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PAIRS OF INVARIANTS OF SURFACE SINGULARITIES

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Abstract

We discuss several invariants of complex normal surface singularities with a special emphasis on the comparison of analytic–topological pairs of invariants. Additionally we also list several open problems related with them.

1 Introduction

Singularity theory aims to study the singular points of algebraic/analytic varieties. It was born together with the classical algebraic geometry, but step by step became an independent discipline within algebraic and complex geometry. Furthermore, it also created formidable connections with other fields, like topology or differential equations. By crucial classification projects in mainstream mathematics (e.g. the Mori program in algebraic geometry targeting classification of varieties, or developments in low-dimensional topology) singularities theory became even more a central area.

In both local and global case the study of surfaces became central. In the global case, after the 'classical' classification of Enriques and the Italian school, a 'modern' classification was provided by Kodaira in 60's based on sheaves, cohomologies and characteristic classes with special emphasis on the relationships of the analytic structures with invariants of the underlying smooth 4–manifolds: e.g. Hodge or Riemann–Roch–Hirzebruch formulas. Crucial open questions were formulated targeting this type of ties: e.g. the purely topological characterization of rational surfaces (as an addendum of Castelnuovo's criterion) or of K3 surfaces (asked by Kodaira). The appearance of Donaldson and Seiberg–Witten theories gave a powerful impetus of such comparison results, and generated a series of new open questions. These were inherited by local singularity theory too. The lack of

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classification results in 3 and 4 dimensional topology put even more in the highlight the possible analytic/algebraic connections.

This is the guiding principle of the present manuscript too: what are the ties between analytic and topological structures of a local complex normal surface singularity.

It is always exciting to understand such bridges between topology and rigid analytic/algebraic geometry. In this note by the style of the presentation we even try to emphasize more the existing parallel pairs of invariants, theorems and constructions from both sides. Though the presented list definitely is not exhaustive, it supports rather well the philosophical conviction that any result in the analytic part must have a topological counterpart, and vice versa.

The guiding pair is $\mathcal{P}(\mathbf{t}) \leftrightarrow Z(\mathbf{t})$, where $\mathcal{P}(\mathbf{t})$ is the multivariable Poincaré series associated with the divisorial filtration given by the irreducible curves of a resolution, while $Z(\mathbf{t})$ is a combinatorial 'zeta' function read from the corresponding resolution graph.

2 Normal surface singularities. Analytic Invariants

2.1 Definitions, notations. Let (X,o) be a complex normal surface singularity. Let $\pi : \widetilde{X} \to X$ be a good resolution with dual graph Γ whose vertices are denoted by $\mathcal{V}$. Set $E := \pi^{-1}(o)$. Let M be the link of (X,o), and we will assume that M is a rational homology sphere. This happens if and only if Γ is a tree and all the irreducible exceptional curves $\{E_v\}_{v\in\mathcal{V}}$ have genus 0.

Set $L := H_2(\widetilde{X},\mathbb{Z})$. It is freely generated by the classes of the irreducible exceptional curves. If L' denotes $H^2(\widetilde{X},\mathbb{Z})$, then the intersection form $(\,,\,)$ on L provides an embedding $L \hookrightarrow L'$ with factor the first homology group H of the link. (In fact, L' is the dual lattice of $(L,(\,,\,))$.) Moreover, $(\,,\,)$ extends to L'. L' is freely generated by the duals E_v^*, where $(E_v^*, E_w) = -1$ for $v = w$ and $= 0$ else.

Effective classes $l = \sum r_v E_v \in L'$ with all $r_v \in \mathbb{Q}_{\geq 0}$ are denoted by $L'_{\geq 0}$ and $L_{\geq 0} := L'_{\geq 0} \cap L$. Denote by $\mathcal{S}'$ the (Lipman's) anti-nef cone $\{l' \in L' \ :\ (l', E_v) \leq 0 \text{ for all } v\}$. It is generated over $\mathbb{Z}_{\geq 0}$ by the base-elements E_v^*. Since all the entries of E_v^* are strict positive, $\mathcal{S}'$ is a sub-cone of $L'_{\geq 0}$, and for any fixed $a \in L'$ the set $\{l' \in \mathcal{S}' \ :\ l' \ngeq a\}$ is finite. Set $\mathfrak{C} := \{\sum l'_v E_v \in L',\ 0 \leq l'_v < 1\}$. For any $l' \in L'$ write its class in H by $[l']$, and or any $h \in H$ let $r_h \in L'$ be its unique representative in $\mathfrak{C}$. Denote by $\theta : H \to \widehat{H}$ the isomorphism $[l'] \mapsto e^{2\pi i (l',\cdot)}$ of H with its Pontrjagin dual $\widehat{H}$.

Denote by $K \in L'$ the *canonical class* satisfying $(K + E_v, E_v) = -2$ for all $v \in \mathcal{V}$. We set $\chi(l') = -(l', l' + K)/2$; by Riemann-Roch theorem $\chi(l) = \chi(\mathcal{O}_l)$ for any $l \in L_{>0}$.

Most of the analytic geometry of $\widetilde{X}$ is described by its line bundles and their cohomology groups. E.g., the geometric genus of (X,o) is $p_g := h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. In this note we mostly target the following numerical invariants (below $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ and $l \in L_{>0}$):

$$(2.1.1) \qquad (a)\ \dim\ H^0(\mathcal{L})\,/\,H^0(\mathcal{L}(-l)) \quad \text{and} \quad (b)\ \dim H^1(\mathcal{L}).$$

Their behaviour for arbitrary line bundles is rather complicated, however for *natural line bundles* we have several (sometimes even topological) descriptions/characterizations. These line bundles are provided by the splitting of the cohomological exponential exact sequence Némethi [n.d.(b), §3]:

$$0 \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to \mathrm{Pic}(\widetilde{X}) \xrightarrow{c_1} L' \to 0.$$

The first Chern class c_1 has an obvious section on the subgroup L, namely $l \mapsto \mathcal{O}_{\widetilde{X}}(l)$. This section has a unique extension $\mathcal{O}(\cdot)$ to L'. We call a line bundle *natural* if it is in the image of this section.

One can recover these bundles via coverings as follows. Let $c : (Y, o) \to (X, o)$ be the universal abelian covering of (X, o), $\pi_Y : \widetilde{Y} \to Y$ the normalized pullback of π by c, and $\widetilde{c} : \widetilde{Y} \to \widetilde{X}$ the morphism which covers c. Then the action of H on (Y, o) lifts to $\widetilde{Y}$ and one has an H–eigensheaf decomposition (Némethi [ibid., (3.7)] or Okuma [2008, (3.5)]):

$$(2.1.2) \qquad \widetilde{c}_*\mathcal{O}_{\widetilde{Y}} = \bigoplus_{l' \in \mathfrak{C}} \mathcal{O}(-l') \quad (\mathcal{O}(-l') \text{ being the } \theta([l'])\text{-eigenspace of } \widetilde{c}_*\mathcal{O}_{\widetilde{Y}}).$$

Note that the geometric genus of Y is $h^1(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})$, hence, by Equation (2.1.2), $\{h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h))\}_{h \in H}$ are the dimensions of the H–eigenspaces; we call them equivariant geometric genera of (X, o).

2.2 Series associated with the divisorial filtration. For those natural line bundles which appear in Equation (2.1.2), the dimensions from Equation (2.1.1)(a) can be organized in a generating function. Indeed, once a resolution π is fixed, $\mathcal{O}_{Y,o}$ inherits the *divisorial multi-filtration* (cf. Némethi [2008b, (4.1.1)]):

$$(2.2.1) \qquad \mathcal{F}(l') := \{f \in \mathcal{O}_{Y,o} \mid \mathrm{div}(f \circ \pi_Y) \geq \widetilde{c}^*(l')\}.$$

Let $\mathfrak{h}(l')$ be the dimension of the $\theta([l'])$-eigenspace of $\mathcal{O}_{Y,o}/\mathcal{F}(l')$. Then, one defines the *equivariant divisorial Hilbert series* by

$$(2.2.2) \qquad \mathcal{H}(\mathbf{t}) = \sum_{l' \in L'} \mathfrak{h}(l') t_1^{l'_1} \cdots t_s^{l'_s} = \sum_{l' \in L'} \mathfrak{h}(l')\mathbf{t}^{l'} \in \mathbb{Z}[[L']] \;\; (l' = \sum_i l'_i E_i).$$

Notice that the terms of the sum reflect the H-eigenspace decomposition too: $\mathfrak{h}(l')\mathbf{t}^{l'}$ contributes to the $\theta([l'])$-eigenspace. For example, $\sum_{l \in L} \mathfrak{h}(l)\mathbf{t}^l$ corresponds to the H-invariants, hence it is the *Hilbert series* of $\mathcal{O}_{X,o}$ associated with the $\pi^{-1}(o)$-divisorial multi-filtration (considered and intensively studied, see e.g. Cutkosky, Herzog, and Reguera [2004] and the citations therein, or Campillo, Delgado, and Gusein-Zade [2004]).

The 'graded version' associated with the Hilbert series is defined (cf. Campillo, Delgado, and Gusein-Zade [2004] and Guseĭn-Zade, Delgado, and Kampil'o [2008]) as

$$\mathcal{P}(\mathbf{t}) = -\mathcal{H}(\mathbf{t}) \cdot \prod_v (1 - t_v^{-1}) \in \mathbb{Z}[[L']]. \tag{2.2.3}$$

Although the multiplication by $\prod_v (1 - t_v^{-1})$ in $\mathbb{Z}[[L']]$ is not injective, hence apparently $\mathcal{P}$ contains less information then $\mathcal{H}$, they, in fact, determine each other as we will see in Equation (2.4.2).

If we write the series $\mathcal{P}(\mathbf{t})$ as $\sum_{l'} \mathfrak{p}(l')\mathbf{t}^{l'}$, then

$$\mathfrak{p}(l') = \sum_{I \subseteq \mathcal{V}} (-1)^{|I|+1} \dim \frac{H^0(\widetilde{X}, \mathcal{O}(-l'))}{H^0(\widetilde{X}, \mathcal{O}(-l' - E_I))} \tag{2.2.4}$$

and $\mathcal{P}$ is supported in the cone $\mathcal{S}'$.

2.3 Quasipolynomials and the periodic constants associated with series. The following definitions are motivated by properties of Hilbert–Samuel functions and also by Ehrhart theory and the properties of its qusipolynomials. The periodic constant of one–variable series was introduced in Némethi and Okuma [2009], Okuma [2008], and Braun and Némethi [2010], the multivariable generalization is treated in László and Némethi [2014].

Let $S(t) = \sum_{l \geq 0} c_l t^l \in \mathbb{Z}[[t]]$ be a formal power series with one variable. Assume that for some $p \in \mathbb{Z}_{>0}$ the counting function $Q^{(p)}(n) := \sum_{l=0}^{pn-1} c_l$ is a polynomial $\mathfrak{Q}^{(p)}$ in n. Then the constant term $\mathfrak{Q}^{(p)}(0)$ is independent of p and it is called the *periodic constant* $\mathrm{pc}(S)$ of the series S. E.g., if $S(t)$ is a finite polynomial, then $\mathrm{pc}(S)$ exists and it equals $S(1)$. If the coefficients of $S(t)$ are given by a Hilbert function $l \mapsto c(l)$, which admits a Hilbert polynomial $H(l)$ with $c(l) = H(l)$ for $l \gg 0$, then $S^{reg}(t) = \sum_{l \geq 0} H(l)t^l$ has zero periodic constant and $\mathrm{pc}(S) = \mathrm{pc}(S - S^{reg}) + \mathrm{pc}(S^{reg}) = (S - S^{reg})(1)$, measuring the difference between the Hilbert function and Hilbert polynomial.

For the multivariable case we consider a (negative) definite lattice $L = \mathbb{Z}\langle E_v \rangle_v$, its dual lattice L', and a series $S(\mathbf{t}) \in \mathbb{Z}[[L']]$, $S(\mathbf{t}) = \sum_{l' \in L'} s(l')\mathbf{t}^{l'}$. We decompose S as $S = \sum_{h \in H} S_h$, where $S_h(\mathbf{t}) = \sum_{[l']=h} s(l')\mathbf{t}^{l'}$, and we consider the following 'counting function of the coefficients'

$$Q_h : L'_h := \{x \in L' : [x] = h\} \to \mathbb{Z}, \quad Q_h(x) = \sum_{l' \ngeq x,\, [l']=h} s(l'). \tag{2.3.1}$$

Assume that there exist a real cone $\mathcal{K} \subset L' \otimes \mathbb{R}$ whose affine closure is top–dimensional, $l'_* \in \mathcal{K}$, a sublattice $\widetilde{L} \subset L$ of finite index, and a quasipolynomial $\mathfrak{Q}_h(l)$ ($l \in \widetilde{L}$) such that $Q_h(l + r_h) = \mathfrak{Q}_h(l)$ for any $l + r_h \in (l'_* + \mathcal{K}) \cap (\widetilde{L} + r_h)$. Then we say that the counting function Q_h (or just $S_h(\mathbf{t})$) admits a quasipolynomial in $\mathcal{K}$, namely $\mathfrak{Q}_h(l)$, and also an (equivariant, multivariable) *periodic constant* associated with $\mathcal{K}$, which is defined by

$$\mathrm{pc}^{\mathcal{K}}(S_h(\mathbf{t})) := \mathfrak{Q}_h(0). \tag{2.3.2}$$

The definition does not depend on the choice of the sublattice $\widetilde{L}$, which corresponds to the choice of p in the one–variable case. This is responsible for the name 'periodic' in the definition. The definition is independent of the choice of l'_* as well.

By general theory of multivariable Ehrhart-type quasipolynomials, for a nicely defined series one can construct a conical chamber decomposition of the space $L' \otimes \mathbb{R}$, such that each cone satisfies the above definition (hence provides a periodic constant), for details see László and Némethi [ibid.] or Szenes and Vergne [2003]. However, it turns out, that in all our situations the whole $\mathcal{S}'_{\mathbb{R}}$ (the real Lipman cone) will be a unique chamber.

2.4 The quasipolynomial of $\mathcal{P}$. Let $\mathcal{P}$ and $\mathcal{H}$ be the series defined in SubSection 2.2.

Theorem 2.4.1. *For any $l' \in L'$ one has*

$$\mathfrak{h}(l') = \sum_{a \in L,\, a \ngeq 0} \mathfrak{p}(l' + a). \tag{2.4.2}$$

Furthermore, there exists a constant $\mathrm{const}_{[-l']}$, *depending only on the class of* $[-l'] \in H$, *such that*

$$-h^1(\widetilde{X}, \mathcal{O}(-l')) = \sum_{a \in L,\, a \ngeq 0} \mathfrak{p}(l' + a) + \mathrm{const}_{[-l']} + \frac{(K + 2l')^2 + |\mathcal{V}|}{8}. \tag{2.4.3}$$

By taking $l' = r_h$ one can identify $\mathrm{const}_{[-l']}$ with the equivariant geometric genus, that is $-h^1(\widetilde{X}, \mathcal{O}(-r_h)) = \mathrm{const}_{-h} + ((K + 2r_h)^2 + |\mathcal{V}|)/8$. In particular, the coefficients of the series $\mathcal{P}(\mathbf{t})$ determine the invariants Equation (2.1.1)((a)-(b)) for all natural line bundles.

Write $l' = l + r_h$ for $l \in L$. Note that if $l \in L_{\leq 0}$, then in Equation (2.4.3) the sum will not appear (check the support of $\mathcal{P}$). On the other hand, if in Equation (2.4.3) $l' \in -K + \mathcal{S}'$ then by the vanishing of $h^1(\mathcal{O}(-l'))$ we get that $\sum_{a \in L,\, a \ngeq 0} \mathfrak{p}(l' + a)$ is the multivariable quadratic function $\chi(l) - (r_h, l) + h^1(\widetilde{X}, \mathcal{O}(-r_h))$. This quadratic function is the quasipolynomial of $\mathcal{P}$ in $\mathcal{K} = \mathcal{S}'_{\mathbb{R}}$, and its periodic constant is $h^1(\widetilde{X}, \mathcal{O}(-r_h))$.

2.5 Reductions of $\mathcal{P}$. Let $\mathcal{U}$ be a nonempty subset of $\mathcal{V}$, and fix $h \in H$ as well. One can introduce the reduction of the series $\mathcal{P}_h$ to the variables $\{t_v\}_{v\in\mathcal{U}}$ in two different ways. First, we can consider the multivariable divisorial filtration induced by the exceptional divisors $\{E_v\}_{v\in\mathcal{U}}$, define the Hilbert series $\mathcal{H}_h^{\mathcal{U}}$ and $\mathcal{P}_h^{\mathcal{U}}$ in variables $\{t_v\}_{v\in\mathcal{U}}$ and associated with $\theta(h)$–eigendecomposition similarly as $\mathcal{H}_h$ and $\mathcal{P}_h$ was defined in Section 2.2. The second possibility is to reduced the variables in the original $\mathcal{P}_h$. The point is that the two constructions have the same output: $\mathcal{P}_h^{\mathcal{U}} = \mathcal{P}_h(\mathbf{t})|_{t_v=1 \text{ for all } v\notin\mathcal{U}}$. In this article we mostly discuss the case $\mathcal{U} = \{v\}$ for a certain vertex v, however all the discussions can be extended for arbitrary $\mathcal{U}$.

2.6 Surgery formulae for $h^1(\widetilde{X}, \mathcal{L})$. Let (X, o) and $\pi : \widetilde{X} \to X$ as above. We fix a vertex $v \in \mathcal{V}$. Let $\cup_{j\in J}\Gamma_j$ be the connected components of the graph obtained from Γ by deleting v and its adjacent edges. Let X' be the space obtained from $\widetilde{X}$ by contracting all irreducible exceptional curves except E_v to normal points. It has $|J|$ normal singular points $\{o_j\}_j$, the images of the connected components of $E - E_v$. Let X_j be a small Stein neighbourhood of o_j in X', $\widetilde{X}_j = \tau^{-1}(X_j)$ its pre-image via the contraction $\tau : \widetilde{X} \to X'$, and $\tau(E) = E' \subset X'$. We denote the local singularities by (X_j, o_j).

We say that the Assumption (*C*) is satisfied if $nE' \subset X'$ is a Cartier divisor for certain $n > 0$.

Theorem 2.6.1. *Okuma [2008] Set $\mathcal{U} = \{v\}$ and fix $h \in H$. Under the Assumption (C)*

$$h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)) = \mathrm{pc}(\mathcal{P}_h^{\{v\}}(t_v)) + \sum_j h^1(\widetilde{X}_j, \mathcal{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}).$$

E.g., this applied for $\mathcal{O}_{\widetilde{X}}$ gives $p_g(X, o) = \mathrm{pc}(\mathcal{P}_h^{\{v\}}(t_v)) + \sum_j p_g(X_j, o_j)$. In general, $\mathcal{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}$ is not a natural line bundle on $\widetilde{X}_j$, however e.g. for splice quotient singularities it is the line bundle associated with the cohomology restriction of $-r_h$ (for details see Section 3.3). Hence, Theorem 2.6.1 provides an ideal inductive procedure for the computation of the cohomology of natural line bundles.

3 Topological invariants. The series $Z(\mathbf{t})$

3.1 The series $Z(\mathbf{t})$. The *multivariable topological series* is the Taylor expansion $Z(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'} \in \mathbb{Z}[[L']]$ at the origin of the rational function

$$f(\mathbf{t}) = \prod_{v\in\mathcal{V}} (1 - \mathbf{t}^{E_v^*})^{\delta_v - 2}. \tag{3.1.1}$$

It is supported in $\mathcal{S}'$. Similarly as $\mathcal{P}$, it decomposes as $Z(\mathbf{t}) = \sum_{h\in H} Z_h(\mathbf{t})$. In fact,

$$Z_h(\mathbf{t}) := \frac{1}{|H|} \cdot \sum_{\rho\in\widehat{H}} \rho(h)^{-1} \cdot \prod_{v\in\mathcal{V}} \left(1 - \rho([E_v^*])\mathbf{t}^{E_v^*}\right)^{\delta_v - 2}. \tag{3.1.2}$$

3.2 Seiberg–Witten invariants of the link M. Let $\widetilde{\sigma}_{can}$ be the *canonical $spin^c$–structure on $\widetilde{X}$* identified by $c_1(\widetilde{\sigma}_{can}) = -K$, and let $\sigma_{can} \in \mathrm{Spin}^c(M)$ be its restriction to M, called the *canonical $spin^c$–structure on M*. $\mathrm{Spin}^c(M)$ is an H–torsor with action denoted by $*$.

We denote by $\mathfrak{sw}_\sigma(M) \in \mathbb{Q}$ the *Seiberg–Witten invariant* of M indexed by the $spin^c$–structures $\sigma \in \mathrm{Spin}^c(M)$ (cf. Lim [2000] and Nicolaescu [2004]). We will use the sign convention of Braun and Némethi [2010] and Némethi [2011].

In the last years several combinatorial expressions were established for the Seiberg–Witten invariants. For rational homology spheres, Nicolaescu [2004] showed that $\mathfrak{sw}(M)$ is equal to the Reidemeister–Turaev torsion normalized by the Casson–Walker invariant. In the case when M is a negative definite plumbed rational homology sphere, combinatorial formula for Casson–Walker invariant in terms of the plumbing graph can be found in Lescop [1996], and the Reidemeister–Turaev torsion is determined by Némethi and Nicolaescu [2002] using Dedekind–Fourier sums. A different combinatorial formula of $\{\mathfrak{sw}_\sigma(M)\}_\sigma$ was proved in Némethi [2011] using qualitative properties of the coefficients of the series $Z(\mathbf{t})$.

Theorem 3.2.1. *Némethi [ibid.] The counting function of $Z_h(\mathbf{t})$ in the cone $S'_{\mathbb{R}}$ admits the (quasi)polynomial*

$$\mathfrak{Q}_h(l) = -\frac{(K + 2r_h + 2l)^2 + |\mathcal{V}|}{8} - \mathfrak{sw}_{-h*\sigma_{can}}(M), \tag{3.2.2}$$

whose periodic constant is

$$\mathrm{pc}^{S'_{\mathbb{R}}}(Z_h(\mathbf{t})) = \mathfrak{Q}_h(0) = -\mathfrak{sw}_{-h*\sigma_{can}}(M) - \frac{(K + 2r_h)^2 + |\mathcal{V}|}{8}. \tag{3.2.3}$$

The right hand side of Equation (3.2.3) with opposite sign is called the *r_h–normalized Seiberg–Witten invariant* of M.

3.3 Surgery formulae for the normalized Seiberg–Witten invariants. Surgery formulae for a certain 3–manifold invariant, in general, compare the invariant of M with the invariants of different surgery modifications of M. In the case of plumbed 3–manifolds, one compares the invariants associated with 3–manifolds obtained by different modifications of the graph. The ‘standard’ topological surgery formulae for the Seiberg–Witten invariant (induced by exact triangles of certain cohomology theories, cf. Ozsváth and Szabó [2004a], Greene [2013], and Némethi [2011]) compare the invariants of three such 3–manifolds (see also Section 5.1 here). Furthermore, in these approaches, one cannot separate a certain fixed $spin^c$ structure, the theory mixes always several of them. (See also Turaev [2001].) The next formula is different: it compares the Seiberg–Witten invariant of two 3–manifolds via the periodic constant of a series, and they split according to the $spin^c$–structures.

Let us fix $v \in \mathcal{V}$ and consider the notations of Section 2.6. For each $j \in J$ we consider the inclusion operator $\iota_j : L(\Gamma_j) \to L(\Gamma)$, $E_v(\Gamma_j) \mapsto E_v(\Gamma)$; let $\iota_j^* : L'(\Gamma) \to L'(\Gamma_j)$ be its dual (the cohomological restriction defined by $\iota_j^*(E_v^*(\Gamma)) = E_v^*(\Gamma_j)$ if $v \in \mathcal{V}(\Gamma_j)$, and $= 0$ otherwise).

Next, consider an arbitrary $spin^c$–structure $\widetilde{\sigma}$ on $\widetilde{X}$. Since $\mathrm{Spin}^c(\widetilde{X})$ is an L'–torsor, there is a unique $l' \in L'$ such that $\widetilde{\sigma} = l' * \widetilde{\sigma}_{can}$. Its restriction to $\mathrm{Spin}^c(M)$ is $\sigma = [l'] * \sigma_{can}$. We write $\widetilde{\sigma}_j$ the restriction of $\widetilde{\sigma}$ to each $\widetilde{X}_j$. Since the canonical $spin^c$–structure of $\widetilde{X}$ restricts to the canonical $spin^c$–structure $\widetilde{\sigma}_{can,j}$ of $\widetilde{X}_j$, $\widetilde{\sigma} = l' * \widetilde{\sigma}_{can}$ restricts to $\widetilde{\sigma}_j := \iota_j^*(l') * \widetilde{\sigma}_{can,j} \in \mathrm{Spin}^c(\widetilde{X}_j)$, whose restriction to the boundary $M_j = M(X_j, o_j) = \partial \widetilde{X}_j$ is $\sigma_j = [\iota_j^*(l')] * \sigma_{can,j}$.

Theorem 3.3.1. *Braun and Némethi [2010] Fix $\mathcal{U} = \{v\}$ and $h \in H$. Extend $h * \sigma_{can} \in \mathrm{Spin}^c(M)$ as $\widetilde{\sigma} := r_h * \widetilde{\sigma}_{can} \in \mathrm{Spin}^c(\widetilde{X})$ and consider the corresponding restrictions. Then*

$$\mathfrak{sw}_{-h*\sigma_{can}}(M) + \frac{(K+2r_h)^2 + |\mathcal{V}|}{8} =$$
$$= \sum_j \Big(\mathfrak{sw}_{-[\iota_j^*(r_h)]*\sigma_{can,j}}(M_j) + \frac{(K(\Gamma_j) + 2\iota_j^*(r_h))^2 + |\mathcal{V}(\Gamma_j))|}{8}\Big) - \mathrm{pc}(Z_h^{\{v\}}(t_v)).$$

For a generalization to an arbitrary $\mathcal{U}$ and to an arbitrary extension $\widetilde{\sigma} := l' * \widetilde{\sigma}_{can}$ see László, Nagy, and Némethi [n.d.].

4 Topological invariants. The lattice cohomology of M

The $spin^c$–structures of M can also be indexed as follows. Set $\mathrm{Char} = \{k \in L' : (k + x, x) \in 2\mathbb{Z}$ for all $x \in L\}$, the set of characteristic elements of $\widetilde{X}$. Then L acts on Char by $l * k = k + 2l$, and the set of orbits $[k] = k + 2L$ is an H torsor identified with $\mathrm{Spin}^c(M)$.

For any $k \in \mathrm{Char}$ one also defines $\chi_k : L' \to \mathbb{Q}$ by $\chi_k(l') := -(l', l' + k)/2$. We write χ for χ_K.

The Seiberg–Witten invariant is the (normalized) Euler-characteristic of the Seiberg–Witten monopole Floer homology of Kronheimer–Mrowka, or equivalently, of the Heegaard–Floer homology of Ozsváth and Szabó. These theories had an extreme influence on the modern mathematics, solving (or disproving) a long list of old conjectures (e.g. Thom Conjecture, or conjectures regarding classification of 4-manifolds, or famous old problems in knot theory); see the long list of distinguished articles of Kronheimer–Mrowka or Ozsváth–Szabó. In Ozsváth and Szabó [2003b] Ozsváth and Szabó provided a computation of the Heegaard–Floer homology for some special plumbed 3-manifolds. This computation resonated incredibly with the theory of computation sequences used in Artin–Laufer program (see e.g. Laufer [1977] and Némethi [1999a,b]). These two facts influenced considerably the definition of the lattice cohomology.

4.1 Short review of Heegaard–Floer homology $HF^+(M)$. We assume that M is an oriented rational homology 3–sphere, and we restrict ourselves to the +–theory of Ozsváth and Szabó. The Heegaard–Floer homology $HF^+(M)$ is a $\mathbb{Z}[U]$–module with a $\mathbb{Q}$–grading compatible with the $\mathbb{Z}[U]$–action, where $\deg(U) = -2$. Additionally, $HF^+(M)$ has another $\mathbb{Z}_2$–grading; $HF^+(M)_{even}$, respectively $HF^+(M)_{odd}$ denote the graded parts. Moreover, $HF^+(M)$ has a natural direct sum decomposition of $\mathbb{Z}[U]$–modules (compatible with all the gradings): $HF^+(M) = \oplus_\sigma HF^+(M,\sigma)$ indexed by the $spin_c$–structures σ of M. For any σ one has

$$HF^+(M,\sigma) = \mathcal{T}^+_{d(M,\sigma)} \oplus HF^+_{red}(M,\sigma),$$

a graded $\mathbb{Z}[U]$–module isomorphism, where $\mathcal{T}^+_r$ denotes $\mathbb{Z}[U^{-1}]$ as a $\mathbb{Z}[U]$–module, in which the degree of 1 is r; and $HF^+_{red}(M,\sigma)$ has finite $\mathbb{Z}$–rank and an induced $\mathbb{Z}_2$–grading. One also considers

$$\chi(HF^+(M,\sigma)) := \operatorname{rank}_{\mathbb{Z}} HF^+_{red,even}(M,\sigma) - \operatorname{rank}_{\mathbb{Z}} HF^+_{red,odd}(M,\sigma).$$

Then via $\chi(HF^+(M,\sigma)) - d(M,\sigma)/2$ one gets the Seiberg–Witten invariant of (M,σ). By changing the orientation one has $\chi(HF^+(M,\sigma)) = -\chi(HF^+(-M,\sigma))$ and $d(M,\sigma) = -d(-M,\sigma)$.

4.2 Lattice cohomology of M. Now we review some facts from the lattice cohomology theory, introduced by the author in Némethi [2008a]. The construction captures the structure of lattice points inside of some real ellipsoids. $L\otimes\mathbb{R}$ has a natural cellular decomposition into cubes. The set of zero–dimensional cubes is provided by the lattice points L. Any $l \in L$ and subset $I \subset \mathcal{V}$ of cardinality q defines a q–dimensional cube (l, I), which has its vertices in the lattice points $(l + \sum_{i\in I'} E_j)_{I'}$, where I' runs over all subsets of I.

Next, we fix $k \in \mathrm{Char}$, we set $\chi_k(l) = -(l, l+k)/2$ and $m_k := \min_{l\in L} \chi_k(l)$. Finally, for any fixed integer $n \geq m_k$ we denote by S_n the union of all q–cubes in the real ellipsoid $\{l \in L \otimes \mathbb{R} \,:\, \chi_k(l) \leq n\}$. Then one defines

$$\mathbb{H}^p(\Gamma, k) := \oplus_{n\geq m_k} H^p(S_n, \mathbb{Z}).$$

For each fixed p, the module $\mathbb{H}^p$ is in a natural way $\mathbb{Z}$– (in fact, $2\mathbb{Z}$)–graded: $H^p(S_n, \mathbb{Z})$ consists of the $2n$–homogeneous elements. Also, it is a $\mathbb{Z}[U]$–module; the U–action is induced by the restriction $H^p(S_{n+1}, \mathbb{Z}) \to H^p(S_n, \mathbb{Z})$. Moreover, there is an augmentation decomposition

$$\mathbb{H}^0(\Gamma, k) = (\oplus_{n\geq m_k} \mathbb{Z}) \oplus (\oplus_{n\geq m_k} \widetilde{H}^0(S_n, \mathbb{Z})) = \mathcal{T}^+_{2m_k} \oplus \mathbb{H}^0_{red}(\Gamma, k).$$

This module $\mathbb{H}^*(\Gamma, k)$ is independent of the choice of the resolution (or plumbing) graph Γ (hence depends only on the 3-manifold M), and it depends only on the class $[k] = k + 2L$ (i.e. only on the corresponding $spin^c$–structure) up to a shift in grading. In order to fix one module in each class we take one $k' \in [k]$ with $m_{k'} = 0$, and we set $\mathbb{H}^*(\Gamma, [k]) := \mathbb{H}(\Gamma, k')$ (which is independent of the choice). For any rational number r we denoted by $\mathbb{H}^*(\Gamma, [k])[r]$ the module isomorphic to $\mathbb{H}^*(\Gamma, [k])$, but whose grading is shifted by r. (The $(d+r)$–homogeneous part of $\mathbb{H}^*(\Gamma, [k])[r]$ is isomorphic with the d–homogeneous part of $\mathbb{H}^*(\Gamma, [k])$.) It is also convenient to redefine $\mathbb{H}^p_{red} := \mathbb{H}^p$ for $p \geq 1$; this is motivated by the fact that $\mathbb{H}^*_{red} = \oplus_{p\geq 0}\mathbb{H}^p_{red}$ has finite $\mathbb{Z}$–rank.

5 The relations of the lattice cohomology with other invariants

5.1 Exact sequence. Let us fix a vertex v_0, we consider the graphs $\Gamma \setminus v_0$ and $\Gamma^+_{v_0}$, where the first one is obtained from Γ by deleting the vertex v_0 and adjacent edges, while the second one is obtained from Γ by increasing the decoration of the vertex v_0 by 1. We will assume that $\Gamma^+_{v_0}$ is still negative definite. Then there exists an exact sequence of $\mathbb{Z}[U]$–modules of the following type:

$$\cdots \longrightarrow \mathbb{H}^q(\Gamma^+_{v_0}) \longrightarrow \mathbb{H}^q(\Gamma) \longrightarrow \mathbb{H}^q(\Gamma \setminus v_0) \longrightarrow \mathbb{H}^{q+1}(\Gamma^+_{v_0}) \longrightarrow \cdots$$

The first 3 terms of the exact sequence (i.e. the $\mathbb{H}^0$–part) appeared in Ozsváth and Szabó [2003b] and Némethi [2005], the exact sequence over $\mathbb{Z}_2$–coefficients was proved in Greene [2013], the general case in Némethi [2011]. For more properties see Némethi [ibid.].

5.2 Rational singularities. By definition, (X, o) is rational if $p_g(X, o) = 0$. This is an analytic property, however, Artin replaced the vanishing of p_g by a topological criterion formulated in terms of χ: (X, o) is rational if and only if $\chi(l) > 0$ for any $l \in L_{>0}$ (independently of the resolution) Artin [1962, 1966]. For a different, the so–called 'Laufer's rationality criterion', see Laufer [1972]. Any connected negative definite graph with this property is called rational graph. The set of rational graphs include e.g. all the graphs with the following property: if e_v and δ_v denote the Euler decoration and the valency of a vertex, then one requires $-e_v \geq \delta_v$ for any $v \in \mathcal{V}$. Hence, if we decrease sufficiently the Euler decorations of any graph we get a rational graph. The class of rational graphs is closed while taking subgraphs and decreasing the Euler numbers.

Rationality in terms of the lattice cohomology is characterized as follows:

Theorem 5.2.1. *Némethi [2005] The following facts are equivalent:*

(a) Γ *is rational;*

(b) $\mathbb{H}^0(\Gamma, K) = \mathcal{T}_0^+$*;*

(c) $\mathbb{H}^0(\Gamma, K) = \mathcal{T}_m^+$ *for some* $m \in \mathbb{Z}$*; or equivalently,* $\mathbb{H}^0_{red}(\Gamma, K) = 0$*;*

(d) $\mathbb{H}^*_{red}(\Gamma, k) = 0$ *for all* $k \in \mathrm{Char}$.

Moreover, if Γ *is rational then* $\min \chi_{k_r} = 0$.

5.3 (Weakly) elliptic singularities. A normal surface singularity (X, o) is called *elliptic*, if (one of its) resolution graph is elliptic. A graph Γ is elliptic if $\min_{l>0} \chi(l) = 0$ (cf. Wagreich [1970] and Laufer [1977], see also Némethi [1999b]).

The set of elliptic singularities includes all the singularities with $p_g = 1$, and all the Gorenstein singularities with $p_g = 2$. But an elliptic singularity might have arbitrary large p_g.

Ellipticity in terms of the lattice cohomology is characterized as follows:

Theorem 5.3.1. *The following facts are equivalent:*

(a) Γ *is elliptic;*

(b) $\mathbb{H}^0(\Gamma, K) = \mathcal{T}_0^+ \oplus \big(\mathcal{T}_0(1)\big)^{\oplus \ell}$ *for some* $\ell \geq 1$*. Here* $\mathcal{T}_0(1)$ *is a free* $\mathbb{Z}$*–module of rank one with trivial* U*–action and concentrated at degree 0. The integer* ℓ *above can be identified with the length of the elliptic sequence. In particular, if the graph* Γ *is minimally elliptic then* $\ell = 1$.

5.4 'Bad' vertices, AR graphs. We fix an integer $n \geq 0$. We say that a negative definite graph has at most n 'bad' vertices if we can find n vertices $\{v_k\}_{1 \leq k \leq n}$, such that by decreasing their Euler decorations we get a rational graph (this fact makes sense because of Section 5.2). In general, the choice of the bad vertices is not unique. A graph with at most one bad vertex is called *almost rational*, or AR, cf. Némethi [2005, 2008a]. Here are some AR graphs:

1) All rational and elliptic graphs are AR.

2) Any star-shaped graph is AR (modify the central vertex).

3) The rational surgery 3–manifolds $S^3_{-p/q}(\mathcal{K})$ ($\mathcal{K}$ algebraic knot, $p/q > 0$) are AR.

4) The class of AR graphs is closed while taking subgraphs and decreasing the Euler numbers.

Theorem 5.4.1. *Némethi [2011] If Γ has at most n bad vertices then $\mathbb{H}^q_{red}(\Gamma) = 0$ for $q \geq n$. In particular, if Γ is AR then $\mathbb{H}^0(\Gamma)$ is the only nonzero module.*

If Γ has at most $n \geq 2$ bad vertices $\{v_k\}_{1\leq k\leq n}$ such that $\Gamma \setminus v_1$ has at most $(n-2)$ bad vertices then $\mathbb{H}^q_{red}(\Gamma) = 0$ for $q \geq n-1$.

See Némethi [2005, (8.2)(5.b)] for a graph Γ with 2 bad vertices $\{v_1, v_2\}$ such that $\Gamma \setminus v_1$ has only rational components.

5.5 Relation with Heegaard–Floer theory.

In Némethi [2008a] the author formulated the following

Conjecture 5.5.1. *For any plumbed rational homology sphere associated with a connected negative definite graph Γ, and for any $k \in$ Char, one has*

$$d(M,[k]) = \max_{k'\in[k]} \frac{(k')^2 + |\mathcal{V}|}{4} = \frac{k^2 + |\mathcal{V}|}{4} - 2\cdot \min \chi_k.$$

Furthermore,

$$HF^+_{red,even}(-M,[k]) = \bigoplus_{p\ even} \mathbb{H}^p_{red}(\Gamma,[k])[-d], \ and$$
$$HF^+_{red,odd}(-M,[k]) = \bigoplus_{p\ odd} \mathbb{H}^p_{red}(\Gamma,[k])[-d].$$

Both parts of the Conjecture were verified for almost rational graphs in Némethi [ibid.], for two bad vertices in Ozsváth, Stipsicz, and Szabó [2014], see Némethi [2008a, p. 8.4] too. Otherwise, the Conjecture is still open.

Note that (conjecturally) $\mathbb{H}^*$ has a richer structure: its p–filtration $\mathbb{H}^* = \oplus_p \mathbb{H}^p$ collapses at the level of HF^+ to a $\mathbb{Z}_2$ odd/even filtration.

5.6 Relation with Seiberg–Witten invariant.

For any $k \in$ Char the (normalized) Euler characteristic of the lattice cohomology is defined as

$$\mathrm{eu}(\mathbb{H}^*(\Gamma,k)) := -\min \chi_k + \sum_p (-1)^p \operatorname{rank} \mathbb{H}^p_{red}(\Gamma,k).$$

Then, it turns out that the (normalized) Euler characteristic of the lattice cohomology equals the (normalized) Seiberg–Witten invariant Némethi [2011] (this fact supports Conjecture 5.5.1 too):

$$-\mathfrak{sw}_{[k]}(M) - \frac{k^2 + |\mathcal{V}|}{8} = -\min \chi_k + \sum_p (-1)^p \operatorname{rank} \mathbb{H}^p_{red}(\Gamma, k).$$

5.7 Relation with L–spaces. By Section 5.2 Γ is rational if and only if $\mathbb{H}^*_{red}(\Gamma) = 0$. On the other hand, following Ozsváth and Szabó, M is an L–space by definition if and only if $HF^+_{red} = 0$. Their equivalence is predicted by Conjecture 5.5.1; in fact this 'tip of the iceberg' statement was proved in Némethi [2017]:

Theorem 5.7.1. *The following facts are equivalent:*
(i) (X, o) is a rational singularity (or, Γ is a rational graph),
(ii) the link M is an L–space.

$(i) \Rightarrow (ii)$ follows from lattice cohomology theory Némethi [2005, 2008a], while $(ii) \Rightarrow (i)$ uses partly the following equivalence $(ii) \Leftrightarrow (iii)$, where (iii) means that $\pi_1(M)$ is not a left–orderable group. The equivalence $(ii) \Leftrightarrow (iii)$ was proved in Hanselman, J. Rasmussen, S. D. Rasmussen, and Watson [n.d.] for any graph–manifolds. For arbitrary 3–manifolds was conjectured by Boyer, Gordon, and Watson [2013], for different developments and other references see Hanselman, J. Rasmussen, S. D. Rasmussen, and Watson [n.d.] and Némethi [2017].

Problem 5.7.2. Characterize elliptic singularities by a certain property of $\pi_1(M)$.

5.8 Reductions. In the lattice cohomology computations it is convenient to take a special representative k for the $spin^c$–structure $[k]$. Indeed, if $s_h \in L'$ is the minimal representative of $h \in H$ in $\mathcal{S}'$, and we take $k_r := K + 2s_h$ as representative for $[k]$, and we define the weighted cubes with the weight function χ_{k_r}, then one has the following 'homotopical identity': $H^p(S_n, \mathbb{Z}) = H^p(S_n \cap L_{\geq 0}, \mathbb{Z})$, where $S_n \cap L_{\geq 0}$ denotes the subset of S_n consisting of cubes with all vertices in $L_{\geq 0}$. Hence, with the natural notations (after notational modification $\mathbb{H}^*(L, k) = \mathbb{H}^*(\Gamma, k)$) we have

Theorem 5.8.1. *László and Némethi [2015]* $\mathbb{H}^*(L, \chi_{k_r}) = \mathbb{H}^*(L_{\geq 0}, \chi_{k_r})$.

This can be reduced even further. Fix k_r as above, that is $k_r = K + 2s_h$ for some h, and rewrite s_h as $s_{[k]}$. Assume that $\overline{\mathcal{V}} \subset \mathcal{V}$ is a set of bad vertices. Set also $\mathcal{V}^* := \mathcal{V} \setminus \overline{\mathcal{V}}$. Let $\overline{L}$ be the free $\mathbb{Z}$–submodule of L spanned by the (base elements of) $\overline{\mathcal{V}}$, and let $\overline{L}_{\geq 0}$ be its first quadrant. Next we introduce a special weight function for the points of $\overline{L}_{\geq 0}$ (which, in general, is not a Riemann–Roch type formula, it is not even quadratic).

For any $\mathbf{i} = (i_{v_1}, \dots) \in \overline{L}_{\geq 0}$ we define the element $x(\mathbf{i}) \in L$ by the following universal property:

(i) the coefficient of E_v in $x(\mathbf{i})$ is i_v for any $v \in \overline{\mathcal{V}}$,
(ii) $(x(\mathbf{i}) + s_{[k]}, E_v) \leq$ for every $v \in \mathcal{V}^*$,
(iii) $x(\mathbf{i})$ is minimal with the properties (i)-(ii).

The definition is motivated by the theory of generalized computation sequences used in singularity theory Laufer [1972, 1977] and Némethi [1999a,b].

Set $w_{k_r}(\mathbf{i}) := \chi_{k_r}(x(\mathbf{i}))$, the weight function of $\overline{L}_{\geq 0}$. Using this weight function, one defines the weight of any cube of $\overline{L}_{\geq 0}$ as the maximum of the weights of the vertices of the cube, and one also repeats the definition of S_n and of the lattice cohomology similarly as above.

Theorem 5.8.2. Reduction Theorem. *László and Némethi [2015]* $\mathbb{H}^*(L_{\geq 0}, \chi_{k_r}) = \mathbb{H}^*(\overline{L}_{\geq 0}, w_{k_r})$.

In particular, if Γ is AR (and $\mathbb{H}^p = 0$ for any $p > 0$ by Theorem 5.4.1) $\mathbb{H}^0(\Gamma, [k])$ for any $spin^c$–structure $[k]$ is determined by the sequence of integers $\{\tau(i)\}_{i \in \mathbb{Z}_{\geq 0}}$, $\tau(i) := \chi_{k_r}(x(i))$, where i is the coordinate of the bad vertex v. This was intensively used in concrete computations. The star–shaped graphs are AR, their τ–functions are described in terms of Seifert invariants in Némethi [2005].

If $\mathcal{K} \subset S^3$ is an algebraic knot (that is, one of its representative can be cut out by an isolated irreducible plane curve singularity germ $f_{\mathcal{K}}$), then the surgery 3–manifold $S^3_{-p/q}(\mathcal{K})$ for $p/q \in \mathbb{Q}_{>0}$ is a plumbed 3–manifold associated with a negative definite AR graph. For their τ–function see Némethi [2007]. This can be determined in terms of the semigroup of $f_{\mathcal{K}}$.

More generally, if $\{\mathcal{K}_i\}_{i=1}^N$ are algebraic knots then the graph of $S^3_{-p/q}(\#_i \mathcal{K})$ and the reduced weights together with the lattice cohomology in terms of the semigroups are determined in Némethi and Román [2012].

Theorem 3.2.1 has its 'reduction' as well. Indeed, one defines the reduction $Z^{\overline{\mathcal{V}}}$ of the series $Z(\mathbf{t})$ to the variables $\{t_v\}_{v \in \overline{\mathcal{V}}}$ (similarly as in Sections 2.5 and 3.3, by taking $t_v = 1$ for any $v \notin \overline{\mathcal{V}}$), and László and Némethi [2015] proves for it the analogue of Theorem 3.2.1. In particular, the Seiberg–Witten invariants can also be recovered as the periodic constants of the reduces series.

5.9 Ehrhart theory. In László and Némethi [2014] the Seiberg–Witten invariant (of a negative definite plumbed 3–manifold) is identified with the third coefficient of certain equivariant Ehrhart polynomial.

5.10 $Z(\mathbf{t})$ and $\mathfrak{eu}$ in terms of weighted cubes. Above in the definition of the lattice cohomology we used for each $spin^c$–structure a different weighted lattice. This can be unified in a common weighted lattice Némethi [2011] and Ozsváth, Stipsicz, and Szabó [2014]. Here we follow Némethi [2011]. The set of p–cubes consists of pairs (k, I), where $k \in \mathrm{Char}$ and $I \subset \mathcal{V}$, $|I| = p$. This pair can be identified with a cube in $L \otimes \mathbb{R}$ with vertices $\{k + 2E_{I'})_{I' \subset I}$. One defines the weight function induced by the intersection form $w : \mathrm{Char} \to \mathbb{Q}$ by $w(k) := -(k^2 + |\mathcal{V}|)/8$, which extends to a weight–function of the q–cubes via $w((k, I)) = \max_{I' \subset I}\{w(k + 2E_{I'})\}$. (The correspondence between the two languages, for any fixed $[k]$, is realized by $k = K + 2s_{[k]} + 2l \leftrightarrow l$ and a universal constant shift in the weights.) The point is that in this language of weighted cubes the topological series $Z(\mathbf{t})$ can also be expressed Némethi [ibid.] as:

$$Z(\mathbf{t}) = \sum_{k \in \mathrm{Char}} \sum_{I \subset \mathcal{V}} (-1)^{|I|+1}\, w((k, I)) \cdot \mathbf{t}^{\frac{1}{2}(k-K)}. \tag{5.10.1}$$

Note that $\mathrm{Char} = K + 2L' \subset L'$, hence $(k - K)/2$ runs over L' when k runs over Char.

The normalized Seiberg–Witten invariant, or the normalized Euler characteristic of the lattice cohomology can also be determined directly by a weighted cube counting (this is really the analogue how Euler defined the classical Euler characteristic via alternating sum of p–cells/cubes). Since we have infinitely many cubes in $L \otimes \mathbb{R}$, we need to consider a ‘truncation’.

Let us fix the class $[k]$. For any two $k_1, k_2 \in [k] \subset \mathrm{Char}$ with $k_1 \leq k_2$ let $R(k_1, k_2)$ denote the rectangle $\{x \in L \otimes \mathbb{R} \, : \, k_1 \leq x \leq k_2\}$. Then we can consider the set of cubes (k, I) with $k \in [k]$ and situated with all vertices in $R(k_1, k_2)$ and the corresponding lattice cohomology $\mathbb{H}^*(R(k_1, k_2), w)$.

Theorem 5.10.2. *For ‘good’ choices of k_1, k_2 (with $k_1 \ll 0$ and $k_2 \gg 0$), and with the abridgment $R = R(k_1, k_2)$, one has the isomorphism of $\mathbb{Z}[U]$–modules $\mathbb{H}^*(\Gamma, k_r) = \mathbb{H}^*(R, w)$. Furthermore,*

$$\begin{aligned}\mathfrak{eu}(\mathbb{H}^*(\Gamma, k_r)) &= -\min(w|R) + \sum_p (-1)^p \operatorname{rank}_{\mathbb{Z}} \mathbb{H}^p_{red}(R, w)\\ &= \sum_{(k,I) \subset R} (-1)^{|I|+1} w((k, I)).\end{aligned}$$

Note that $Z(\mathbf{t})$ determines the intersection form $(\, , \,)$ of the lattice L, hence the lattice cohomology as well (note also the direct ‘periodic constant formula’ for the Seiberg–Witten invariant).

Problem 5.10.3. Find a universal construction, which assigns to any series $S(\mathbf{t})$ a graded $\mathbb{Z}[U]$–module, such that it applied to $Z(\mathbf{t})$ we recover $\mathbb{H}^*$.

5.11 The fundamental group of M. In fact, the same question can be asked for the fundamental group of M. Note that $\pi_1(M)$ determines M in a unique way (except for lens spaces, a case which can be disregarded, since they have trivial $\mathbb{H}^*_{red}$).

Problem 5.11.1. Find a universal construction, which assigns to any group a graded $\mathbb{Z}[U]$–module, such that it applied to $\pi_1(M)$ we recover $\mathbb{H}^*$.

5.12 Another surgery formula for $\mathbb{H}^*$. Recall the surgery formulae 3.3.1 valid for the Seiberg–Witten invariant involving the periodic constant as a 'correction term'.

Problem 5.12.1. Find its analogue at the level of $\mathbb{H}^*$. (This is related with Problem 5.10.3 too).

5.13 Path lattice cohomology. Consider the situation of Section 4.2 with fixed Γ, and $k = K$ and weight function $\chi = \chi_K$. Furthermore, consider a sequence $\gamma := \{x_i\}_{i=0}^t$ so that $x_0 = 0$, $x_{i+1} = x_i + E_{v(i)}$ for certain $v(i) \in \mathcal{V}$ for each $0 \leq i < t$, and $x(t) \in -K + \mathcal{S}'$. Then for the union of 0–cubes marked by the points $\{x_i\}_i$ and segments (1–cubes) of type $[x_i, x_{i+1}]$ we can repeat the definition of the lattice cohomology (associated with χ_k), and we get a graded $\mathbb{Z}[U]$–module $\mathbb{H}^*(\gamma, K)$. In fact, only $\mathbb{H}^0$ is nonzero, and $\mathbb{H}^0(\gamma, K) = \mathcal{T}^+_{2\min_i\{\chi(x_i)\}} \oplus \mathbb{H}^0_{red}(\gamma, K)$. This is called the path–cohomology associated with the 'path' γ and χ. Similarly as in Section 5.6, we consider its normalized Euler characteristic

$$eu(\mathbb{H}^0(\gamma, K)) := -\min_i\{\chi(x_i)\} + \operatorname{rank}_{\mathbb{Z}} \mathbb{H}^0_{red}(\gamma, K).$$

One shows that

$$eu(\mathbb{H}^0(\gamma, K)) = \sum_{i=0}^{t-1} \max\{\chi(x_i), \chi(x_{i+1})\} - \chi(x_{i+1}) = \sum_{i=0}^{t-1} \max\{(x_i, E_{v(i)}) - 1, 0\}.$$

It is convenient to introdce the following notation as well (cf. Section 5.6):

$$eu(\mathbb{H}^0(\Gamma, K)) := -\min\chi + \operatorname{rank} \mathbb{H}^0_{red}(\Gamma, K).$$

It turns out (cf. Némethi and Sigurdsson [2016]) that

$$\min_\gamma\, eu(\mathbb{H}^0(\gamma, K)) \leq eu(\mathbb{H}^0(\Gamma, K)).$$

5.14 Graded roots. For each Γ and $k \in \mathrm{Char}$, the author in Némethi [2005] constructed a *graded root*, from which one recovers by a natural procedure $\mathbb{H}^0(\Gamma, k)$. This is tree whose vertices are $\mathbb{Z}$–graded, the weight n–vertices correspond to the connected components of S_n, and the edges of the tree to the possible inclusions of the components of S_n to the components of S_{n+1} (this at the $\mathbb{H}^0(\Gamma, k)$ level is codified in the U–action). For AR graphs it contains all the lattice cohomology (hence Heegaard–Floer) information, but it codifies completely the inclusions of the S_n–components (a fact, which from the U–action cannot be recovered).

There is a parallel characterization of rational and elliptic graphs in terms of graded roots too.

5.15 Classification of singularities. The 'Artin-Laufer program' starts the classification of singularities with the class of rational and elliptic singularities — it identifies topologically the families, and then (under certain analytic conditions in the elliptic case, e.g. Gorenstein property) determines certain analytic invariants (multiplicity, Hilbert function) by uniform formulae for each family.

In order to continue this program, we first have to identify subfamilies for which one can show that they share 'common properties'. We propose to identify these subfamilies by graded roots, or by the (slightly weaker) $\mathbb{Z}[U]$–module $\mathbb{H}^0(\Gamma, K)$ (or, we can even consider the whole $\mathbb{H}^*(\Gamma, K)$) Némethi [2007].

Though we know all the possible topological types (namely, the singularity resolution graphs are exactly the connected negative definite graphs), it is not clear at all what graded tree might appear as a graded root associated with a resolution graph.

Problem 5.15.1. Classify all the possible graded roots associated with all (Γ, K) of normal surface singularities. Is there any hidden structure property carried by the $\mathbb{Z}[U]$–modules $\mathbb{H}^*(\Gamma, k)$ associated with topological type of singularities? It is possible to describe all the possible modules produced in this way?

6 Analytic – topological connections. The Seiberg–Witten Invariant Conjecture

In this section we treat a set of possible properties connecting the analytic invariants with the topological ones, namely, the equivariant geometric genera with the Seiberg–Witten invariants of the link. When they are valid they provide a topological description of the equivariant geometric genera. The identities are generalizations of the statement of the Casson Invariant Conjecture of Neumann and Wahl to the case of normal surface singularities with rational homology sphere links.

Conjecture 6.0.1. Seiberg–Witten Invariant Conjecture/Coincidence.

We say that (X,o) satisfy the equivariant SWIC if for any $h\in H$ the following identity holds

$$h^1(\widetilde{X},\mathcal{O}(-r_h)) = -\mathfrak{sw}_{-h*\sigma_{can}}(M) - \frac{(K+2r_h)^2+|\mathcal{V}|}{8}. \tag{6.0.2}$$

Its validity automatically extends to arbitrary natural line bundles as follows:

$$h^1(\widetilde{X},\mathcal{O}(l')) = -\mathfrak{sw}_{[l']*\sigma_{can}}(M) - \frac{(K-2l')^2+|\mathcal{V}|}{8}. \tag{6.0.3}$$

We say that (X,o) satisfies the SWIC if the above identity holds for $h=0$, that is, if

$$p_g(X,o) = -\mathfrak{sw}_{\sigma_{can}}(M) - \frac{K^2+|\mathcal{V}|}{8}. \tag{6.0.4}$$

The identity SWIC was formulated as a conjecture in Némethi and Nicolaescu [2002], while the equivariant case in Némethi [2007]: the expectation was that it holds for any $\mathbb{Q}$–Gorenstein singularity. Now we know that this is not true for this large class of singularities (see Luengo-Velasco, Melle-Hernández, and Némethi [2005]), although it is valid for large number of smaller families of singularities. But even in the case of those families when it fails, it still indicates interesting 'virtual' properties. The limits of the validity of the properties are not clarified at this moment. Having in mind the existence of cases when the identity does not hold, one might say that it is not totally justified the name SWI 'Conjecture', although this was its name in the literature in the last ten years. Hence, the reader might read the abbreviation SWIC as SWI 'Coincidence' too.

Example 6.0.5. CIC of Neumann and Wahl [1990] Assume that (X,o) is Gorenstein and it admits a smoothing with smooth nearby Milnor fiber F. Then the signature of F satisfies $\sigma(F)+8p_g+K^2+|\mathcal{V}|=0$, hence the SWIC for $h=0$ reads as $\sigma(F)/8=\mathfrak{sw}_{\sigma_{can}}(M)$. (In this case, usually, $\sigma(F)/8$ is not an integer.) Additionally, if (X,o) is a complete intersection with integral homology sphere link, then $\mathfrak{sw}_{\sigma_{can}}(M)$ equals the Casson invariant $\lambda(M)$ of M, hence the above identity reads as $\sigma(F)/8=\lambda(M)$. This is the Casson Invariant Conjecture of Neumann and Wahl, predicted for any complete intersection with integral homology sphere link Neumann and Wahl [ibid.].

The CIC was solved for weighted homogeneous singularities and hypersurface suspension singularities ($(X,o)=\{f(x,y)+z^N=0\}$) in Neumann and Wahl [ibid.], for splice singularities (which includes the weighted homogeneous case as well) in Némethi and Okuma [2009]. The general case is still open.

6.1 Regarding the validity of SWIC.

We have the following statement:

Theorem 6.1.1. *The equivariant SWIC was verified in the following cases: rational singularities Némethi and Nicolaescu [2002], weighted homogeneous singularities Némethi and Nicolaescu [2002, 2004], splice quotient singularities Némethi and Okuma [2008].*

Additionally, the SWIC (for $h = 0$) was verified for suspensions $\{f(x, y) + z^N = 0\}$ with f irreducible Némethi and Nicolaescu [2005], for hypersurface Newton non–degenerate singularities Sigurdsson [2016] and superisolated singularities with one cusp Fernández de Bobadilla, Luengo-Velasco, Melle-Hernández, and Némethi [2006, 2007] and Borodzik and Livingston [2014]. Since the identity of the SWIC is stable with respect to equisingular deformations, the SWIC remains valid for such deformations of any of the above cases.

Recall that (X, o) is a hypersurface superisolated singularity if

$$(X, o) = \{f(x_1, x_2, x_3) = 0\}$$

where f is a hypersurface singularity with isolated singularity and the homogeneous terms $f_d + f_{d+1} + \cdots$ of f satisfy the following properties: $C := \{f_d = 0\}$ is reduced and it defines in $\mathbb{CP}^2$ an irreducible rational cuspidal curve C; furthermore, the intersection $\{f_{d+1} = 0\} \cap \mathrm{Sing}\{f_d = 0\}$ in $\mathbb{CP}^2$ is empty. The restriction regarding f_d implies that the link of (X, o) is a rational homology sphere. One shows that the minimal good graph of (X, o) has ν bad vertices, where ν is the number of cusps of C.

In all cases $p_g = d(d-1)(d-2)/6$, hence it depends only on d, however the Seiberg–Witten invariant (and the plumbing graph too) depend essentially on the type of cusps of C (see Section 6.4 below).

For superisolated singularities in certain cases when C is not unicuspidal the SWIC ($h = 0$ case) is not true Luengo-Velasco, Melle-Hernández, and Némethi [2005].

6.2 The path lattice cohomology bound of p_g. The failure of the SWIC in the case of superisolated singularities motivated a parallel deeper study of these germs. Surprisingly, in this case a rather natural universal bound of p_g will become equality.

Consider the set of paths $\mathfrak{P}$, $\gamma := \{x_i\}_{i=0}^t$ so that $x_0 = 0$, $x_{i+1} = x_i + E_{v(i)}$ for certain $v(i) \in \mathcal{V}$ for each $0 \leq i < t$, and $x(t) \in -K + \mathcal{S}'$. (If Γ is numerically Gorenstein we can even take $x(t) = -K$.) Then, considering the cohomology exact sequences $0 \to \mathcal{O}_{E_{v(i)}}(-x_i) \to \mathcal{O}_{x_{i+1}} \to \mathcal{O}_{x_i} \to 0$, we obtain that $p_g \leq \mathrm{eu}(\mathbb{H}^0(\gamma, K))$. Therefore,

$$p_g \leq \min_{\gamma \in \mathfrak{P}} \mathrm{eu}(\mathbb{H}^0(\gamma, K)). \tag{6.2.1}$$

This inequality is in the spirit of computational sequences initiated by Laufer and used intensively in Laufer [1972, 1977], Némethi [1999a], and László and Némethi [2015]. From this point of view, this relation, in some sense, is even more natural then that required by the SWIC. Note that in Equation (6.2.1) equality holds if and only if all the cohomology exact sequence (along a path which minimalizes the right hand side) split. The point is that this happens for certain key families.

Theorem 6.2.2 (Némethi and Sigurdsson [2016]). *The identity*

$$p_g = \min_{\gamma \in \mathfrak{P}} \mathrm{eu}(\mathbb{H}^0(\gamma), K)$$

holds in the following cases:

(a) if $p_g = \mathrm{eu}(\mathbb{H}^0(\Gamma, K))$ *(this happens e.g. if* $\mathbb{H}^q(\Gamma, K) = 0$ *for* $q \geq 1$ *and* (X, o) *satisfies the SWIC). In particular,* $p_g = \min_{\gamma \in \mathfrak{P}} \mathrm{eu}(\mathbb{H}^0(\gamma), K)$ *holds for all weighted homogeneous and minimally elliptic singularities.*

(b) rational and Gorenstein elliptic singularities;

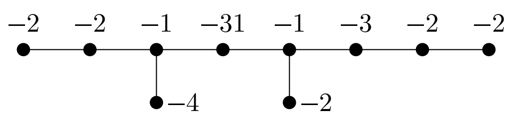
(c) for superisolated singularities (with arbitrary number of cusps);

(d) for hypersurface singularities with non-degenerate Newton principal part.

Again, since the identity is stable with respect to equisingular deformations, it remains valid for such deformations of any of the above cases.

The next example shows that in the case of superisolated singularities the non-vanishing of $\mathbb{H}^q(M)$ $(1 \leq q < \nu)$ obstructs the validity of the SWIC.

Example 6.2.3. Némethi and Sigurdsson [ibid.] Assume that (X, o) is a superisolated singularity with C of degree $d = 5$ and two cusps, both with one Puiseux pair: $(3, 4)$ and $(2, 7)$ respectively. The graph Γ is



We set $k = K$. One shows that $\min \chi = -5$, $\mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^0_{red}) = 5$, $\mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^1) = 2$. Hence $\mathrm{eu}(\mathbb{H}^*) = 8$. Since for the superisolated germ with $d = 5$ one has $p_g = 10$, using Section 5.6 we get that the SWIC is not valid. On the other hand, $\min_\gamma \mathrm{eu}(\mathbb{H}^0(\gamma, K)) = 10$ as well, hence $p_g = \min_\gamma \mathrm{eu}(\mathbb{H}^0(\gamma, K))$, as it is predicted by the above theorem.

If we take any other analytic structure supported by Γ, by Equation (6.2.1) $p_g \leq 10$ still holds.

This graph (that is, topological type) supports another natural analytic structure as well, namely a splice quotient analytic type (compare with Section 7.2): it is the $\mathbb{Z}_5$–factor of the complete intersection $\{z_1^3 + z_2^4 + z_3^5 z_4 = z_3^7 + z_4^2 + z_1^4 z_2 = 0\} \subset (\mathbb{C}^4, 0)$ by the diagonal action $(\alpha^2, \alpha^4, \alpha, \alpha)$ $(\alpha^5 = 1)$. By Theorem 6.1.1 it satisfies the SWIC, hence $p_g = 8$.

In particular, in their choices of the topological characterization of their geometric genus, some analytic structures prefer $eu(\mathbb{H}^*(\Gamma, K))$, some of them the extremal

$$\min_{\gamma} eu(\mathbb{H}^0(\gamma, K))$$

(and there might exists even other choices, as parts/versions of the lattice cohomology). From this point of view the abridgement SWIC might also mean 'SWI Choice'.

6.3 What is the optimal topological lower/upper bounds of p_g? We are guided by the following key question: If one fixes a topological type (say, a minimal good resolution graph) and varies the possible analytic structures supported on this fixed topological type, then what are the possible values of the geometric genus p_g? A more concrete version is formulated as follows:

Problem 6.3.1. Associate combinatorially a concrete integer $\mathrm{MAX}(\Gamma)$ to any resolution graph Γ, such that for any analytic type supported by Γ one has $p_g \leq \mathrm{MAX}(\Gamma)$, and furthermore, for certain analytic structure one has equality.

Obviously, one can ask for the symmetric $\mathrm{MIN}(\Gamma)$ as well. But for the optimal lower bound we know the answer. A possible candidate is the 'artihmetical genus' $p_a(X, o) = 1 - \min \chi$ Wagreich [1970]. Indeed, for any analytic structure, whenever $p_g > 0$, one also has $1 - \min\chi \leq p_g$ Wagreich [ibid., p. 425].

The inequality looks not very sharp, however we have the following statement:

Theorem 6.3.2. *For any non–rational topological type $p_g = 1 - \min\chi$ for the generic analytic structure. (In particular, for non–rational graphs $\mathrm{MIN}(\Gamma)$ is exactly $1 - \min\chi$.)*

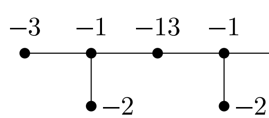
This was proved for elliptic singularities in Laufer [1977], the general case in Némethi and Nagy [n.d.].

A possible upper bound for p_g, hence a candidate for $\mathrm{MAX}(\Gamma)$, is $\min_{\gamma\in\mathfrak{P}} eu(\mathbb{H}^0(\gamma))$, cf. Equation (6.2.1).

However, for the next graph, $p_g \leq \min_{\gamma\in\mathfrak{P}} eu(\mathbb{H}^0(\gamma))$ is not sharp. Indeed

$$\min_{\gamma\in\mathfrak{P}} eu(\mathbb{H}^0(\gamma)) = 4$$

while for *any* analytic type $p_g \leq 3$ Némethi and Okuma [2017]. (For any Gorenstein structure one has $p_g = 3$, nevertheless, $p_g = 3$ can be realized by non–Gorenstein structure as well.)



6.4 Superisolated singularities revisited. Assume that the superisolated germ f is associated with the projective rational cuspidal curve C, as in Section 6.1. Here two main parts of the algebraic/analytic geometry meat: the classification of projective plane curves with the theory of local singularities. One of the aims of the classification of projective plane curves is to list the set of local topological types of plane curve singularities which can be realized as singularities of a degree d projective cuspidal curve. The strategy is to impose restrictions, obstructions for such realizations from the theory of singularities applied for the corresponding superisolated singularities. E.g., if $\{\mathcal{K}_i \subset S^3\}_{i=1}^{\nu}$ are the local algebraic knots of the plane curve singularities $\{(C, p_i)\}_{i=1}^{\nu}$ of C, and $\deg(C) = d$, then the link of the superisolated germ (X, o) is $M = S^2_{-d}(\#_i \mathcal{K}_i)$. Furthermore, if μ_i is the Milnor number of $(\mathcal{K}_i \subset S^3)$, then by the genus formula and the rationality of C implies $\sum_i \mu_i = (d-1)(d-2)$, hence d is uniquely determined by the local knot–types. The question whether we can impose any other restriction on M from the existence of C.

In the next presentation we follow Bodnár and Némethi [2016] and Fernández de Bobadilla, Luengo-Velasco, Melle-Hernández, and Némethi [2006, 2007]. Let us introduce the following notations. For each (C, p_i) (or $\mathcal{K}_i \subset S^3$) let Δ_i denote the Alexander polynomial (normalized as $\Delta_i(1) = 1$) and $\Gamma_i \subset \mathbb{Z}_{\geq 0}$ the semigroup of (C, p_i). Recall that $\Delta_i(t) = (1-t)\cdot\sum_{k\in\Gamma_i} t^k$ Guseĭn-Zade, Delgado, and Kampil'o [1999], and $\#\{\mathbb{Z}_{\geq 0}\setminus\Gamma_i\} = \delta_i = \mu_i/2$ (the so-called delta–invariant, or genus of (C, p_i)), and write $\delta := \sum_i \delta_i = (d-1)(d-2)/2$.

Furthermore, consider the product of Alexander polynomials:

$$\Delta(t) := \Delta_1(t)\Delta_2(t)\cdots\Delta_\nu(t).$$

There is a unique polynomial Q for which $\Delta(t) = 1 + \delta(t-1) + (t-1)^2 Q(t)$. Write $Q(t) = \sum_{j=0}^{2\delta-2} q_j t^j$. For $\nu = 1$ one shows

$$Q(t) = \sum_{s\notin\Gamma_1} (1 + t + \cdots + t^{s-1}), \text{ hence } q_j = \#\{s \notin \Gamma_1 \,:\, s > j\} \quad (\text{if } \nu = 1). \tag{6.4.1}$$

Next, set the rational function

$$R(t) := \frac{1}{d}\sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} - \frac{1-t^d}{(1-t)^3}. \tag{6.4.2}$$

In Fernández de Bobadilla, Luengo-Velasco, Melle-Hernández, and Némethi [2006, (2.4)] is proved that $R(t)$ is a symmetric polynomial ($R(t) = t^{d-3}R(1/t)$), and

$$R(t) = \sum_{j=0}^{d-3} \Big(q_{(d-3-j)d} - \frac{(j+1)(j+2)}{2}\Big)\, t^{d-3-j}. \tag{6.4.3}$$

In fact, if we reduce the variables of $Z(\mathbf{t})$ and $\mathcal{P}(\mathbf{t})$ to the variable t corresponding to $\mathcal{U} := \{C\} \subset \mathcal{V}$ then $R(t) = Z^{\mathcal{U}}_{h=0}(t) - \mathcal{P}^{\mathcal{U}}_{h=0}(t)$, hence, by surgery formulae 2.6.1 and 3.3.1 we get that

$$(6.4.4) \qquad R(1) = -\mathfrak{sw}_{\sigma_{can}}(M) - (K^2 + |\mathcal{V}|)/8 - p_g(X, o).$$

Hence the vanishing of $R(1)$ is equivalent with the SWIC for (X, o). Moreover, in Fernández de Bobadilla, Luengo-Velasco, Melle-Hernández, and Némethi [ibid.] is also proved that for $\nu = 1$ all the coefficients of $R(t)$ are nonnegative, hence $(R(1) \leq 0) \Leftrightarrow (R(1) = 0) \Leftrightarrow$ all coefficients of $R(t)$ are zero. On the other hand, by Theorem 6.1.1, for $\nu = 1$ one has $R(1) = 0$ indeed. This implies the vanishing of all coefficients of $R(t)$; hence Equations (6.4.1) and (6.4.3) combined provide the strong *semigroup distribution property* of Γ_1, which must be satisfied by any collection of local knot types and degree d whenever the data is realized by a curve C.

In Fernández de Bobadilla, Luengo-Velasco, Melle-Hernández, and Némethi [ibid.] we conjectured that for any rational cuspidal plane curve $C \subset \mathbb{C}P^2$ of degree d with arbitrary number of cusps all the coefficients of $R(t)$ are non–positive. In Bodnár and Némethi [2016] is proved that this is indeed true for any $\nu \leq 2$, but it fails, in general, for $\nu \geq 3$. The corrected conjecture, as formulated in Bodnár and Némethi [ibid.] is the following:

Conjecture 6.4.5. Bodnár and Némethi [ibid.] For any superisolated germ

$$R(1) \leq 0, \quad \text{that is,} \quad p_g \geq -\mathfrak{sw}_{\sigma_{can}}(M) - (K^2 + |\mathcal{V}|)/8.$$

In Bodnár and Némethi [ibid.] this conjecture is proved for $\nu \leq 2$ and verified for all 'known' rational cuspidal curves with $\nu \geq 3$. The next reformulation of this conjecture in terms of the lattice cohomology emphasize once again the differences and resemblances between $\mathfrak{eu}(\mathbb{H}^*(\Gamma, K))$ and $\mathfrak{eu}(\mathbb{H}^0(\Gamma, K))$.

Conjecture 6.4.6. *Bodnár and Némethi [ibid.] For the link $M = S^3_{-d}(\#_i \mathcal{K}_i)$ of a superisolated surface singularity corresponding to a rational cuspidal projective plane curve of degree d we have:*

$$\mathfrak{eu}(\mathbb{H}^*(M, K)) \leq \mathfrak{eu}(\mathbb{H}^0(M, K)).$$

In fact, $\mathfrak{eu}(\mathbb{H}^0(M, K)) = d(d-1)(d-2)/6$ by Borodzik and Livingston [2014] and Bodnár and Némethi [2016]. From this reformulation it is clear its validity for $\nu \leq 2$, however a conceptual argument for its validity for $\nu \geq 3$ is still missing.

6.5 Coverings. Let (X,o) be a normal surface singularity with rational homology sphere link, and let (Y,o) be its universal abelian covering, cf. Section 2.1. Then, from Equation (2.1.2) automatically one has the analytic identity

$$p_g(Y,o)=\sum_{h\in H} h^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-r_h)), \tag{6.5.1}$$

that is, the sum of the equivariant geometric genera of (X,o) equals the geometric genus of (Y,o).

Let M and N be the links of (X,o) and (Y,o) respectively. Then N is the regular universal abelian covering of M (associated with the representation $\pi_1(M)\to H$). Let $\Gamma(M)$ and $\Gamma(N)$ be the corresponding negative definite resolution/plumbing graphs as well. In the next discussion we also assume that N is a rational homology sphere too.

Having in mind the equivariant SWIC identities for (X,o) and the SWIC for (Y,o), following Bodnár and Némethi [2017] we say that M and its universal abelian covering N satisfy the '*covering additivity property*' (CAP) if

$$\mathfrak{sw}_{\sigma_{can}}(N)+\frac{K(\Gamma(N))^2+|\mathcal{V}(\Gamma(N))|}{8}=$$
$$=\sum_{h\in H}\mathfrak{sw}_{-h*\sigma_{can}}(M)+\frac{(K(\Gamma(M))+2r_h)^2+|\mathcal{V}(\Gamma(N))|}{8}.$$

Clearly, if (X,o) satisfies the equivariant SWIC and (Y,o) the SWIC, then (CAP) for M and its universal abelian covering N holds. However, since (CAP) is a totally topological identity, we might ask its validity for any singularity link M (whose universal abelian covering N is a rational homology sphere), independently of the existence of any nice analytic structure, or independently of singularity theory.

The point is that the property (CAP) in general is not true Bodnár and Némethi [ibid.]. But, what is really surprising is that (CAP) is true for the surgery 3–manifolds $M=S^3_{-p/q}(\#_i\mathcal{K}_i)$, even though (some of) these 3–manifolds appear as the links of superisolated singularities, and the superisolated singularities are the basic couterexamples for SWIC.

Theorem 6.5.2. *Bodnár and Némethi [ibid.] Let $M=S^3_{-p/q}(\mathcal{K})$ be a manifold obtained by a negative rational Dehn surgery of S^3 along a connected sum of algebraic knots $\mathcal{K}=\mathcal{K}_1\#\dots\#\mathcal{K}_\nu$ ($p,q>0$, $\gcd(p,q)=1$). Assume that N, the universal abelian covering of M, is a rational homology sphere. Then (CAP) holds.*

Both statements Equation (6.5.1) and Theorem 6.5.2 remain valid for any abelian covering.

Problem 6.5.3. Characterize those 3–manifolds M (or, singularity links) for which (CAP) holds.

6.6 Newton non–degenerate germs revisited. Note that for a hypersurface Newton non–degenerate isolated singularity (X, o), with rational homology sphere link, both identities are true: $p_g = \mathfrak{eu}(\mathbb{H}^*(\Gamma, K))$ by Theorem 6.1.1, and

$$p_g = \min_{\gamma \in \mathfrak{P}} \mathfrak{eu}(\mathbb{H}^0(\gamma, K))$$

by Theorem 6.2.2. In particular, the link of such a singularity satisfy a very strong topological restiction: $\mathfrak{eu}(\mathbb{H}^*(\Gamma, K)) = \min_{\gamma \in \mathfrak{P}} \mathfrak{eu}(\mathbb{H}^0(\gamma, K))$. Though we know that such 3–manifolds are rather special (for the algorithm which determines the plumbing graph from the Newton diagram see Oka [1987, 1997], or Braun and Némethi [2010]), still, this topological identity is a mystery for us. (Maybe it is worth to mention that in this case, p_g also equals the lattice points of $(\mathbb{Z}_{>0})^3$ below the Newton diagram.)

Finally, we end this section with the following

Problem 6.6.1. Find the Heegaard–Floer theoretical interpretation of

$$\min_{\gamma \in \mathfrak{P}} \mathfrak{eu}(\mathbb{H}^0(\gamma, K))$$

7 Analytic – topological connections. $\mathcal{P}(\mathbf{t})$ versus $Z(\mathbf{t})$

7.1 The $\mathcal{P}(\mathbf{t}) = Z(\mathbf{t})$ identity. Recall that by Section 2.4 the periodic constant of $\mathcal{P}_h$ is the equivarint geometric genus $h^1(\widetilde{X}, \mathcal{O}(-r_h))$, while by Theorem 3.2.1 the periodic constant of Z_h is the opposite of the r_h–normalized Seiberg–Witten invariant. Hence the equivariant SWIC says that the periodic constants of $\mathcal{P}_h$ and Z_h are equal. Hence, it is natural to ask for the validity of an even stronger identity, namely for $\mathcal{P}(\mathbf{t}) = Z(\mathbf{t})$. If this identity holds, then it provides a topological characterisation of the multivariable Hilbert function of the divisorial filtration. For some simple singularities, e.g. for cyclic quotient singularities one can compute directly both sides, and their equality follows visibly.

Theorem 7.1.1. *The equality $\mathcal{P}(\mathbf{t}) = Z(\mathbf{t})$ is true in the following cases: (a) rational singularities Campillo, Delgado, and Gusein-Zade [2004] and Guseĭn-Zade, Delgado, and Kampil'o [2008] (for a different proof see Némethi [2008b]); (b) minimally elliptic singularities Némethi [ibid.]; (c) splice quotient singularities Némethi [2012] (this includes e.g. the weighted homogeneous case as well).*

Even the 'reduced identity' $\mathcal{P}^{\mathcal{U}} = Z^{\mathcal{U}}$, for some subset $\mathcal{U} \subset \mathcal{V}$, can be interesting, even if $\mathcal{U}$ contains only one element v. In the splice quotient case $\mathcal{P}^{\{v\}} = Z^{\{v\}}$ for certain nodes was proved and used in Okuma [2008]. In the weighted homogeneous case, when the minimal good graph is star–shaped and v is the central vertex, then $\mathcal{P}^{\{v\}}_{h=0}$ coincides also with the Poincaré series of the $\mathbb{Z}_{\geq 0}$–graded algebra of the singular point (grading induced by the $\mathbb{C}^*$–action). For this series Pinkham provided a topological expression in terms of the Seifert invariants of the Seifert 3–manifold link Pinkham [1977], which coincides with $Z^{\{v\}}_{h=0}$ (see e.g. Némethi and Nicolaescu [2004]). In the case of suspension singularities the difference $\mathcal{P}^{\{C\}}_{h=0} - Z^{\{C\}}_{h=0}$ is captured by the polynomial R, see Section 6.4.

We note also that the identity $P_h = Z_h$ is much stronger than SWIC for h: one can construct examples when $P_h \neq Z_h$ but the SWIC holds. On the other hand, in any situation when the SWIC fails, the identity $\mathcal{P} = Z$ also fails (e.g. for certain superisolated singularities, see also Némethi [2008b]).

7.2 A closer look at splice quotient singularities. Splice quotient singularities were introduced by Neumann and Wahl [2005b,a]. From any fixed graph Γ (whose plumbed manifold is a rational homology sphere and which has some additional special arithmetical properties, see below) one constructs a family of singularities with common equisingularity type, such that any member admits a distinguished resolution, whose dual graph is exactly Γ. The construction suggests that the analytic properties of the singularities constructed in this way are strongly linked with a fixed resolution and with its graph Γ. (Hence, the expectation is that certain analytic invariants might be computable from Γ.)

In present, there are three different approaches how one can introduce and study splice quotient singularities; each of them is based on a different geometric property. They are: (a) the 'original' construction of Neumann–Wahl, (b) the 'modified' version by Okuma [2008], and (c) considering singularities satisfying the 'end–curve condition'. It turns out that all these approaches are equivalent.

In the first two cases we start with a topological type (that is, with Γ), which satisfies certain restrictions, and we endow it with an analytic structure, the 'splice quotient' analytic type. In the third case we start with an analytic structure, which satisfies a certain analytic property.

The construction of Neumann and Wahl [2005a] imposes two combinatorial restriction on Γ, the *semigroup and congruence conditions*. The congruence condition is empty if $\det(\, , \,) = 1$. Using the first condition one writes the equations of a complete intersection (Y, o). The equations depend only on the splice diagram associated with the graph, in particular they are called 'splice diagram equations'. Then one defines an action of H on this complete intersection, free off o, (here the congruence condition is needed), and sets $(X, o) = (Y, o)/G$. It turns out that (X, o) has a resolution with dual graph Γ and (Y, o) is the universal abelian covering of (X, o).

Okuma replaces the semigroup and congruence conditions by the *monomial conditions*, otherwise the construction and the output is the same. Singularities constructed in this way (either Neumann–Wahl version or Okuma version) are called splice quotient singularities.

The third approach defines a family of singularities with a special analytic property, with the *end curve condition*. This requires the existence of a resolution which has the following property: For each exceptional irreducible component E_v, which corresponds to an end–vertex of Γ, there exists an analytic function whose reduced strict transform is irreducible, it intersects the exceptional curve only along E_v, and this intersection is transversal. These are called 'end curve functions'.

Theorem 7.2.1. *(I) (Topological part) Fix a graph* Γ*. The following facts are equivalent:*

(a) There exists a splice quotient singularity with resolution graph Γ*,*

(b) Γ *satisfies the semigroup and congruence conditions;*

(c) Γ *satisfies the monomial condition.*

(II) (Analytic part) Fix a normal surface singularity (X, o)*. The following facts are equivalent:*

(a) (X, o) *is splice quotient (in the sense of Neumann–Wahl or Okuma);*

(b) (X, o) *satisfies the end curve condition;*

(c) $\mathcal{P}(\mathbf{t}) = Z(\mathbf{t})$.

(I) follows from Neumann and Wahl [ibid.], the equivalence $(I)(b) \Leftrightarrow (c)$ was proved in Neumann and Wahl [ibid., §13]). Regarding (II), the fact that splice quotient singularities satisfy the 'end curve condition' follows basically from the construction of the singularities: some powers of the coordinate functions of (Y, o) are 'end curve function'. The converse is the subject of the 'End Curve Theorem' Neumann and Wahl [2010] and Okuma [2010]. Part $(b) \Rightarrow (c)$ was proved in Némethi [2012], part $(c) \Rightarrow (b)$ follows from definitions.

Example 7.2.2. The end curve condition is satisfied in the following cases: (a) rational singularities, where π is an arbitrary resolution; (b) minimally elliptic singularities, and π is a minimal resolution; (c) weighted homogeneous singularities, where π is the minimal good resolution.

As we already said, we do not know the 'limits' of the SWIC, but the stronger version $\mathcal{P}(\mathbf{t}) = Z(\mathbf{t})$ occurs exactly when (X, o) is splice quotient (provided that M is a rational homology sphere). In this case the analytic $\mathcal{P}$ make the topological choice Z.

Problem 7.2.3. Find other topological candidates for $\mathcal{P}(\mathbf{t})$ — or, transformed into a question: what other topological choices might have $\mathcal{P}(\mathbf{t})$ when we vary the analytic structure of (X, o) ?

What is the 'multivariable series lift' of $\min_{\gamma \in \mathfrak{P}} \mathfrak{eu}(\mathbb{H}^0(\gamma, K))$? What is $\mathcal{P}_{h=0}(\mathbf{t})$ for hypersurface superisolated or Newton non–degenerate germs ? (In the last cases it can even happen that $\mathcal{P}_{h=0}(\mathbf{t})$ is not constant along the equisingular strata, then find $\mathcal{P}$ for some 'normal form'.) Describe/characterize the universal abelian covers of hypersurface superisolated or Newton non–degenerate singularities.

7.3 The analytic semigroup of (X, o). In Section 2.1 we introduced the 'topological semigroup' $\mathcal{S}'$ determined from the lattice L'. Its analytic counterpart is defined as

$$\mathcal{S}'_{an,h} = \{ r_h + \mathrm{div}_E(s), \text{ where } s \in H^0(\mathcal{O}_{\widetilde{X}}(-r_h))\}$$

for any $h \in H$, and $\mathcal{S}'_{an} = \cup_h \mathcal{S}'_{an,h}$. One shows that $\mathcal{S}'_{an} \subset \mathcal{S}'$, and $\mathcal{P}(\mathbf{t})$ is supported in $\mathcal{S}'_{an}$, while $Z(\mathbf{t})$ is supported in $\mathcal{S}'$. In both cases, usually the supports are much smaller then the corresponding semigroups. However, both semigroups conceptually guide important geometric properties of the analytic/topological type of (X, o). For rational or minimally elliptic singularities $\mathcal{S}'_{an} = \mathcal{S}'$, however the computation of $\mathcal{S}'_{an}$ usually is extremely hard.

Problem 7.3.1. Find efficient methods for the computation of $\mathcal{S}'_{an}$. Describe $\mathcal{S}'_{an}$ for certain key families of singularities.

7.4 Linear subspace arrangements, as 'lifts' of the series $\mathcal{P}(\mathbf{t})$ and $Z(\mathbf{t})$. Fix (X, o), a resolution π and the filtration $\{\mathcal{F}(l')\}_{l' \in L'}$ as in Section 2.2. For any $l' \in L'$, the linear space

$$(\mathcal{F}(l')/\mathcal{F}(l'+E))_{\theta([l'])} = H^0(\mathcal{O}_{\widetilde{X}}(-l'))/H^0(\mathcal{O}_{\widetilde{X}}(-l'-E))$$

naturally embeds into

$$T(l') := H^0(\mathcal{O}_E(-l')).$$

Let its image be denoted by $A(l')$. Furthermore, for every $v \in \mathcal{V}$, consider the linear subspace $T_v(l')$ of $T(l')$ given by

$$T_v(l') := H^0(\mathcal{O}_{E-E_v}(-l'-E_v)) = \ker\big(H^0(\mathcal{O}_E(-l')) \to H^0(\mathcal{O}_{E_v}(-l'))\big) \subset T(l').$$

Then the image $A_v(l')$ of $H^0(\mathcal{O}_{\widetilde{X}}(-l'-E_v)/H^0(\mathcal{O}_{\widetilde{X}}(-l'-E))$ in $T(l')$ satisfies $A_v(l') = A(l') \cap T_v(l')$.

The point is that one can show that the vector space $T(l')$ and the linear subspace arrangement $\{T_v(l')\}_v$ in $T(l')$ depends only on the resolution graph.

Definition 7.4.1. The (finite dimensional) arrangement of linear subspaces $\mathcal{A}_{\mathrm{top}}(l') = \{T_v(l')\}_v$ in $T(l')$ is called the 'topological arrangement' at $l' \in L'$. The arrangement of linear subspaces $\mathcal{A}_{\mathrm{an}}(l') = \{A_v(l') = T_v(l') \cap A(l')\}_v$ in $A(l')$ is called the 'analytic arrangement' at $l' \in L'$. The corresponding projectivized arrangement complements will be denoted by $\mathbb{P}(T(l') \setminus \cup_v T_v(l'))$ and $\mathbb{P}(A(l') \setminus \cup_v A_v(l'))$ respectively.

If $l' \notin \mathcal{S}'$ then there exists v such that $(E_v, l') > 0$, that is $h^0(\mathcal{O}_{E_v}(-l')) = 0$, proving that $T_v(l') = T(l')$. Hence $A_v(l') = A(l')$ too. In particular, both arrangement complements are empty. In fact, if $l' \notin \mathcal{S}'_{an}$, then by similar argument, the analytic arrangement complement is empty too.

The connection with the series $\mathcal{P}(\mathbf{t})$ and $\mathbb{Z}(\mathbf{t})$ is given by the following topological Euler characteristic formulae associated with all the linear subspace arrangements for all l'.

Theorem 7.4.2.

$$\mathcal{P}(\mathbf{t}) = \sum_{l' \in \mathcal{S}'_{an}} \chi_{\text{top}}(\,\mathbb{P}(A(l') \setminus \cup_v A_v(l'))\,) \cdot \mathbf{t}^{l'};$$

$$Z(\mathbf{t}) = \sum_{l' \in \mathcal{S}'} \chi_{\text{top}}(\,\mathbb{P}(T(l') \setminus \cup_v T_v(l'))\,) \cdot \mathbf{t}^{l'}.$$

For proof see Némethi [n.d.(a)], the analytic case (in the language of $\{\mathcal{F}(l')\}_{l'}$) already appeared in Campillo, Delgado, and Gusein-Zade [2004] and Guseĭn-Zade, Delgado, and Kampil'o [2008].

The corresponding dimensions of the linear subspaces in $\mathcal{A}_{\text{an}}(l')$ are as follows. For any $l' \in L'$ and $I \subset \mathcal{V}$ one has $\dim A(l') = \mathfrak{h}(l' + E) - \mathfrak{h}(l')$, $\dim \cap_{v \in I} A_v(l') = \mathfrak{h}(l' + E) - \mathfrak{h}(l' + E_I)$. Thus, the analytic arrangement is rather sensitive to the modification of the analytic structure, and in general, does not coincide with the topological arrangement. The corresponding dimensions in the topological case are computed in Némethi [n.d.(a)], they are slightly more technical. Examples show that $\chi_{\text{top}}(\mathbb{P}(A(l') \setminus \cup_v A_v(l'))) = \chi_{\text{top}}(\mathbb{P}(T(l') \setminus \cup_v T_v(l')))$ can happen even if $A(l') \neq T(l')$.

7.4.3. Note that the analytic subspace arrangement $\mathcal{A}_{an}(l')$, naturally determined by the divisorial filtration, exists even without its embedding into $T(l')$. On the other hand, if one wishes to find its topological analogue, its embedding into the topological $T(l')$ is the most natural possibility, that is, the choice of $\mathcal{A}_{top}(l')$ is the most natural universal object, which might include all the possible analytic arrangements indexed by different analytic structures.

In this way, $(A(l'), \{A_v(l')\}_v) \subset (T(l'), \{T_v(l')\}_v)$ looks a perfect pairing. This immediately induces (by taking the Euler characteristic of the corresponding spaces) the two series $Z(\mathbf{t})$ and $\mathcal{P}(\mathbf{t})$. Though these two series looked artificially paired at the beginning, now, after Theorem 7.4.2, this fact is totally motivated and justified. Furthermore, taking the periodic constants of the series Z and P, we get that the pairing predicted by the SWIC is indeed very natural and totally justified.

In particular, these steps provide a totally conceptual explanation for the appearance of the Seiberg–Witten invariant in the theory of complex surface singularities.

7.4.4. Extensions. The above picture allows to extend the series $\mathcal{P}(\mathbf{t})$ and $Z(\mathbf{t})$ to capture some additional information from the corresponding Hodge or Grothendieck ring structures as well. In the analytic case the extension of $\mathcal{P}(\mathbf{t})$ to the series $\sum_{l' \in \mathcal{S}'} [\mathbb{P}(A(l') \setminus$

$\cup_v A_v(l'))] \cdot \mathbf{t}^{l'}$ with coefficients in the Grothendieck ring was already considered e.g. in Campillo, Delgado, and Gusein-Zade [2007]. Once we have the topological arrangement in hand, we can also introduce the following series with coefficients in the Grothendieck ring

$$Z(\mathbb{L}, \mathbf{t}) = \sum_{l' \in \mathcal{S}'} [\mathbb{P}(T(l') \setminus \cup_v T_v(l'))] \cdot \mathbf{t}^{l'}. \tag{7.4.5}$$

It is remarkable that this series has a closed expression in terms of lattice too. Indeed, if $\mathcal{E}$ denotes the set of edges of Γ, then (see Némethi [n.d.(a)])

$$Z(\mathbb{L}, \mathbf{t}) = \frac{\prod_{(u,v)\in\mathcal{E}} (1 - \mathbf{t}^{E_u^*} - \mathbf{t}^{E_v^*} + \mathbb{L}\mathbf{t}^{E_u^*+E_v^*})}{\prod_{v\in\mathcal{V}} (1 - \mathbf{t}^{E_v^*})(1 - \mathbb{L}\mathbf{t}^{E_v^*})}. \tag{7.4.6}$$

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