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TOEPLITZ METHODS IN COMPLETENESS AND SPECTRAL PROBLEMS

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Abstract

We survey recent progress in the gap and type problems of Fourier analysis obtained via the use of Toeplitz operators in spaces of holomorphic functions. We discuss applications of such methods to spectral problems for differential operators.

1 Introduction

This note is devoted to a discussion of several well-known problems from the area of the Uncertainty Principle in Harmonic Analysis (UP) and their reformulations in terms of kernels of Toeplitz operators. The Toeplitz approach first appeared in a series of papers by Hruščёv, Nikolskii, and Pavlov [1981], Nikolskii [1986], and Pavlov [1979], where it was applied to problems on Riesz bases and sequences of reproducing kernels in model spaces. It was extended to problems on completeness and spectral analysis of differential operators in Makarov and Poltoratski [2005, 2010] and used in several recent papers in the area.

The Beurling-Malliavin (BM) theory, created in the early 1960s to solve the problem on completeness of exponential functions in $L^2([0,1])$ Beurling and Malliavin [1962], Beurling and Malliavin [1967], and Beurling [1989], remains one of the deepest ingredients of UP. Although the theory did solve the classical problem it aimed to solve, the need for expansions to broader classes of function spaces, other systems of functions and related problems immediately appeared. At present, most of such problems remain open and suitable extensions of BM theory are yet to be found.

A search for such an extension in the settings of completeness problems and spectral problems for differential operators served as initial motivation for Makarov and Poltoratski [2005, 2010]. The first paper contained a list of problems of UP which could be translated into problems on injectivity of Toeplitz operators. It included problems on completeness

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of Airy and Bessel functions and spectral problems for 1D Schrödinger equation. The main result of the second paper gives a condition for the triviality of a kernel of a Toeplitz operator with a unimodular symbol, which can be viewed as an extension of the BM theorem.

In this note we will discuss the BM problem as well as the so-called gap and type problems of Fourier analysis, along with their reformulations in the Toeplitz language. We will show how such reformulations become a part of a new circle of problems on the partial ordering relations for the set of meromorphic inner functions (MIF) induced by Toeplitz operators. At the end of the paper we include applications to inverse spectral problems for Krein's canonical systems.

2 Completeness problems and spectral gaps

2.1 Beurling-Malliavin, gap and type problems. One of the canonical questions of Harmonic Analysis is whether any function from a given space can be approximated by linear combinations of functions from a selected collection of harmonics. The most common choices for the space are weighted L^p-spaces or spaces with weighted uniform norm of Bernstein's type. The role of harmonics can be played by polynomials, complex exponentials, solutions to various differential equations or special functions, such as Airy or Bessel functions, etc. A system of functions is called complete if finite linear combinations of its functions are dense in the space.

Let $\Lambda = \{\lambda_n\}$ be a discrete (without finite accumulation points) sequence of complex numbers. Denote by $\mathcal{E}_\Lambda$ the system of complex exponentials with frequencies from Λ:

$$\mathcal{E}_\Lambda = \{e^{i\lambda_n z},\ \lambda_n \in \Lambda\}.$$

The main question answered by BM theory is for what Λ will $\mathcal{E}_\Lambda$ be complete in $L^2([0,a])$. More precisely, let us define *the radius of completeness* of Λ as

$$R(\Lambda) = \sup\{a \mid \mathcal{E}_\Lambda \text{ is complete in } L^2([0,a])\}$$

and as 0 if the set is empty. BM theory provided a formula for $R(\Lambda)$ in terms of the exterior density of the sequence, defined as follows.

A sequence of disjoint intervals I_n on the real line is called long if

$$\sum \frac{|I_n|^2}{1+\operatorname{dist}^2(0,I_n)} = \infty, \tag{2-1}$$

and short otherwise. For a discrete sequence of real points Λ we define its exterior (BM) density as

$$D(\Lambda) = \sup\{d \mid \exists \text{ long sequence } \{I_n\} \text{ such that },\ \forall n,\ \#(\Lambda \cup I_n) > d|I_n|\}.$$

If Λ is a complex sequence, assuming without loss of generality that it has no purely imaginary points, we define $D(\Lambda)$ to be the density of the real sequence $1/(\Re\frac{1}{\lambda_n})$.

Theorem 1 (Beurling and Malliavin [1962] and Beurling and Malliavin [1967]).

$$R(\Lambda) = 2\pi D(\Lambda).$$

Since its appearance in the early 1960s, the BM theorem above and several ingredients of its proof had major impact on Harmonic Analysis. At present, new applications of the theorem continue to emerge in adjacent fields including Fourier analysis and spectral theory. At the same time, the search for generalizations of the BM theorem to other function spaces, several variables, other families of functions, etc., still continues with most of such problems remaining open.

A similar, and in some sense 'dual' completeness problem, is the so-called type problem which can be formulated as follows. For $a > 0$ denote by $\mathcal{E}_a$ the system of exponentials

$$\mathcal{E}_a = \{e^{isz},\ s \in [-a, a]\}.$$

For a finite positive Borel measure μ on $\mathbb{R}$ the exponential type of μ is defined as

$$\mathbf{T}_\mu = \inf\{a \mid \mathcal{E}_a \text{ is complete in } L^2(\mu)\}.$$

The problem of finding $\mathbf{T}_\mu$ in terms of μ appears in several adjacent areas of analysis and was studied by N. Wiener, A. Kolmogorov and M. Krein in connection with prediction theory and spectral problems for differential operators. For further information on such connections and problem's history see for instance Borichev and Sodin [2011] and Poltoratski [2015b]. We will return to the type problem in Section 2.5.

If the system $\mathcal{E}_a$ is incomplete in $L^p(\mu),\ p > 1$ then there exists $f \in L^q(\mu),\ \frac{1}{p}+\frac{1}{q} = 1$, annihilating all functions from $\mathcal{E}_a$. Equivalently for the Fourier transform $\widehat{f\mu}$ of the measure $f\mu$ we have

$$\widehat{f\mu}(s) = \frac{1}{\sqrt{2\pi}}\int e^{-ist} f(t)d\mu(t) = 0$$

for all $s \in [-a, a]$. Hence, the problem translates into finding an L^q-density f such that $f\mu$ has a gap in the Fourier spectrum (spectral gap) containing the interval $[-a, a]$. It turns out that before solving the type problem one needs to solve this version of the gap problem for $q = 1$, which is no longer equivalent to L^p-completeness. This is the so-called gap problem, whose recent solution we discuss in Section 2.3.

2.2 Toeplitz kernels. Recall that the Toeplitz operator T_U with a symbol $U \in L^\infty(\mathbb{R})$ is the map

$$T_U : H^2 \to H^2, \qquad F \mapsto P_+(UF),$$

where P_+ is the Riesz projection, i.e. the orthogonal projection from $L^2(\mathbb{R})$ onto the Hardy space H^2 in the upper half-plane $\mathbb{C}_+$. Passing from a function in H^2 to its non-tangential boundary values on $\mathbb{R}$, H^2 can be identified with a closed subspace of $L^2(\mathbb{R})$ which makes the Riesz projection correctly defined.

We will use the following notation for kernels of Toeplitz operators (or *Toeplitz kernels*) in H^2:

$$N[U] = \ker T_U.$$

A bounded analytic function in $\mathbb{C}_+$ is called inner if its boundary values on $\mathbb{R}$ have absolute value 1 almost everywhere. Meromorphic inner functions (MIFs) are those inner functions which can be extended as meromorphic functions in the whole complex plane. MIFs play a significant role in applications to spectral problems for differential operators, see for instance Makarov and Poltoratski [2005].

If θ is an inner function we denote by K_θ the so-called model space of analytic functions in the upper half-plane defined as the orthogonal complement in H^2 to the subspace θH^2, $K_\theta = H^2 \ominus \theta H^2$. An important observation is that for the Toeplitz kernel with the symbol $\bar\theta$ we have $N[\bar\theta] = K_\theta$ for any inner θ.

Along with H^2-kernels of Toeplitz operators, one may consider kernels $N^p[U]$ in other Hardy classes H^p, the kernel $N^{1,\infty}[U]$ in the 'weak' space $H^{1,\infty} = H^p \cap L^{1,\infty}$, $0 < p < 1$, or the kernel in the Smirnov class $\mathfrak{N}^+(\mathbb{C}_+)$, defined as

$$N^+[U] = \{f \in \mathfrak{N}^+ \cap L^1_{loc}(\mathbb{R}) : \bar U \bar f \in \mathfrak{N}^+\}$$

for $\mathfrak{N}^+$ and similarly for other spaces. If θ is a meromorphic inner function, $K^+_\theta = N^+[\bar\theta]$ can also be considered. For more on such kernels see Makarov and Poltoratski [2005] and Poltoratski [2015b].

Now let us discuss reformulations of the BM, gap and type problems mentioned above in the language of Toeplitz kernels, starting with the gap problem. One of the ways to state the gap problem is as follows.

Denote by M the set of all finite complex measures on $\mathbb{R}$. For a closed subset of real line X define its gap characteristic $\mathbf{G}_X$ as

$$\mathbf{G}_X = \sup\{a \mid \exists\, \mu \in M,\ \mu \not\equiv 0,\ \operatorname{supp}\mu \subset X,\ \text{ such that } \hat\mu = 0 \text{ on } [0,a]\}. \tag{2-2}$$

The problem is to find a formula for $\mathbf{G}_X$ in terms of X. Notice that the version of the problem discussed in the previous section, where for a fixed measure μ one looks for the supremum of the size of the spectral gap of $f\mu$ taken over all $f \in L^1(\mu)$ is equivalent to the last version of the problem with $X = \operatorname{supp}\mu$, see Proposition 1 in Poltoratski [2012] or Poltoratski [2015b].

While we postpone the formula for $\mathbf{G}_X$ until Section 2.3, here is the connection with the problem on injectivity of Toeplitz operators.

Here and throughout the paper for $a > 0$ we denote by S^a the exponential MIF, $S^a(z) = e^{iaz}$. For an inner function θ in the upper half-plane we denote by $\operatorname{spec}_\theta$ the closed subset of $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of points at which the non-tangential limits of θ are equal to 1. Note that θ is a MIF if and only if $\operatorname{spec}_\theta$ is a discrete set.

If once again $X \subset \mathbb{R}$ is a closed set, denote

$$N_X = \sup\{a \mid N[\bar{\theta} S^a] \neq 0 \text{ for some meromorphic inner } \theta,\ \operatorname{spec}_\theta \subset X\}.$$

Theorem 2 (Makarov and Poltoratski [2005], Section 2.1)**.**

$$\mathbf{G}_X = N_X.$$

Similar translations can be given for the BM and type problems. If $\Lambda \subset \mathbb{C}$ is a complex sequence of frequencies, denote by $\Lambda' \subset \mathbb{C}_+$ the sequence in the upper half-plane obtained from Λ by replacing all points from the lower half-plane with their complex conjugates and replacing every real point $\lambda \in \Lambda$ with $\lambda + i$. Note that if Λ' does not satisfy the Blaschke condition in $\mathbb{C}_+$ then the radius of completeness of Λ, $R(\Lambda)$ defined in the last section, is infinite. If Λ' does satisfy the Blaschke condition, denote by B_Λ the Blaschke product with zeros at Λ'. Then the radius of completeness of Λ satisfies

$$R(\Lambda) = \sup\{a \mid N[S^a B_\Lambda] = 0\}.$$

This formula provides a reformulation of the BM problem in the language of injectivity of Toeplitz operators and can be used to translate the BM theorem into a result in this area. Such a translation can then be used in applications and point to further generalizations of BM theory, see Makarov and Poltoratski [2005, 2010].

Let μ be a finite positive singular measure on $\mathbb{R}$ and let us denote by $K\mu$ its Cauchy integral

$$K\mu(z) = \frac{1}{2\pi i}\int \frac{d\mu(t)}{t - z}.$$

Let $\theta = \theta_\mu$ denote the Clark inner function corresponding to μ, i.e., the inner function in $\mathbb{C}_+$ defined as

$$\theta(z) = \frac{K\mu(z) - 1}{K\mu(z) + 1}.$$

The Toeplitz version of the type problem is obtained via the following formula for the type of μ (defined in the last section):

$$\mathbf{T}_\mu = \sup\{a \mid N[\bar{\theta} S^a] \neq 0\}.$$

Comparing this equation with the Toeplitz version of the formula for the radius of completeness above, one can see the 'duality' relation between the BM and type problems, which translate into problems on injectivity of Toeplitz operators with complex conjugate symbols. Such a connection between the two problems was known to the experts on the intuitive level for a long time, but now can be expressed in precise mathematical terms using the Toeplitz language.

For a more detailed discussion of the results mentioned in this section and further references see Poltoratski [2015b].

2.3 A formula for the gap characteristic of a set. To give a formula for $\mathbf{G}_X$ defined in Section 2.2 we need to start with the following definition.

Let

$$\ldots < a_{-2} < a_{-1} < a_0 = 0 < a_1 < a_2 < \ldots$$

be a two-sided sequence of real points. We say that the intervals $I_n = (a_n, a_{n+1}]$ form a short partition of $\mathbb{R}$ if $|I_n| \to \infty$ as $n \to \pm\infty$ and the sequence $\{I_n\}$ is short, i.e. the sum in (2-1) is finite.

Let $\Lambda = \{\lambda_n\}$ be a discrete sequence of distinct real points and let d be a positive number. We say that Λ is a d-uniform sequence if there exists a short partition $\{I_n\}$ such that

$$\Delta_n = \#(\Lambda \cap I_n) = d|I_n| + o(|I_n|) \text{ as } n \to \pm\infty \text{ (density condition)} \tag{2-3}$$

and

$$\sum_n \frac{\Delta_n^2 \log|I_n| - E_n}{1 + \operatorname{dist}^2(0, I_n)} < \infty \quad \text{(energy condition)} \tag{2-4}$$

where

$$E_n = E(\Lambda \cap I_n) = \sum_{\lambda_k, \lambda_l \in I_n,\ \lambda_k \neq \lambda_l} \log|\lambda_k - \lambda_l|.$$

Notice that the series in the energy condition is positive since every term in the sum defining E_n is at most $\log|I_n|$ and there are less than Δ_n^2 terms. Convergence of positive series is usually easier to analyze.

The quantity E_n admits a physical interpretation as the potential energy of a system of 'flat electrons' placed at points of $\Lambda \cap I_n$. The term $\Delta_n^2 \log |I_n|$ corresponds to the energy of Δ_n electrons spread uniformly over I_n, up to a $O(|I_n|^2)$-term, which is negligible in (2-4) due to the shortness of $\{I_n\}$. Hence (2-4) can be viewed as the condition of finite work, needed to transform our sequence into an arithmetical progression.

In regard to the gap problem, d-uniform sequences have the property that any such sequence can support a measure with a spectral gap of the size $d - \varepsilon$ for any $\varepsilon > 0$. Conversely, any discrete sequence with this property must contain a d-uniform sequence. Moreover, for a general closed set we have

Theorem 3 (Poltoratski [2012, 2015b]).

$$\mathbf{G}_X = \sup\{d \mid X \textit{ contains a } d\textit{-uniform sequence}\},$$

if the set on the right is non-empty and $\mathbf{G}_X = 0$ *otherwise.*

One of the main ingredients of the proof is the Toeplitz approach to the gap problem discussed in the last section and used earlier in Mitkovski and Poltoratski [2010]. De Branges' "Theorem 66" (Theorem 66, de Branges [1968]) in Toeplitz form, which in its turn uses the Krein-Milman result on the existence of extreme points in a convex set, provides a key step of the proof allowing to discretize the problem. Another key component is the idea by Beurling and Malliavin to set up an extremal problem in the real Dirichlet space in $\mathbb{C}_+$ to construct an extremal measure with the desired spectral gap, see Poltoratski [2012, 2015b].

2.4 Bernstein's weighted uniform approximation. We say that a function $W \geqslant 1$ on $\mathbb{R}$ is a *weight* if W is lower semi-continuous and $W(x) \to \infty$ as $|x| \to \infty$. Denote by C_W the space of all continuous functions f on $\mathbb{R}$ such that $f/W \to 0$ as $x \to \pm\infty$ with the norm

$$||f||_W = \sup_{\mathbb{R}} \frac{|f|}{W}. \tag{2-5}$$

The weighted approximation problem posted by Sergei Bernstein in 1924 Bernstein [1924] asks to describe the weights W such that polynomials are dense in C_W. Similar questions can be formulated for exponentials and other families of functions in place of polynomials. Further information and references on the history of Bernstein's problem can be found in two classical surveys by Akhiezer Ahiezer [1956] and Mergelyan [1956], a recent one by Lubinsky [2007], or in the first volume of Koosis' book Koosis [1988].

It turns out that to obtain a formula for the type of a finite positive measure it makes sense to follow the historic path and first consider the problem on completeness of exponentials in Bernstein's settings. As a byproduct, for the original question on completeness of polynomials one obtains the following formula.

If Λ is a discrete real sequence we will assume that it is enumerated in the natural order, i.e. $\lambda_n < \lambda_{n+1}$, non-negative elements are indexed with non-negative integers and negative elements with negative integers.

We say that a sequence $\Lambda = \{\lambda_n\}$ has (two-sided) upper density d if

$$\limsup_{A\to\infty} \frac{\#[\Lambda \cap (-A, A)]}{2A} = d.$$

If $d = 0$ we say that the sequence has zero density.

A discrete sequence $\Lambda = \{\lambda_n\}$ is called *balanced* if the limit

$$\lim_{N\to\infty} \sum_{|n|<N} \frac{\lambda_n}{1+\lambda_n^2} \tag{2-6}$$

exists.

Observe that any even sequence (any sequence Λ satisfying $-\Lambda = \Lambda$) is balanced. So is any two-sided sequence sufficiently close to even. At the same time, a one-sided sequence has to tend to infinity fast enough to be balanced (the series $\sum \lambda_n^{-1}$ must converge).

Let $\Lambda = \{\lambda_n\}$ be a balanced sequence of finite upper density. For each n, $\lambda_n \in \Lambda$, put

$$p_n = \frac{1}{2}\left[\log(1+\lambda_n^2) + \sum_{n\neq k,\ \lambda_k\in\Lambda} \log \frac{1+\lambda_k^2}{(\lambda_k-\lambda_n)^2}\right],$$

where the sum is understood in the sense of principle value, i.e. as

$$\lim_{N\to\infty} \sum_{0<|n-k|<N} \log \frac{1+\lambda_k^2}{(\lambda_k-\lambda_n)^2}.$$

We will call the sequence of such numbers $P = \{p_n\}$ the *characteristic sequence* of Λ.

Note that for a sequence of finite upper density the last limit exists for every n if and only if it exists for some n, if and only if the sequence is balanced.

Theorem 4 (Poltoratski [2015a,b]). *Let W be a weight such that C_W contains all polynomials. Polynomials are not dense in C_W if and only if there exists a balanced sequence $\Lambda = \{\lambda_n\}$ of zero density such that Λ and its characteristic sequence $P = \{p_n\}$ satisfy*

$$\sum W(\lambda_n) \exp(p_n) < \infty. \tag{2-7}$$

The proof is elementary in nature and the result is similar to de Branges' theorem from de Branges [1959] where the condition of completeness of polynomials is formulated in terms of existence of a certain entire function. In the next section we will pass from Bernstein's problem to the type problem mentioned in the introduction by first replacing the polynomials with exponentials. Such a replacement complicates the problem significantly. In particular, its solution presented below requires advanced tools of BM theory, which are not required for the above result.

2.5 Type formulas. Continuing our discussion of Bernstein's weighted uniform approximation from the last section, for a weight W we define the type of W as

$$\mathbf{T}_W = \inf\{a \mid \mathcal{E}_a \text{ is complete in } C_W\}.$$

We put $\mathbf{T}_W = 0$ if the last set is empty.

Theorem 5 (Poltoratski [2015b]).

$$\mathbf{T}_W = \sup\left\{d \mid \sum \frac{\log W(\lambda_n)}{1+\lambda_n^2} < \infty \textit{ for some } d\textit{-uniform sequence } \Lambda\right\},$$

if the set is non-empty, and 0 *otherwise.*

Via the connection between Bernstein's and L^p-approximation problems found by A. Bakan in Bakan [2008], the last statement immediately yields the following L^p-statement. For $p > 1$ We define the L^p-type of a measure, $\mathbf{T}^p_\mu$, similarly to the definition of $\mathbf{T}_\mu$ given in Section 2.1, but with $L^2(\mu)$ replaced with $L^q(\mu)$. In these notations, $\mathbf{T}_\mu = \mathbf{T}^2_\mu$. We say that W is a μ-weight if $\int W d\mu < \infty$.

Corollary 1 (Poltoratski [2013]). *Let μ be a finite positive measure on the line. Let $1 < p \leqslant \infty$ and $a > 0$ be constants.*

Then $\mathbf{T}^p_\mu \geqslant a$ if and only if for any μ-weight W and any $0 < d < a$ there exists a d-uniform sequence $\Lambda = \{\lambda_n\} \subset \operatorname{supp} \mu$ such that

$$\sum \frac{\log W(\lambda_n)}{1+\lambda_n^2} < \infty. \tag{2-8}$$

As one can see from this statement, $\mathbf{T}_\mu = \mathbf{T}^p_\mu$ for any $p > 1$, which came as a surprise to some of the experts. In view of this property, it makes sense to return to the notation $\mathbf{T}_\mu$ in our future statements. Note that the case of $p = 1$ constitutes the gap problem discussed above with a different solution.

A more convenient L^p-statement was recently given in Poltoratski [n.d.]. If $\Lambda = \{\lambda_n\} \subset \mathbb{R}$ is a discrete sequence of distinct points we denote by Λ^* the sequence of intervals λ^*_n such that each λ^*_n is centered at λ_n and has the length equal to one-third of the distance from λ_n to the rest of Λ. Note that then the intervals λ^*_n are pairwise disjoint.

Theorem 6. *Let μ be a finite positive measure on the line. Then*

$$\mathbf{T}_\mu = \max\{d \,|\, \exists\, d\text{-uniform } \Lambda \text{ such that } \sum \frac{\log \mu(\lambda_n^*)}{1+n^2} > -\infty\}$$

if the set is non-empty and $\mathbf{T}_\mu = 0$ otherwise.

2.6 Toeplitz order. Toeplitz operators provide universal language which can put many seemingly different problems from the area of UP into the same scale. Translations of known results and open problems from different areas into this universal language reveal surprising connections, point to correct generalizations and may indicate further directions for research. As was mentioned before, first such translations were found in the series of papers by Hruschev, Nikolski and Pavlov Hruščëv, Nikolskii, and Pavlov [1981] and Pavlov [1979] where problems on Riesz sequences and bases in model spaces were studied in terms of *invertibility* of related Toeplitz operators. In Makarov and Poltoratski [2005, 2010] connections between problems on completeness, uniqueness sets and spectral problems for differential operators with *injectivity* of Toeplitz operators were established. A recent attempt to systematize the problems on Toeplitz operators which emerge via this approach was undertaken in Poltoratski [2017]. The main idea is to define a partial order on the set of MIFs induced by Toeplitz operators and view several general questions of UP as questions on the properties of such an order. In this section we present a short overview of this approach.

Definition 1. *If θ is an inner function we define its (Toeplitz) dominance set $\mathfrak{D}(\theta)$ as*

$$\mathfrak{D}(\theta) = \{I \text{ inner} \,|\, N[\bar{\theta} I] \neq 0\}.$$

Every collection of sets admits natural partial ordering by inclusion. In our case, we consider dominance sets $\mathfrak{D}(\theta)$ as subsets of the set of all inner functions in the upper half-plane and the partial order $\subset$ on this collection. This partial order induces a preorder on the set of all inner functions in $\mathbb{C}_+$. Proceeding in a standard way, we can modify this preorder into a partial order by introducing equivalence classes of inner functions. The details of this definition are as follows.

Definition 2. *We will say that two inner functions I and J are Toeplitz equivalent, writing $I \overset{\text{T}}{\sim} J$, if $\mathfrak{D}(I) = \mathfrak{D}(J)$. This equivalence relation divides the set of all inner functions in $\mathbb{C}_+$ into equivalence classes. We call this relation* ***Toeplitz equivalence*** *(TE).*

Further, we introduce a partial order on these equivalence classes defining it as follows.

Definition 3. *We write $I \overset{\text{T}}{\leqslant} J$ (meaning that the equivalence class of I is 'less or equal' than the equivalence class of J) if $\mathfrak{D}(I) \subset \mathfrak{D}(J)$. We call this partial order on the set of inner functions in $\mathbb{C}_+$* ***Toeplitz order*** *(TO).*

Let B_n and B_k be Blaschke products of degree n and k correspondingly. Then $B_n \overset{\mathrm{T}}{\sim} B_k$ iff $n = k$ and $B_n \overset{\mathrm{T}}{<} B_k$ iff $n < k$. The relation becomes more interesting for infinite Blaschke products and singular inner functions. For instance, if J_μ and J_ν are two singular functions then $\nu - \mu \geq 0$ implies $J_\mu \overset{\mathrm{T}}{\leqslant} J_\nu$ but not vice versa as follows from an example by A. Alexandrov.

Now let us present some of the translations of the known problems mentioned in previous sections in the language of TO.

Recall that any MIF I has the form $I = B_\Lambda S^a$ where B is a Blaschke product with a discrete sequence of zeros Λ and $S^a = e^{iaz}$ is the exponential function. Put $r(I) = D^*(\Lambda) + a$. The BM Theorem 1 discussed in Section 2.1 is equivalent to the following statement.

Theorem 7. *For any MIF I,*

$$I \overset{\mathrm{T}}{\leqslant} S^b \;\Rightarrow\; r(I) \leq b$$

and

$$r(I) < b \;\Rightarrow\; I \overset{\mathrm{T}}{<} S^b.$$

As we can see, the Beurling-Malliavin formula gives a metric condition for TO in the very specific case when one of the functions is the exponential function. Similar descriptions for more general classes of inner functions, especially those appearing in applications to completeness problems and spectral analysis remain mostly open. Below we present one of such extensions found in Makarov and Poltoratski [2010].

As was shown in Makarov and Poltoratski [2005] the class of MIFs with polynomially growing arguments appears naturally in a number of applications including completeness problems for Airy and Bessel functions, spectral problems for regular Schrödinger operators and Dirac systems, etc. An analog of Theorem 1 proved in Makarov and Poltoratski [2010] can be applied to some of such problems. Here we present an equivalent reformulation similar to Theorem 7.

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\gamma(\mp\infty) = \pm\infty$. i.e.,

$$\lim_{x\to-\infty} \gamma(x) = +\infty, \qquad \lim_{x\to+\infty} \gamma(x) = -\infty.$$

Define γ^* to be the smallest non-increasing majorant of γ:

$$\gamma^*(x) = \max_{t\in[x,+\infty)} \gamma(t).$$

The family of intervals $BM(\gamma) = \{I_n\}$ is defined as the collection of the connected components of the open set

$$\{x \in \mathbb{R} |\; \gamma(x) \neq \gamma^*(x)\}.$$

Let $\kappa \geq 0$ be a constant. We say that γ is *κ-almost decreasing* if

$$\sum_{I_n \in BM(\gamma)} (\operatorname{dist}(I_n, 0) + 1)^{\kappa-2} |I_n|^2 < \infty. \tag{2-9}$$

We define an argument of a MIF I on $\mathbb{R}$ is a real analytic function ψ such that $I = e^{i\psi}$.

Theorem 8 (Makarov and Poltoratski [2010] and Poltoratski [2017]). *Let U be a MIF with $|U'| \asymp x^\kappa$,*
$\kappa \geq 0$, $\gamma = \arg U$ on $\mathbb{R}$. Let J be another MIF, $\sigma = \arg J$ on $\mathbb{R}$.

I) If $\sigma - (1-\varepsilon)\gamma$ is κ-almost decreasing, then $J \overset{\mathrm{T}}{<} U$;
II) If $J \overset{\mathrm{T}}{\leqslant} U$ then $\sigma - (1+\varepsilon)\gamma$ is κ-almost decreasing.

Let us point out that even finding an analog for the above statement for $\kappa < 0$ presents an open problem. MIFs with $\kappa < 0$ appear in some of the applications mentioned in Makarov and Poltoratski [2010].

In regard to the type problem we have the following translation. We will denote by θ_μ the inner function with Clark measure μ as defined in Section 2.2.

Theorem 9 (Poltoratski [2017]).

$$\mathbf{T}_\mu = \sup\{a | S^a \overset{\mathrm{T}}{\leqslant} \theta_\mu\}.$$

As one can see, the solutions to BM and type problems give formulas which can be used to compare MIFs with respect to TO in several particular situations. In both cases the functions are compared with the exponential function S^a. The extension found in Makarov and Poltoratski [2010] replaces S^a with a function with polynomially growing argument. Apart from the results mentioned here and a few elementary examples contained in Poltoratski [2017], giving metric conditions on MIFs I, J necessary or sufficient for $I \overset{\mathrm{T}}{\sim} J$ present a collection of open problems with applications in several areas of UP.

3 Inverse spectral problems and truncated Toeplitz operators

3.1 Canonical systems. Consider a 2×2 differential system with a spectral parameter $z \in \mathbb{C}$:

$$\Omega \dot{X} = zH(t)X - Q(t)X, \quad -\infty < t_- < t < t_+ \tag{3-1}$$

where

$$X(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We assume the (real-valued) coefficients to satisfy

$$H,\ Q \in L^1_{loc}((t_-, t_+) \to \mathbb{R}^{2\times 2}).$$

By definition, a solution $X = X_z(t)$ is a $C^2((t_-, t_+))$-function satisfying the equation. An initial value problem (IVP) for the system (3-1) can be given via an initial condition $X(t_-) = x,\ x \in \mathbb{C}^2$. Let us immediately point out the following well-known property.

Theorem 10. *Every IVP for* (3-1) *has a unique solution on* (t_-, t_+). *For each fixed* t*, this solution presents an entire function* $E_t(z) = u_z(t) - iv_z(t)$ *of exponential type.*

Let us further assume that $H(t),\ Q(t)$ are real symmetric locally summable matrix-valued functions and that $H(t) \geq 0$. The Hilbert space $L^2(H)$ consists of (equivalence classes) of vector-functions with

$$||f||^2_H = \int_{t_-}^{t_+} < Hf, f > dt < \infty.$$

The system (3-1) is an eigenvalue equation $DX = zX$ for the (formal) differential operator

$$D = H^{-1}\left[\Omega\frac{d}{dt} + Q\right].$$

Many important classes of second order differential equations can be rewritten in the form of (3-1). Consider, for instance, the Schrödinger equation $-\ddot{u} = zu - qu$ on an interval. Put $v = -\dot{u}$ and $X = (u, v)^T$ to obtain

$$\Omega\dot{X} = z\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}X - \begin{pmatrix}q & 0\\ 0 & -1\end{pmatrix}X.$$

Similarly one can use (3-1) to rewrite string, Dirac and several other known equations. A discrete version of (3-1) where the function H consists of so-called jump intervals can be used to express difference equations and Jacobi matrices. Well studied Dirac systems are usually written in the form of a 2×2 system. In that case $H \equiv I$ and the general form is

$$\Omega\dot{X} = z\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}X - \begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}X,\ q_{12} = q_{21}.$$

The "standard form" of a Dirac system is with $Q = \begin{pmatrix}-q_2 & -q_1\\ -q_1 & q_2\end{pmatrix}$. In this case $f = q_1 + iq_2$ is the potential function.

Among all self-adjoint systems discussed above we single out a subclass of so-called canonical systems with $Q \equiv 0$:

(3-2) $$\Omega\dot{X} = zH(t)X.$$

This turns out to be the right object for the theory. The first key observation is that a general self-adjoint system can be reduced to canonical form. To perform such a reducion first solve $\Omega\dot{V} = -QV$ for a 2×2 matrix valued function V and make a substitution $X = VY$. Then (3-1) becomes

$$\Omega\dot{Y} = z\,[V^*HV]\,Y.$$

For instance, a Dirac system with real potential f becomes a canonical system with Hamiltonian

$$H^{CS} = \begin{pmatrix} e^{-2\int_0^t f} & 0 \\ 0 & e^{-2\int_0^t f} \end{pmatrix}.$$

When analyzing a second order equation one starts with a self-adjoint form of the system (3-1) into which the equation can be easily converted. Further conversion into the canonical form is necessary in the theory because only for canonical systems (after some additional normalizations) the theory provides a one-to-one correspondence with de Branges spaces of entire functions defined below. In the preliminary (3-1)-form two different systems may correspond to the same space.

3.2 De Branges spaces. We call an entire function E an an Hermite-Biehler (de Branges) function if it satisfies

$$|E(z)| > |E(\bar{z})|$$

for all $z \in \mathbb{C}_+$. As before we denote by H^2 the Hardy space in $\mathbb{C}_+$.

For an entire funcion $G(z)$ we denote $G^\#(z) = \bar{G}(\bar{z})$. For every Hermite-Biehler (HB) function E we define the space $B(E)$ to be the Hilbert space of entire functions F such that

$$F/E,\; F^\#/E \in H^2.$$

The Hilbert structure in the space is inherited from H^2, i.e. if $F, G \in B(E)$ then

$$< F, G >_{B(E)} = < F/E, G/E >_{H^2} = \int_{\mathbb{R}} F(x)\bar{G}(x)\frac{dx}{|E|^2}.$$

The well-known Paley-Wiener spaces of entire functions PW_a give a particular example of de Branges spaces $B(E)$ with $E(z) = e^{-iaz},\ a > 0$.

One of the fundamental properties of de Branges' spaces is that they admit an equivalent axiomatic definition. A similar definition with a slightly different set of axioms (for a slightly different space) was earlier found by Krein.

Theorem 11 (de Branges [1968]).

Suppose that H is a Hilbert space of entire functions that satisfies

(A1) $F \in H, F(\lambda) = 0 \Rightarrow F(z)(z - \bar{\lambda})/(z - \lambda) \in H$ *with the same norm*
(A2) $\forall \lambda \notin \mathbb{R}$, *the point evaluation is bounded*
(A3) $F \to F^{\#}$ *is an isometry*
Then $H = B(E)$ *for some* $E \in HB$.

Let E be an Hermite-Biehler function. Consider entire functions

$$A = (E + E^{\#})/2 \text{ and } B = (E - E^{\#})/2i.$$

The space $B(E)$ is a reproducing kernel space, i.e. any $\lambda \in \mathbb{C}$, there exists $K_\lambda \in B(E)$ such that $F(\lambda) =< F, K_\lambda >$ for any $F \in B(E)$. The kernels are given by the formula

$$K_\lambda(z) = \frac{1}{\pi} \frac{B(z)\bar{A}(\lambda) - A(z)\bar{B}(\lambda)}{z - \bar{\lambda}}.$$

Each HB-function E gives rise to a MIF $\theta_E(z) = \frac{E^{\#}(z)}{E(z)}$. Conversely, for each MIF θ there exists an HB-function E such that $\theta = \theta_E$. The model space K_θ defined in Section 2.2 is related to the de Branges space $B(E)$ via the simple identity $B(E) = EK_\theta$ with the map $f \mapsto Ef$ defining an isometric isomorphism between the spaces.

The following connection between canonical systems and de Branges spaces translates spectral problems for various classes of second order differential equations into the language of complex analysis.

For the sake of brevity here we will consider only canonical systems (3-2) without ”jump intervals”, i.e. without intervals where H is a constant matrix of rank 1. This assumption is made in many survey articles on the subject as it simplifies the main statements in the theory. At the same time, the case of jump intervals allows one to include discrete models, such us difference equations, Jacobi matrices, orthogonal polynomials, etc., into the scope of Krein - de Branges theory.

Consider a canonical system (without jump intervals) with any real initial condition at t_-. Denote the solution by $X_z(t) = (A_t(z), B_t(z))$. For each fixed t consider the entire function $E_t(z) = A_t(z) - iB_t(z)$. The following statement connects canonical systems with HB-functions and de Branges spaces, see de Branges [1968] and Makarov and Poltoratski [2005].

Theorem 12. *For any fixed* t, $E_t(z)$ *is a Hermit-Biehler entire function. The map* W *defined as* $WX_z = K^t_{\bar{z}}$ *extends unitarily to the map from* $L^2(H, (t_-, t))$ *to the de Branges space* $B(E_t)$ *(Weyl Transform).*

The formula for W*:*

$$Wf(z) =< Hf, X_{\bar{z}} >_{L^2(H,(t_-,t))} = \int_{t_-}^{t} < H(t)f(t), X_{\bar{z}}(t) > dt.$$

Here K_z^t denotes the reproducing kernel in $B(E_t)$. The Weyl transform can be viewed as a general form of Fourier transform that puts Krein's canonical systems into one-to-one correspondence with chains of de Branges' spaces $B(E_t)$, $t_- \leq t < t_+$. The case of free Dirac system ($Q = 0$) produces the standard Fourier transform and Payley-Wiener spaces PW_t.

One of the key results of the Krein-de Branges theory says that for any (regular) chain of de Branges spaces there exists a canonical system generating that chain as in the last statement. For some such chains the corresponding systems will have jump intervals, the case we do not discuss in this note. For instance, orthogonal polynomials in $L^2(\mu)$ satisfy difference equations that can be rewritten as a Krein system with jump intervals. In that case $B(E_t) = B_n$ will be a space of polynomials of degree less than n. The space will remain the same, as a set, on each jump interval.

We will write $B(E_1) \doteq B(E_2)$ if the two de Branges spaces $B(E_1)$ and $B(E_2)$ are equal as sets, with possibly different norms. Such relations occur in spectral theory when the difference between corresponding Hamiltonians is locally summable or similar. For instance, the following is an important observation in the theory of Dirac systems:

Theorem 13. *Let $B(E_t)$, $t \in (t_-, t_+)$ be the chain of de Branges' spaces corresponding to a Dirac system on (t_-, t_+) with an L^1_{loc}-potential. Then $B(E_t) \doteq PW_t$.*

A more general Gelfand-Levitan theory can be viewed a subset of Krein - de Branges theory in the case when the system corresponds to the chain of de Branges spaces that are equal to Payley-Wiener spaces as sets. Together with regular Schrödinger and Dirac operators such theory will contain a broader class of canonical systems, appearing in applications, see Makarov and Poltoratski [n.d.].

The following important question arises in connection with Gelfand-Levitan theory. For what functions E will the de Brnanges space $B(E)$ coincide with the Payley-Wiener space as a set (with a different but equivalent norm)? This question is equivalent to characterizing Riesz bases of reproducing kernels in Payley-Wiener spaces or Riesz bases of exponential functions in $L^2(a, b)$. It is related to many similar questions in harmonic analysis, such as problems on frames, sampling and interpolation.

A more general question, that can be similarly translated, is to describe pairs of HB-functions such that $B(E_1) \doteq B(E_2)$. Even in the Payley-Wiener case above, despite a large number of deep results (see for instance Ortega-Cerdà and Seip [2002] and Seip [2004] for such results and further references), the problem is not completely finished. Very little is known in the general case.

In the language of Toeplitz Order defined in Section 2.6 we have the following connection.

Theorem 14 (Poltoratski [2017]).

$$B(E_1) = B(E_2) \quad \Leftrightarrow \quad \theta_{E_1} \overset{\mathrm{r}}{\sim} \theta_{E_2}.$$

3.3 Inverse spectral problems. Once again, let us consider a canonical system (3-2) with no jump intervals. Let us fix a boundary condition $X(t_-) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We call a positive measure μ a spectral measure for the system (3-2) (corresponding to the chosen boundary condition at t_-) if for every $t \in [t_-, t_+)$ the space $B(E_t)$ is isometrically embedded into $L^2(\mu)$. Such a spectral measure may be unique (limit point case) or belong to a one-parameter family of measures with similar property (limit circle case). The limit point case occurs when the integral

$$\int_{t_-}^{t_+} \text{trace } H(t)dt$$

is infinite and the limit circle case corresponds to the finite integral.

Equivalently, a spectral measure can be defined with the condition that the Weyl transform is a unitary operator $L^2(H, (t_-, t)) \to L^2(\mu)$ for any $t \in [t_-, t_+)$.

An inverse spectral problem for the canonical system (3-2) asks to recover the system, i.e., the Hamiltonian H, from its spectral measure μ. Classical results by Borg, Marchenko (Schrödinger case) and de Branges (general canonical system) establish uniqueness of solution for the inverse spectral problem. In the case of canonical systems, there is also a remarkable existence result by de Branges de Branges [1968] which says that any Poisson-finite positive measure is a spectral measure of a canonical system. We call a measure μ on $\mathbb{R}$ Poisson-finite if

$$\int \frac{d|\mu|}{1+x^2} < \infty.$$

As it often happens, the existence theorem does not provide an algorithm for the recovery of H from μ. In fact, after many decades of research only several elementary examples of explicit solutions to the inverse spectral problem for canonical systems have been recorded in the literature. Our methods based on the use of truncated Toeplitz operators provide such an algorithm which yields a number of new interesting examples for the inverse problem. Our next goals is to describe our methods and present examples.

3.4 PW-measures and systems. This part of the note is based on our recent work with N. Makarov, Makarov and Poltoratski [n.d.].

Let μ be a Poisson-finite positive measure on $\mathbb{R}$. We say that μ is a sampling measure for the Paley-Wiener space PW_a if there exist constants $0 < c < C$ such that for any $f \in PW_a$,

$$c||f||_{PW_a} < ||f||_{L^2(\mu)} < C||f||_{PW_a}.$$

We say that μ is a Payley-Wiener (PW) measure if it is sampling for all spaces PW_a, $0 < a < \infty$.

Note that any PW-measure defines equivalent norms in all PW_a spaces. By verifying the axioms from Theorem 11 one can show the chain of PW-spaces with norms inherited from $L^2(\mu)$ is a chain of de Branges spaces $B(E_t)$ for some unknown HB-functions E_t, $B(E_t) \doteq PW_t$.

On the other hand, by a theorem from de Branges [1968], μ is a spectral measure of a canonical system (3-2) with a locally summable Hamiltonian H. By uniqueness of regular de Branges chains isometrically embedded in L^2-spaces of Poisson-finite measures, the chain induced by the canonical system must coincide with the chain $B(E_t)$ obtained above.

Conversely, if one starts with a canonical system on $[0, \infty)$ whose de Branges chain satisfies $B(E_t) \doteq PW_t$, its spectral measure μ is a PW-measure. We will call such systems PW-systems. This is a broad class of canonical systems which includes all of the equations considered in the classical Gelfand-Levitan theory and more.

As we saw, PW-systems and measures are in one-to-one correspondence with each other (after standard normalization of the time variable in the system). A study of spectral problems for PW-systems can be viewed as a generalization of the Gelfand-Levitan theory.

Let us start with the following analytic description of PW-measures. Let δ be a positive constant. We say that an interval $I \subset \mathbb{R}$ is a (μ, δ)-interval if

$$\mu(I) > \delta \text{ and } |I| > \delta.$$

Theorem 15 (Makarov and Poltoratski [n.d.]). *A positive Poisson-finite measure μ is a Paley-Wiener measure if and only if*

1) $\sup_{x\in\mathbb{R}} \mu((x, x+1)) < \infty$.

2) For any $d > 0$ there exists $\delta > 0$ such that for all sufficiently large intervals I there exist at least $d|I|$ disjoint (μ, δ)-intervals intersecting I.

3.5 Truncated Toeplitz operators. Let $\phi \in L^\infty(\mathbb{R})$. The truncated Toeplitz operator with symbol ϕ is defined as

$$L_\phi : PW_a \to PW_a,\ L_\phi f = P(\phi f),$$

where P denotes the orthogonal projection $L^2(\mathbb{R}) \to PW_a$. If μ is a measure on $\mathbb{R}$ one can define L_μ via quadratic forms with the operator $L_\mu : PW_a \to PW_a$ given by the relation

$$\int_{\mathbb{R}} f\bar{g}dx = \int f\bar{g}d\mu.$$

Notice that if $d\mu(x) = \phi(x)dx$ for $\phi \in L^\infty(\mathbb{R})$, then $L_\mu = L_\phi$.

Lemma 1 (Makarov and Poltoratski [ibid.]). *L_μ is a positive invertible operator in PW_a if and only if μ is a sampling measure for PW_a. Consequently, L_μ is a positive invertible operator in every PW_a, $0 < a < \infty$, if and only if μ is a PW-measure.*

Truncated Toeplitz operators corresponding to PW-measures appear in inverse spectral problems for canonical systems in the following way. For simplicity, let us consider the case of Krein's string, i.e., a canonical system (3-2) with a diagonal locally summable Hamiltonian

$$H(t) = \begin{pmatrix} h_{11}(t) & 0 \\ 0 & h_{22}(t) \end{pmatrix}.$$

Via a proper change of variable, one can normalize the problem so that $\det H = 1$ a.e. on (t_-, t_+). After performing such a normalization and fixing a boundary condition at t_-, such systems are put in one-to-one correspondence with even Poisson-finite positive measures on the real line (spectral measures).

Let now μ be a spectral measure of a PW-type Krein's string. Define the truncated Toeplitz operator L_μ and notice that by the last lemma L_μ is invertible in every PW_a. The key relation which solves the inverse spectral problem in this case is that the reproducing kernel in the de Branges space $B(E_t)$ corresponding to the system is the image of the sinc function, the reproducing kernel in PW_a, under L_μ^{-1}:

$$K_0^t(z) = L_\mu^{-1}\left(\frac{\sin tz}{z}\right).$$

After the reproducing kernel K_0^t is recovered, the Hamiltonian of the system can be found from

$$h_{11}(t) = \pi \frac{d}{dt} K_0^t(0) \tag{3-3}$$

and $h_{22} = 1/h_{11}$ (since $\det H = 1$).

In the case of general, non-diagonal Hamiltonians, the problem requires several additional steps, which can be completed via similar methods. The key ingredient in the general case, which we have no space to discuss here, is the so-called generalized Hilbert transform, which maps the de Branges chain $B(E_t)$ into the conjugate chain $B(\tilde{E}_t)$ corresponding to the Hamiltonian

$$\tilde{H} = \begin{pmatrix} h_{11}(t) & -h_{12}(t) \\ -h_{21}(t) & h_{22}(t) \end{pmatrix},$$

see Example 3 below. This operator reduces to the standard Hilbert transform in the free case. For more detailed account and the proofs see Makarov and Poltoratski [ibid.].

3.6 Examples of inverse spectral problems via truncated Toeplitz operators. We finish with the following examples of solutions to the inverse spectral problem.

Example 1. *Let $\mu = \sqrt{2\pi}\delta_0 + \frac{1}{\sqrt{2\pi}}m$, where m stands for the Lebesgue measure on $\mathbb{R}$ and δ_a is the unit mass at the point a. This is probably the simplest new example in our theory. The measure satisfies conditions of Theorem 15 and therefore is a PW-measure. The measure is even, hence H must be diagonal*

$$H = \begin{pmatrix} h(x) & 0 \\ 0 & \frac{1}{h}(x) \end{pmatrix}.$$

Our goal is to find h using the formula (3-3).

Denote by f_t the Fourier transform of $K_0^t(x)$. Then f_t is supported on $[-t,t]$ and satisfies

$$f_t * \hat{\mu} = 1 \text{ on } [-t,t]$$

because $\frac{1}{\sqrt{2\pi}}\chi_{[-t,t]}$ is the Fourier transform of the sinc function, the reproducing kernel of PW_t, and

$$\widehat{K_0^t\mu} = \frac{1}{\sqrt{2\pi}} f_t * \hat{\mu}.$$

Since $\hat{\mu} = \delta_0 + m$, the Fourier transform of the reproducing kernel satisfies

$$1 = f_t(x) + \int_{-t}^{t} f_t(y)dy \text{ on } [-t,t].$$

It follows that $f_t(x) = c(t)\chi_{[-t,t]}(x)$ where

$$1 = c(t) + 2tc(t), \text{ i.e., } c(t) = 1/(1+2t).$$

Then

$$h(t) = \frac{d}{dt}K_0^t(0) = \frac{d}{dt}\int_{-t}^{t} c(t)dx = \frac{d}{dt}\frac{2t}{1+2t} = \frac{2}{(1+2t)^2}$$

and

$$H = \begin{pmatrix} \frac{2}{(1+2t)^2} & 0 \\ 0 & \frac{(1+2t)^2}{2} \end{pmatrix}.$$

for $0 \le t < \infty$.

Example 2. *Consider the Krein system with the spectral measure $\mu = \frac{1}{\sqrt{2\pi}}(1+\cos x)m$. Once again, the measure is a PW-measure. It is even and therefore H must be diagonal.*

As before, the Fourier transform of K_0^t, f_t, satisfies

$$f_t * \hat{\mu} = 1 \text{ on } [-t,t].$$

Inserting the Fourier transform of μ *and solving the last equation one obtains* f_t *as a step function described below.*

For $\frac{n}{2} < t < \frac{n+1}{2}$, *define* f_t *as follows. Consider the infinite Toeplitz matrix*

$$J = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & \dots \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \dots \\ 0 & 0 & 0 & \frac{1}{2} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the diagonals are equal to the pointmasses of $\widehat{\mu}$ *at* n, $n \in \mathbb{Z}$. *Denote by* J_n *the* $n \times n$ *sub-matrix in the upper left corner of* J.

Consider the intervals $I_k = (a_k, b_k), 1 \le k \le n+1$, *of length* $t - \frac{n}{2}$ *centered at*

$$\frac{-n}{2}, \frac{-n+2}{2}, \dots, \frac{n-2}{2}, \frac{n}{2}$$

enumerated in the natural left-to-right order. Denote by J_k *complementary intervals* $J_k = (b_k, a_{k+1})$. *On each* I_k *define* $f_t = \frac{1}{2}a_k^{n+1}$ *and on each* J_k *define* $f_t = \frac{1}{2}a_k^n$, *where*

$$\begin{pmatrix} a_1^m \\ a_2^m \\ \vdots \\ a_m^m \end{pmatrix} = J_m^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Notice that

$$h(t) = \frac{d}{dt} K_0^t(0) = \frac{d}{dt} \int_{-t}^{t} f_t(x)dx = \Sigma(J_{n+1}^{-1}) - \Sigma(J_n^{-1}) \text{ on } \left(\frac{n}{2}, \frac{n+1}{2}\right],$$

where Σ *denotes the sum of all elements of the matrix, with* $\Sigma(J_0^{-1})$ *defined as* 0. *Elementary calculations show the values of* $h(t)$ *on* $(0, \frac{1}{2}], (\frac{1}{2}, 1], (1, \frac{3}{2}], \dots$ *to be*

$$1, \frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \frac{3}{5}, \frac{3}{7}, \frac{4}{7}, \frac{4}{9}, \frac{5}{9}, \frac{5}{11}, \frac{6}{11}, \frac{6}{13},$$

$$\frac{7}{13}, \frac{7}{15}, \frac{8}{15}, \frac{8}{17}, \frac{9}{17}, \frac{9}{19}, \frac{10}{19}, \frac{10}{21}, \frac{11}{21}, \dots$$

$$\dots, \frac{n}{2n-1}, \frac{n}{2n+1}, \frac{n+1}{2n+1}, \frac{n+1}{2n+3}, \frac{n+2}{2n+3}, \dots$$

Example 3. *Let now* $\mu = \frac{1}{\sqrt{2\pi}}(1+\sin x)m$. *Note that* μ *is not even and hence the Hamiltonian has the general form*

$$H = \begin{pmatrix} \alpha(x) & \beta(x) \\ \beta(x) & \gamma(x) \end{pmatrix},$$

with non-zero β. *Clearly,* μ *is a* PW*-measure and hence this example can be treated within the extended Gelfand-Levitan theory. The entries* α *and* γ *can be calculated as in the last example. The Toeplitz matrix for the pointmasses of* $\widehat{\mu} = \delta_0 + \frac{i}{2}(\delta_{-1} - \delta_1)$ *is*

$$J = \begin{pmatrix} 1 & -\frac{i}{2} & 0 & 0 & 0 & \dots \\ \frac{i}{2} & 1 & -\frac{i}{2} & 0 & 0 & \dots \\ 0 & \frac{i}{2} & 1 & -\frac{i}{2} & 0 & \dots \\ 0 & 0 & \frac{i}{2} & 1 & -\frac{i}{2} & \dots \\ 0 & 0 & 0 & \frac{1}{2} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$\alpha(t) = \Sigma(J_{n+1}^{-1}) - \Sigma(J_n^{-1}) \text{ on } \left(\frac{n}{2}, \frac{n+1}{2}\right].$$

Elementary calculations give the following values of α *on on* $(0, \frac{1}{2}], (\frac{1}{2}, 1], (1, \frac{3}{2}], \dots$ *:*

$$i1, 5/3,\ 4/3,\ 4/5,\ 13/15,\ 25/21,\ 8/7,\ 8/9,\ 41/45,\ 61/55,\ 12/11,\ 12/13,$$

$$85/91,\ 113/105,\ 16/15,\ 16/17,\ 145/153,\ 181/171,\ 20/19,\ 20/21, \dots$$

For γ *we need to consider*

$$\sigma = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2}m + \frac{1}{2}\sum \delta_{2\pi n - \frac{\pi}{2}}\right),$$

the so-called dual Clark measure for μ *(i.e.,* σ *is the Clark measure for* $-\theta$*, while* μ *is the Clark measure for* θ*, as defined is Section 2.2). After calculating*

$$\widehat{\sigma} = \frac{1}{2}\delta_0 + \frac{1}{2}\sum_{n\in\mathbb{Z}} (i)^n \delta_n,$$

the corresponding Toeplitz matrix is

$$L = \begin{pmatrix} 1 & \frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} & \frac{1}{2} & \cdots \\ -\frac{i}{2} & 1 & \frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} & \cdots \\ -\frac{1}{2} & -\frac{i}{2} & 1 & \frac{i}{2} & -\frac{1}{2} & \cdots \\ \frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} & 1 & \frac{i}{2} & \cdots \\ \frac{1}{2} & \frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$\gamma(t) = \Sigma(L_{n+1}^{-1}) - \Sigma(L_n^{-1}) \text{ on } \left(\frac{n}{2}, \frac{n+1}{2}\right],$$

which produces the values

$$1, 5/3,\ 17/6,\ 5/2,\ 5/3,\ 37/21,\ 65/28,\ 9/4,\ 9/5,\ 101/55,\ 145/66,\ 13/6,$$

$$13/7,\ 197/105,\ 257/120,\ 17/8,\ 17/9,\ 325/171,\ 401/190,\ 21/10, \ldots$$

For the calculation of β we need to utilize the generalized Hilbert transform of the kernel K_0^t (at 0) mentioned in Section 3.5.

We define the Hilbert transform as

$$Hf = K(f\mu) - K(f\mu) + cf,$$

where

$$Kf\mu = \frac{1}{i\pi}\int \frac{1}{t-z} f(t)d\mu(t) = P(f\mu) + iQ(f\mu).$$

Here P and Q stand for the Poisson and the conjugate Poisson transforms correspondingly. The constant c is to be determined at the end of the calculation from the condition $\det H = 1$.

One of the main formulas of the theory gives the remaining coefficient $\beta(t)$ as

$$\beta(t) = \frac{d}{dt} HK_0^t(0).$$

We have

$$\widehat{PK_0^t\mu} = \widehat{K_0^t\mu} \text{ and } \widehat{QK_0^t\mu} = (-i)\operatorname{sign} x \cdot \widehat{K_0^t\mu},$$

and therefore

$$\widehat{KK_0^t\mu} = K_0^t\mu + \operatorname{sign} x \cdot \widehat{K_0^t\mu}$$

For $K_0^t K\mu$ we have

$$\widehat{K_0^t K\mu} = \widehat{K_0^t\mu} + \widehat{K_0^t} * (\operatorname{sign} x \cdot \widehat{\mu}).$$

Altogether we get

$$\beta(t) = \frac{d}{dt}\int_{\mathbb{R}} [\widehat{K_0^t\mu} + \text{sign } x \cdot \widehat{K_0^t\mu} - \widehat{K_0^t\mu} - \widehat{K}_0^t * (\text{sign } x \cdot \widehat{\mu}) + c\widehat{K}_0^t] =$$

$$\frac{d}{dt}\int_{\mathbb{R}} [\text{sign } x \cdot \widehat{K_0^t\mu} \ - \widehat{K}_0^t * (\text{sign } x \cdot \widehat{\mu}) + c\widehat{K}_0^t].$$

Put $n = [2t]$. Let, like in the first section,

$$\begin{pmatrix} a_1^m \\ a_2^m \\ \vdots \\ a_m^m \end{pmatrix} = J_m^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

By our construction, $\widehat{K_0^t\mu}$ is equal to

$$\widehat{K_0^t\mu} = \begin{cases} -\frac{i}{2}a_1^n \text{ on } (-t-1, -1-n/2) \\ -\frac{i}{2}a_1^{n-1} \text{ on } (-1-n/2, -t) \\ 1/2 \text{ on } (-t, t) \\ \frac{i}{2}a_{n-1}^{n-1} \text{ on } (t, n/2+1) \\ \frac{i}{2}a_n^n \text{ on } (n/2+1, t+1) \end{cases}.$$

Hence,

$$\frac{d}{dt}\int_{\mathbb{R}} \widehat{K_0^t\mu} = 1 + \frac{i}{2}(a_n^n - a_{n-1}^{n-1} - a_1^n + a_1^{n-1}).$$

Taking into account the relation $a_1^m = \bar{a}_m^m$, we get

$$\frac{d}{dt}\int_{\mathbb{R}} \widehat{K_0^t\mu} = 1 + \Im a_1^n - \Im a_1^{n-1}.$$

Since $\mu = (1 + \sin x)m$,

$$\int_{\mathbb{R}} \widehat{K_0^t\mu} = \int_{\mathbb{R}} \widehat{K_0^t} + \frac{i}{2}\int_{\mathbb{R}} \widehat{K_0^t} - \frac{i}{2}\int_{\mathbb{R}} \widehat{K_0^t} = \int_{\mathbb{R}} \widehat{K_0^t}.$$

and we get

$$\frac{d}{dt}\int_{\mathbb{R}} \widehat{K_0^t} = 1 + \Im(a_1^n - a_1^{n-1}).$$

Also,

$$\frac{d}{dt}\int_{\mathbb{R}} \widehat{K_0^t\mu} \text{ sign } x = \frac{1}{2}(a_n^n - a_{n-1}^{n-1} + a_1^n - a_1^{n-1}) =$$

$$\Re(a_1^n - a_1^{n-1}).$$

Finally,

$$\int_{\mathbb{R}} \widehat{K}_0^t * (\operatorname{sign} x \cdot \widehat{\mu}) = \frac{i}{2}\int_{\mathbb{R}} \widehat{K_0^t} + \frac{i}{2}\int_{\mathbb{R}} \widehat{K_0^t} = i\int_{\mathbb{R}} \widehat{K_0^t}.$$

For β we get

$$\beta(t) = \Re(a_1^n - a_1^{n-1}) + i(1 + \Im(a_1^n - a_1^{n-1})) + c(1 + \Im(a_1^n - a_1^{n-1})).$$

Numerical calculation and the condition $\alpha\gamma - \beta^2 = 1$ suggest $c = -1 - i$ and

$$\beta = \Re(a_1^n - a_1^{n-1}) - \Im(a_1^n - a_1^{n-1}) - 1$$

on $((n-1)/2, n/2]$.

This produces the values of β on $(0, \frac{1}{2}], (\frac{1}{2}, 1], (1, \frac{3}{2}], \ldots$:

$$0, -4/3,\ -5/3,\ -1,\ -2/3,\ -22/21,\ -9/7,\ -1,\ -4/5,\ -56/55,\ -13/11,$$

$$-1,\ -6/7,\ -106/105,\ -17/15,\ -1,\ -8/9,\ -172/171,\ -21/19,$$

$$-1,\ -10/11,\ -254/253,\ -25/23, -1\ldots$$

Note that the sequence for $-\beta$ displays the following pattern. Each entry number $4n$, $n = 1, 2, \ldots$, is equal to 1. The $4n + 1$ entry, $n = 0, 1, 2, \ldots$ has denominator $2n + 1$ and numerator $2n$. The $4n + 3$ entry has the denominator $4n + 3$ and numerator $4n + 5$. Finally the $4n + 2$ entry, between the last two, has the denominator equal to the product of their denominators, $(2n + 1)(4n + 3)$ and the numerator $(2n + 1)(4n + 3) + 1$.

These and further examples of inverse spectral problems along with the necessary proofs can be found in Makarov and Poltoratski [n.d.].

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