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MEASURABLE EQUIDECOMPOSITIONS

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Abstract

The famous Banach–Tarski paradox and Hilbert's third problem are part of story of paradoxical equidecompositions and invariant finitely additive measures. We review some of the classical results in this area including Laczkovich's solution to Tarski's circle-squaring problem: the disc of unit area can be cut into finitely many pieces that can be rearranged by translations to form the unit square.

We also discuss the recent developments that in certain cases the pieces can be chosen to be Lebesgue measurable or Borel: namely, a measurable Banach–Tarski 'paradox' and the existence of measurable/Borel circle-squaring.

1 Paradoxical equidecompositions and invariant measures

In the plane, any polygon can be transformed into any other polygon of the same area by cutting it into polygonal pieces and recomposing these after applying translations and rotations (isometries of the plane). This is the Bolyai–Gerwien–Wallace theorem from the nineteenth century. The analogous problem about polyhedra in $\mathbb{R}^3$ became known as Hilbert's third problem in 1900. This was solved by Dehn by inventing an algebraic invariant that shows that the unit cube cannot be cut into finitely many polyhedral pieces that, after applying isometries, reassemble into the regular tetrahedron of unit volume.

It makes sense to study analogous questions where we may cut geometric objects into arbitrary sets, not just polygons or polyhedra. For sets $A, B \in \mathbb{R}^n$ let $A \cong B$ denote that they are congruent; that is, there is a distance-preserving bijection from A to B, or equivalently, there is an Euclidean motion (isometry) moving A to B.

Definition 1.1. We say that sets $A, B \in \mathbb{R}^n$ are *equidecomposable* if there are finite partitions $A = \cup_{i=1}^k A_i$ and $B = \cup_{i=1}^k B_i$ such that $A_i \cong B_i$ for every $i = 1, \dots, k$.

The most famous result about equidecompositions is the following.

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Theorem 1.2 (Banach–Tarski paradox Banach and Tarski [1924]). *If $A, B \subset \mathbb{R}^n$, $n \geq 3$, are bounded sets with non-empty interior, then A and B are equidecomposable.*

Their result is based on earlier work of Hausdorff that essentially says that the unit sphere in $\mathbb{R}^3$ is equidecomposable to the disjoint union of two unit spheres, modulo countable sets.

Theorem 1.3 (Hausdorff paradox, 1914). *Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Then there are partitions*

$$\mathbb{S}^2 = A_1 \cup A_2 \cup C_1 \quad \textit{and} \quad \mathbb{S}^2 = A_3 \cup A_4 \cup A_5 \cup C_2$$

where the sets A_i are congruent to each other and C_1, C_2 are countable.

Hausdorff's proof is based on his discovery that $SO(3)$, the group of rotations of $\mathbb{S}^2$, contains a free subgroup of rank 2.

The interest in equidecompositions originates from questions about the existence of certain invariant measures. In 1905 Vitali proved that there are no non-trivial isometry-invariant σ-additive measures defined on all subsets of $\mathbb{R}$ (that assign measure 1 to the unit interval). Hausdorff raised the question whether at least *finitely additive* isometry-invariant measures defined on all subsets of $\mathbb{R}^n$ exist. An immediate corollary of his Theorem 1.3 is that the analogous question for $\mathbb{S}^2$ has a negative answer. His question, for $\mathbb{R}$ and $\mathbb{R}^2$, was solved by Banach in 1923.

Theorem 1.4 (Existence of Banach measures in $\mathbb{R}$ and $\mathbb{R}^2$ Banach [1923].). *In $\mathbb{R}$ and $\mathbb{R}^2$ the Lebesgue measure can be extended to all subsets as an isometry-invariant finitely additive measure.*

An immediate corollary of this theorem is that if two Lebesgue measurable sets $A, B \subset \mathbb{R}$ or $\mathbb{R}^2$ are equidecomposable, then they must have equal Lebesgue measure. Similarly, the Banach–Tarski paradox immediately implies that there are no non-trivial isometry-invariant finitely additive measures defined on all subsets of $\mathbb{R}^n$ for $n \geq 3$ (that assign positive and finite measure to the unit cube).

So clearly, the existence of paradoxical equidecompositions imply the non-existence of invariant measures. It is a fundamental theorem of Tarski that the other direction holds as well in a very general setting.

Definition 1.5. Assume that a group G acts on a set X. We say that $A, B \subset X$ are G-equidecomposable if there are finite partitions $A = \cup_{i=1}^k A_i$ and $B = \cup_{i=1}^k B_i$ such that, for each i, $B_i = g_i(A_i)$ for some $g_i \in G$.

Theorem 1.6 (Tarski 1929). *Assume that a group G acts on a set X and $A \subset X$. Then there is a G-invariant finitely additive measure μ defined on all subsets of X satisfying $\mu(A) = 1$ if and only if A cannot be written as the union of disjoint sets A', A'' such that A is G-equidecomposable to both A' and A''.*

1.1 Remark on amenability. The Banach–Tarski paradox holds in $\mathbb{R}^n$ for $n \geq 3$, while Banach measures exist in $\mathbb{R}$ and $\mathbb{R}^2$. This contrast was better understood after von Neumann (1929) studied the behaviour of the isometry groups of these spaces. A group G is called *amenable* if there is a finitely additive probability measure μ defined on all subsets of G that is left invariant under the action of G on itself:

$$\mu(\gamma A) = \mu(A) \quad \text{for every } A \subset G,\ \gamma \in G.$$

He proved that the isometry group of $\mathbb{R}$ and $\mathbb{R}^2$ are amenable (in fact, solvable, and as he proved, every solvable group is amenable), whereas the isometry group of $\mathbb{R}^n$ $(n \geq 3)$ and $SO(3)$ are not amenable.

When G is amenable, the existence of G-invariant measures carries to spaces X on which G acts.

Theorem 1.7 (Mycielski [1979]). *Assume that an amenable group G is acting on a set X, and let μ be a G-invariant finitely additive measure defined on a G-invariant algebra $\mathfrak{A}$ of subsets of X. Then μ can be extended to be a G-invariant finitely additive measure defined on all subsets of X.*

1.2 Equidecompositions using sets of the Baire property. Recall that a set $A \subset \mathbb{R}^n$ is meager (or of the first Baire category) if it is a union of countably many nowhere dense sets. A set is said to have the Baire property if it is the symmetric difference of an open set and a meager set. (All Borel sets have the Baire property.) A set A is called Jordan measurable if it is bounded and the boundary ∂A has Lebesgue measure zero. Its Jordan measure is the same as its Lebesgue measure.

Marczewski proved an analogue of Banach's Theorem 1.4.

Theorem 1.8 (Existence of Marczewski measures in $\mathbb{R}$ and $\mathbb{R}^2$). *In $\mathbb{R}$ and $\mathbb{R}^2$ there is an isometry-invariant finitely additive measure μ defined on all subsets such that μ extends the Jordan measure and vanishes on meager sets.*

Marczewski posed the question, in 1930, whether the same holds in $\mathbb{R}^n$, for $n \geq 3$. This was unsolved until 1992 when Dougherty and Foreman proved that these measures cannot exist in higher dimensions. In fact, they proved the striking result that the Banach–Tarski paradox works with pieces that are Baire measurable.

Theorem 1.9 (Dougherty and Foreman [1992, 1994]). *Let $n \geq 3$. The unit ball in $\mathbb{R}^n$ can be equidecomposed to the union of two disjoint unit balls by sets that have the Baire property.*

The underlying statement is that a dense open subset of the unit ball is equidecomposable by open pieces to a dense open subset of the union of two disjoint unit balls. Theorem 1.9 is then obtained by "combining" this equidecomposition with the one given by the Banach–Tarski paradox restricted to a suitable meager set.

1.3 The Banach–Ruziewicz problem and the measurable Banach–Tarski paradox.

In this section we consider finitely additive measures that are defined only on the σ-algebra of Lebesgue measurable sets, not for all subsets of $\mathbb{R}^n$.

Question 1.10 (Banach–Ruziewicz problem). *Let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ be the unit sphere.*

1. *Assume that μ is a rotation invariant finitely additive probability measure defined on Lebesgue measurable sets of $\mathbb{S}^{n-1}$. Does μ necessarily coincide with the normalized Lebesgue measure?*

2. *Assume that μ is an isometry invariant finitely additive measure defined on bounded Lebesgue measurable sets of $\mathbb{R}^n$, assigning measure 1 to the unit cube. Does μ necessarily coincide with the Lebesgue measure?*

For $\mathbb{S}^1$, $\mathbb{R}$ and $\mathbb{R}^2$, the questions have a negative answer. These easily follow from the existence of Marczewski measures μ on $\mathbb{R}$ and $\mathbb{R}^2$ (Theorem 1.8), as μ differs from Lebesgue measure on meager sets of positive Lebesgue measure.

On the other hand, for $n \geq 3$, the answers to both questions are positive. This was independently proved by Margulis [1980] and Sullivan [1981] for $n \geq 5$ in 1980 and then by Drinfel'd [1984] for $n = 2, 3$ in 1984. The proofs rely on Kazhdan's property (T) and the existence of a spectral gap for certain averaging operators, see Section 2 for details.

The Banach–Ruziewicz problem also has a connection to paradoxical equidecompositions. The following theorem can be regarded as the measurable version of the Banach–Tarski paradox. It is easy to see that it implies the positive answers to Question 1.10 for $n \geq 3$.

Theorem 1.11 (Grabowski, Máthé, and Pikhurko [2016]). *Let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ be the unit sphere, and let $n \geq 3$.*

1. *Let $A, B \subset \mathbb{S}^{n-1}$ be measurable sets with non-empty interior and equal Lebesgue measure. Then A is equidecomposable to B using rotations with* measurable pieces.

2. *Let $A, B \subset \mathbb{R}^n$ be bounded measurable sets with non-empty interior and equal Lebesgue measure. Then A is equidecomposable to B with* measurable pieces.

The proof is based on work by Lyons and Nazarov [2011], Elek and Lippner [2010] and the spectral gap results of Margulis, Sullivan and Drinfeld. We review this proof in detail in Section 2.

1.4 Tarski's circle-squaring problem.

Question 1.12 (Tarski's circle-squaring problem, 1925). *Are the disc and a square of the same area equidecomposable?*

Dubins, Hirsch, and Karush [1963] proved in 1963 that circle-squaring is not possible by pieces that are Jordan domains (that is, topological discs), even if their boundaries can be ignored.

In 1985 Gardner proved that circle-squaring is not possible if the pieces are arbitrary but they are moved by isometries that generate a locally discrete group. In fact, he proved the following theorem.

Theorem 1.13 (Gardner [1985]). *Let G be a locally discrete group of isometries of $\mathbb{R}^n$. If a convex polytope and a convex set in $\mathbb{R}^n$ are G-equidecomposable, then they are G-equidecomposable with convex pieces.*

Finally, in 1990, Tarski's circle-squaring problem was solved by Laczkovich in the affirmative.

Theorem 1.14 (Laczkovich [1990]). *The disc is equidecomposable to a square of the same area; in fact, it is enough to use translations only.*

His proof extends to more general sets, assuming a condition on their boundary. Recall that the upper Minkowski (or box) dimension of a (non-empty bounded) set $X \subset \mathbb{R}^n$ is

$$\overline{\dim}_M(X) = \limsup_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}$$

where $N_\delta(X)$ denotes the minimum number of (grid) cubes of side δ that cover the set X.

Theorem 1.15 (Laczkovich [1992a]). *Let A and B be bounded measurable sets in $\mathbb{R}^n$ with equal positive Lebesgue measure such that $\overline{\dim}_M(\partial A) < n$ and $\overline{\dim}_M(\partial B) < n$. Then A and B are equidecomposable using translations; that is, there exist partitions $A = \cup_{i=1}^k A_i$, $B = \cup_{i=1}^k B_i$ and translation vectors $x_i \in \mathbb{R}^n$ such that $B_i = A_i + x_i$.*

The condition on the Minkowski dimension is satisfied, for example, when the sets are bounded convex sets, or if they are Jordan domains of the plane with rectifiable boundaries (of finite length).

The condition on the Minkowski dimension of the boundary is necessary. Laczkovich [1993] proved that there exist intervals converging to zero such that their union A is not equidecomposable to an interval. (The boundary of A is a sequence of points converging to zero — such sequences can have Minkowski dimension one, and it has in this case.)

It has turned out recently that circle-squaring is possible with "nice" pieces as well.

Theorem 1.16 (Grabowski, Máthé, and Pikhurko [2017]). *Circle-squaring is possible using translations with pieces that are Lebesgue measurable and have the Baire property.*

This result was quickly superseded by a stronger result of Marks and Unger.

Theorem 1.17 (A. S. Marks and S. T. Unger [2017]). *Circle-squaring is possible using translations with Borel pieces.*

In fact, both Grabowski, Máthé, and Pikhurko [2017] and A. S. Marks and S. T. Unger [2017] prove that these equidecompositions exist not just for the disc and square, but for any sets A, B satisfying the assumptions of Theorem 1.15. Both papers build on Laczkovich's work.

In the next section we discuss the connection of equidecompositions to perfect matchings in (infinite) bi-partite graphs. To briefly summarise these proofs: Theorem 1.15 is proved by checking that Hall's condition holds in the bi-partite graph implying the existence of a perfect matching; Theorem 1.16 considers (an algorithm involving) augmenting paths to provide a measurable perfect matching; and Theorem 1.17 uses (Borel) flows to find a Borel perfect matching.

We will discuss Laczkovich's solution in Section 3, and the proof of Theorem 1.16 (for measurable pieces) in Section 5. (For the elegant proof Theorem 1.17, see the original paper A. S. Marks and S. T. Unger [ibid.].)

Remark 1.18. Before the results on the measurable and Borel circle-squaring, it was already known that Laczkovich's non-measurable circle-squaring implies circle-squaring by *measurable functions*. This was proved independently by Wehrung and Laczkovich. We review the exact statement and its proof in Section 4.

1.5 Equidecompositions and perfect matchings in bi-partite graphs.

When we are looking for equidecompositions, the best way is to first fix the finitely many isometries that we are going to use to move the pieces, and then try to find the pieces. This way the problem of finding equidecompositions reduces to finding perfect matchings in certain bi-partite graphs. Let A, B be arbitrary subsets of a set X, and let H be a finite set of bijections $X \to X$. (For example, $X = \mathbb{R}^n$ and H is a finite set of isometries.) To avoid confusion later, let us assume that A and B are disjoint. Define the bi-partite graph

$$\Gamma_H(A, B) = \{(a, b) \in A \times B \, : \, b = f(a) \text{ for some } f \in H\}.$$

Then the two sets of vertices of the bi-partite graph are A and B, and a vertex $a \in A$ is connected to a vertex $b \in B$ by an edge if one of the bijections $f \in H$ move a to b.

Recall that a set $\mathfrak{M}$ of edges of a bi-partite graph is a *matching* if no vertex is covered by two edges in $\mathfrak{M}$, and it is a *perfect matching* if every vertex is covered exactly once. The following is obvious.

Lemma 1.19.

1. *There is a perfect matching in the graph* $\Gamma_H(A, B)$ *if and only if* A *is equidecomposable to* B *by using bijections of* H*.*
2. *Let* G *act on* X*. Then* $A, B \subset X$ *are* G*-equidecomposable if there is a finite set* $H \subset G$ *such that* $\Gamma_H(A, B)$ *contains a perfect matching.*

As it is noted by Laczkovich [2002], the connection of equidecomposability and perfect matchings was already known and used by König and Tarski.

Hall's marriage theorem extend to the case of infinite bi-partite graphs if all degrees are finite (Rado [1949]). In particular, we obtain the following.

Lemma 1.20. *The graph* $\Gamma_H(A, B)$ *contains a perfect matching if and only if*

$$|N(U)| \geq |U| \tag{1-1}$$

for every finite set of vertices U*, where* $N(U)$ *denotes the set of neighbours of* U*.*

Therefore proving that Hall's condition (1-1) holds is enough to prove the existence of an equidecomposition. However, it does not yield measurable or Borel equidecompositions as Rado's result relies on the Axiom of Choice.

Remark 1.21. For the full story of equidecompositions see Wagon's excellent book on the Banach–Tarksi paradox Wagon [1985] and the new edition by Tomkowicz and Wagon [2016]. Another excellent reading is Laczkovich's monograph on paradoxes in measure theory Laczkovich [2002].

2 Measurable Banach–Tarski

In this section we sketch the proof of Theorem 1.11 Grabowski, Máthé, and Pikhurko [2016] following Grabowski, Máthé, and Pikhurko [2014]. We focus on the technically easier case of the sphere $\mathbb{S}^{n-1}$. First we consider measurable equidecompositions modulo nullsets.

Theorem 2.1. *Let* $n \geq 3$ *and let* $A, B \subset \mathbb{S}^{n-1}$ *be measurable sets with non-empty interiors and of the equal Lebesgue measure. Then there are measurable sets* $A' \subset A$ *and* $B' \subset B$ *that are equidecomposable with measurable pieces such that* $A \setminus A'$ *and* $B \setminus B'$ *have measure zero.*

Sketch of proof. Let us assume, to simplify notation, that A and B are disjoint. We may assume that A, B are Borel sets (as we can forget about nullsets). Let λ denote the normalised Lebesgue measure on $\mathbb{S}^{n-1}$. The key ingredient of the proof is the following *spectral gap property*.

Lemma 2.2 (Margulis [1980], Sullivan [1981] for $n \geq 5$, Drinfel'd [1984] for $n \geq 3$). *There exist rotations $\gamma_1, \dots, \gamma_k \in SO(n)$ and $\varepsilon > 0$ such that the averaging operator $T : L^2(\mathbb{S}^{n-1}, \lambda) \to L^2(\mathbb{S}^{n-1}, \lambda)$ defined by*

$$(Tf)(x) = \frac{1}{k} \sum_{i=1}^{k} f(\gamma_i(x)) \qquad (f \in L^2(\mathbb{S}^{n-1}, \lambda),\ x \in \mathbb{S}^{n-1})$$

satisfies $\|Tf\|_2 \leq (1-\varepsilon)\|f\|_2$ for every $f \in L^2(\mathbb{S}^{n-1}, \lambda)$ with $\int f\, d\lambda = 0$.

It is easy to show that this lemma implies the following *expansion property*. For any large constant $C > 1$ there is a finite set S of rotations such that for every measurable set $U \subset \mathbb{S}^{n-1}$ we have

$$\lambda\big(\cup_{\gamma \in S}\, \gamma(U)\big) \geq \min\big(C\lambda(U),\ 1 - C^{-1}\big). \tag{2-1}$$

Since A and B have non-empty interior in the sphere, we can find a finite set of rotations T such that $\mathbb{S}^{n-1} = \cup_{\gamma \in T}\, \gamma(A) = \cup_{\gamma \in T}\, \gamma(B)$. Choose C so that $C \geq 2|T|$ and $C^{-1} \leq \lambda(A)/3$ and with the obtained $S = S(C)$ define

$$R = T^{-1}S \cup S^{-1}T = \{\tau^{-1}\gamma : \tau \in T, \gamma \in S\} \cup \{\gamma^{-1}\tau : \tau \in T, \gamma \in S\} \subset SO(n).$$

Then R is closed under taking inverses, $R^{-1} = R$.

As in Section 1.5, consider the bi-partite graph Γ whose set of vertices is $A \cup B$ and there is an edge between $x \in A$ and $y \in B$ if $y = \gamma(x)$ for some $\gamma \in R$. (We will also use the notation $xy \in E(\Gamma)$ in this case.) Then the following expansion property holds.

Claim 2.3. *Let $U \subset A \cup B$ and let $N(U)$ be the set of neighbours of U in G. Then*

$$\lambda(N(U)) \geq \min\big(2\lambda(U),\ \tfrac{2}{3}\lambda(A)\big). \tag{2-2}$$

Proof. It is enough to prove the claim when $U \subset A$ or $U \subset B$. We may assume $U \subset A$. Let $S.U$ denote $\cup\{\gamma(U) : \gamma \in S\}$. By (2-1), we have $\lambda(S.U) \geq 2|T|\lambda(U)$ or $\lambda(S.U) \geq 1 - \lambda(A)/3$. First assume the former. Since the sets $\tau(B)$ ($\tau \in T$) cover the sphere, there is $\tau \in T$ such that

$$2\lambda(U) \leq \lambda(S.U \cap \tau(B)) = \lambda(\tau^{-1}(S.U) \cap B) \leq \lambda(N(U)).$$

Now assume that $\lambda(S.U) \geq 1 - \lambda(A)/3$. Then, for any $\tau \in T$,

$$\tfrac{2}{3}\lambda(B) \leq \lambda(S.U \cap \tau(B)) = \lambda(\tau^{-1}(S.U) \cap B) \leq \lambda(N(U)). \qquad \square$$

A matching $\mathfrak{M}$ in the graph Γ is called *Borel* if there exist disjoint Borel subsets $A_\gamma \subset A$ indexed by $\gamma \in R$ such that

$$\mathfrak{M} = \cup_{\gamma \in R}\big\{\{x, \gamma(x)\} : x \in A_\gamma\big\}. \tag{2-3}$$

Clearly, in order to finish the proof it is enough to find a Borel matching in Γ such that the set of unmatched vertices has measure zero. As noted in Lyons and Nazarov [2011, Remark 2.6], the expansion property (2-2) suffices for this. In the following, we just outline their strategy.

Recall that an *augmenting path* for a matching $\mathfrak{M}$ is a path which starts and ends at an unmatched vertex and such that every second edge belongs to $\mathfrak{M}$. A *Borel augmenting family* is a Borel subset $U \subset A \cup B$ and a finite sequence $\gamma_1, \dots, \gamma_l$ of elements of R such that (i) for every $x \in U$ the sequence $y_0, \dots, y_l$, where $y_0 = x$ and $y_j = \gamma_j(y_{j-1})$ for $j = 1, \dots, l$, forms an augmenting path and (ii) for every distinct $x, y \in U$ the corresponding augmenting paths are vertex-disjoint.

As shown by Elek and Lippner [2010], there exists a sequence $(\mathfrak{M}_i)_{i\in\mathbb{N}}$ of Borel matchings such that $\mathfrak{M}_i$ admits no augmenting path of length at most $2i - 1$ and $\mathfrak{M}_{i+1}$ can be obtained from $\mathfrak{M}_i$ by iterating the following at most countably many times: pick some Borel augmenting family $(U, \gamma_1, \dots, \gamma_l)$ with $l \le 2i + 1$ and flip (i.e. augment) the current matching along all paths given by the family. See Elek and Lippner [ibid.] for more details.

Our task now is to show, using Claim 2.3, that the measure of of vertices not matched by $\mathfrak{M}_i$ tends to zero as $i \to \infty$ and that the sequence $(\mathfrak{M}_i)_{i\in\mathbb{N}}$ stabilises almost everywhere (that is, for almost every vertex, the edge in $\mathfrak{M}_i$ containing the vertex stabilises as $i \to \infty$).

Lemma 2.4. *Let $i \ge 1$. Then the measure of vertices of A and B that are not covered by $\mathfrak{M}_i$ is at most*

$$2\lambda(A) \cdot 2^{-\lfloor (i-1)/2 \rfloor}.$$

Before proving this lemma, let us finish the proof of the theorem.

As we noted before, $\mathfrak{M}_{i+1}$ arises from $\mathfrak{M}_i$ by flipping augmenting paths of length at most $2i+1$ in a Borel way. When one such path is flipped, two vertices are removed from the current set of unmatched vertices. Using this observation and the fact that each rotation is measure-preserving, one can show that the set of vertices covered by the symmetric difference $\mathfrak{M}_{i+1} \triangle \mathfrak{M}_i$ has measure at most $(2i + 2)$ times the measure of unmatched vertices by $\mathfrak{M}_i$. We know from the lemma that this goes to 0 exponentially fast with i; in particular, it is summable over $i \in \mathbb{N}$. The Borel–Cantelli Lemma implies that the sequence of matchings $(\mathfrak{M}_i)_{i\in\mathbb{N}}$ stabilises almost everywhere.

Proof of Lemma 2.4. Let us fix $i \ge 1$ and let X_0 be the subset of A consisting of vertices that are not matched by $\mathfrak{M}_i$. An *alternating path of length l* is a sequence of distinct vertices $x_0, \dots, x_l$ such that (i) $x_0 \in X_0$, (ii) for odd j we have $x_j x_{j+1} \in \mathfrak{M}_i$, and (iii) for even j we have $x_j x_{j+1} \in E(\Gamma) \setminus \mathfrak{M}_i$. Let X_j consist of the end-vertices of alternating paths of length at most j. Clearly for all j we have $X_j \subset X_{j+1}$ and so, in particular, $\lambda(X_{j+1}) \geq \lambda(X_j)$. For $j \geq 1$, let $X'_j = X_j \setminus X_{j-1}$.

It's not difficult to show the following.

Claim 2.5. *For every odd $j \leq 2i - 1$ we have $\lambda(X'_j) = \lambda(X'_{j+1})$ and $\lambda(X_j \cap B) \leq \lambda(X_{j+1} \cap A)$.*

Proof of Claim. All vertices in X'_j are covered by the matching $\mathfrak{M}_i$, for otherwise we would have an augmenting path of length j. It follows that $\mathfrak{M}_i$ gives a bijection between X'_j and X'_{j+1}. If we take the sets A_γ that represent $\mathfrak{M}_i$ as in Equation (2-3), then the partitions $\cup_{\gamma \in R} A_\gamma$ and $\cup_{\gamma \in R} \gamma(A_\gamma)$ induce a Borel equidecomposition between X'_j and X'_{j+1}, so these sets have the same measure, as required.

The second part (i.e. the inequality) follows analogously from the fact that $\mathfrak{M}_i$ gives an injection of $X_j \cap B$ into $X_{j+1} \cap A$ (with X_0 being the set of vertices missed by this injection). □

Let k be even, with $2 \leq k \leq 2i-2$. Let $U = X_k \cap A$. We have that $N(U) = X_{k+1} \cap B$. By Claim 2.3,

$$\lambda(X_{k+1} \cap B) = \lambda(N(U)) \geq \min\left(\frac{2}{3}\lambda(A), 2\lambda(U)\right).$$

If $\lambda(X_{k+1} \cap B) \geq \frac{2}{3}\lambda(A)$ then, by Claim 2, $\lambda(X_{k+2} \cap A) \geq \lambda(X_{k+1} \cap B) \geq \frac{2}{3}\lambda(A)$ and thus

$$\lambda(X_{k+2}) = \lambda(X_{k+1} \cap B) + \lambda(X_{k+2} \cap A) \geq \frac{4}{3}\lambda(A).$$

Now, suppose that $\lambda(X_{k+1} \cap B) \geq 2\lambda(U)$. By applying Claim 2.5 for $j = k - 1$ we obtain

$$\lambda(X'_{k+1}) = \lambda(X_{k+1} \cap B) - \lambda(X_{k-1} \cap B) \geq 2\lambda(U) - \lambda(U) = \lambda(U).$$

Again, by Claim 2, $\lambda(X'_{k+2}) = \lambda(X'_{k+1})$ and $\lambda(X_k) = \lambda(X_{k-1} \cap B) + \lambda(U) \leq 2\lambda(U)$. Thus

$$\lambda(X_{k+2}) = \lambda(X_k) + \lambda(X'_{k+1}) + \lambda(X'_{k+2}) \geq \lambda(X_k) + 2\lambda(U) \geq 2\lambda(X_k).$$

Thus the measure of X_k expands by factor at least 2 when we increase k by 2, unless $\lambda(X_{k+2}) \geq \frac{4}{3}\lambda(A)$. Also, this conclusion formally holds for $k = 0$, when $X_1 = N(X_0)$.

By using induction, we conclude that, for all even k with $0 \leq k \leq 2i$,

$$\lambda(X_k) \geq \min\left(\frac{4}{3}\lambda(A), 2^{k/2}\lambda(X_0)\right). \tag{2-4}$$

In the same fashion we define Y_0 to be the subset of B consisting of vertices not matched by $\mathfrak{M}_i$ and let Y_j consist of the end-vertices of alternating paths that start in Y_0 and have length at most j. As before, we obtain that the sets Y_j satisfy the analogue of Equation (2-4).

The sets X_{i-1} and Y_i are disjoint for otherwise we would find and augmenting path of length at most $2i - 1$. It follows that they cannot each have measure more than $\lambda(A) = \frac{1}{2}\lambda(A \cup B)$. Since $\lambda(X_0) = \lambda(Y_0)$ we conclude that

$$\lambda(X_0 \cup Y_0) \leq 2\,\lambda(A) \cdot \left(\frac{1}{2}\right)^{\lfloor (i-1)/2 \rfloor}. \tag{2-5}$$

This proves the Lemma. □

This finishes the proof of Theorem 2.1. □

Theorem 2.6. *Let $n \geq 3$ and let $A, B \subset \mathbb{S}^{n-1}$ be measurable sets with non-empty interiors and of the equal Lebesgue measure. Then there are measurable sets $A' \subset A$ and $B' \subset B$ that are equidecomposable with measurable pieces such that $A \setminus A'$ and $B \setminus B'$ have measure zero.*

Sketch of proof. The argument that leads to Claim 2.3 can be adopted to show that $|N(X)| \geq 2\,|X|$ for every *finite* subset X of A (and of B). By a result of Rado [1949], this guarantees that Γ has a perfect matching. The (exact) measurable equidecomposition of A and B can be obtained by modifying the equidecomposition from Theorem 2.1 on suitably chosen sets of measure zero (containing $A \setminus A'$ and $B \setminus B'$) where we use Rado's theorem and the Axiom of Choice. This way we obtain a measurable equidecomposition between A and B. □

Remark 2.7. The spectral gap property as stated in Lemma 2.2 fails in $\mathbb{R}^n$. However, one can argue that a suitable reformulation of the expansion property still holds Grabowski, Máthé, and Pikhurko [2016], and that is enough to prove Theorem 1.11 for $\mathbb{R}^n$, $n \geq 3$. Alternatively, one can use the notion of a local spectral gap Boutonnet, Ioana, and Golsefidy [2017], see also Grabowski, Máthé, and Pikhurko [2016].

3 Laczkovich's circle-squaring

The aim of this section is to decribe the main steps in Laczkovich's proof of Tarski's circle-squaring problem, providing an equidecomposition between the disc and a square (using non-measurable pieces).

Instead of looking at the problem in $\mathbb{R}^n$, we may assume that A, B are subsets of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Indeed, we may assume that $A, B \subset [0, 1/3]^n$, and in this case, any equidecomposition of A to B on the torus yields an equidecomposition of A to B in $\mathbb{R}^n$, with the same (number of) pieces. Let λ denote the probability Lebesgue measure in $\mathbb{T}^n$.

Let d be a sufficiently large positive integer, and fix translation vectors $v_1, v_2, \dots, v_d \in \mathbb{T}^n$ that are linearly independent over the rationals and generate a dense subgroup (isomorphic to $\mathbb{Z}^d$) of $\mathbb{T}^n$. (It will turn out later that a random choice of these vectors is what we need.) Considering how the subgroup's cosets intersect A and B we define, for $u \in \mathbb{T}^n$,

$$A_u^v = \Big\{(k_1, \dots, k_d) \in \mathbb{Z}^d \ : \ u + k_1 v_1 + \dots k_d v_d \in A\Big\},$$

$$B_u^v = \Big\{(k_1, \dots, k_d) \in \mathbb{Z}^d \ : \ u + k_1 v_1 + \dots k_d v_d \in B\Big\}.$$

The following lemma implies that if we can find bijections from A_u^v to B_u^v that move every point by at most a fixed distance, then A and B are equidecomposable using translation vectors that are integer linear combinations of the vectors v_j.

Lemma 3.1. *The following statements are equivalent for every constant C.*

(i) A and B are equidecomposable using the translation vectors

$$V_C = \left\{ \sum_{j=1}^{d} k_j v_j \in \mathbb{T}^n \ : \ |k_j| \le C \right\}.$$

(ii) For every $u \in \mathbb{T}^n$ there exist a bijection $f_u : A_u^v \to B_u^v$ such that

$$\| f_u(k) - k \|_\infty \le C \quad (k \in \mathbb{Z}^d).$$

Proof. Notice that A and B are equidecomposable using vectors $w_1, \dots, w_m$ if and only if there is a bijection $f : A \to B$ such that for every $x \in A$, $f(x) - x \in \{w_1, \dots, w_m\}$.

So (i) is equivalent to saying that there is a bijection $f : A \to B$ such that for every $x \in A$, $f(x) - x \in V_C$. On the other hand, it is easy to see that (ii) is equivalent to saying that for every coset C_u of the group generated by the vectors v_j there is a bijection $f : A \cap C_u \to B \cap C_u$ with $f(x) - x \in V_C$ for $x \in A \cap C_u$.

Clearly, (i) implies (ii). For the opposite direction, observe that many choices of u determine the very same coset. Still, we can pick exactly one u from each coset by the Axiom of Choice. Therefore (ii) implies (i). □

In order to prove Theorem 1.15, it is enough to prove that (ii) of Lemma 3.1 holds for some choice of vectors $v_1, \dots, v_d$. By the pointwise ergodic theorem, we can expect that the densities of the sets A_u^v and B_u^v in $\mathbb{Z}^d$ will be the same (and equal to $\lambda(A) = \lambda(B)$). This is obviously necessary for the existence of the bijections f_u^v. However, we need to know much more about these sets than just the densities. In particular, we need fine estimates on the number of points of A_u^v and B_u^v inside any cube of $\mathbb{Z}^d$.

First consider the case that $A = X \subset \mathbb{T}^n$ is a box, that is, the product of sub-intervals of $[0, 1)$. An application of the Erdős–Turán–Koksma inequality yields the following.

Lemma 3.2. *Let $v_1, \ldots, v_d \in \mathbb{T}^n$ be uniformly distributed independent random vectors. Then with probability 1, there is a constant c such that for every box $X \subset \mathbb{T}^n$, every $u \in \mathbb{T}^n$, every cube $Q \subset \mathbb{Z}^d$ (containing more than 1 point) we have that*

$$\Big| |X_u^v \cap Q| - \lambda(X)\,|Q| \Big| \le c \log^{n+d+1} |Q|.$$

This lemma implies a similar discrepancy result (with weaker upper bounds) for sets A of (upper) Minkowski dimension less than n.

Lemma 3.3. *Let $A \subset \mathbb{T}^n$ satisfy $\overline{\dim}_M(A) < n$. If d is large enough, the following is true. Let $v_1, \ldots, v_d \in \mathbb{T}^n$ be uniformly distributed independent random vectors. With probability 1, there is a constant c (depending on A and the vectors v_j) such that for every $u \in \mathbb{T}^n$, every cube $Q \subset \mathbb{Z}^d$ with side length N we have that*

$$\Big| |A_u^v \cap Q| - \lambda(A)\,|Q| \Big| \le c N^{d-2}.$$

(Instead of the bound N^{d-2}, any exponent less than $d-1$ would be sufficient.)

Lemma 3.3 follows from Lemma 3.2 by a result of Niederreiter and Wills [1975]. The outline of the proof is the following. For some $\delta > 0$, cover A by grid cubes of side δ. Consider those cubes that are in the interior of A. If we apply Lemma 3.2 to all of these boxes, we cannot obtain a good bound on the discrepancy. Instead, we merge some of these cubes into boxes in the following way. If two cubes (in the interior of A) share an $(n-1)$-dimensional face and have the same projection to the first $n-1$ coordinates, we merge them into the same box. Applying Lemma 3.2 to these boxes and using trivial bounds for the grid cubes that intersect the boundary of A, we obtain a bound on the discrepancy. The last step in the proof is to choose δ so that we minimize this bound on the discrepancy.

The key and most difficult part in Laczkovich's proof is the following theorem.

Theorem 3.4 (Laczkovich [1992b]). *Let $A^*, B^* \subset \mathbb{Z}^d$ and suppose that there are $\alpha, \varepsilon, c > 0$ such that*

$$\Big| |A^* \cap Q| - \alpha |Q| \Big| \le c N^{d-1-\varepsilon},$$

$$\Big| |B^* \cap Q| - \alpha |Q| \Big| \le c N^{d-1-\varepsilon}$$

for every cube $Q \subset \mathbb{Z}^d$ of side length N. Then there is a bijection $f : A^* \to B^*$ such that $\|f(k) - k\|_\infty \le C$ for every $k \in A^*$, where the constant C depends only on α, ε, c and d.

By Lemma 1.19, the existence of this bijection $f : A^* \to B^*$ is equivalent to the existence of a perfect matching in the bi-partite graph $\Gamma(A^*, B^*)$ where $a \in A^*$ is connected to $b \in B^*$ if $\|a - b\|_\infty \le C$. The main part of Laczkovich's proof is checking that Hall's condition is satisfied (for large enough C).

Finally, it is easy to see that Theorem 3.4 and Lemma 3.3 imply Theorem 1.15.

Remark 3.5. Laczkovich's estimate is that about 10^{40} pieces are enough to equidecompose the disc to the square.

4 Circle-squaring with measurable functions

The following result was proved independently by Laczkovich [1996] and Wehrung [1992]. See also Laczkovich [2002, Theorem 9.6].

Theorem 4.1. *Suppose A and B are Lebesgue measurable sets in $\mathbb{R}^n$. If A and B are equidecomposable under isometries g_1, ..., g_m from an amenable group G (for example, they are all translations), then there are non-negative Lebesgue measurable functions $f_1, \dots, f_m$ such that*

$$1_A = f_1 + \dots + f_m$$
$$1_B = f_1 \circ g_1^{-1} + \dots + f_m \circ g_m^{-1}.$$

(In such cases we say that A and B are continuously equidecomposable with Lebesgue measurable functions $f_1, \dots, f_m$.)

The proof is very short and enlightening so we include it here. The idea is that one can approximate non-measurable sets by Lebesgue measurable functions by considering convolutions with Lebesgue measurable mollifiers where the integration is with respect to a (finitely additive) G-invariant measure defined on all subsets of $\mathbb{R}^n$.

Proof. Since the Lebesgue measure is isometry invariant, Theorem 1.7 gives us a G-invariant finitely additive measure μ defined on all subsets of $\mathbb{R}^n$ that extends Lebesgue measure. (For $n = 1, 2$, this measure μ can be taken to be the Banach measure.)

By the assumption on equidecomposability, there is a partition $A = A_1 \cup \dots \cup A_m$ such that $B = g_1(A_1) \cup \dots \cup g_m(A_m)$. Let $B(x, r)$ denote the open ball around x of radius r.

Consider the sequences of densities

$$f_i^k(x) = \frac{\mu(B(x, 1/k) \cap A_i)}{\mu(B(x, 1/k))}$$

where k is a positive integer, and the denominator does not depend on x. The functions f_i^k are measurable, because they are Lipschitz. Notice that

$$\sum_{i=1}^{m} f_i^k(x) = \frac{\mu(B(x,1/k) \cap A)}{\mu(B(x,1/k))}$$

and, since μ is invariant under the isometries g_i,

$$\sum_{i=1}^{m} f_i^k(g_i^{-1}(x)) = \frac{\mu(B(x,1/k) \cap B)}{\mu(B(x,1/k))}.$$

Take a subsequence (k_j) such that the weak-* limits exist:

$$f_i = \lim_{j \to \infty} f_i^{k_j}.$$

We obtain measurable functions f_i (defined almost everywhere) that satisfy

$$\sum_{i=1}^{m} f_i = 1_A \tag{4-1}$$

and

$$\sum_{i=1}^{m} f_i \circ g_i^{-1} = 1_B \tag{4-2}$$

almost everywhere (since A and B are Lebesgue measurable). We claim that one can modify these functions on a nullset such that the equalities hold everywhere, using the original equidecompositions. Indeed, consider a nullset that is closed under the countable group generated by the isometries g_i and contains the points where (4-1) or (4-2) fail or the functions f_i are not defined. On this nullset, redefine f_i to be the characteristic function of A_i. □

5 Measurable circle-squaring

Theorem 5.1 (Grabowski, Máthé, and Pikhurko [2017]). *Let A and B be bounded measurable sets in $\mathbb{R}^n$ with equal positive Lebesgue measure such that $\overline{\dim}_M(\partial A) < n$ and $\overline{\dim}_M(\partial B) < n$. Then A and B are equidecomposable using translations with measurable pieces; that is, there exist partitions $A = \cup_{i=1}^k A_i$, $B = \cup_{i=1}^k B_i$ and translation vectors $x_i \in \mathbb{R}^n$ such that $B_i = A_i + x_i$.*

(Note that under the same assumptions, A and B are equidecomposable with Borel pieces by the result of A. S. Marks and S. T. Unger [2017].)

Note that it is enough to prove that A and B are equidecomposable up to nullsets with measurable pieces. Indeed, having such an equidecomposition, we can extend it and modify it on a nullset (which is invariant under our translations) using Theorem 1.15 to obtain a measurable equidecomposition of A to B with measurable pieces.

As in Section 3, we may assume that $A, B \subset \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We may also assume that A and B are disjoint.

Definition 5.2. Given a finite set $V = \{v_1, \dots, v_d\} \subset \mathbb{T}^n$ and a positive integer C, let

$$V_C = \left\{ \sum_{j=1}^{d} k_j v_j \ : \ |k_j| \le C \right\}.$$

As in Section 1.5, consider the bi-partite graph

$$\Gamma_{V_C}(A, B) = \{(a, b) \ : \ a \in A,\ b \in B,\ b - a \in V_C\}.$$

We may simply write $\Gamma(A, B)$ when V_C is clear from the context.

Similarly to Lemma 3.1, we have the following.

Lemma 5.3. *For any finite set of vectors $V = \{v_1, \dots, v_d\} \subset \mathbb{T}^n$ and any constant C, the following are equivalent.*

(i) A and B are equidecomposable with measurable pieces using translation vectors from V_C.

(ii) There exists a measurable bijection $f : A \to B$ such that

$$f(x) - x \in V_C \ \textit{ for every } x \in A.$$

In other words, there is a measurable perfect matching in $\Gamma_{V_C}(A, B)$. □

Recall that Laczkovich's proof of Theorem 1.15 relied on Hall's and Rado's theorem to conclude the existence of a perfect matching in the graph Γ_{V_C}. This graph has continuum many connected components as every connected component is contained by a coset of the subgroup generated by V. If there was a measurable set that intersected every coset in exactly one point, then the same proof would yield a measurable equidecomposition. Of course, such measurable set does not exist.

One of the ideas of the proof of Theorem 5.1 is to consider small Borel sets in $\mathbb{T}^n$ that intersect every coset in a sufficiently sparse but non-empty set, and use the points of these sparse sets as the origins of (local) coordinate systems in the cosets (that are isomorphic to $\mathbb{Z}^d$).

The measurable perfect matching in Γ_{V_C} (to be precise, the measurable almost perfect matching) is obtained by taking a limit of a sequence of measurable matchings that stabilises almost everywhere. These matchings are provided by an algorithm that improves our matchings by augmenting paths. The measurability of the matchings is an immediate consequence of the fact that our algorithm is local: whether there is an edge (a, b) in the i^{th} matching only depends on how the sets A, B and the graph $\Gamma_{V_C}(A, B)$ look like in the R_i neighbourhood of a (or b). Of course, $R_i \to \infty$.

The existence of this algorithm and the matchings rely on sufficient conditions on the discrepancy of the sets A and B in the cosets of V, similar to those needed by Laczkovich's proof. An extra ingredient that we need is the existence of short augmenting paths. We summarise these tools below.

Definition 5.4. Given $v = (v_1, \ldots, v_d) \in (\mathbb{T}^n)^d$, for $p \in \mathbb{Z}^d$, let

$$\langle v, p\rangle = v_1 p_1 + \ldots + v_d p_d \in \mathbb{T}^n.$$

For a set $P \subset \mathbb{Z}^d$, let

$$\langle v, P\rangle = \{\langle v, p\rangle \,:\, p \in P\} \subset \mathbb{T}^n.$$

When P is a product of intervals (i.e. sets of consecutive integers), we may refer to both P and $\langle v, P\rangle$ as a rectangle.

Lemma 5.5. *Fix any $\varepsilon > 0$. Let d be sufficiently large and let $v_1, \ldots, v_d$ be random independent uniformly chosen vectors in $\mathbb{T}^n$. Then, with probability 1, there is a positive integer C and $c > 0$ such that the following statements hold.*

1. *For any $x \in \mathbb{T}^n$ and any rectangle $R \subset \mathbb{Z}^d$ with maximal side length N,*

$$\Big| |A \cap (x + \langle v, R\rangle)| - |B \cap (x + \langle v, R\rangle)| \Big| \le c N^{d-1-\varepsilon}.$$

2. *Let R be a rectangle in $\mathbb{Z}^d$ with maximal side length N. Then, for every $x \in \mathbb{T}^n$, there is a matching inside $x + \langle v, R\rangle$, that is, inside*

$$\Gamma_{V_C}(A \cap (x + \langle v, R\rangle),\ B \cap (x + \langle v, R\rangle))$$

such that at most $c N^{d-1-\varepsilon}$ points are unmatched.

3. *Let Q be any cube in $\mathbb{Z}^d$ of side length N. Let $\mathfrak{M}$ be a matching inside $x + \langle v, Q\rangle$. If there are points both in A and B that are unmatched by $\mathfrak{M}$ inside $x + \langle v, Q\rangle$ then there is an augmenting path connecting two unmatched points of length at most N.*

Proofs of these (or similar) statements can be found in Grabowski, Máthé, and Pikhurko [2017]. All these are generalizations and strengthenings of statements of Laczkovich [1992b]. The third statement uses the fact that not only Hall's condition

$$|N(X)| \geq |X| \quad (X \subset A \text{ finite})$$

holds in the graph, but it holds relative to a large enough cube, moreover, it can be replaced (essentially) by the stronger inequality

$$|N(X)| \geq |X| + c'|\partial X| \geq |X| + |X|^{\frac{d-1}{d}}. \tag{5-1}$$

Here ∂X can be understood as those points x of X for which $x + V_1 \not\subset X$. Note that the exponent $(d-1)/d$ is optimal by the isoperimetric inequality. (To be correct, ∂X in (5-1) should be replaced by boundary of a smoothened version of X.)

Corollary 5.6. *Let Q be any cube in $\mathbb{Z}^d$ of side length N. Let M be a matching inside $x + \langle v, Q \rangle$ such that the number of unmatched points is t. Using augmenting paths we can define a new matching M' such that only $cN^{d-1-\varepsilon}$ points will be unmatched and that $|M \triangle M'| \leq tN$.*

Proof of Corollary 5.6. We can improve the matching by augmenting paths; combining part Item 2. and Item 3. of Lemma 5.5 concludes the proof. □

6 Open problems

Question 6.1 (Borel Banach–Tarski)**.** *Let $n \geq 3$. Let $A, B \in \mathbb{R}^n$ be bounded Borel sets of non-empty interior of equal Lebesgue measure. Is A equidecomposable to B using Borel pieces?*

Note that the answer is affirmative if A and B have nice boundaries by the theorem of A. S. Marks and S. T. Unger [2017]; that is, if the boundaries have upper Minkowski dimension less than n. (On the other hand, for $n = 1, 2$ the answer is negative. Laczkovich [2003] gave examples of Jordan domains in the plane that are not even equidecomposable with arbitrary pieces.)

The disc and the square are equidecomposable using Borel pieces, in particular, with pieces that are both Lebesgue and Baire measurable. However, in some sense, the "true" combination of Lebesgue and Baire measurability is Jordan measurability. Recall that a bounded set is Jordan measurable if its boundary has Lebesgue measure zero.

Question 6.2. *Is it possible to equidecompose the disc to a square by Jordan measurable pieces?*

Note that the result of Dubins, Hirsch, and Karush [1963] says that the pieces cannot be Jordan domains.

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References

Stefan Banach (1923). "Sur le problème de la mesure". *Fundamenta Mathematicae* 4.1, pp. 7–33 (cit. on p. 1732).

Stefan Banach and Alfred Tarski (1924). "Sur la décomposition des ensembles de points en parties respectivement congruentes". *Fund. math* 6.1, pp. 244–277 (cit. on p. 1732).

Bachir Bekka, Pierre de la Harpe, and Alain Valette (2008). *Kazhdan's property (T)*. Vol. 11. New Mathematical Monographs. Cambridge University Press, Cambridge, pp. xiv+472. MR: 2415834.

Rémi Boutonnet, Adrian Ioana, and Alireza Salehi Golsefidy (2017). "Local spectral gap in simple Lie groups and applications". *Invent. Math.* 208.3, pp. 715–802. MR: 3648974 (cit. on p. 1741).

Randall Dougherty and Matthew Foreman (1992). "Banach-Tarski paradox using pieces with the property of Baire". *Proc. Nat. Acad. Sci. U.S.A.* 89.22, pp. 10726–10728. MR: 1190902 (cit. on p. 1733).

– (1994). "Banach-Tarski decompositions using sets with the property of Baire". *J. Amer. Math. Soc.* 7.1, pp. 75–124. MR: 1227475 (cit. on p. 1733).

V. G. Drinfel'd (1984). "Finitely-additive measures on S^2 and S^3, invariant with respect to rotations". *Funktsional. Anal. i Prilozhen.* 18.3, p. 77. MR: 757256 (cit. on pp. 1734, 1738).

Lester Dubins, Morris W. Hirsch, and Jack Karush (1963). "Scissor congruence". *Israel J. Math.* 1, pp. 239–247. MR: 0165424 (cit. on pp. 1735, 1748).

Gábor Elek and Gábor Lippner (2010). "Borel oracles. An analytical approach to constant-time algorithms". *Proc. Amer. Math. Soc.* 138.8, pp. 2939–2947. MR: 2644905 (cit. on pp. 1734, 1739).

R. J. Gardner (1985). "Convex bodies equidecomposable by locally discrete groups of isometries". *Mathematika* 32.1, pp. 1–9. MR: 817100 (cit. on p. 1735).

Łukasz Grabowski, András Máthé, and Oleg Pikhurko (Aug. 2014). "Measurable equidecompositions via combinatorics and group theory". arXiv: 1408.1988 (cit. on p. 1737).

– (Jan. 2016). "Measurable equidecompositions for group actions with an expansion property". arXiv: 1601.02958 (cit. on pp. 1734, 1737, 1741).

– (2017). "Measurable circle squaring". *Ann. of Math. (2)* 185.2, pp. 671–710. MR: 3612006 (cit. on pp. 1736, 1745, 1748).

M. Laczkovich (1990). "Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem". *J. Reine Angew. Math.* 404, pp. 77–117. MR: 1037431 (cit. on p. 1735).

– (2003). "Equidecomposability of Jordan domains under groups of isometries". *Fund. Math.* 177.2, pp. 151–173. MR: 1992530 (cit. on p. 1748).

Miklós Laczkovich (1992a). "Decomposition of sets with small boundary". *J. London Math. Soc. (2)* 46.1, pp. 58–64. MR: 1180882 (cit. on p. 1735).

– (1992b). "Uniformly spread discrete sets in $\mathbf{R}^d$". *J. London Math. Soc. (2)* 46.1, pp. 39–57. MR: 1180881 (cit. on pp. 1743, 1748).

– (1993). "Decomposition of sets of small or large boundary". *Mathematika* 40.2, pp. 290–304. MR: 1260893 (cit. on p. 1735).

– (1996). "Decomposition using measurable functions". *C. R. Acad. Sci. Paris Sér. I Math.* 323.6, pp. 583–586. MR: 1411046 (cit. on p. 1744).

– (2002). "Paradoxes in measure theory". In: *Handbook of measure theory, Vol. I, II*. North-Holland, Amsterdam, pp. 83–123. MR: 1953492 (cit. on pp. 1737, 1744).

Russell Lyons and Fedor Nazarov (2011). "Perfect matchings as IID factors on non-amenable groups". *European J. Combin.* 32.7, pp. 1115–1125. MR: 2825538 (cit. on pp. 1734, 1739).

G. A. Margulis (1980). "Some remarks on invariant means". *Monatsh. Math.* 90.3, pp. 233–235. MR: 596890 (cit. on pp. 1734, 1738).

Andrew S. Marks and Spencer T. Unger (2017). "Borel circle squaring". *Ann. of Math. (2)* 186.2, pp. 581–605. MR: 3702673 (cit. on pp. 1736, 1746, 1748).

Andrew Marks and Spencer Unger (2016). "Baire measurable paradoxical decompositions via matchings". *Adv. Math.* 289, pp. 397–410. MR: 3439691.

Jan Mycielski (1979). "Finitely additive invariant measures. I". *Colloq. Math.* 42, pp. 309–318. MR: 567569 (cit. on p. 1733).

Heiner Niederreiter and Jörg M. Wills (1975). "Diskrepanz und Distanz von Maßen bezüglich konvexer und Jordanscher Mengen". *Math. Z.* 144.2, pp. 125–134. MR: 0376588 (cit. on p. 1743).

R. Rado (1949). "Factorization of even graphs". *Quart. J. Math., Oxford Ser.* 20, pp. 95–104. MR: 0030202 (cit. on pp. 1737, 1741).

Dennis Sullivan (1981). "For $n > 3$ there is only one finitely additive rotationally invariant measure on the n-sphere defined on all Lebesgue measurable subsets". *Bull. Amer. Math. Soc. (N.S.)* 4.1, pp. 121–123. MR: 590825 (cit. on pp. 1734, 1738).

Alfred Tarski (1925). "Probléme 38". *Fund. Math* 7, p. 381.

Grzegorz Tomkowicz and Stan Wagon (2016). *The Banach-Tarski paradox*. Second. Vol. 163. Encyclopedia of Mathematics and its Applications. With a foreword by Jan Mycielski. Cambridge University Press, New York, pp. xviii+348. MR: 3616119 (cit. on p. 1737).

Stan Wagon (1985). *The Banach-Tarski paradox*. Vol. 24. Encyclopedia of Mathematics and its Applications. With a foreword by Jan Mycielski. Cambridge University Press, Cambridge, pp. xvi+251. MR: 803509 (cit. on p. 1737).

Friedrich Wehrung (1992). "Injective positively ordered monoids. I, II". *J. Pure Appl. Algebra* 83.1, pp. 43–82, 83–100. MR: 1190444 (cit. on p. 1744).

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