PROC. INT. CONG. OF MATH. – 2018
Rio de Janeiro, Vol. 3 (2381–2408)

SMALL SCALES AND SINGULARITY FORMATION IN FLUID DYNAMICS

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Abstract

We review recent advances in understanding singularity and small scales formation in solutions of fluid dynamics equations. The focus is on the Euler and surface quasi-geostrophic (SQG) equations and associated models.

1 Introduction

Fluids are all around us, and attempts to mathematically understand fluid motion go back many centuries. Anyone witnessing a dramatic phenomena like tornado or hurricane or even an everyday river flow or ocean wave breaking can easily imagine the complexity of the task. There has been tremendous accumulation of knowledge in the field, yet it is remarkable that some of the fundamental properties of key equations of fluid mechanics remain poorly understood.

A special role in fluid mechanics is played by the incompressible Euler equation, first formulated in 1755 Euler [1755]. Amazingly, it appears to be the second partial differential equation ever derived (the first one is wave equation derived by D'Alembert 8 years earlier). The incompressible Euler equation describes motion of an inviscid, volume preserving fluid; fluid with such properties is often called "ideal". The Euler equation is a nonlinear and nonlocal system of PDE, with dynamics near a given point depending on the flow field over the entire region filled with fluid. This makes analysis of these equations exceedingly challenging, and the array of mathematical methods applied to their study has been very broad.

The basic purpose of an evolution PDE is solution of Cauchy problem: given initial data, find a solution that can then be used for prediction of the modelled system. This is

This work has been supported by the NSF-DMS award 1712294.
MSC2010: primary 35Q35; secondary 76B03, 35L67.
Keywords: Euler equation, SQG equation, two-dimensional incompressible flow, small scale creation, vorticity gradient growth, singularity formation, hyperbolic flow.

exactly how weather forecasting works, or how new car and airplane shapes are designed. Therefore, one of the first questions one can ask about a PDE is existence and uniqueness of solutions in appropriate functional spaces. The PDE is called globally regular if there exists a unique, sufficiently smooth solution for reasonable classes of initial data. On the other hand, singularity formation - meaning that some quantities associated with fluid motion become infinite - can indicate spontaneous creation of intense fluid motion. Singularities are also important to understand since they may indicate potential breakdown of the model, may lead to loss of uniqueness and predictive power, and are very hard to resolve computationally. More generally, one can ask a related and broader question about creation of small scales in fluids - coherent structures that vary sharply in space and time, and contribute to phenomena such as turbulence (see e.g. Eyink [2008] for further references).

For the Euler equation, the global regularity vs finite time blow up story is very different depending on the dimension. Let us recall that the incompressible Euler equation in a domain $D \subset \mathbb{R}^d$, $d = 2$ or 3 with natural no penetration boundary conditions is given by

$$\partial_t u + (u \cdot \nabla) u = \nabla p, \quad \nabla \cdot u = 0, \quad u \cdot n|_{\partial D} = 0, \tag{1}$$

along with the initial data $u(x,0) = u_0$. Here $u(x,t)$ is the vector field describing fluid velocity, p is pressure, and n is the normal at the boundary ∂D. The equation (1) is just the second Newton's law written for ideal fluid. The difference between dimensions becomes clear if we rewrite the equation in vorticity $\omega = \mathrm{curl} u$:

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u, \quad \omega(x,0) = \omega_0(x), \tag{2}$$

along with Biot-Savart law which allows to recover u from ω. For instance, in the case of a smooth domain $D \subset \mathbb{R}^2$ one gets $u = \nabla^\perp(-\Delta_D)^{-1}\omega$ - where $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$ and Δ_D is the Dirichlet Laplacian. In the form (2), one can observe that the term $(\omega \cdot \nabla)u$ on the right hand side vanishes in dimension two. The resulting equation conserves the L^∞ norm and in fact any L^p norm of a regular solution, which helps prove global regularity. This result has been known since 1930s works by Wolibner [1933] and Hölder [1933]. We will focus on the 2D Euler equation in Section 3 below. We note that another feature of the Euler equation made obvious by the vorticity representation (2) (in any dimension) is nonlocality: the inverse Laplacian in the Biot-Savart law is a manifestly nonlocal operator, involving integration over the entire domain.

In three dimensions, the "vortex-stretching" term $(\omega \cdot \nabla)u$ is present and can affect the intensity of vorticity. Local well-posedness results in a range of natural spaces are well known; one can consult Majda and Bertozzi [2002] or Marchioro and Pulvirenti [1994] for proofs and further references. However, the global regularity vs finite time singularity formation question remains open. In fact, this problem for the Euler equation is a close relative of the celebrated Clay Institute Millennium problem on the 3D Navier-Stokes equation C. L. Fefferman [2006]. Indeed, the Navier-Stokes equation only differs from (1) by the presence of a regularizing, linear term Δu on the right hand side modeling viscosity (and by different boundary conditions if boundaries are present). The nonlinearity - the principal engine of possible singular growth - is identical in both equations.

A host of numerical experiments sought to discover a scenario for singularity formation in solutions of 3D Euler equation (see e.g. Bhattacharjee, Ng, and Wang [1995], Boratav and R. B. Pelz [1994], E and Shu [1994], Grauer and Sideris [1991], Hou and R. Li [2006], Kerr [1993], Larios, Petersen, Titi, and Wingate [n.d.], Ohkitani and Gibbon [2000], R. Pelz and Gulak [1997], and Pumir and Siggia [1992], or a detailed review by Gibbon [2008] where more references can be found). In the analytic direction, a complete review would be too broad to attempt here. Let us mention the classical work of Beale, Kato, and Majda [1984] on criteria for global regularity, papers Caflisch [1993] and Siegel and Caflisch [2009] on singularities for complex-valued solutions to Euler equations, as well as regularity criteria by Constantin, C. Fefferman, and Majda [1996] and by Hou and collaborators Hou and R. Li [2006], Hou and C. Li [2005], and Deng, Hou, and Yu [2005] which involve more subtle geometric conditions sufficient for regularity. See Constantin [2007] for more history and analytical aspects of this problem.

There have been several recent developments in classical problems on regularity and solution estimates for the fundamental equations of fluid mechanics. First, Hou and Luo produced a new set of careful numerical experiments suggesting finite time singularity formation for solutions of 3D Euler equation Luo and Hou [2014]. The scenario of Hou and Luo is axi-symmetric. Very fast vorticity growth is observed at a ring of hyperbolic stagnation points of the flow located on the boundary of a cylinder. None of the available regularity criteria such as Constantin, C. Fefferman, and Majda [1996], Hou and R. Li [2006], Deng, Hou, and Yu [2005], and Hou and C. Li [2005] seem to apply to scenario's geometry. The scenario has a close analog for the 2D inviscid Boussinesq system, for which the question of global regularity is also open; it is listed as one of the "eleven great problems of hydrodynamics" by V. I. Yudovich [2003]. More details on the scenario will be provided in Section 2.

The hyperbolic stagnation point on the boundary scenario also leads to interesting phenomena in solutions of the 2D Euler equation. The best known upper bound on the growth of the gradient of vorticity, as well as higher order Sobolev norms, is double exponential in time:

$$\|\nabla\omega(\cdot,t)\|_{L^\infty} \le (1+\|\nabla\omega_0\|_{L^\infty})^{\exp(C\|\omega_0\|_{L^\infty}t)}. \tag{3}$$

This result has appeared explicitly in V. I. Yudovich [1962], though related bounds can be traced back to Wolibner [1933]. The question whether such upper bounds are sharp has been open for a long time. Kiselev and Šverák [2014] provided an example of smooth initial data in the disk such that the corresponding solution exhibits double exponential growth in the gradient of vorticity for all times, establishing qualitative sharpness of (3). The construction is based on the hyperbolic point at the boundary scenario, and will be described in more detail in Section 3.

Further attempts to rigorously understand the Hou-Luo scenario involved construction of 1D and 2D models retaining some of the analytical structure of the original problem. We will discuss some of these models in Section 4.

The surface quasi-geostrophic (SQG) equation is similar to the 2D Euler equation in vorticity form, but is more singular:

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \;\; u = \nabla^{\perp}(-\Delta)^{-1+\alpha}\omega, \;\; \alpha = 1/2, \;\; \omega(x,0) = \omega_0(x). \tag{4}$$

The value $\alpha = 0$ corresponds to the 2D Euler equation, while $0 < \alpha < \frac{1}{2}$ is called the modified SQG range. The SQG and modified SQG equations come from atmospheric science. They model evolution of temperature near the surface of a planet and can be derived by formal asymptotic analysis from a larger system of rotating 3D Navier-Stokes equations coupled with temperature equation through buoyancy force Held, Pierrehumbert, Garner, and Swanson [1995], Majda [2003], Pedlosky [1987], and Pierrehumbert, Held, and Swanson [1994]. In mathematical literature, the SQG equation was first considered by Constantin, Majda, and Tabak [1994], where a parallel between the structure of the SQG equation and the 3D Euler equation was drawn. The SQG and modified SQG equations are perhaps simplest looking equations of fluid mechanics for which the question of global regularity vs finite time blow up remains open. The equation (4) can be considered with smooth initial data, but another important class of initial data is patches, where $\theta_0(x)$ equals linear combination of characteristic functions of some disjoint domains $\Omega_j(0)$. The resulting evolution yields time dependent regions $\Omega_j(t)$. The regularity question in this context addresses the regularity class of the boundary $\partial\Omega_j(t)$ and lack of contact between different components. Existence and uniqueness of patch solution for 2D Euler equation follows from Yudovich theory Judovič [1963], Majda and Bertozzi [2002], and Marchioro and Pulvirenti [1994]. The global regularity question has been settled affirmatively by Chemin [1993] (Bertozzi and Constantin [1993] provided a different proof). For the SQG and modified SQG equations patch dynamics is harder to set up. Local well-posedness has been shown by Rodrigo in C^∞ class Rodrigo [2005] and by Gancedo in Sobolev spaces Gancedo [2008] in the whole plane setting. Numerical simulations by Córdoba, Fontelos, Mancho, and Rodrigo [2005] suggest that finite time singularities – in particular formation of corners and different components touching each other – is possible, but rigorous understanding of this phenomena remained missing. In Kiselev, Ryzhik, Yao, and Zlatoš [2016], Kiselev, Yao, and Zlatoš [2017], we considered modified SQG and 2D Euler patches in half-plane, with the no penetration boundary conditions. The initial patches are regular and do not touch each other but may touch the boundary. We proved a kind

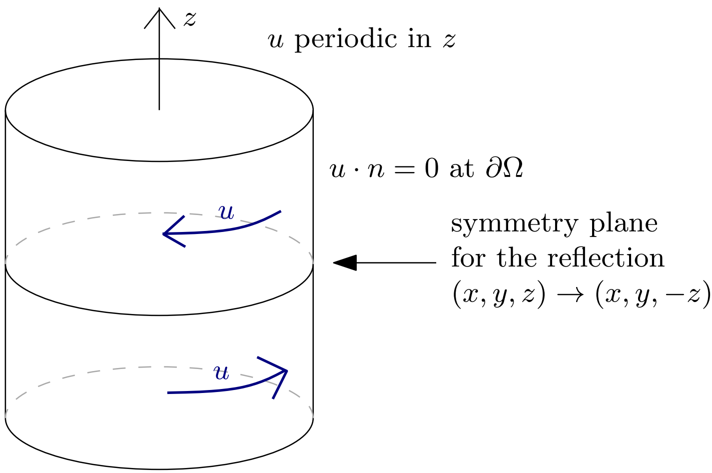

Figure 1: The initial data for Hou-Luo scenario

of phase transition in this setting: the 2D Euler patches stay globally regular, while for a range of small $\alpha > 0$ some initial data lead to blow up in finite time. The blow up scenario again involves a stagnation hyperbolic point of the flow on the boundary. This result will be described in more detail in Section 5.

2 The 3D Euler equation and the 2D Boussinesq system: the hyperbolic scenario

In Luo and Hou [2014] the authors study 3D axi-symmetric solutions of incompressible Euler equation with roughly the initial configuration shown on Figure 1: only swirl u^{ϕ} is initially non-zero, and it is odd and periodic in z variable.

One of the standard forms of the axi-symmetric Euler equations in the usual cylindrical coordinates (r, ϕ, z) is

(5a) $$\partial_t \left(\frac{\omega^{\phi}}{r}\right) + u^r \partial_r \left(\frac{\omega^{\phi}}{r}\right) + u^z \partial_z \left(\frac{\omega^{\phi}}{r}\right) = \partial_z \left(\frac{(ru^{\phi})^2}{r^4}\right)$$

(5b) $$\partial_t (ru^{\phi}) + u^r \partial_r (ru^{\phi}) + u^z \partial_z (ru^{\phi}) = 0\,,$$

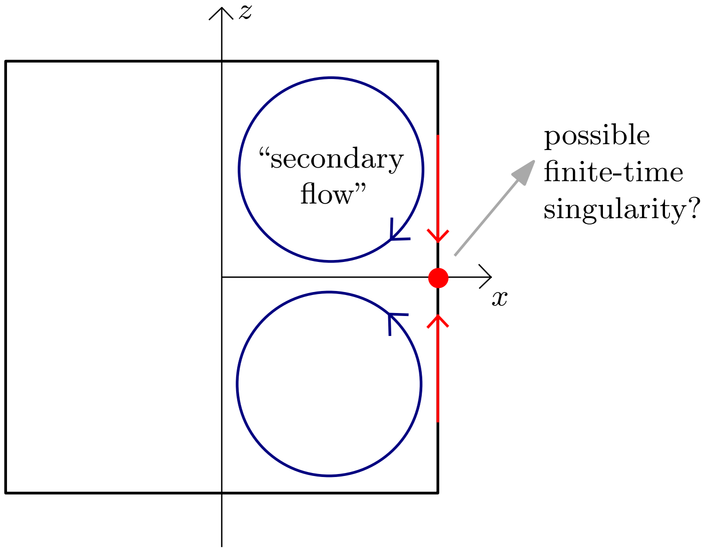

Figure 2: The secondary flows in fixed ϕ section

with the understanding that u^r, u^z are given from ω^ϕ via the Biot-Savart law which in the setting of Hou-Luo scenario takes form

$$u^r = -\frac{\partial_z \psi}{r}, \quad u^z = \frac{\partial_r \psi}{r}, \quad L\psi = \frac{\omega^\phi}{r}, \quad L\psi = -\frac{1}{r}\partial_r\left(\frac{1}{r}\partial_r \psi\right) - \frac{1}{r^2}\partial_{zz}^2 \psi.$$

From (5), it is clear that the swirl will spontaneously generate toroidal rolls corresponding to non-zero ω^ϕ. These are the so-called "secondary flows", Prandtl [1952]; its effect on river flows was studied by Einstein [1926]. Thus the initial condition leads to the (schematic) picture in the xz–plane shown on Figure 2, in which we also indicate the point where a conceivable finite-time singularity (or at least an extremely strong growth of vorticity) is observed numerically. In the three-dimensional picture, the points with very fast growth form a ring on the boundary of the cylinder.

A somewhat similar scenario can be considered for the 2D inviscid Boussinesq system in a half-space $\mathbb{R}^+ = \{(x, y) \in \mathbb{R} \times (0, \infty)\}$ (or in a flat half-cylinder $\mathbf{S}^1 \times (0, \infty)$), which we will write in the vorticity form:

$$\partial_t \omega + u_1 \partial_x \omega + u_2 \partial_y \omega = \partial_x \theta \tag{6a}$$

$$\partial_t \theta + u_1 \partial_x \theta + u_2 \partial_y \theta = 0\,. \tag{6b}$$

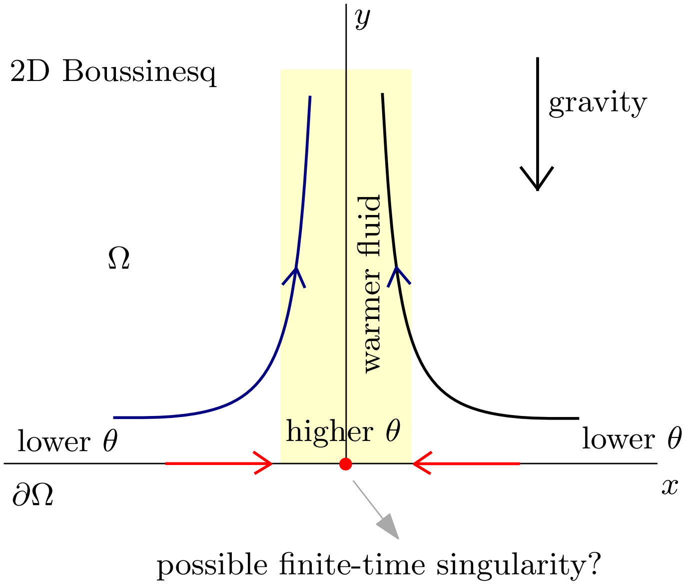

Figure 3: The 2D Boussineq singularity scenario

Here $u = (u_1, u_2)$ is obtained from ω by the usual Biot-Savart law $u = \nabla^{\perp}(-\Delta)^{-1}\omega$, with appropriate boundary conditions on Δ, and θ represents the fluid temperature or density.

It is well-known (see e.g. Majda and Bertozzi [2002]) that this system has properties similar to the 3D axi-symmetric Euler (5), at least away from the rotation axis. Indeed, comparing (5) with (6), we see that θ essentially plays the role of the square of the swirl component ru^{ϕ} of the velocity field u, and ω replaces ω^{ϕ}/r. The real difference between the two systems only emerges near the axis of rotation, where the factors of r can conceivably change the nature of dynamics. For the purpose of comparison with the axi-symmetric flow, the last picture should be rotated by $\pi/2$, after which it resembles the picture relevant for (6), see Figure 3.

In both the 3D axi-symmetric Euler case and in the 2D Boussinesq system case the best chance for possible singularity formation seems to be at the points of symmetry at the boundary, which numerical simulations suggest are fixed hyperbolic points of the flow.

3 The 2D Euler equation

A reasonable first step to understand the hyperbolic stagnation point on the boundary blow up scenario is to consider the case of constant density in the Boussinesq system first. Of course, this reduces the system to the 2D Euler equation:

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \;\; u = \nabla^{\perp}(-\Delta_D)^{-1}\omega, \;\; \omega(x,0) = \omega_0(x). \tag{7}$$

Here Δ_D stands for Dirichlet Laplacian; such choice of the boundary condition corresponds to no-penetration property $u \cdot n|_{\partial D} = 0$. Of course, for the 2D Euler equation solutions are globally regular. Let us state this result, going back to 1930s Wolibner [1933] and Hölder [1933].

Theorem 3.1. *Let $D \subset \mathbb{R}^2$ be a compact domain with smooth boundary, and $\omega_0(x) \in C^1(D)$Then there exists a unique smooth solution $\omega(x,t)$ of the equation (7) corrsponding to the initial data ω_0, which moreover satisfies*

$$1 + \log\left(1 + \frac{\|\nabla\omega(x,t)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right) \le \left(1 + \log\left(1 + \frac{\|\nabla\omega_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)\right)\exp(C\|\omega_0\|_{L^\infty}t) \tag{8}$$

for some constant C which may depend only on the domain D.

A key ingredient of the proof is the Kato inequality Kato [1986]: for every $1 > \alpha > 0$, we have

$$\|\nabla u(x,t)\|_{L^\infty} \le C(\alpha, D)\|\omega_0\|_{L^\infty}\left(1 + \log\frac{\|\omega(x,t)\|_{C^\alpha}}{\|\omega_0\|_{L^\infty}}\right). \tag{9}$$

Note that the matrix ∇u consists of double Riesz transforms $\partial^2_{ij}(-\Delta_D)^{-1}\omega$. Riesz transforms are well known to be bounded on L^p, $1 < p < \infty$ (see e.g. Stein [1970]), but the bound fails at the endpoints and we have to pay a logarithm of the higher order norm to obtain a correct bound. It is exactly the extra log in (9) that leads to the double exponential upper bound as opposed to the single one (see e.g. Kiselev and Šverák [2014] for more details).

The question of whether such upper bounds are sharp has been open for a long time. Judovič [1974] and V. I. Yudovich [2000] provided an example showing infinite growth of the vorticity gradient at the boundary of the domain, by constructing an appropriate Lyapunov-type functional. These results were further improved and generalized in Morgulis, Shnirelman, and V. Yudovich [2008], leading to description of a broad class of flows with infinite growth in their vorticity gradient. Nadirashvili [1991] proved a more quantitative linear in time lower bound for a "winding" flow in an annulus. A variant of the example due to Bahouri and Chemin [1994] provides singular stationary solution of the 2D

Euler equation defined on $\mathbb{T}^2$ with fluid velocity which is just log-Lipschitz in spacial variables. Namely, if we set $\mathbb{T}^2 = [-\pi, \pi) \times [-\pi, \pi)$, the solution is equal to -1 in the first quadrant $[0, \pi) \times [0, \pi)$ and is odd with respect to both coordinate axes. Note that the solution is just L^∞ but existence and uniqueness of solutions in this class is provided essentially by Yudovich theory Judovič [1963]. The origin is a fixed hyperbolic point of the fluid velocity, with x_1 being the contracting direction, and the velocity satisfies $u_1(x_1, 0) = \frac{4}{\pi} x_1 \log x_1 + O(x_1)$ for small x_1. The trajectory starting at a point $(x_1, 0)$ on a horizontal separatrix will therefore converge to the origin at a double exponential rate in time. If a smooth passive scalar ψ initially supported away from the origin is advected by a flow generated by singular cross, $\partial_{x_1}\psi$ will grow at a double exponential rate in time if $\psi(x_1, 0)$ is not identically zero. Of course, derivative growth does not make sense for the singular cross solution itself since it is stationary and already singular. But this solution shows a blueprint of how double exponential growth can be conceivably generated in smooth solution: it needs to approach the discontinuous configuration similar to the singular cross, while at the same time the solution should be nonzero on the contracting direction. This turns out to be hard to implement, especially without boundary.

In recent years, there has been a series of works by Denisov on this problem. In Denisov [2009], he constructed an example with superlinear growth in vorticity gradient of the solution in the periodic case. In Denisov [2015], he showed that for any time T, one can arrange smooth initial data so that the corresponding solution will experience double exponential burst of growth over $[0, T]$. The example is based on smoothing out Bahouri-Chemin example, abandoning odd symmetry to put a ripple on a separatrix, and controlling the resulting solution over finite time interval.

In Kiselev and Šverák [2014], we proved

Theorem 3.2. *Consider two-dimensional Euler equation on a unit disk D. There exists a smooth initial data ω_0 with $\|\nabla\omega_0\|_{L^\infty}/\|\omega_0\|_{L^\infty} > 1$ such that the corresponding solution $\omega(x,t)$ satisfies*

$$\frac{\|\nabla\omega(x,t)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}} \geq \left(\frac{\|\nabla\omega_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)^{c\exp(c\|\omega_0\|_{L^\infty} t)} \tag{10}$$

for some $c > 0$ and for all $t \geq 0$.

The theorem shows that double exponential growth in the gradient of vorticity can actually happen for all times, so the double exponential upper bound is sharp. As in Hou-Luo blow up scenario, growth happens near a hyperbolic fixed point of the flow at the boundary. The result has been generalized to the case of any compact sufficiently regular domain with symmetry axis by Xu [2016]. The question of whether double exponential growth can happen in the bulk of the fluid remains open; Zlatoš [2015] has improved the techniques behind Theorem 3.2 to construct examples of smooth solutions with exponential growth of $\|\nabla^2\omega\|_{L^\infty}$ in periodic setting. The question of whether double exponential growth is at all possible in the bulk of the fluid remains wide open.

A key step in the proof is understanding the structure of fluid velocity near the hyperbolic point. Let

$$D^+ = \{x \in D \,|x_1 \geq 0.\}$$

We will choose the initial data that is odd in x_1, and $-1 \leq \omega_0(x) < 0$ for $x \in D^+$. Let us set the origin of our coordinate system at the bottom of the disc, where interesting things will be happening. Given the symmetry of ω, we have

$$u(x,t) = -\nabla^\perp \int_D G_D(x,y)\omega(y,t)\,dy \tag{11}$$

$$= -\frac{1}{2\pi}\nabla^\perp \int_{D^+} \log\left(\frac{|x-y||\tilde{x}-\bar{y}|}{|x-\bar{y}||\tilde{x}-y|}\right)\omega(y,t)\,dy, \tag{12}$$

where G_D is the Green's function of Dirichlet Laplacian and $\tilde{x} = (-x_1, x_2)$. For each point $(x_1, x_2) \in D^+$, let us introduce the region

$$Q(x_1,x_2) = \{(y_1,y_2) \in D^+ :\ \ x_1 \leq y_1,\ \ x_2 \leq y_2\},$$

and set

$$\Omega(x_1,x_2,t) = -\frac{4}{\pi}\int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4}\omega(y,t)\,dy_1 dy_2. \tag{13}$$

Finally, for any $0 < \gamma < \pi/2$, let ϕ be the usual polar angle coordinate of point x, and denote

$$D_1^\gamma = \{x \in D^+ \,|0 \leq \phi \leq \pi/2 - \gamma\},\ \ D_2^\gamma = \{x \in D^+ \,|\gamma \leq \phi \leq \pi/2\}.$$

The following Lemma is crucial for the proof of Theorem 3.2.

Lemma 3.3. *Suppose that ω_0 is odd with respect to x_1. Fix a small $\gamma > 0$. There exists $\delta > 0$ so that for all $x \in D_1^\gamma$ such that $|x| \leq \delta$ we have*

$$u_1(x_1,x_2,t) = -x_1\Omega(x_1,x_2,t) + x_1 B_1(x_1,x_2,t), \tag{14}$$

where $|B_1(x_1,x_2,t)| \leq C(\gamma)\|\omega_0\|_{L^\infty}$.

Similarly, for all $x \in D_2^\gamma$ such that $|x| \leq \delta$ we have

$$u_2(x_1,x_2,t) = x_2\Omega(x_1,x_2,t) + x_2 B_2(x_1,x_2,t), \tag{15}$$

where $|B_2(x_1,x_2,t)| \leq C(\gamma)\|\omega_0\|_{L^\infty}$.

Proof of the lemma is based on analysis of (11), full details can be found in Kiselev and Šverák [2014]. The terms on the right hand sides of (14), (15) involving Ω can be thought of as main terms in certain regimes. Indeed, observe that as support of the set where $\omega(x,t) \geq c > 0$ approaches the origin, the size of Ω given by (13) may grow as a logarithm of this distance. Thus Lemma 3.3 provides a sort of quantitative version of Bahouri-Chemin log-Lipschitz singular flow for smooth setting. In the regime where the Ω terms dominate, the flow trajectories described by (14), (15) are close to precise hyperbolas. Another interesting feature of the formulas (14), (15) is a hidden comparison principle: the influence region $Q(x_1,x_2)$ tends to be larger for points closer to the origin. The comparison principle is not precise, but it turns out to be true up to Lipschitz errors. This feature is key in the construction of the example with double exponential growth.

Let us now sketch the construction of such example. Fix small $\gamma > 0$, and take $\epsilon > 0$ smaller than the corresponding value of δ from Lemma 3.3. We also take ϵ sufficiently small so that $\log \epsilon^{-1}$ is much larger than the constant $C(\gamma)$ from Lemma 3.3. Take the initial data ω_0 such that $\omega_0(x_1,x_2) = -1$ if $x_1 \geq \epsilon^{10}$, odd with respect to x_1, and satisfying $-1 \leq \omega_0(x) \leq 0$ for $x \in D^+$. This leaves certain ambiguity in how we define ω_0 in the region close to x_2 axis. As we will see, it does not matter for the construction exactly how ω_0 is defined there.

The first observation is that with such choice of the initial data,

$$\Omega(x,t) \geq C \log \epsilon^{-1} \tag{16}$$

for every $x \in D^+$ with $|x| \leq \epsilon$. Indeed, due to incompressibility of the flow the measure of the points $x \in D^+$ where $\omega(x,t) > -1$ does not exceed 2ϵ for any time t. It is then straightforward to show that even if all these points are pushed by the dynamics into the region where the size of the kernel in (13) is maximal, the estimate (16) still holds.

Next, for $0 < x_1' < x_1'' < 1$ we denote

$$\mathcal{O}(x_1', x_1'') = \left\{(x_1,x_2) \in D^+\,,\; x_1' \leq x_1 \leq x_1''\,,\; x_2 \leq x_1\right\}\,. \tag{17}$$

We would like to analyze the evolution in time of the region $\mathcal{O}_{\epsilon^{10}\epsilon}$. For this purpose, for $0 < x_1 < 1$ we let

$$\underline{u}_1(x_1,t) \quad = \quad \min_{(x_1,x_2)\in D^+,\, x_2 \leq x_1} u_1(x_1,x_2,t) \tag{18}$$

and

$$\overline{u}_1(x_1,t) \quad = \quad \max_{(x_1,x_2)\in D^+,\, x_2 \leq x_1} u_1(x_1,x_2,t)\,. \tag{19}$$

Define $a(t)$ by

$$a' = \overline{u}_1(a,t)\,, \quad a(0) = \epsilon^{10} \tag{20}$$

and $b(t)$ by

$$b' = \underline{u}_1(b,t)\,, \quad b(0) = \epsilon. \tag{21}$$

Let

$$\mathcal{O}_t = \mathcal{O}(a(t), b(t))\,. \tag{22}$$

We claim that $\omega(x,t) = -1$ for every $x \in \mathcal{O}_t$, and every $t \geq 0$. Indeed, if this is not the case, then there must exist some time s and a trajectory $\Phi_s(y)$ such that $y \notin \mathcal{O}_0$, and $\Phi_s(y) \in \partial\mathcal{O}_s$ for the first time. But the trajectory $\Phi_s(y)$ cannot enter $\mathcal{O}_s$ through the boundary of D due to the no-penetration boundary condition. It is also not hard to see it cannot enter at the $x_1 = a(s)$ or $x_1 = b(s)$ pieces of $\partial\mathcal{O}_s$ due to the definition of $a(t), b(t)$ and $\overline{u}$, $\underline{u}$. This leaves the diagonal $x_1 = x_2$. However, due to Lemma 3.3 and the estimate (16), we have that

$$\frac{C\,\log\epsilon^{-1} - C(\gamma)}{C\,\log\epsilon^{-1} + C(\gamma)} \leq \frac{-u_1(x_1,x_1,t)}{u_2(x_1,x_1,t)} \leq \frac{C\,\log\epsilon^{-1} + C(\gamma)}{C\,\log\epsilon^{-1} - C(\gamma)} \tag{23}$$

for every $t \geq 0$ and $x_1 \leq \epsilon$. Due to choice of ϵ, we have that the ratio $-u_1/u_2$ on the diagonal part of the boundary of $\mathcal{O}_s$ is close to 1. Thus the vector field u points outside of the region $\mathcal{O}_s$ on the diagonal part of the boundary at all times and so the trajectory cannot enter through the diagonal either.

Next, we are going to estimate how quickly $a(t)$ approaches the origin. The constant C below depends only on γ and may change from line to line. By Lemma 3.3, we have

$$\underline{u}_1(b(t),t) \geq -b(t)\,\Omega(b(t), x_2(t)) - Cb(t),$$

for some $x_2(t) \leq b(t)\,,\ (x_2(t), b(t)) \in D^+$ as $\|\omega(x,t)\|_{L^\infty} \leq 1$ by our choice of the initial datum ω_0. A straightforward calculation shows that

$$\Omega(b(t), x_2(t)) \leq \Omega(b(t), b(t)) + C.$$

Thus we get

$$\underline{u}_1(b(t),t) \geq -b(t)\,\Omega(b(t), b(t)) - Cb(t). \tag{24}$$

Similarly,

$$\overline{u}_1(a(t),t) \leq -a(t)\,\Omega(a(t), \tilde{x}_2(t)) \leq -a(t)\,\Omega(a(t), 0) + Ca(t).$$

These estimates establish a form of comparison principle, up to Lipschitz errors, of the fluid velocities of the front and back of the region $\mathcal{O}_{a(t),b(t)}$.

Now observe that

$$\Omega(a(t),0) \geq -\frac{4}{\pi}\int_{\mathcal{O}_t}\frac{y_1y_2}{|y|^4}\omega(y,t)\,dy_1dy_2 + \Omega(b(t),b(t)). \tag{25}$$

Since $\omega(y,t) = -1$ on $\mathcal{O}_t$, a direct estimate shows that the integral in (25) is bounded from below by $\kappa(-\log a(t) + \log b(t))$ for some $\kappa > 0$. Applying the above estimates to evolution of $a(t)$ and $b(t)$ we obtain

$$\frac{d}{dt}(\log a(t) - \log b(t)) \leq \kappa(\log a(t) - \log b(t)) + C. \tag{26}$$

Applying Gronwall lemma and choosing ϵ small enough leads to $\log a(t) \leq \exp(\kappa t)\log \epsilon$. To arrive at (10), it remains to note that we can arrange $\|\nabla\omega_0\|_{L^\infty} \lesssim \epsilon^{-10}$.

4 The one-dimensional models

One-dimensional models in fluid mechanics have a long history. We briefly review some of the results most relevant to our narrative. In the context of modeling finite time blow and global regularity issues, Constantin, Lax, and Majda [1985] considered the model

$$\partial_t\omega = \omega H\omega, \;\; \omega(x,0) = \omega_0(x), \tag{27}$$

where H is the Hilbert transform, $H\omega(x,t) = \frac{1}{\pi}P.V.\int_{\mathbb{R}}\frac{\omega(y,t)}{x-y}\,dy$. The equation (27) is designed to model the vortex stretching term on the right hand side of (2); the advection term is omitted. Surprisingly, the model (27) is explicitly solvable due to special properties of Hilbert transform. Finite time blow up happens for a broad class of initial data - specifically, near the points where $\omega_0(x)$ vanishes and the real part of $H\omega_0$ has the right sign.

A more general model has been proposed by De Gregorio [1990], De Gregorio [1996]:

$$\partial_t\omega + u\partial_x\omega = \omega H\omega, \;\; u_x = H\omega, \;\; \omega(x,0) = \omega_0(x). \tag{28}$$

This model includes the advection term. Amazingly, the question of whether the solutions to (28) are globally regular or can blow up in finite time is currently open. Numerical simulations appear to suggest global regularity Okamoto, Sakajo, and Wunsch [2008], but the mechanism for it is not well understood. Recently, global regularity near a manifold of equilibria as well as other interesting features of the solutions of (28) have been shown in Jia, Stewart, and Sverak [n.d.]. Variants of (28) and other related models appear in for example Bauer, Kolev, and Preston [2016], Castro and Córdoba [2010], Elgindi and Jeong [n.d.(c)], Escher and Kolev [2014], and Wunsch [2011], where further references can be found.

Already in Luo and Hou [2014], Hou and Luo proposed a simplified one-dimensional model specifically designed to gain insight into the singularity formation process in the scenario described in Section 2. This model is given by

$$\begin{aligned} &\partial_t \omega + u\partial_x \omega = \partial_x \theta, \\ &\partial_t \theta + u\partial_x \theta = 0, \ \ u_x = H\omega. \end{aligned} \tag{29}$$

Here as above H is the Hilbert transform, and the setting can be either periodic or the entire axis with some decay of the initial data. The model (29) can be thought of as an effective equation on the $x_2 = 0$ axis in the Boussinesq case (see (6) and Figure 3) or on the boundary of the cylinder in the 3D axi-symmetric Euler case. The model can be derived from the full equations under certain boundary layer assumption: that $\omega(x,t)$ is concentrated in a boundary layer of width a near $x_2 = 0$ axis and is independent of x_2, that is $\omega(x_1, x_2, t) = \omega(x_1, t)\chi_{[0,a]}(x_2)$. Such assumption is necessary to close the equation and reduces the half-plane Biot-Savart law to $u_x = H\omega$ in the main order; the parameter a enters into the additional term that is non-singular and is dropped from (29). See Luo and Hou [ibid.], Choi, Hou, Kiselev, Luo, Sverak, and Yao [2017] for more details. We will call the system (29) the HL model.

The HL model is still fully nonlocal. A further simplification was proposed in Choi, Kiselev, and Yao [2015], where the Biot-Savart law has been replaced with

$$u(x,t) = -x\int_x^1 \frac{\omega(y,t)}{y}\,dy. \tag{30}$$

Here the most natural setting is on an interval $[0,1]$ with smooth initial data supported away from the endpoints. The law (30) is motivated by the velocity representation in Lemma 3.3 above, as it is the simplest one dimensional analog of such representation. This law is "almost local" - if one divides u by x and differentiates, one gets local expression. We will call the model (30) the CKY model.

For both HL and CKY models, local well-posedness in a reasonable family of spaces (such as sufficiently regular Sobolev spaces) is not difficult to obtain. In Choi, Kiselev, and Yao [ibid.], finite time blow up has been proved for the CKY model. The proof used analysis of the trajectories and of the nonlinear feedback loop generated by the forcing term $\partial_x \theta$. We will sketch a very similar argument below. The proof does not provide a detailed blow picture. In a later work Hou and Liu [2015], more precise picture of blow up was established with aid of computer assisted proof. It shows self-similar behavior near the origin properly matched with the outside region. For the original HL model, finite time blow up has been proved in Choi, Hou, Kiselev, Luo, Sverak, and Yao [2017]. For the model including the additional term obtained from the boundary layer assumption into Biot-Savart law, finite time blow up has been proved in Do, Kiselev, and Xu [n.d.]. We now sketch a variant of the blow up proof for the HL model (29).

Let us consider an HL model on $[0, L]$ with periodic boundary conditions. In this setting, using the expression for periodic Hilbert transform, the Biot-Savart law becomes

$$u_x(x) = H\omega(x) = \frac{1}{L} P.V. \int_0^L \omega(y) \cot[\mu(x-y)]\, dy,$$

where $\mu = \pi/L$. Integration leads to

$$u(x) = \frac{1}{\pi} \int_0^L \omega(y) \log|\sin[\mu(x-y)]|\, dy. \tag{31}$$

The initial data will be chosen as follows: ω_0 is odd, which together with periodicity implies that it is also odd with respect to $x = L/2$, and satisfies $\omega_0(x) \geq 0$ if $x \in [0, L/2]$. The initial density θ_0 is even with respect to both 0 and $L/2$, and satisfies $\theta_0' \geq 0$ for $x \in [0, L/2]$. The solution, while it exists, will satisfy the same properties. The symmetry assumptions on ω lead to the following version of the Biot-Savart law, which can be verified by direct computation.

Lemma 4.1. *Let ω be periodic with period L and odd at $x = 0$ and let u be defined by (31). Then for any $x \in [0, \frac{1}{2}L]$,*

$$u(x)\cot(\mu x) = -\frac{1}{\pi} \int_0^{L/2} K(x,y)\omega(y)\cot(\mu y)\, dy, \tag{32}$$

where

$$K(x,y) = s \log\left|\frac{s+1}{s-1}\right| \qquad \text{with} \qquad s = s(x,y) = \frac{\tan(\mu y)}{\tan(\mu x)}. \tag{33}$$

Furthermore, the kernel $K(x,y)$ has the following properties:

1. *$K(x,y) \geq 0$ for all $x,\ y \in (0, \frac{1}{2}L)$ with $x \neq y$;*
2. *$K(x,y) \geq 2$ and $K_x(x,y) \geq 0$ for all $0 < x < y < \frac{1}{2}L$;*

The key observation is a certain positivity property that will help us control the behavior of trajectories Choi, Hou, Kiselev, Luo, Sverak, and Yao [2017].

Lemma 4.2. *Let the assumptions in Lemma 4.1 be satisfied and assume in addition that $\omega \geq 0$ on $[0, \frac{1}{2}L]$. Then for any $a \in [0, \frac{1}{2}L]$,*

$$\int_a^{L/2} \omega(x)\big[u(x)\cot(\mu x)\big]_x\, dx \geq 0. \tag{34}$$

With these two lemmas, the rest of the proof proceeds as follows. Towards a contradiction, let us assume that there exists a global solution (θ, ω) to (29) with the initial data as described in the beginning of this section and denote $A := \theta_0(\frac{1}{2}L) > 0$. Since θ_0 is assumed to be increasing on $[0, \frac{1}{2}L]$, we can choose a decreasing sequence x_n in $(0, \frac{1}{2}L)$ with $n \geq 0$ such that $\theta_0(x_n) = [2^{-1} + 2^{-(n+2)}]A$. Note that $x_0 < \frac{1}{2}L$ since $\theta_0(x_0) < \theta_0(\frac{1}{2}L)$.

For x_n defined as above, let $\Phi_n(t)$ denote the characteristics of (29) originating from x_n, that is, let $\frac{d}{dt}\Phi_n(t) = u(\Phi_n(t), t)$ with $\Phi_n(0) = x_n$. Lemma 4.1 then implies the following estimate on the evolution of Φ_n:

$$\frac{d}{dt}\Phi_n(t) = u(\Phi_n(t), t) \leq -\frac{2}{\pi}\tan(\mu\Phi_n(t)) \int_{\Phi_n(t)}^{L/2} \omega(y,t)\cot(\mu y)\,dy \tag{35}$$

$$\leq -\frac{2\mu}{\pi}\,\Phi_n(t)\Omega_n(t), \tag{36}$$

where for simplicity we have written

$$\Omega_n(t) := \int_{\Phi_n(t)}^{L/2} \omega(y,t)\cot(\mu y)\,dy. \tag{37}$$

Introducing the new variable $\psi_n(t) := -\log\Phi_n(t)$, we may write (36) as

$$\frac{d}{dt}\psi_n(t) \geq \frac{2\mu}{\pi}\,\Omega_n(t). \tag{38}$$

Then for each $n \geq 1$, we have

$$\begin{aligned}
\frac{d}{dt}\Omega_n(t) &= \int_{\Phi_n(t)}^{L/2} \omega(y,t)\big[u(y,t)\cot(\mu y)\big]_y\,dy + \int_{\Phi_n(t)}^{L/2} \theta_y(y,t)\cot(\mu y)\,dy \\
&\geq \int_{\Phi_n(t)}^{\Phi_{n-1}(t)} \theta_y(y,t)\cot(\mu y)\,dy \\
&\geq \cot(\mu\Phi_{n-1}(t))\big[\theta_0(x_{n-1}) - \theta_0(x_n)\big] = 2^{-(n+2)}A\cot(\mu\Phi_{n-1}(t)),
\end{aligned}$$

where in the second step we have used Lemma 4.2 and the fact that $\theta_x \geq 0$ on $[0, \frac{1}{2}L]$. To find a lower bound for the right hand side, note that for any fixed $z \in (0, \frac{1}{2}\pi)$, there exists some constant $c > 0$ depending only on z such that $\cot(x) > cx^{-1}$ for any $x \in (0, z]$. In our situation, we have $\mu\Phi_{n-1}(t) \leq \mu\Phi_0(t) \leq \mu x_0 < \frac{1}{2}\pi$, and as a result there exists some constant $c_0 > 0$ depending only on μ and x_0 such that $\cot(\mu\Phi_{n-1}(t)) \geq c_0[\Phi_{n-1}(t)]^{-1}$. This leads to the estimate

$$\frac{d}{dt}\Omega_n(t) \geq 2^{-(n+2)}c_0 A e^{\psi_{n-1}(t)}. \tag{39}$$

Once we have (38), (39), the proof of finite time blow is fairly straightforward. Details of a similar argument can be found in Choi, Kiselev, and Yao [2015]. We can choose A large enough to show inductively that $\psi_n(t_n) \geq bn + a$ for some suitably chosen $a \in \mathbb{R}$, $b > 0$ and an increasing sequence $t_n \to T < \infty$. This implies that θ has to develop a shock at $x = 0$ by time T - unless blow up happens before that in some other fashion (invalidating regularity assumptions underlying our estimates such as integration by parts). In fact, the informal flavor of bounds (38), (39) is that of $F'' \geq ce^{cF}$ differential inequality, leading to dramatically fast growth.

5 The SQG patch problem: a blow up blueprint

Part of the difficulty in securing rigorous understanding of the Hou-Luo blow up scenario for 3D axi-symmetric Euler or 2D inviscid Boussinesq equation lies in growth of ω, which destroys the estimate of error terms in Lemma 3.3. The error terms may no longer be of smaller order than the main term. In fact, heuristic computations taking ω that behaves approximately like some inverse power of x_1 in a certain region near origin - an ansatz that appears to be in agreement with the numerical simulations - indicate that the error terms will now be of the same order as the main term obtained by integration over the bulk. In this section, we discuss a different setting in which this situation is the case - the portion of the Biot-Savart integral pushing the solution towards blow up has the same order as the part pushing in the opposing direction. Nevertheless, the conclusion is finite time blow, essentially due to presence of a parameter that can be used to overcome the error term. The setting is that of modified SQG patch solutions in the half-plane.

Namely, let us in this section set $D = \mathbb{R}^2_+ = \{(x_1, x_2) \,|\, x_2 \geq 0\}$. The Bio-Savart law for the patch evolution on the half-plane $D := \mathbb{R} \times \mathbb{R}^+$ is

$$u = \nabla^\perp (-\Delta_D)^{-1+\alpha} \omega,$$

with Δ_D being the Dirichlet Laplacian on D, which can also be written as

$$u(x,t) := \int_D \left(\frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-\bar{y})^\perp}{|x-\bar{y}|^{2+2\alpha}} \right) \omega(y,t)dy \tag{40}$$

The case $\alpha = 0$ corresponds to the 2D Euler equation, while $\alpha = 1/2$ to the SQG equation; the range $0 < \alpha < 1$ is called modified SQG. Note that u is divergence free and tangential to the boundary. A traditional approach to the 2D Euler ($\alpha = 0$) vortex patch evolution, going back to Yudovich (see Marchioro and Pulvirenti [1994] for an exposition) is via the corresponding flow map. The active scalar ω is advected by u from Equation (40) via

$$\omega(x,t) = \omega\left(\Phi_t^{-1}(x), 0\right), \tag{41}$$

where

$$\frac{d}{dt}\Phi_t(x) = u\left(\Phi_t(x), t\right) \qquad \text{and} \qquad \Phi_0(x) = x. \tag{42}$$

The initial condition ω_0 for (40)-(42) is patch-like,

$$\omega_0 = \sum_{k=1}^{N} \theta_k \chi_{\Omega_{0k}}, \tag{43}$$

with $\theta_1, \dots, \theta_N \neq 0$ and $\Omega_{01}, \dots, \Omega_{0N} \subseteq D$ bounded open sets, whose closures $\overline{\Omega_{0,k}}$ are pairwise disjoint and whose boundaries $\partial\Omega_{0k}$ are simple closed curves. The question of regularity of solution in patch setting becomes the question of the conservation of regularity class of the patch boundary, as well as lack of self-intersection or collisions between different patches.

One reason the Yudovich theory works for the 2D Euler equations is that for ω which is (uniformly in time) in $L^1 \cap L^\infty$, the velocity field u given by Equation (40) with $\alpha = 0$ is log-Lipschitz in space, and the flow map Φ_t is everywhere well-defined (see e.g. Majda and Bertozzi [2002] and Marchioro and Pulvirenti [1994]). In our situation, when ω is a patch solution and $\alpha > 0$, the flow u from Equation (40) is smooth away from the patch boundaries $\partial\Omega_k(t)$ but is only $1-\alpha$ Hölder constinuous at $\partial\Omega_k(t)$, which is exactly where one needs to use the flow map. This creates significant technical difficulties in proving local well-posedness of patch evolution in some reasonable functional space. For the case without boundaries, local well-posedness has been proved in Rodrigo [2005] for C^∞ patches and for Sobolev H^3 patches in Gancedo [2008] for $0 < \alpha \leq 1$. A naive intuition on why patch evolution can be locally well-posed for $\alpha > 0$ is that the below-Lipschitz loss of regularity only affects the tangential component of the fluid velocity at patch boundary. The normal to patch component, that intuitively should determine the evolution of the patch, retains stronger regularity.

In presence of boundaries, the problem is harder. Intuitively, one reason for the difficulties can be explained as follows. In the simplest case of half-plane the reflection principle implies that the boundary can be replaced by a reflected patch (or patches) of the opposite sign. If the patch is touching the boundary, then the reflected and original patch are touching each other, and the low regularity tangential component of the velocity field generated by the reflected patch has strong influence on the boundary of the original patch near touch points. Even in the 2D Euler case, the global regularity for patches in general domains with boundaries is currently open (partial results for patches not touching the boundary or with loss of regularity can be found in Depauw [1999], Dutrifoy [2003]). In the half-plane, a global regularity result has been recently established in Kiselev, Ryzhik, Yao, and Zlatoš [2016]:

Theorem 5.1. *Let $\alpha = 0$ and $\gamma \in (0, 1]$. Then for each $C^{1,\gamma}$ patch-like initial data ω_0, there exists a unique global $C^{1,\gamma}$ patch solution ω to (41), (40), (42) with $\omega(\cdot, 0) = \omega_0$.*

In the case $\frac{1}{24} > \alpha > 0$ with boundary, even local well-posedness results are highly non-trivial. The following result has been proved in Kiselev, Yao, and Zlatoš [2017] for the half-plane.

Theorem 5.2. *If $\alpha \in (0, \frac{1}{24})$, then for each H^3 patch-like initial data ω_0, there exists a unique local H^3 patch solution ω with $\omega(\cdot, 0) = \omega_0$. Moreover, if the maximal time T_ω of existence of ω is finite, then at T_ω a singularity forms: either two patches touch, or a patch boundary touches itself or loses H^3 regularity.*

We note that one has to be careful in the definition of solutions in this case as trajectories (42) may not be unique. Solutions can be defined in a weak sense by pairing with a test function, or in an appealing geometric way by specifying evolution of patch boundary with velocity (40) in the sense of Hausdorff distance; see Kiselev, Yao, and Zlatoš [ibid.] for more details. The constraint $\alpha < \frac{1}{24}$ appears due to estimates near boundary; it is not clear if it is sharp.

On the other hand, in Kiselev, Ryzhik, Yao, and Zlatoš [2016], it was proved that for any $\alpha > 0$, there exist patch-like initial data leading to finite time blow up.

Theorem 5.3. *Let $\alpha \in (0, \frac{1}{24})$. Then there are H^3 patch-like initial data ω_0 for which the unique local H^3 patch solution ω with $\omega(\cdot, 0) = \omega_0$ becomes singular in finite time (i.e., its maximal time of existence T_ω is finite).*

Together, Theorems 5.1 and 5.3 give rigorous meaning to calling the 2D Euler equation critical. In the half-plane patch framework $\alpha = 0$ is the exact threshold for phase transition from global regularity to possibility of finite time blow up.

In what follows, we will sketch proof of the blow up Theorem Theorem 5.3. We concentrate on the main ideas only; full details can be found in Kiselev, Ryzhik, Yao, and Zlatoš [ibid.]. Let us describe the initial data set up.Denote $\Omega_1 := (\varepsilon, 4) \times (0, 4)$, $\Omega_2 := (2\varepsilon, 3) \times (0, 3)$, and let $\Omega_0 \subseteq D^+ \equiv \mathbb{R}^+ \times \mathbb{R}^+$ be an open set whose boundary is a smooth simple closed curve and which satisfies $\Omega_2 \subseteq \Omega_0 \subseteq \Omega_1$. Here ϵ is a small parameter depending on α that will be chosen later.

Let $\omega(x, t)$ be the unique H^3 patch solution corresponding to the initial data

$$\omega(x, 0) := \chi_{\Omega_0}(x) - \chi_{\tilde{\Omega}_0}(x) \tag{44}$$

with maximal time of existence $T_\omega > 0$. Here, $\tilde{\Omega}_0$ is the reflection of Ω_0 with respect to the x_2-axis. Then

$$\omega(x, t) = \chi_{\Omega(t)}(x) - \chi_{\tilde{\Omega}(t)}(x) \tag{45}$$

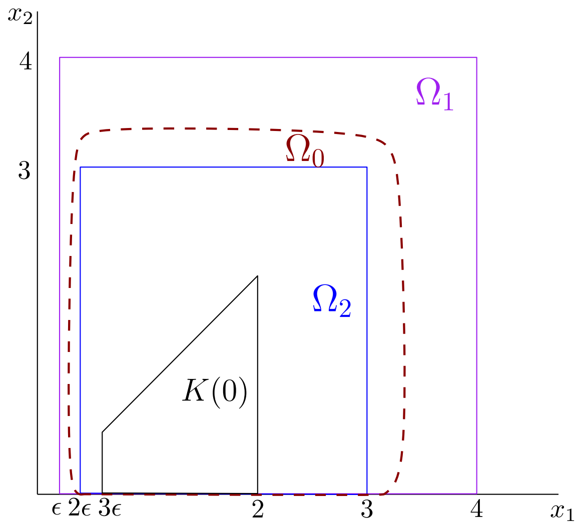

Figure 4: The domains $\Omega_1, \Omega_2, \Omega_0$, and $K(0)$ (with $\omega_0 = \chi_{\Omega_0} - \chi_{\tilde{\Omega}_0}$).

for $t \in [0, T_\omega)$, with $\Omega(t) := \Phi_t(\Omega_0)$. It can be seen from (40) that the rightmost point of the left patch on the x_1-axis and the leftmost point of the right patch on the x_1-axis will move toward each other. In the case of the 2D Euler equations $\alpha = 0$, Theorem 5.1 shows that the two points never reach the origin. When $\alpha > 0$ is small, however, it is possible to control the evolution sufficiently well to show that — unless the solution develops another singularity earlier — both points will reach the origin in a finite time. The argument yielding such control is fairly subtle, and the estimates do not extend to all $\alpha < \frac{1}{2}$, even though one would expect singularity formation to persist for more singular equations. This situation is not uncommon in the field: there is plenty of examples with the infinite in time growth of derivatives for the smooth solutions of 2D Euler equation, while none are available for the more singular SQG equation Kiselev and Nazarov [2012].

To show finite time blow up, we will deploy a barrier argument. Define

$$K(t) := \{x \in D^+ \ : \ x_1 \in (X(t), 2) \text{ and } x_2 \in (0, x_1)\} \tag{46}$$

for $t \in [0, T]$, with $X(0) = 3\epsilon$. Clearly, $K(0) \subset \Omega(0)$. Set the evolution of the barrier by

$$X'(t) = -\frac{1}{100\alpha} X(t)^{1-2\alpha}. \tag{47}$$

Then $X(T) = 0$ for $T = 50(3\epsilon)^{2\alpha}$. So if we can show that $K(t)$ stays inside $\Omega(t)$ while the patch solution stays regular, then we obtain that singularity must form by time T : the different patch components will touch at the origin by this time unless regularity is lost before that.

The key step in the proof involves estimates of the velocity near origin. In particular, u_1 needs to be sufficiently negative to exceed the barrier speed (47); u_2 needs to be sufficiently positive in order to ensure that $\Omega(t)$ cannot cross the barrier along its diagonal part. Note that it suffices to consider the part of the barrier that is very close to the origin, on the order $\sim \epsilon^{2\alpha}$. Indeed, the time T of barrier arrival at the origin has this order, and the fluid velocity satisfies uniform L^∞ bound that follows by a simple estimate which uses only $\alpha < 1/2$. Thus the patch $\Omega(t)$ has no time to reach more distant boundary points of the barrier before formation of singularity.

Let us focus on the estimates for u_1. For $y = (y_1, y_2) \in \bar{D}^+ = \mathbb{R}^+ \times \mathbb{R}^+$, we denote $\bar{y} := (y_1, -y_2)$ and $\tilde{y} := (-y_1, y_2)$. Due to odd symmetry, (40) becomes (we drop t from the notation in this sub-section)

$$u_1(x) = -\int_{D^+} K_1(x, y)\omega(y)dy, \tag{48}$$

where

$$K_1(x, y) = \underbrace{\frac{y_2 - x_2}{|x - y|^{2+2\alpha}}}_{K_{11}(x,y)} - \underbrace{\frac{y_2 - x_2}{|x - \tilde{y}|^{2+2\alpha}}}_{K_{12}(x,y)} - \underbrace{\frac{y_2 + x_2}{|x + y|^{2+2\alpha}}}_{K_{13}(x,y)} + \underbrace{\frac{y_2 + x_2}{|x - \bar{y}|^{2+2\alpha}}}_{K_{14}(x,y)}, \tag{49}$$

Analyzing (49), it is not hard to see that we can split the region of integration in the Biot-Savart law according to whether it helps or opposes the bounds we seek. Define

$$u_1^{bad}(x) := -\int_{\mathbb{R}^+ \times (0, x_2)} K_1(x, y)\omega(y)dy \tag{50}$$

$$u_1^{good}(x) := -\int_{\mathbb{R}^+ \times (x_2, \infty)} K_1(x, y)\omega(y)dy. \tag{51}$$

The following two lemmas contain key estimates.

Lemma 5.4 (Bad part). *Let $\alpha \in (0, \frac{1}{2})$ and assume that ω is odd in x_1 and $0 \le \omega \le 1$ on D^+. If $x \in \overline{D^+}$ and $x_2 \le x_1$, then*

$$u_1^{bad}(x) \le \frac{1}{\alpha}\left(\frac{1}{1 - 2\alpha} - 2^{-\alpha}\right) x_1^{1-2\alpha}. \tag{52}$$

The proof of this lemma uses (49) and after cancellations leads to the bound

$$u_1^{bad}(x) \le -\int_{(0,2x_1) \times (0, x_2)} \frac{y_2 - x_2}{|x - y|^{2+2\alpha}} dy, \tag{53}$$

which gives (52)

In the estimate of the good part, we need to use a lower bound on ω that will be provided by the barrier. Define

$$A(x) := \{y \,:\, y_1 \in (x_1, x_1+1) \text{ and } y_2 \in (x_2, x_2 + y_1 - x_1)\}. \tag{54}$$

Lemma 5.5 (Good part). *Let $\alpha \in (0, \frac{1}{2})$ and assume that ω is odd in x_1 and for some $x \in \overline{D^+}$ we have $\omega \geq \chi_{A(x)}$ on D^+, with $A(x)$ from (54). There exists $\delta_\alpha \in (0,1)$, depending only on α, such that the following holds.*

If $x_1 \leq \delta_\alpha$, then

$$u_1^{good}(x) \leq -\frac{1}{6\cdot 20^\alpha \alpha} x_1^{1-2\alpha}.$$

Here analysis of (49) leads to

$$u_1^{good}(x) \leq \underbrace{-\int_{A_1} \frac{y_2 - x_2}{|x-y|^{2+2\alpha}}\,dy}_{T_1} + \underbrace{\int_{A_2} \frac{y_2 - x_2}{|x-y|^{2+2\alpha}}\,dy}_{T_2},$$

with the domains

$$\begin{aligned} A_1 &:= \{y \,:\, y_2 \in (x_2, x_2+1) \text{ and } y_1 \in (x_1 + y_2 - x_2, 3x_1 + y_2 - x_2)\}, \\ A_2 &:= (x_1+1, 3x_1+1) \times (x_2, x_2+1). \end{aligned}$$

The term T_2 can be estimated by Cx_1, since the region of integration A_2 lies at a distance ~ 1 from the singularity. A relatively direct estimate of the term T_1 leads to the result of the Lemma.

A distinctive feature of the problem is that estimates for the "bad" and "good" terms appearing in Lemmas 5.4 and 5.5 above have the same order of magnitude $x_1^{1-2\alpha}$. This is unlike the 2D Euler double exponential growth construction, where we were able to isolate the main term. To understand the balance in the estimates for the "bad" and "good" terms, note that the "bad" term estimate comes from integration of the Biot-Savart kernel over rectangle $(0, 2x_1) \times (0, x_2)$, while the good term estimate from integration of the same kernel over the region A_1 above. When α is close to zero, the kernel is longer range, and the more extended nature of the region A_1 makes the "good" term dominate. In particular, the coefficient $\frac{1}{\alpha}\left(\frac{1}{1-2\alpha} - 2^{-\alpha}\right)$ in front of $x_1^{1-2\alpha}$ in Lemma 5.4 converges to to finite limit as $\alpha \to 0$, while the coefficient $\frac{1}{6\cdot 20^\alpha \alpha}$ in Lemma 5.5 tends to infinity. On the other hand, when $\alpha \to \frac{1}{2}$, the singularity in the Biot-Savart kernel is strong and getting close to non-integrable. Then it becomes important that the "bad" term integration region contains an angle π range near the singularity, while the "good" region only $\frac{\pi}{4}$. For this reason, controlling the "bad" term for larger values of α is problematic - although there is no reason why there cannot be a different, more clever argument achieving this goal.

It is straightforward to check that the dominance of the "good" term over "bad" one extends to the range $\alpha \in (0, \frac{1}{24})$, and that in this range we get as a result

(55) $$u_1(x,t) \leq -\frac{1}{50\alpha} x_1^{1-2\alpha}$$

for $x = (x_1, x_2)$ such that $x_1 \leq \delta_\alpha$ and $x_1 \geq x_2$. A similar bound can be proved showing that

(56) $$u_2(x,t) \geq \frac{1}{50\alpha} x_2^{1-2\alpha}$$

for $x = (x_1, x_2)$ such that $x_2 \leq \delta_\alpha$ and $x_1 \leq x_2$.

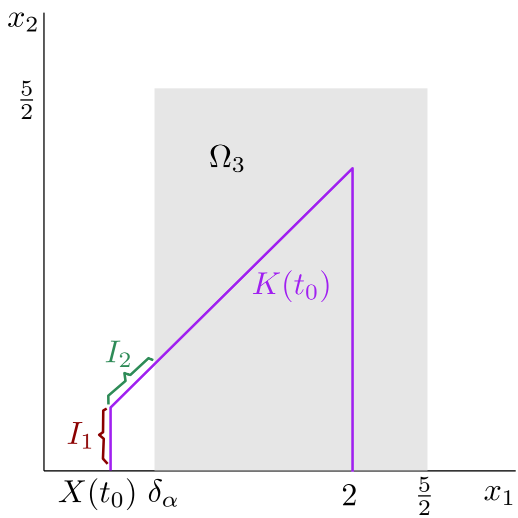

Figure 5: The segments I_1 and I_2 and the sets Ω_3 and $K(t_0)$.

The proof is completed by a contradiction argument, where we assume that the barrier $K(t)$ catches up with the patch $\Omega(t)$ at some time $t = t_0 < T$ of first contact. Taking ϵ sufficiently small compared to δ_α from Lemma 5.5, we can make sure the contact can only happen on the intervals I_1 and I_2 along the boundary of the barrier $K(t_0)$ appearing on Figure 5. But then bounds (55), (56) and the evolution of the barrier prescription (47) lead to the conclusion that the barrier should have been crossed at $t < t_0$, yielding a contradiction; full details can be found in Kiselev, Ryzhik, Yao, and Zlatoš [2016]).

6 Discussion

There are a few more recent papers that have contributed towards understanding the hyperbolic point blow up scenario. Two-dimensional simplified models of the 2D Boussinesq system have been considered in Hoang, Orcan-Ekmekci, Radosz, and Yang [n.d.] and in Kiselev and Tan [2018]. In both cases, the derivative forcing term in (6) is replaced by a simpler sign-definite approximation $\frac{\theta}{x_1}$, and the Biot-Savart law is replaced by a simpler version $u = (-x_1\Omega(x,t), x_2\Omega(x,t))$. In Hoang, Orcan-Ekmekci, Radosz, and Yang [n.d.], Ω takes form similar to the 2D Euler example (13). In Kiselev and Tan [2018], Ω is closely related but is also chosen to keep u incompressible. Both papers prove finite time blow up, Hoang, Orcan-Ekmekci, Radosz, and Yang [n.d.] by a sort of barrier argument while the argument Kiselev and Tan [2018] deploys an appropriate Lyapunov-type functional.

A very interesting recent work by Elgindi and Jeong takes a different approach Elgindi and Jeong [n.d.(b)], Elgindi and Jeong [n.d.(a)]. In Elgindi and Jeong [n.d.(b)], they look at a class of scale invariant solutions for the 2D Boussinesq system that satisfy $\frac{1}{\lambda}u(\lambda x,t) = u(x,t)$ and $\frac{1}{\lambda}\theta(\lambda x,t) = \theta(x,t)$. Observe that this class allows velocity and density that grow linearly at infinity. Also, the solution is not regular at the origin: for example the vorticity is just L^∞. The setting is a sector which has size $\frac{\pi}{2}$ (and some results can be generalized to other angles $< \pi$). First, they prove a local well-posedness theorem in a class of solutions that includes scale invariant solutions; additional symmetry assumptions are needed for this result. Secondly, for such solutions, they obtain an effective one-dimensional equation, some solutions of which are shown to blow up in finite time. These are not the first examples of infinite energy solutions (see Childress, Ierley, Spiegel, and Young [1989], Constantin [2000], and Sarria and Wu [2015]). However in Elgindi and Jeong [n.d.(b)] a procedure to cut off the solution at infinity while maintaining finite time blow up property is carried out. This yields finite energy solutions leading to finite time blow up - in the sense that $\int_0^T \|\nabla u(\cdot,t)\|_{L^\infty}\,dt \to \infty$ at blow up time. The vertex of the sector is a hyperbolic stagnation point of the flow, making connection to the Hou-Luo scenario. In Elgindi and Jeong [n.d.(a)], related results are announced and partly proved in the 3D axi-symmetric Euler case; here the domain is given by $z^2 \le c(|x|^2 + |y|^2)$ with a sufficiently small c. The finite time singularity formation in this setting remains open as the effective one-dimensional system turns out to be more complex.

The main challenge to analyzing smooth solutions to 2D Boussinesq and 3D axi-symmetric Euler equations in this context stems from difficulties estimating the velocity produced by Biot-Savart law with growing vorticity. It does not appear that there is a clear main term, as in 2D Euler example, or a clear small parameter to play in the same order of magnitude opposing terms, as in modified SQG patches. On the other hand, all model problems point to finite time blow up outcome in the original Hou-Luo scenario. The challenge is finding enough controllable structures to carry through rigorous analysis.

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Received 2017-11-16.

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