

SPECIAL GEOMETRY ON CALABI–YAU MODULI SPACES AND Q -INVARIANT MILNOR RINGS

ALEXANDER BELAVIN

Abstract

The moduli spaces of Calabi–Yau (CY) manifolds are the special Kähler manifolds. The special Kähler geometry determines the low-energy effective theory which arises in Superstring theory after the compactification on a CY manifold. For the cases, where the CY manifold is given as a hypersurface in the weighted projective space, a new procedure for computing the Kähler potential of the moduli space has been proposed by Konstantin Aleshkin and myself. The method is based on the fact that the moduli space of CY manifolds is a marginal subspace of the Frobenius manifold which arises on the deformation space of the corresponding Landau–Ginzburg superpotential. I review this approach and demonstrate its efficiency by computing the Special geometry of the 101-dimensional moduli space of the quintic threefold around the orbifold point.

1 Introduction

To compute the low-energy Lagrangian of the string theory compactified on a CY manifold [Candelas, Horowitz, Strominger, and Witten \[1985\]](#), one needs to know the Special geometry of the corresponding CY moduli space [Candelas, Green, and Hübsch \[1989\]](#), [Strominger \[1990\]](#), [Candelas, Green, and Hübsch \[1990\]](#), and [Candelas and de la Ossa \[1991\]](#).

More precisely, the effective Lagrangian of the vector multiplets in the superspace contains $h^{2,1}$ supermultiplets. Scalars from these multiplets take value in the target space \mathcal{M} , which is a moduli space of complex structures on a CY manifold and is a special Kähler manifold. Metric $G_{a\bar{b}}$ and Yukawa couplings κ_{abc} on this space are given by the following

formulae in the special coordinates z^a :

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K, \quad e^{-K} = -i \int_X \Omega \wedge \bar{\Omega},$$

$$\kappa_{abc} = \int_X \Omega \wedge \partial_a \partial_{\bar{b}} \partial_{\bar{c}} \Omega = \frac{\partial^3 F}{\partial z^a \partial z^b \partial z^c},$$

where

$$z^a = \int_{A_a} \Omega, \quad \frac{\partial F}{\partial z^a} = \int_{B^a} \Omega$$

are the period integrals of the holomorphic volume form Ω on X . Here A_a and B^a form the symplectic basis in $H_3(X, \mathbb{Z})$.

We can rewrite the expression (1) for the Kähler potential using the periods as

$$e^{-K} = -i \Pi \Sigma \Pi^\dagger, \quad \Pi = (\partial F, z),$$

where matrix $(\Sigma)^{-1}$ is an intersection matrix of cycles A_a, B^a equal to the symplectic unit.

The computation of periods in the symplectic basis appears to be very non-trivial. It was firstly performed for the case of the quintic CY manifold in the distinguished paper [Candelas, de la Ossa, Green, and Parkes \[1991\]](#).

Here I present an alternative approach to the computation of Kähler potential for the case where CY manifold is given by a hypersurface $W(x, \phi) = 0$ in a weighted projective space. The approach is based on the connection of CY manifold with a Frobenius ring which arises on the deformations of the singularity defined by the superpotential $W_0(x)$ [Lerche, Vafa, and N. P. Warner \[1989\]](#), [Martinec \[1989\]](#), and [Vafa and N. Warner \[1989\]](#).

Let a CY manifold X be given as a solution of an equation

$$W(x, \phi) = W_0(x) + \sum_{s=1}^{h^{2,1}} \phi_s e_s(x) = 0$$

in some weighted projective space, where $W_0(x)$ is a quasihomogeneous function in \mathbb{C}^5 of degree d that defines an isolated singularity at $x = 0$. The monomials $e_s(x)$ also have degree d and are in a correspondence to deformations of the complex structure of X .

Polynomial $W_0(x)$ defines a Milnor ring R_0 . Inside R_0 there exists a subring R_0^Q which is invariant under the action of the so-called quantum symmetry group Q that acts on \mathbb{C}^5 diagonally, and preserves $W(x, \phi)$. In many cases $\dim R_0^Q = \dim H^3(X)$ and the ring

itself has a Hodge structure $R_0^Q = (R_0^Q)^0 \oplus (R_0^Q)^1 \oplus (R_0^Q)^2 \oplus (R_0^Q)^3$ in correspondence with the elements of degrees $0, d, 2d, 3d$.

Another important group is the subgroup of phase symmetries G , which acts diagonally on \mathbb{C}^5 , commutes with the quantum symmetry Q and preserves $W_0(x)$. It acts naturally on the invariant ring R_0^Q , and this action respects the Hodge decomposition of R_0^Q . This allows to choose a basis $e_\mu(x)$ in each of the Hodge decomposition components of R_0^Q to be eigenvectors for the G group action.

On the ring R_0^Q we introduce the invariant pairing η . The pairing turns the ring to a Frobenius algebra [Dubrovin \[1992\]](#). The pairing η plays an important role for our construction of the explicit expression for the volume of the Calabi-Yau manifold.

Using the invariant ring R_0^Q and differentials $D_\pm = d \pm dW_0 \wedge$ we construct two Q -invariant cohomology groups $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$. These groups inherit the Hodge structure from R_0^Q . We can choose in $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$ the eigenbases $e_\mu(x) d^5x$ which are also invariant under the phase symmetry action.

As shown in [Candelas \[1988\]](#), elements of these cohomology groups are in correspondence with the harmonic forms of $H^3(X)$. This isomorphism allows to define the antilinear involution $*$ on the invariant cohomology $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$ that corresponds to the complex conjugation on the space of the harmonic forms of $H^3(X)$.

It turns out, that in the basis $e_\mu(x)$ it reads

$$*e_\mu(x) d^5x = M_\mu^\nu e_\nu(x) d^5x, \quad M_\mu^\nu = \delta_{e_\mu \cdot e_\nu, e_\rho} A^\mu$$

where $e_\rho(x)$ is the unique element of degree $3d$ in R_0^Q , and $\delta_{e_\mu \cdot e_\nu, e_\rho}$ is 1 if $e_\mu \cdot e_\nu = e_\rho$ and 0 otherwise.

Having $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$ we define the relative invariant homology subgroups $\mathcal{H}_5^{\pm, inv} := H_5(\mathbb{C}^5, W_0 = L, \text{Re}L \rightarrow \pm\infty)_{inv}$ inside the relative homology groups $H_5(\mathbb{C}^5, W_0 = L, \text{Re}L \rightarrow \pm\infty)$. To do this we will use the oscillatory integrals and their pairing with elements of $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$. Using this pairing we define a cycle Γ_μ^\pm in the basis of relative invariant homology to be dual to $e_\mu(x) d^5x$.

At last we define periods $\sigma_\mu^\pm(\phi)$ to be oscillatory integrals over the basis of cycles Γ_μ^\pm . They are equal to periods of the holomorphic volume form Ω on X in a special basis of cycles of $H_3(X, \mathbb{C})$ with complex coefficients.

It follows from the phase symmetry invariance that in the chosen basis of cycles Γ_μ^\pm the formula for Kähler potential has the diagonal form:

$$e^{-K(\phi)} = \sum_\mu (-1)^{|\nu|} \sigma_\mu^+(\phi) A^\mu \overline{\sigma_\mu^-(\phi)}.$$

On the other hand, as shown in [Aleshkin and A. Belavin \[n.d.\(a\)\]](#), matrix $A = \text{diag}\{A^\mu\}$ is equal to the product of the matrix of the invariant pairing η in the Frobenius algebra $R_0^{\mathcal{Q}}$ and the real structure matrix M such that

$$e^{-K(\phi)} = \sum_{\mu, \nu} \sigma_\mu^+(\phi) \eta^{\mu\lambda} M_\lambda^\nu \overline{\sigma_\mu^-(\phi)}.$$

The real structure matrix is nothing but matrix M from (1). Using this we are able to explicitly compute the diagonal matrix elements A^μ and to obtain the explicit expression for the whole e^{-K} .

2 The special geometry on the CY moduli space

It was shown in in [Strominger \[1990\]](#), [Candelas, Green, and Hübsch \[1989, 1990\]](#), and [Candelas and de la Ossa \[1991\]](#) that the moduli space \mathcal{M} of complex (or Kähler) structures of a given CY manifold is a special Kähler manifold.

Namely on \mathcal{M} there exist so-called special (projective) coordinates $z^1 \dots z^{n+1}$ and a holomorphic homogeneous function $F(z)$ of degree 2 in z , called a prepotential, such that the Kähler potential $K(z)$ of the moduli space metric is given by

$$e^{-K(z)} = \int_X \Omega \wedge \bar{\Omega} = z^a \cdot \frac{\partial \bar{F}}{\partial \bar{z}^a} - \bar{z}^a \cdot \frac{\partial F}{\partial z^a}$$

To obtain this formula, we choose Poincare dual symplectic bases $\alpha_a, \beta^b \in H^3(X, \mathbb{Z})$ and $A^a, B_b \in H_3(X, \mathbb{Z})$ and define the periods as

$$z^a = \int_{A^a} \Omega, \quad F_b = \int_{B_b} \Omega.$$

Then using the Kodaira Lemma

$$\partial_a \Omega = k_a \Omega + \chi_a,$$

we can show that

$$F_a(z) = \frac{1}{2} \partial_a(F(z)),$$

where $F(z) = 1/2 z^b F_b(z)$.

Therefore, according to the definition (2) metric $G_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K(z)$ is a special Kähler metric with prepotential $F(z)$ and with the special coordinates given by the period vector

$$\Pi = (F_a, z^b)$$

we write the expression for the Kähler potential as

$$e^{-K(z)} = \Pi_\mu \Sigma^{\mu\nu} \bar{\Pi}_\nu,$$

where Σ is a symplectic unit, which is an inverse intersection matrix for cycles A^a and B_b .

Using formula (2), we can rewrite this expression in a basis of periods defined as integrals over arbitrary basis of cycles $q_\mu \in H_3(X, \mathbb{Z})$

$$\omega_\mu = \int_{q_\mu} \Omega.$$

Such that

$$e^{-K} = \omega_\mu C^{\mu\nu} \bar{\omega}_\nu,$$

where $C^{\mu\nu}$ is the inverse matrix of the intersection of the cycles q_μ .

So to find the Kähler potential, we must compute the periods over a basis of cycles on CY manifold and find their intersection matrix.

3 Hodge structure on the middle cohomology of the quintic

Now let us specialize to the case where X is a quintic threefold:

$$X = \{(x_1 : \dots : x_5) \in \mathbb{P}^4 \mid W(x, \phi) = 0\},$$

and

$$W(x, \phi) = W_0(x) + \sum_{t=0}^{100} \phi_t e_t(x), \quad W_0(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$$

and $e_t(x)$ are the degree 5 monomials such that each variable has the power that is a non-negative integer less than four. Let us denote monomials $e_t(x) = x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4} x_5^{t_5}$ by its

degree vector $t = (t_1, \dots, t_5)$. Then there are precisely 101 of such monomials, which can be divided into 5 sets in respect to the permutation group S_5 : $(1, 1, 1, 1, 1)$, $(2, 1, 1, 1, 0)$, $(2, 2, 1, 0, 0)$, $(3, 1, 1, 0, 0)$, $(3, 2, 0, 0, 0)$. In these groups there are correspondingly 1, 20, 30, 30, 20 different monomials. We denote $e_0(x) := e_{(1,1,1,1,1)}(x) = x_1x_2x_3x_4x_5$ to be the so-called fundamental monomial, which will be somewhat distinguished in our picture.

For this CY $\dim H_3(X) = 204$ and period integrals have the form

$$\omega_\mu(x) = \int_{q_\mu} \frac{x_5 dx_1 dx_2 dx_3}{\partial W(x, \phi) / \partial x_4} = \int_{Q_\mu} \frac{dx_1 \cdots dx_5}{W(x, \phi)},$$

where $q_\mu \in H_3(X, \mathbb{Z})$ and the corresponding cycles $Q_\mu \in H_5(\mathbb{C}^5 \setminus (W(x, \phi) = 0), \mathbb{Z})$.

Cohomology groups of the Kähler manifold X possess a Hodge structure $H^3(X) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$. Period integrals measure variation of the Hodge structure on $H^3(X)$ as the complex structure on X varies with ϕ .

This Hodge structure variation is in correspondence with a Frobenius ring which we will now describe.

4 Hodge structure on the invariant Milnor ring

Now we will consider $W_0(x)$ as an isolated singularity in \mathbb{C}^5 and the associated with it Milnor ring

$$R_0 = \frac{\mathbb{C}[x_1, \dots, x_5]}{\langle \partial_i W_0 \rangle}.$$

We can choose its elements to be unique smallest degree polynomial representatives. For the quintic threefold X its Milnor ring R_0 is generated as a vector space by monomials where each variable has degree less than four, and $\dim R_0 = 1024$.

Since the polynomial $W_0(x)$ is homogeneous one of the fifth degree it follows that $W_0(\alpha x_1, \dots, \alpha x_5) = W_0(x_1, \dots, x_5)$ for $\alpha^5 = 1$. This action preserves $W_0(x)$ and is trivial in the corresponding projective space and on X . Such a group with this action is called a *quantum symmetry* Q , in our case $Q \simeq \mathbb{Z}_5$. Q obviously acts on the Milnor ring R_0 .

We define a subring R_0^Q to be a Q -invariant part of the Milnor ring

$$R_0^Q := \{e_\mu(x) \in R_0 \mid e_\mu(\alpha x) = e_\mu(x)\}, \alpha^5 = 1.$$

$R_0^{\mathcal{Q}}$ is multiplicatively generated by 101 fifth-degree monomials $e_r(x)$ from (3) and consists of elements of degree 0, 5, 10 and 15. The dimensions of the corresponding subspaces are 1, 101, 101 and 1.

This degree filtration defines a Hodge structure on $R_0^{\mathcal{Q}}$. Actually, $R_0^{\mathcal{Q}}$ is isomorphic to $H^3(X)$ and this isomorphism sends the degree filtration on $R_0^{\mathcal{Q}}$ to the Hodge filtration on $H^3(X)$ [Candelas \[1988\]](#).

Let us denote $\chi_{\bar{j}}^i = g^{i\bar{k}} \chi_{\bar{k}\bar{j}}$ as an extrinsic curvature tensor and $g_{i\bar{k}}$ is a metric for the hypersurface $W(x, \phi) = 0$ in \mathbb{P}^4 . Then the isomorphism above can be written as a map from $R_0^{\mathcal{Q}}$ to closed differential forms in $H^3(X)$:

$$\begin{aligned} 1 &\rightarrow \Omega_{ijk} \in H^{3,0}(X), \\ e_{\mu}(x) &\rightarrow e_{\mu}(x(y)) \chi_{\bar{i}}^l \Omega_{ljk} \in H^{2,1}(X) \text{ if } |\mu| = 5, \\ e_{\mu}(x) &\rightarrow e_{\mu}(x(y)) \chi_{\bar{i}}^l \chi_{\bar{j}}^m \Omega_{lmk} \in H^{1,2}(X) \text{ if } |\mu| = 10, \\ e_{\rho}(x) &= x_1^3 x_2^3 x_3^3 x_4^3 x_5^3 \rightarrow \chi_{\bar{i}}^l \chi_{\bar{j}}^m \chi_{\bar{k}}^p \Omega_{lmp} = \kappa \bar{\Omega} \in H^{0,3}(X) \end{aligned}$$

The details of this map can be found in [Candelas \[ibid.\]](#). We also introduce the notation $e_{\mu}(x)$ for elements of the monomial basis of $R_0^{\mathcal{Q}}$, where $\mu = (\mu_1, \dots, \mu_5)$, $\mu_i \in \mathbb{Z}_+^5$, $e_{\mu}(x) = \prod_i x_i^{\mu_i}$ and the degree of $e_{\mu}(x)$ $|\mu| = \sum \mu_i$ is equal to zero module 5. In particular, $\rho = (3, 3, 3, 3, 3)$, that is $e_{\rho}(x)$ is the unique degree 15 element of $R_0^{\mathcal{Q}}$.

The phase symmetry group \mathbb{Z}_5^5 acts diagonally on \mathbb{C}^5 : $\alpha \cdot (x_1, \dots, x_5) = (\alpha_1 x_1, \dots, \alpha_5 x_5)$, $\alpha_i^5 = 1$. This action preserves $W_0 = \sum_i x_i^5$. The mentioned above quantum symmetry \mathcal{Q} is a diagonal subgroup of the phase symmetries. Basis $\{e_{\mu}(x)\}$ consists of the eigenvectors of the phase symmetry and each $e_{\mu}(x)$ has a unique weight. Note that the action of the phase symmetry preserves the Hodge decomposition.

Another important fact is that on the invariant ring $R_0^{\mathcal{Q}}$ there exists a natural invariant pairing turning it into a Frobenius algebra [Dubrovin \[1992\]](#):

$$\eta_{\mu\nu} = \text{Res} \frac{e_{\mu}(x) e_{\nu}(x)}{\prod_i \partial_i W_0(x)}.$$

Up to an irrelevant constant for the monomial basis it is $\eta_{\mu\nu} = \delta_{\mu+\nu, \rho}$. This pairing plays a crucial role in our construction.

Let us introduce a couple of Saito differentials as in [Aleshkin and A. Belavin \[n.d.\(a\)\]](#) on differential forms on \mathbb{C}^5 : $D_{\pm} = d \pm dW_0(x) \wedge$. They define two cohomology groups $H_{D_{\pm}}^*(\mathbb{C}^5)$. The cohomologies are only nontrivial in the top dimension $H_{D_{\pm}}^5(\mathbb{C}^5) \stackrel{J}{\simeq} R_0$.

The isomorphism J has an explicit description

$$J(e_\mu(x)) = e_\mu(x) d^5x, \quad e_\mu(x) \in R_0.$$

We see, that $Q = \mathbb{Z}_5$ naturally acts on $H_{D_\pm}^5(\mathbb{C}^5)$ and J sends the elements of Q -invariant ring R_0^Q to Q -invariant subspace $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$. Therefore, the latter space obtains the Hodge structure as well. Actually, this Hodge structure naturally corresponds to the Hodge structure on $H^3(X)$.

The complex conjugation acts on $H^3(X)$ so that $\overline{H^{p,q}(X)} = H^{q,p}(X)$, in particular $\overline{H^{2,1}(X)} = H^{1,2}(X)$. Through the isomorphism between R_0^Q and $H^3(X)$ the complex conjugation acts also on the elements of the ring R_0^Q as $*e_\mu(x) = p_\mu e_{\rho-\mu}(x)$, where $p_\mu p_{\rho-\mu} = 1$ and p_μ is a constant to be determined. In particular, differential form built from the linear combinations $e_\mu(x) + p_\mu e_{\rho-\mu}(x) \in H^3(X, \mathbb{R})$ is real.

5 Oscillatory representation and computation periods $\sigma_\mu(\phi)$

Relative homology groups $H_5(\mathbb{C}^5, W_0 = L, \text{Re}L \rightarrow \pm\infty)$ have a natural pairing with Q -invariant cohomology groups $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$ defined as

$$\langle e_\mu(x) d^5x, \Gamma^\pm \rangle = \int_{\Gamma^\pm} e_\mu(x) e^{\mp W_0(x)} d^5x, \quad H_5(\mathbb{C}^5, W_0 = L, \text{Re}L \rightarrow \pm\infty).$$

Using this we introduce two Q -invariant homology groups¹ $\mathcal{H}_5^{\pm,inv}$ as quotient of $H_5(\mathbb{C}^5, W_0 = L, \text{Re}L \rightarrow \pm\infty)$ with respect to the subgroups orthogonal to $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$. Now we introduce bases Γ_μ^\pm in the homology groups $\mathcal{H}_5^{\pm,inv}$ using the duality with the bases in $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$:

$$\int_{\Gamma_\mu^\pm} e_\nu(x) e^{\mp W_0(x)} d^5x = \delta_{\mu\nu}$$

and the corresponding periods

$$\sigma_{\alpha\mu}^\pm(\phi) := \int_{\Gamma_\mu^\pm} e_\alpha(x) e^{\mp W(x,\phi)} d^5x,$$

$$\sigma_\mu^\pm(\phi) := \sigma_{0\mu}^\pm(\phi)$$

¹We are grateful to V. Vasiliev for explaining to us the details about these homology groups and their connection with the middle homology of X .

which are understood as series expansions in ϕ around zero.

The periods $\sigma_\mu^\pm(\phi)$ satisfy the same differential equation as periods $\omega_\mu(\phi)$ of the holomorphic volume form on X . Moreover, these sets of periods span same subspaces as functions of ϕ . Therefore we can define cycles $Q_\mu^\pm \in \mathcal{H}_5^{\pm,inv}$ such that

$$\int_{Q_\mu^\pm} e^{\mp W(x,\phi)} d^5x = \int_{q_\mu} \Omega = \int_{Q_\mu} \frac{d^5x}{W(x,\phi)}.$$

So the periods $\omega_{\alpha\mu}^\pm(\phi)$ are given by the integrals over these cycles analogous to (5).

With these notations the idea of computation of periods [A. Belavin and V. Belavin \[2016\]](#)

$$\sigma_\mu^\pm(\phi) = \int_{\Gamma_\mu^\pm} e^{\mp W(x,\phi)} d^5x$$

can be stated as follows.

To explicitly compute $\sigma_\mu^\pm(\phi)$, first we expand the exponent in the integral (5) in ϕ representing $W(x,\phi) = W_0(x) + \sum_s \phi_s e_s(x)$

$$\sigma_\mu^\pm(\phi) = \sum_m \left(\prod_s \frac{(\pm\phi_s)^{m_s}}{m_s!} \right) \int_{\Gamma_\mu^\pm} \prod_s e_s(x)^{m_s} e^{\mp W_0(x)} d^5x.$$

We note, that $\sigma_\mu^-(\phi) = (-1)^{|\mu|} \sigma_\mu^+(\phi)$, so we focus on $\sigma_\mu(\phi) := \sigma_\mu^+(\phi)$.

For each of the summands in (5) the form $\prod_s e_s(x)^{m_s} d^5x$ belongs to $H_{D_\pm}^5(\mathbb{C}^5)_{inv}$, because it is Q -invariant. Therefore, we can expand it in the basis $e_\mu(x) d^5x \in H_{D_\pm}^5(\mathbb{C}^5)_{inv}$. Namely we can find such a polynomial 4-form U , that

$$\prod_s e_s(x)^{m_s} d^5x = \sum_v C_v(m) e_v(x) d^5x + D_+ U.$$

In result we obtain for the integral in (5)

$$\int_{\Gamma_\mu^\pm} \prod_s e_s(x)^{m_s} e^{\mp W_0(x)} d^5x = C_\mu(m).$$

So from (5) we have

$$\sigma_\mu(\phi) = \sum_m \left(\prod_s \frac{\phi_s^{m_s}}{m_s!} \right) \int_{\Gamma_\mu^+} \prod_{s,i} x_i^{\sum_s m_s s_i} e^{-W_0(x)} d^5x.$$

We can rewrite the sum in the exponent of x_i as $\sum_s m_s s_i = 5n_i + \nu_i$, $\nu_i < 5$. Therefore we need to compute the coefficients c_ν^m in the equations

$$\prod x_i^{5n_i + \nu_i} d^5 x = \sum_\nu c_\nu^m e_\nu(x) d^5 x + D_+ U.$$

Note that

$$\begin{aligned} D_+ \left(\frac{1}{5} x_1^{5n+k-4} f(x_2, \dots, x_5) dx_2 \wedge \dots \wedge dx_5 \right) &= \\ &= \left[x_1^{5n+k} + \left(n + \frac{k-4}{5} \right) x_1^{5(n-1)+k} \right] f(x_2, \dots, x_5) d^5 x \end{aligned}$$

Therefore in D_+ cohomology we have

$$\prod_i x_i^{5n_i + \nu_i} d^5 x = - \left(n_1 + \frac{\nu_1 - 4}{5} \right) x_1^{5(n_1-1) + \nu_1} \prod_{i=2}^5 x_i^{5n_i + \nu_i} d^5 x, \nu_i < 5.$$

By induction we obtain

$$\prod_i x_i^{5n_i + \nu_i} d^5 x = (-1)^{\sum_i n_i} \prod_i \left(\frac{\nu_i + 1}{5} \right)_{n_i} \prod_i x_i^{\nu_i} d^5 x, \nu_i < 5.$$

where $(a)_n = \Gamma(a + n) / \Gamma(a)$.

Using (5) once again, we see that if any $\nu_i = 4$ then the differential form is trivial and the integral is zero. Hence, rhs of (5) is proportional to $e_\nu(x)$ and gives the desired expression.

Plugging (5) into (5) and integrating over Γ_μ^+ we obtain the answer

$$\sigma_\mu(\phi) = \sigma_\mu^+(\phi) = \sum_{n_i \geq 0} \prod_i \left(\frac{\mu_i + 1}{5} \right)_{n_i} \sum_{m \in \Sigma_n} \prod_s \frac{\phi_s^{m_s}}{m_s!},$$

where

$$\Sigma_n = \{ m \mid \sum_s m_s s_i = 5n_i + \mu_i \}$$

Further we will also use the periods with slightly different normalization, which turn out to be convenient

$$\hat{\sigma}_\mu(\phi) = \prod_i \Gamma\left(\frac{\mu_i + 1}{5}\right) \sigma_\mu(\phi) = \sum_{n_i \geq 0} \prod_i \Gamma\left(n_i + \frac{\mu_i + 1}{5}\right) \sum_{m \in \Sigma_n} \prod_s \frac{\phi_s^{m_s}}{m_s!}.$$

6 Computation of the Kähler potential

Pick any basis Q_μ^\pm of cycles with integer or real coefficients as in (5). Then for the Kähler potential we have the formula

$$e^{-K} = \omega_\mu^+(\phi) C^{\mu\nu} \overline{\omega_\nu^-(\phi)}$$

in which the matrix $C^{\mu\nu}$ is related with the Frobenius pairing η as

$$\eta_{\alpha\beta} = \omega_{\alpha\mu}^+(0) C^{\mu\nu} \overline{\omega_{\beta\nu}^-(0)}.$$

The derivation of the last relation is given in Cecotti and Vafa [1991] and Chiodo, Iritani, and Ruan [2014].

Let also T^\pm be the matrix that connects the cycles Q_μ^\pm and Γ_ν^\pm . That is

$$Q_\mu^\pm = (T^\pm)_\mu^\nu \Gamma_\nu^\pm$$

. Then $M = (T^-)^{-1} \overline{T^+}$ is a real structure matrix, that is $M \bar{M} = 1$ and by construction M doesn't depend on the choice of basis Q_μ^\pm . M is only defined by our choice of Γ_μ^\pm .

In Aleshkin and A. Belavin [n.d.(a)] we deduced from (6) and (6) the formula

$$e^{-K(\phi)} = \sigma_\mu^+(\phi) \eta^{\mu\lambda} M_\lambda^\nu \overline{\sigma_\nu^-(\phi)} = \sigma_\mu A^{\mu\nu} \overline{\sigma_\nu},$$

where $\eta^{\mu\nu} = \eta_{\mu\nu} = \delta_{\mu, \rho - \nu}$.

Now we show that the matrix $A^{\mu\nu}$ in (6) is diagonal. To do this we extend the action of the phase symmetry group to the action \mathcal{A} on the parameter space $\{\phi_s\}$ such that $W = W_0 + \sum_s \phi_s e_s(x)$ is invariant under this new action. It easy to see that each $e_s(x)$ has an unique weight under this group action. Action \mathcal{A} can be compensated using the coordinate transformation and therefore is trivial on the moduli space of the quintic (implying that point $W = W_0$ is an orbifold point of the moduli space).

In particular, $e^{-K} = \int_X \Omega \wedge \bar{\Omega}$ is \mathcal{A} invariant. Consider

$$e^{-K} = \sigma_\mu A^{\mu\nu} \overline{\sigma_\nu}$$

as a series in ϕ_s and $\overline{\phi}_t$. Each monomial has a certain weight under \mathcal{A} . For the series to be invariant, each monomial must have weight 0. But weight of $\sigma_\mu \overline{\sigma}_\nu$ equals to $\mu - \nu$ and due to non-degeneracy of weights of σ_μ only the ones with $\mu = \nu$ have weight zero. Thus, (6) becomes

$$e^{-K} = \sum_{\mu} A^{\mu} |\sigma_{\mu}(\phi)|^2.$$

Moreover, the matrix A should be real and, because $A = \eta \cdot M$, $M\overline{M} = 1$ and $\eta_{\mu\nu} = \delta_{\mu+\nu,\rho}$, we have

$$A^{\mu} A^{\rho-\mu} = 1.$$

Monodromy considerations. To fix finally the real numbers A^{μ} we use monodromy invariance of e^{-K} around $\phi_0 = \infty$. Pick some $t = (t_1, t_2, t_3, t_4, t_5)$ with $|t| = 5$ and let $\phi_s|_{s \neq t, 0} = 0$. We will consider only the first order in ϕ_t . Then the condition that period $\sigma_{\mu}(\phi)$ contains only non-zero summands of the form $\phi_0^{m_0} \phi_t$ implies that $\mu = t + const \cdot (1, 1, 1, 1, 1) \pmod{5}$. For each t from the table below the only such possibilities are $\mu = t$ and $\mu = \rho - t' = (3, 3, 3, 3, 3) - t'$, where t' denotes a vector obtained from t by permutation (written explicitly in the table below) of its coordinates.

Therefore, in this setting (6) becomes

$$e^{-K} = \sum_{k=0}^3 a_k |\hat{\sigma}_{(k,k,k,k,k)}|^2 + a_t |\hat{\sigma}_t|^2 + a_{\rho-t'} |\hat{\sigma}_{\rho-t'}|^2 + O(\phi_t^2),$$

here we use periods $\hat{\sigma}$ from (5) and denote $a_t = A^t / \prod_i \Gamma((t_i + 1)/5)^2$. And the coefficients a_k , $k = 0, 1, 2, 3$ are already known from [Candelas, de la Ossa, Green, and Parkes \[1991\]](#). This expression has to be monodromy invariant under the transport of ϕ_0 around ∞ . From the formula (5) we have

$$F_1 = \hat{\sigma}_k(\phi_t, \phi_0) = g_t \phi_k F(a, b; a + b | (\phi_0/5)^5) + O(\phi_t^6),$$

$$F_2 = \hat{\sigma}_{\rho-t'}(\phi_t, \phi_0) = g_{\rho-t'} \phi_t \phi_0^{1-a-b} F(1 - a, 1 - b; 2 - a - b | (\phi_0/5)^5) + O(\phi_t^6),$$

where $g_t, g_{\rho-t'}$ are some constants. Explicitly for all different labels t

t	$\rho - t'$	(a, b)
(2,1,1,1,0)	(3,2,2,2,1)	(2/5,2/5)
(2,2,1,0,0)	(3,3,2,1,1)	(1/5,3/5)
(3,1,1,0,0)	(0,3,3,2,2)	(1/5,2/5)
(3,2,0,0,0)	(1,0,3,3,3)	(1/5,1/5)

When ϕ_0 goes around infinity

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = B \cdot \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},$$

where

$$B = \frac{1}{is(a+b)} \begin{pmatrix} c(a-b) - e^{i\pi(a+b)} & 2s(a)s(b) \\ 2e^{2\pi i(a+b)}s(a)s(b) & e^{\pi i(a+b)}[e^{2\pi i a} + e^{2\pi i b} - 2]/2 \end{pmatrix}.$$

Here $c(x) = \cos(\pi x)$, $s(x) = \sin(\pi x)$. It is straightforward to show the following

Proposition 1.

$$a_t |\hat{\sigma}_t|^2 + a_{\rho-t'} |\hat{\sigma}_{\rho-t'}|^2 = a_t \prod_i \Gamma\left(\frac{t_i + 1}{5}\right)^2 |\sigma_t|^2 + a_{\rho-t'} \prod_i \Gamma\left(\frac{4 - t_i}{5}\right)^2 |\sigma_{\rho-t'}|^2$$

is B -invariant iff $a_t = -a_{\rho-t'}$.

Due to symmetry we have $a_{\rho-t'} = a_{\rho-t}$ in each case. From (6) it follows that the product of the coefficients at $|\sigma_\mu|^2$ and $|\sigma_{\rho-\mu}|^2$ in the expression for e^{-K} should be 1:

$$A^{\rho-t'} \cdot A^t = a_{\rho-t'} \cdot a_t \prod_i \Gamma\left(\frac{t_i + 1}{5}\right)^2 \Gamma\left(\frac{4 - t_i}{5}\right)^2 = 1.$$

Due to reflection formula $a_t = \pm \prod_i \sin(\pi(t_i + 1)/5)$ up to a common factor of π . The sign turns out to be minus for Kähler metric to be positive definite in the origin. Therefore

$$A^\mu = (-1)^{\deg(\mu)/5} \prod \gamma\left(\frac{\mu_i + 1}{5}\right).$$

Finally the Kähler potential becomes

$$e^{-K(\phi)} = \sum_{\mu=0}^{203} (-1)^{\deg(\mu)/5} \prod \gamma\left(\frac{\mu_i + 1}{5}\right) |\sigma_\mu(\phi)|^2,$$

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$.

7 Real structure on the cycles Γ_μ^\pm

Let cycles $\gamma_\mu \in H_3(X)$ be the images of cycles Γ_μ^+ under the isomorphism $\mathcal{H}_5^{+,inv} \simeq H_3(X)$.

Complex conjugation sends (2, 1)-forms to (1, 2)-forms. Similarly it extends to a mapping on the dual homology cycles γ_μ .

Lemma 1. *Conjugation of homology classes has the following form: $*\gamma_\mu = p_\mu \gamma_{\rho-\mu}$, where $\rho = (3, 3, 3, 3, 3)$ is a unique maximal degree element in the Milnor ring.*

Proof. We perform a proof for the cohomology classes represented by differential forms. For one-dimensional $H^{3,0}(X)$ and $H^{0,3}(X)$ it is obvious. Let

$$\Omega_{2,1} := e_t(x) \chi_i^l \Omega_{ljk} \in H^{2,1}(X).$$

Any element from $H^{1,2}(X)$ is representable by a degree 10 polynomial $P(x)$ as follows from (4) as

$$\overline{\Omega_{2,1}} = \Omega_{1,2} := P(x) \chi_i^l \chi_j^m \Omega_{lmk} \in H^{1,2}(X).$$

The group of phase symmetries modulo common factor acts by isomorphisms on X . Therefore, it also acts on the differential forms. Lhs and rhs of the previous equation should have the same weight under this action, and weight of the lhs is equal $-t$ modulo (1, 1, 1, 1, 1). It follows that $P(x) = p_t e_{\rho-t}(x)$ with some constant p_t . □

Using this lemma and applying the complex conjugation of cycles to the formula (6) to obtain

$$e^{-K} = \sum_\mu A^\mu |\sigma_\mu|^2 = \sum_\mu p_\mu^2 A^\mu |\sigma_{\rho-\mu}|^2,$$

it follows that $A^\mu = \pm 1/p_\mu$. Now formula (6) implies

$$p_\mu = \prod \gamma\left(\frac{4-\mu_i}{5}\right).$$

8 Conclusion

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ALEXANDER BELAVIN
L.D. LANDAU INSTITUTE FOR THEORETICAL PHYSICS
AKADEMIKA SEMENOVA AV. 1-A
CHERNOGOLOVKA, 142432 MOSCOW
RUSSIA
sashabelavin@gmail.com