endowed with the restricted topology is homeomorphic to the unit circle $\mathbb{S}^1$.

[1] Often it is assumed that the two points are different; we do not make this assumption.

[2] For simplicity we abuse notation and denote by γ both the parametrized arc and its image.

A useful observation is that when U is relatively compact in S, every prime end has a representative chain $(D_i)_{i\in\mathbb{N}}$ such that the diameters of the cross-cuts $\partial_U D_i$ tend to 0 as $i \to \infty$ (see Koropecki, Le Calvez, and Nassiri [ibid.]).

If $f : S \to S$ is an orientation preserving homeomorphism, then $f|_U$ extends to an orientation-preserving homeomorphism of the prime ends compactification, which we denote by $f_e : c_{\mathcal{E}}U \to c_{\mathcal{E}}U$. The *prime ends rotation number* of f in U, denoted $\rho(f, U) \in \mathbb{R}/\mathbb{Z}$, is defined as the Poincaré rotation number of the homeomorphism of the circle given by the restriction of f_e to $b_{\mathcal{E}}U \simeq \mathbb{S}^1$.

2.1 Prime ends vs. boundary dynamics. Suppose $U \subset S$ is an open topological disk invariant by the orientation-preserving homeomorphism f. What can be said about $f|_{\partial U}$ in terms of the prime ends rotation number? In particular, is it true that there are periodic points in ∂U if and only if $\rho(f, U)$ is rational? The answer in general is no, in both directions: The example in Figure 3 has no fixed points in ∂U, but the prime ends dynamics is a north pole-south pole map. On the other hand, the example in Figure 4 (see Walker [1991]) has fixed points in the boundary (every point in the outer circle is fixed), but the prime ends dynamics is a (Denjoy) homeomorphism with irrational rotation number.

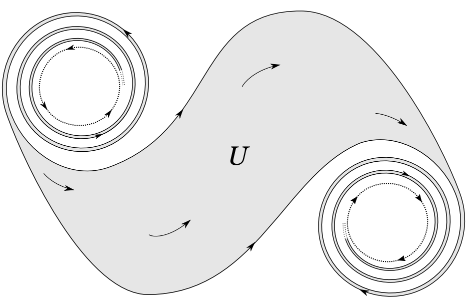

Figure 3: $\rho(f, U) = 0$, no fixed points

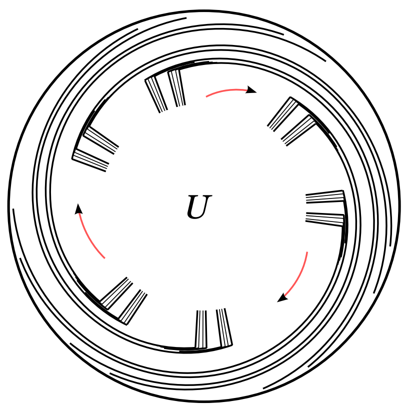

Figure 4: $\rho(f, U) \notin \mathbb{Q}/\mathbb{Z}$

Both examples have attracting or repelling regions near the boundary, and this is not a coincidence. If one excludes this behavior (for instance, if f is area-preserving and U has finite area), the situation changes. To illustrate this fact, we begin with a result proved by Cartwright and Littlewood. A cross-section D is *trapping* if it satisfies $f(D) \subset D$ and

$\overline{\partial_U D}$ is disjoint from its own image. Note that this also implies $\mathrm{cl}_U\, f(D) \subset D$. We also say that a cross-cut is trapping if one of its corresponding cross-sections is trapping.

Theorem 2.1. *If U is relatively compact and $\rho(f, U) = 0$ then either there is a fixed point in ∂U or there are trapping cross-cuts arbitrarily close to ∂U.*

Note that if f is area-preserving in U then one cannot have any trapping cross-cuts, so only the first case may hold.

Proof. Since $\rho(f, U) = 0$ there exists a prime end $\mathfrak{p}$ such that $f_e(\mathfrak{p}) = \mathfrak{p}$. Since U is relatively compact, one may choose a prime chain $(D_n)_{n\in\mathbb{N}}$ representing $\mathfrak{p}$ such that the diameters of the cross-cuts $\alpha_n = \partial_U D_n$ tend to 0 as $n \to \infty$. If infinitely many of these cross-cuts are trapping, then we have trapping cross-cuts arbitrarily close to $\partial_U U$. Thus we may assume that there exists i such that α_n is non-trapping for all $n \geq i$. The fact that $\mathfrak{p}$ is fixed means $(f(D_n))_{n\in\mathbb{N}}$ is also a prime chain representing $\mathfrak{p}$. Thus there is j_0 such that $D_j \subset f(D_i)$ for any $j > j_0$. We claim that if $j > j_1 := \min\{i, j_0\}$, then $f(\overline{\alpha}_j) \cap \overline{\alpha}_j \neq \varnothing$. Indeed, assume on the contrary that $f(\overline{\alpha}_j) \cap \overline{\alpha}_j = \varnothing$. Note that $D_j \subset f(D_i) \cap D_i$. By the previous argument using f^{-1} instead of f, there is $k > j$ such that $D_k \subset f^{-1}(D_j) \cap D_j$. In particular we have $D_k \cup f(D_k) \subset D_j \subset D_i \cap f(D_i)$. This means that both components of $U \setminus \alpha_j$ intersect their own image by f, and in particular letting D be the connected component of $U \setminus \alpha_j$ which does not contain $f(\alpha_j)$ one has $D \cap f(D) \neq \varnothing$. Since D is disjoint from $\partial_U f(D) = f(\alpha_j)$, this implies that $D \subset f(D)$. Note that $U \setminus D = \mathrm{cl}_U D'$, where D' is the cross-section determined by α_j which is not D; thus $f(\mathrm{cl}_U D') \subset \mathrm{cl}_U D'$, and since we assumed that $\overline{\alpha}_j$ is disjoint from $f(\overline{\alpha}_j)$, it follows that $f(\mathrm{cl}_U D') \subset D'$. This means that α_j is trapping, contradicting our assumption.

Thus $\overline{\alpha}_j \cap f(\overline{\alpha}_j) \neq \varnothing$ for all $j > j_1$, and we may choose $z_j \in \overline{\alpha}_j \cap f(\overline{\alpha}_j)$ for each $j \geq j_1$. Since the diameter of α_j tends to 0, we have $d(z_j, f^{-1}(z_j)) \to 0$ as $j \to \infty$, so any limit point of the sequence $(z_j)_{j\geq j_1}$ is a fixed point in ∂U. □

In fact, what the proof of the previous theorem shows is that if $\mathfrak{p}$ is a fixed prime end, then either one may find a prime chain for $\mathfrak{p}$ such that the corresponding cross-cuts are all trapping and their diameters tend to 0, or every *principal point* of the prime end $\mathfrak{p}$ is fixed (and this is particularly the case when f is area-preserving). A principal point is a point of ∂U that is the limit of a sequence of cross-cuts with diameter tending to 0 bounding the cross-sections of a prime chain of $\mathfrak{p}$. The set $\Pi(\mathfrak{p})$ of all principal points of $\mathfrak{p}$ is relevant because it can be characterized in the following alternative way. Consider the family $\mathfrak{F}$ of all arcs $\eta\colon [0, 1) \to U$ such that $\lim_{t\to 1^-} \eta(t) = \mathfrak{p}$ in the topology of $c_{\mathcal{E}} U$. The same limit may fail to exist in the ambient space S (it will only exist if $\mathfrak{p}$ is accessible), but we may consider the limit set $L(\eta) = \bigcap_{0\leq t<1} \overline{\gamma([t, 1))}$. The principal set is equal to

$\bigcap_{\eta \in \mathfrak{F}} L(\eta)$, and moreover it is realized as $L(\eta)$ for some η Mather [1982b]. In particular $\Pi(\mathfrak{p})$ is connected.

There have been other results regarding the realization of fixed points from fixed prime ends under different hypotheses, e.g. Ortega and Ruiz del Portal [2011] and Alligood and Yorke [1992] (see also Section 3). The converse of this result is more subtle, and was proved recently in Koropecki, Le Calvez, and Nassiri [2015]. In order to state it we need another definition.

2.2 The boundary condition. Note that a trapping cross-section can only occur if the prime ends rotation number vanishes. Here we introduce a natural generalization of the notion of trapping cross-section which poses no restrictions on the rotation number.

A (positive) *strong boundary trapping region* of U for f is an open set of the form $W = \bigcup_{D \in \mathfrak{F}} D$ with the following additional properties:

- $\mathfrak{F}$ is a family of pairwise disjoint cross-sections of U;
- The set $\{D \in \mathfrak{F} : \operatorname{diam}(\partial_U D) > c\}$ is finite for each $c > 0$;
- For each $D \in \mathfrak{F}$ there is $D' \in \mathfrak{F}$ such that $f(D) \subset D'$ and the cross-cuts α, α' bounding D and D' in U satisfy $f(\overline{\alpha}) \cap \overline{\alpha}' = \varnothing$.

These conditions in particular imply that $\operatorname{cl}_U f(W) \subset W$. The second condition always holds if $\partial_U W$ is contained in a compact arc. The last item guarantees that when one considers D and D' as subsets of the prime ends compactification $c_{\mathcal{E}} U$, the closure of D is mapped into D'. For instance, a single trapping cross-section is a strong boundary trapping region.

We say that f has the *boundary condition* in U if there is a compact set $K \subset U$ such that, for each $n \in \mathbb{Z}$, $U \setminus K$ does not contain a set of the form $\partial_U W$ where W is a strong boundary trapping region of U for f^n.

The boundary condition is automatically satisfied if any of the following properties hold:

- f is area-preserving and U has finite area;
- f is nonwandering in U;
- there are no wandering cross-cuts of U;
- the dynamics induced in the circle of prime ends is transitive.

One of the reasons this condition is useful is that it is local (it can be verified by examining f in a neighborhood of ∂U), and therefore enables the use of our results in different settings by modifying f in a compact subset of U or outside of U. A particularly useful case is when one wants to extend results to non-simply connected open sets. Most of our results can be applied to isolated topological ends, and to do so one may use a surgery in the open set to “cut away” everything outside a collar neighborhood of the topological end and replace it by a fixed point.

Many results in Koropecki, Le Calvez, and Nassiri [2015] are stated under a slightly stronger condition called the ∂-*nonwandering condition*. However the only part of the article where this condition is crucial (Lemma 4.6 about maximal cross-cuts) remains valid if one assumes instead the boundary condition, and this is clear from the proof.

2.3 The irrational case. The proof of Theorem 1.2 requires a converse of Theorem 2.1. Assume $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving homeomorphism and $U = f(U) \subsetneq \mathbb{R}^2$ is an open topological disk.

Theorem 2.2 (Koropecki, Le Calvez, and Nassiri [ibid.]). *If f has the boundary condition in U and $\rho(f, U) \neq 0$ (mod $\mathbb{Z}$), then* $\mathrm{Fix}(f) \cap \partial U = \varnothing$. *If U is unbounded, then* $\mathrm{Fix}(f) \subset U$.

To prove this we rely on a general result on translation arcs. An *n-translation arc* for f is a simple arc α joining a point z to $f(z)$ such that the concatenation $\alpha * f(\alpha) * \cdots * f^n(\alpha)$ is also a simple arc. As typical example is when α is a fundamental domain of a stable or unstable branch of a hyperbolic saddle. Such α is an ∞-*translation arc* (i.e. it is an n-traslation arc for each $n \in \mathbb{N}$). A simplified version of the lemma on translation arcs Koropecki, Le Calvez, and Nassiri [ibid., Theorem E] states the following:

Lemma 2.3. *Under the hypotheses of the previous theorem, there exist $N \geq 1$ depending only on the rotation number $\rho = \rho(f, U)$ and a compact subset K of U such that any N-translation arc in $\mathbb{R}^2 \setminus K$ is disjoint from ∂U.*

The key is to prove this lemma is to compare the combinatorics of the orbits of cross-cuts in $\alpha \cap U$ seen in the cyclic order of the circle of prime ends $b_{\mathcal{E}} U$ (which corresponds to the combinatorics of the rigid rotation by $\rho(f, U)$ on the circle) with their respective positions in the linear order of the arc $\Gamma = \alpha * f(\alpha) * \cdots * f^N(\alpha)$. Using these two different approaches, if N is chosen sufficiently large, one is able to construct two simple loops which have nonzero algebraic intersection number, which is not possible in a surface of genus 0. In order to construct these simple loops, a fundamental step is to prove a "maximal cross-cut lemma" Koropecki, Le Calvez, and Nassiri [ibid., Lemma 4.6], which is the only part of Koropecki, Le Calvez, and Nassiri [ibid.] and Koropecki, Le Calvez, and Nassiri [2017] where the boundary condition plays a role (see Koropecki, Le Calvez, and Nassiri [2015, §4], Koropecki, Le Calvez, and Nassiri [2017], Koropecki, Le Calvez, and Tal [2017, Lemma 5]).

To illustrate the usefulness of Lemma 2.3, one may easily prove a particular case of Theorem 2.2, namely that *there is no hyperbolic (saddle) fixed point in ∂U*. Suppose that f is differentiable and p is a hyperbolic saddle. Because of the linear hyperbolic behavior near p, for any given N one may find an arbitrarily small neighborhood V of p such that

$V \setminus p$ is covered by N-translation arcs. If N is chosen as in Lemma 2.3, this means that V cannot intersect U and therefore p is not on the boundary of U.

The proof of Theorem 2.2 in the general case requires considerably more work. The main idea is to use a version of Brouwer's arc translation lemma to show that if there is a fixed point in ∂U then one may find arbitrarily close to this fixed point an N-translation arc intersecting ∂U, thus contradicting Lemma 2.3. But this cannot be done directly in $\mathbb{R}^2$; we need to work on a lift to the universal cover of $\mathbb{R}^2 \setminus X$ where X is a special set of fixed points. For the details we refer the reader to Koropecki, Le Calvez, and Nassiri [2015, §5]. We only mention here that an essential component of the proof is the existence of *maximally unlinked* sets of fixed points.

A closed set $X \subset \mathrm{Fix}(f)$ is *maximally unlinked* if it has the property that $f|_{\mathbb{R}^2 \setminus X}$ is isotopic to the identity in $\mathbb{R}^2 \setminus X$, and X is maximal among sets with this property with respect to inclusion. The existence of maximally unlinked sets was established by Jaulent [2014]. A stronger and more useful version of this result was recently obtained by Béguin, Crovisier, and Le Roux [2016]. These results about maximally unlinked sets in combination with the foliated version of Brouwer's plane translation theorem due to Le Calvez [2005] provide a powerful tool in two-dimensional dynamics, and led to many advances in recent years. We mention in particular the forcing theory of Le Calvez and Tal [2017], which produced several outstanding results in surface dynamics.

A version of Theorem 2.2 is also valid on an arbitrary surface, but one needs to make a special exception on the sphere, where it is easy to construct an area-preserving homeomorphism with an invariant disk U such that ∂U looks like a *hedgehog*: it has a single fixed point, with "hairs" which rotate around the fixed point with the combinatorics of an irrational rotation (see Figure 5 ahead). This kind of example has irrational prime ends rotation number but yet has a fixed point. It turns out that this is the only situation where a periodic point may exist if the rotation number is irrational:

Theorem 2.4 (Koropecki, Le Calvez, and Nassiri [2015]). *Let f be an orientation and area-preserving homeomorphism of a closed orientable surface S, and $U \subset S$ an open f-invariant topological disk whose complement has more than one point and such that $\rho(f, U) \notin \mathbb{Q}/\mathbb{Z}$. Then one of the following holds:*

- *(i) ∂U is an inessential annular continuum without periodic points;*
- *(ii) S is a sphere, U is dense in S, and $S \setminus U$ is a non-separating continuum with a unique fixed point and no other periodic points.*

Here by *inessential continuum* in S we mean a compact connected set $K \subset S$ which has a neighborhood D homeomorphic to a disk. By *annular* we mean that K is a decreasing intersection of closed topological annuli $A_1 \supset A_2 \supset \cdots$ such that A_{k+1} is essential in A_k for each k. Equivalently, this means that K has a neighborhood A homeomorphic to the annulus $\mathbb{A}$ such that K is essential in A and $A \setminus K$ has exactly two components. Even more general versions of Theorem 2.4 can be found in Koropecki, Le Calvez, and Nassiri [2015, §6-7].

Note how this result not only gives us dynamical information (no periodic points, with a single exception) but also topological information about the boundary. For instance, a disk as in Figure 6 cannot happen in Theorem 2.4 since its boundary is not annular.

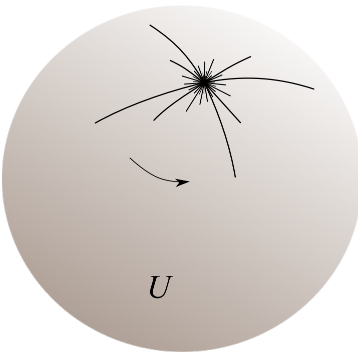

Figure 5: A hedgehog

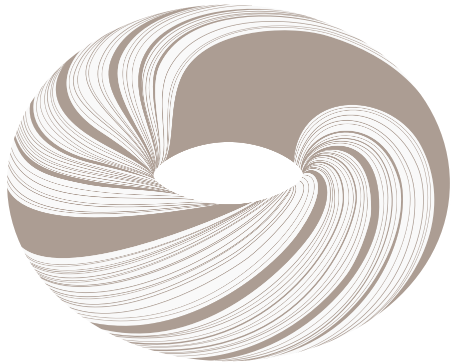

Figure 6: A homotopically unbounded disk

There is room for improvement in this case. We do not know much about the dynamics in ∂U when $\rho(f, U)$ is irrational and f is area-preserving, other than the fact that there are no periodic points (with the exception given in Theorem 2.4). For example, what type of minimal dynamics may appear in subsets of ∂U? Can one have more than one minimal set in ∂U when the map induced on the prime ends is minimal?

2.4 The rational case. We now consider part (i) of Theorem 1.2, which deals with the case when the rotation number is rational. This is contained in Koropecki, Le Calvez, and Nassiri [2017].

Instead of working in the plane, it is more convenient to work on the sphere, so we assume $f : \mathbb{S}^2 \to \mathbb{S}^2$ and $U \subset \mathbb{S}^2$ is an invariant open topological disk with more than one point in its complement. This is equivalent to working in the plane, since the Cartwright-Littlewood theorem Cartwright and Littlewood [1951] guarantees that there is a fixed point in $\mathbb{S}^2 \setminus U$, so by removing this point we are in a similar setting on the plane (and conversely, one may compactify $\mathbb{R}^2$ with one point).

Theorem 2.5 (Koropecki, Le Calvez, and Nassiri [2017]). *If f has the boundary condition in U and $\rho(f, U) = 0$, then the nonwandering set of $f|_{\partial U}$ is contained in* $\mathrm{Fix}(f)$.

The proof of this result has several components. We only mention that it relies on maximal isotopies Béguin, Crovisier, and Le Roux [2016], on a *real* prime ends rotation number associated to these isotopies, and on the study of the dynamics of a lift of f to the universal cover of $\mathbb{A} = \mathbb{S}^2 \setminus \{p, q\}$ where $p \in U$ and $q \notin U$ are fixed points of the maximal isotopy. If $\tilde{U}$ is a lift of $U \setminus \{p\}$ to this covering space, then we are able to show that every cross-cut of $\tilde{U}$ determined by a sufficiently small simple loop around a non-fixed point has endpoints located in a single fundamental domain of the *line of prime ends* (which can be thought as the universal cover of circle of prime ends of U). This essentially tells us that $\tilde{U}$ cannot have "tongues" that come close to two different non-fixed points in the same fiber of the covering map. Combining this with results from Brouwer-Le Calvez theory this leads to a contradiction if some non-fixed point of ∂U is nonwandering. But even without that assumption, this argument is what allows us to show that, unless ∂U has many fixed points, there are strong restrictions on the topology of ∂U. In particular, this is how we prove results such as Theorem 1.5.

A version of this result for arbitrary surfaces is also proved in Koropecki, Le Calvez, and Nassiri [2017], but only in the case where f is isotopic to the identity. The general case remains open and seems to be related to the problem of homotopical boundedness (see Section 2.7).

2.5 On C^r-generic area-preserving diffeomorphisms. Using surgery arguments, one may also obtain a version of Theorem 2.4 for an invariant *complementary domain* (i.e. a connected component of the complement of a continuum), which is not necessarily simply connected (see Koropecki, Le Calvez, and Nassiri [2015, §7]). This is particularly useful for C^r-generic area-preserving diffeomorphisms, since a variation of Mather's arguments Mather [1981] shows that rotation number associated to each topological end of an invariant complementary domain is irrational. This leads to the following (Koropecki, Le Calvez, and Nassiri [2015, Theorem B]):

Theorem 2.6. *If f is a C^r-generic area-preserving diffeomorphism ($r \geq 1$) and U is a periodic complementary domain, then there are no periodic points in ∂U. Moreover, ∂U is the union of finitely many pairwise disjoint annular continua.*

The last claim says that in a way $\overline{U}$ ressembles a surface with boundary. The generic condition required in this theorem can be given explicitly. The theorem holds whenever the following properties hold:

- every periodic point is either hyperbolic or elliptic, and there are no saddle connections;

- every neighborhood of an elliptic periodic point p contains a smaller neighborhood of p bounded by finitely many subarcs of the stable and unstable manifolds of some hyperbolic periodic point q, intersecting transversely.

Both conditions are C^r-generic for any $r \geq 1$ Robinson [1970] and Zehnder [1973].

Theorem 2.6 is a key part of the proof of Theorem 1.1. Let us briefly explain the idea behind that proof. For each area-preserivng diffeomorphism f we define the set K_f as the closure of the union of all stable manifolds of hyperbolic periodic points of f. Our aim is to prove that $K_f = S$ for a C^r-generic f.

Suppose that f satisfies the generic hypotheses listed above, and $K_f \neq S$. Let U be a connected component of $S \setminus K_f$, which must be periodic since $S \setminus K_f$ is invariant and f preserves area. Note that U cannot contain a periodic point, since it would then intersect a stable manifold of some hyperbolic periodic point (due to the generic assumptions). We claim that ∂U also contains no periodic points. Suppose instead that there is a periodic point $p \in \partial U$, and let C be the connected component of K_f containing p. Then the connected component U_0 of $S \setminus C$ containing U is a periodic complementary domain. By Theorem 2.6 there is no periodic point in ∂U_0. But clearly $p \in \partial U_0$, so this is a contradiction.

Thus $\overline{U}$ is aperiodic and compact. By the main theorem of Koropecki [2010] this means that either $\overline{U} = S = \mathbb{T}^2$ or $\overline{U}$ is an annular continuum. In the case that $S = \mathbb{T}^2$ it is known that a C^r-small perturbation creates a periodic point Addas-Zanata [2005, Corollary 2]. On the other hand if $\overline{U}$ is annular continuum, then it is easy to see that U must be homeomorphic to an annulus (otherwise it would contain a periodic point), and by a generalization of the Poincaré–Birkhoff theorem Franks [1988] one may find a C^r-small perturbation which creates a periodic point in U.

Using the lower-semicontinuity of the map $f \mapsto K_f$ in combination with these perturbative arguments one concludes that C^r-generically such a set U cannot exist. See Koropecki, Le Calvez, and Nassiri [2015, §8.5].

These results are useful for the study of dynamics of generic group actions on surfaces which still needs to be explored. For instance, one can prove existence of dense orbits for the semi-group generated by a pair of C^r generic area-preserving diffeomorphisms Koropecki and Nassiri [2010]. This is particularly useful as it has consequences on the instability problem of symplectic dynamics in higher dimensions Nassiri and Pujals [2012].

2.6 The smooth setting. Let us say a few words about Theorem 1.4. Note that the fact that there is a fixed point in ∂U implies that $\rho(f, U) = 0$ due to Theorem 2.2. Since U is bounded, by a simple argument we may reduce the problem to the analogous statement on the sphere with the additional assumption that $\rho(f, U) = 0$. Suppose that $f\colon \mathbb{S}^2 \to \mathbb{S}^2$

is an orientation-preserving and area-preserving C^1-diffeomorphism, and $U \subset \mathbb{S}^2$ is an f-invariant open simply connected set with $\rho(f, U) = 0$.

If $p \in \partial U$ is a fixed point of f, we may then blow-up p to a disk B, bounded by a circle C, where the dynamics corresponds to the map induced by $Df(p)$ on the unit circle by $v \mapsto Df(p)v/\|Df(p)v\|$. Denoting by g the new map obtained in this way, it still preserves a measure of full support on $\mathbb{S}^2 \setminus B$. Moreover, $\partial U \cap C$ is a nonempty compact invariant subset of C, so it should contain some recurrent point z. By Theorem 2.5, the point z must be a fixed point of $g|_C$. This means that $Df(p)$ has a 1-dimensional invariant subspace, with a positive eigenvalue. Since $\det(Df(p)) = 1$, both eigenvalues are positive as claimed in the first item of the theorem.

If f has no fixed point with eigenvalue 1 in ∂U then all fixed points in ∂U must be hyperbolic saddles, and there are finitely many of them, $p_1, \dots, p_k$. Theorem 2.5 implies that every point of ∂U belongs to $W^s(p_i) \cap W^u(p_j)$ for some i, j. By a standard argument using the fact that f is area-preserving, if Γ is a stable or unstable branch of p_i (i.e. a connected component of $W^s(p_i) \setminus \{p_i\}$ or $W^u(p_i) \setminus \{p_i\}$) then $\Gamma \setminus \{p_i\}$ is either disjoint from ∂U or contained in ∂U. Thus ∂U is a union of stable and unstable branches of the points p_i. To prove that these branches are saddle connections, we show that if Γ^s, Γ^u are two branches in ∂U such that $\Gamma^s \cap \Gamma^u \neq \varnothing$ then $\Gamma^s = \Gamma^u$. Indeed, if this is not the case then there exists some simple loop γ bounded by the union of a compact subarc of Γ^s and a compact subarc of Γ^u. Since this loop is contained in ∂U, the set U must be in one of the connected components of $\mathbb{S}^2 \setminus \gamma$. If D denotes the remaining connected component, using the fact that f is area-preserving one may easily deduce that there exists $n > 0$ such that $f^n(\partial D) \cap D \neq \varnothing$, which means that $\partial U \cap D \neq \varnothing$ contradicting the fact that U is disjoint from D. Hence ∂U is a finite union of saddle connections as stated in Theorem 1.4.

We mention that Theorem 1.4 is potentially useful to study certain families of area-preserving maps, such as the standard family or conservative Hénon maps. These maps can be extended to bi-holomorphisms of $\mathbb{C}^2$, and it is known that whenever this happens saddle connections between periodic points cannot occur Ushiki [1980].

2.7 Homotopical boundedness. Consider an open topological disk $U \subset S$ where S is a closed orientable surface, invariant by a homeomorphism $f : S \to S$. The general idea that certain topological properties of ∂U force the presence of many fixed points has already appeared in Section 2.4. When S is an arbitrary surface, a general question inspired by the study of instability regions of area-preserving maps is the homotopical boundedness of U. If S is endowed with a metric of constant curvature we define the *covering diameter* $\mathfrak{D}(U) \in \mathbb{R}_+ \cup \{\infty\}$ as the diameter of any lift of U to the universal covering space of S. It is not difficult to produce examples where this number is infinite (for instance, see Figure 6). However, in the case that f is isotopic to the identiy and area-preserving (or more generally under a boundary condition) one can show that if $\mathfrak{D}(U) = \infty$ then the set of fixed points of f is *essential*.

Theorem 2.7 (Koropecki and Tal [2014, 2017]). *If f is area-preserving, isotopic to the identity, and its fixed point set is inessential, then there is a constant M independent of U such that $\mathfrak{D}(U) \leq M$.*

If f is not isotopic to the identity the result may fail to be true, but we conjecture that either $\mathfrak{D}(U) < \infty$ or $\mathrm{Fix}(f^n)$ is essential for some n (which is likely to depend only on the isotopy class of f). The case where f is in a pseudo-Anosov isotopy class seems to be easy to deal with (using a more direct argument), but the reducible case seems to need a new approach.

The proof Theorem 2.7 relied on Brouwer-Le Calvez theory, but recently a surprisingly simple proof was found using a "triple boundary lemma", which in its simplest form can be stated as follows:

Theorem 2.8 (Koropecki, Le Calvez, and Tal [2017]). *Suppose $f : \mathbb{S}^2 \to \mathbb{S}^2$ is orientation- and area-preserving and U_1, U_2, U_3 are pairwise disjoint open f-invariant topological disks. Then every $x \in \partial U_1 \cap \partial U_2 \cap \partial U_3$ is a fixed point.*

As an example of simple application of this theorem we state the following (which complements Theorem 1.5):

Theorem 2.9 (Koropecki, Le Calvez, and Tal [ibid.]). *If $K \subset \mathbb{R}^2$ is an invariant Wada-type continuum and f is area-preserving then $f^n|_K$ is the identity for some $n > 0$.*

We remark that the notion of homotopical boundedness can also be defined for non-simply connected open sets, and results on the same line as Theorem 2.7 are available Koropecki and Tal [2017].

3 Further results

3.1 Vanishing rotation numbers without fixed points. Consider a bounded open topological disk $U \subset \mathbb{R}^2$ invariant by some orientation-preserving homeomorphism f. We do not assume any additional condition on the dynamics. As we saw in Figure 3, even if $\rho(f, U) = 0$ it may be the case that $\mathrm{Fix}(f) \cap \partial U = \varnothing$. Theorem 2.1 tells us that this implies that there are boundary traps arbitrarily cose to ∂U; but this can be improved to the following statement, which is explicitly proved in Matsumoto and Nakayama [2011] but attributed to Cartwright-Littlewood.

Theorem 3.1. *Suppose there is no fixed point in ∂U and $\rho(f, U) = 0$. Then the dynamics induced by f on the circle of prime ends consists of alternating attracting and repelling fixed points. Moreover, each attracting fixed prime end is attracting in the disk $c_{\mathcal{E}}U$, and similarly for the repelling prime ends.*

What this tells us is that for the induced map f_e in $c_{\mathcal{E}}U$, a neighborhood of the circle of prime ends is covered by the basins of alternating attracting and repelling prime ends (see Figure 7). Although this is a beautiful result, it does not tell us anything about the dynamics of $f|_{\partial U}$. In Koropecki and Passeggi [2017], the authors obtained a translation to the boundary dynamics which tells us that this situation can only occur under very strict conditions. A simplified version of the main result states the following:

Theorem 3.2. *Suppose that $\rho(f, U) = 0$ and* $\mathrm{Fix}(f) \cap \partial U = \varnothing$. *Then there exists a finite pairwise disjoint family of rotational attractors and repellors (at least one of each) such that ∂U is contained in the union of their basins.*

By a *rotational attractor* we mean an invariant non-separating continuum A which is a topological attractor and has nonzero external rotation number (i.e. the prime ends rotation number of the disk $\mathbb{R}^2 \cup \{\infty\} \setminus A$ is nonzero).

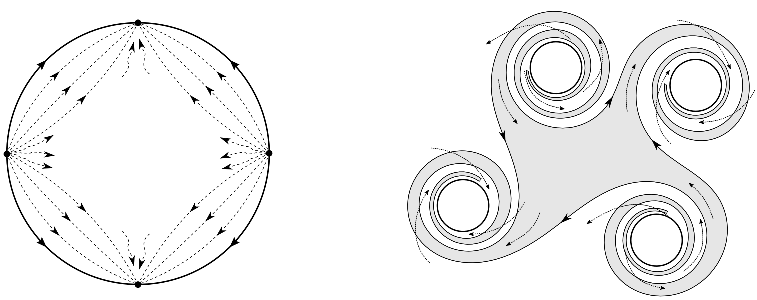

Figure 7: Prime ends and boundary dynamics as in Theorems 3.1 and 3.2

It is also shown that $f|_{\partial U}$ is topologically semiconjugate to a planar graph G where each vertex is an attractor or repellor and every edge is contained in the intersection of the basins of the corresponding vertices. Moreover, the semiconjugation extends to a monotone map from a neighborhood of G (contained in the union of the basins of the vertices) to a neighborhood of ∂U. We refer to Koropecki and Passeggi [ibid., §5.5] for further details. We only mention that a useful result introduced in that article, and essential to the proof of Theorem 3.2, is a Poincaré-Bendixson type theorem for translation lines Koropecki and Passeggi [ibid., Theorem A].

3.2 Rotation sets. As mentioned in the introduction, a common approach to generalize the notion of one-dimensional rotation number is to consider the rotation of orbits along homological directions. We consider the particularly simple case of the annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$. Suppose that $K \subset \mathbb{A}$ is an essential continuum and $f : \mathbb{A} \to \mathbb{A}$ is a homeomorphism isotopic to the identity. Given a lift $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ to the universal cover and $z \in K$, one defines its rotation number as

$$\rho(\tilde{f}, \tilde{z}) = \lim_{n\to\infty} (\tilde{f}^n(\tilde{z}) - \tilde{z})_1/n,$$

for any $\tilde{z}$ in the fiber of z, where $(\cdot)_1$ denotes the first coordinate. This limit may fail to exist, and unlike in the circle, when it exists it generally depends on z. But considering the set of all such possible limits when they exist, one may define the (pointwise) *rotation set* $\rho(\tilde{f}, K) \subset \mathbb{R}$ associated to this lift. If K is a closed annulus, or more generally if K is *annular* (i.e. separates $\mathbb{A}$ into exactly two components), then the rotation set is compact and nonempty Handel [1990] and Koropecki [2016].

One may also define the rotation interval $\overline{\rho}(\tilde{f}, K) = [\inf \rho(\tilde{f}, K), \sup \rho(\tilde{f}, K)]$. It is natural then to ask whether having a rational element p/q in the rotation set or interval implies that one must have a corresponding periodic point of f of period q in K. The first result of this type is the classical Poincaré-Birkhoff theorem, which in a generalization due to Franks [1988] states that if f is area-preserving and K is a closed annulus then any rational element in its rotation interval is realized by a periodic point in K. Moreover, even without the area-preserving assumption, the result holds for elements of the rotation set Handel [1990] and Koropecki [2016].

If $K \subset \mathbb{A}$ is an essential continuum, we denote by $U_-(K)$ and $U_+(K)$ the two unbounded connected components of its complement (the latter being the one unbounded from above). Of particular interest are continua which are minimal with respect to the condition of being essential in $\mathbb{A}$. We call these continua *coboundaries*, and they are characterized by the property that $\partial U_-(K) = K = \partial U_+(K)$. Such a continuum may fail to be annular (as in Wada-type continua), but the continuum $K' = \mathbb{A} \setminus (U_-(K) \cup U_+(K))$, which can be thought as the "filling" of K, is annular and minimal with the property of being annular and essential. Any continuum with the latter property is called a *circloid*. Coboundaries and circloids are interesting because every essential continuum K contains a coboundary, and if K is invariant it also contains an invariant coboundary, which in turn has a corresponding invariant circloid.

One may then consider the (real) *upper* and *lower* prime ends rotation numbers $\rho_-(\tilde{f}, K)$ and $\rho_+(\tilde{f}, K)$ of K, which are defined in terms of the lift $\tilde{f}$ and lifts of the circles of prime ends of $U_\pm(K) \cup \{\pm\infty\}$. See Koropecki [2016] for more details. The relationship between these prime ends rotation numbers and the rotation set was studied in Matsumoto [2012] and Hernández-Corbato [2017], where it was proved that $\rho_\pm(\tilde{f}, K) \subset \overline{\rho}(\tilde{f}, K)$ when K is an annular continuum (see also Franks and Le Calvez [2003]).

3.3 No Birkhoff-like behavior for area-preserving maps. A circloid with empty interior is called a *cofrontier*. It is possible to produce an example of a cofrontier K where rotation interval has more than one point. For instance the Birkhoff attractor Le Calvez [1986] has a nontrivial interval as its rotation set (see also Boroński and Oprocha [2015]). When this happens, in general the dynamics in the cofrontier is rich; for instance there are infinitely many periodic points of arbitrarily large periods, uncountably many ergodic measures Koropecki [2016]. Moreover, if K is an attractor it implies that $f|_K$ has positive topological entropy Passeggi, Potrie, and Sambarino [2017]. It is conjectured that this is true even if K is not an attractor.

The results from Koropecki, Le Calvez, and Nassiri [2015] allow us to show that this kind of behavior is not possible in the area preserving setting: the rotation set is always a single point and coincides with the prime ends rotation numbers:

Theorem 3.3. *Suppose that f is area-preserving and $K \subset \mathbb{A}$ is an essential cofrontier. Then $\rho_-(\tilde{f}, K) = \rho_+(\tilde{f}, K)$ and this number is the only element of $\rho(\tilde{f}, K)$.*

The proof of this fact is explained in Koropecki [2016, Theorem 2.8] in a more general setting. This result also holds for arbitrary circloids (using the same argument combined with Koropecki, Le Calvez, and Tal [2017], for instance).

3.4 A Poincaré-like result for decomposable circloids. In the setting of the previous section, suppose that $K \subset \mathbb{A}$ is an f-invariant essential circloid whose boundary is *decomposable*, i.e. it can be written as the union of two proper subcontinua. The dynamics in a continuum of this type was studied in Jäger and Koropecki [2017], where the authors obtained a general Poincaré-type result without any additional hypothesis:

Theorem 3.4. *If K is an invariant essential circloid with decomposable boundary, then $\overline{\rho}(f, K)$ has a single element α, and*

- *α is rational if and only if there is a periodic point in K;*
- *α is irrational if and only if $f|_K$ is monotonically semiconjugate to the corresponding irrational rotation on the circle.*

Moreover, the semiconjugation in the last item is unique up to post-composition with a rotation.

The number α of course coincides with $\rho_-(f, K)$ and $\rho_+(f, K)$. This result emphasizes the fact that having points with different rotational behavior forces the topology of K to be complicated (which is a trait of indecomposable continua), something already noted in Barge and Gillette [1991].

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Andres Koropecki
Instituto de Matemática e Estatística
Universidade Federal Fluminense
Niterói
Brazil
ak@id.uff.br

Meysam Nassiri
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
Tehran
Iran
nassiri@ipm.ir